Stability Theorems for a Class of Nonlinear Servomechanisms

L. MARKUS & N. WAGNER

Communicated by IRMGARD FLÜGGE-LOTZ

1. Statement of the problem. Let us consider real differential systems of the form

$$(1) \quad \dot{x}^i + f^i(x) = h^i(z, \dot{z}) \quad (i = 1, 2, \dots, n)$$

where f and h are smooth functions and the vector $z(t)$ is selected from some known class I of input functions. We seek conditions on f , h , and I so that the solutions or output vectors $x(t)$ shall have certain stability properties.

A simple example is the servomechanism constructed by suspending a mass m between two springs on a horizontal line. The first spring has one end fixed to the line, say at the origin, and the coordinate of the mass m at time t is $x(t)$. The second spring has one end fastened to a moving point with coordinate $z(t)$. If the first spring has a nonlinear restoring force $-(kx + \beta x^3)$, whereas the second spring produces a linear force $a(z - x)$, and if there is a frictional force $-c\dot{x}$ on m , then the differential equation for the motion of m is

$$m\ddot{x} + c\dot{x} + kx + \beta x^3 = a(z - x),$$

with m , c , k , β , a positive constants. The stability theory for such equations will always refer to that of the equivalent first order system. Here let $x^1 = x$, $x^2 = \dot{x}$, $z^1 = z$ and write

$$\dot{x}^1 - x^2 = 0$$

$$\dot{x}^2 + \frac{c}{m} x^2 + \frac{k+a}{m} x^1 + \frac{\beta}{m} (x^1)^3 = \frac{a}{m} z^1.$$

If $z = s$ is a constant, then it is physically clear that each solution $(x(t), \dot{x}(t)) = x^i(t)$ of this system approaches a certain equilibrium value as $t \rightarrow +\infty$. If $z^i(t)$ is not constant but varies quite slowly, one expects the output $x^i(t)$ to "follow" the input signal $z^i(t)$. That is, fluctuations in $z^i(t)$ will be reproduced, after some specified modification, as fluctuations of $x^i(t)$. We shall formalize this problem and prove a general theorem which yields the desired result for the spring example as a special case.

2. Attraction to a point. Consider

$$(1) \quad \dot{x}^i + f^i(x) = h^i(z, \dot{z}) \quad (i = 1, 2, \dots, n)$$

where f and h are (real) vector-valued functions, defined and in class $C^{(1)}$ for all real n -vectors $x, z, \dot{z}$. Let the class I of input vectors consist of certain real vector functions $z(t)$ in $C^{(2)}$ on $-\infty < t < \infty$.

Definition 1. The differential system

$$(1) \quad \dot{x}^i + f^i(x) = h^i(z, \dot{z}) \quad (i = 1, 2, \dots, n)$$

for a class I of inputs has a vector x_∞ of attraction in case for each input $z(t) \in I$ and all corresponding outputs $x(t)$ one has $\lim_{t \rightarrow +\infty} x(t) = x_\infty$.

In case I consists of the zero vector only, one refers to static attraction to x_∞ .

Theorem 1. Consider

$$(2) \quad \ddot{x} + c\dot{x} + kx + \beta x^3 = b(\dot{z} - \dot{x}) + a(z - x)$$

where the constant coefficients are restricted by $a + k > 0$, $b + c > 0$, $\beta > 0$. For each input class I_s , consisting of the constant function $z(t) = s$ only, the equation (2) has the vector of attraction $x_\infty^1 = A(s)$, $x_\infty^2 = 0$ where $A(s)$ is the unique real root of the cubic equation $\beta A^3 + (a + k)A - as = 0$.

Proof. First consider the polynomial $P(A) = \beta A^3 + (a + k)A - as$. Since $P'(A) = 3\beta A^2 + (a + k) > 0$, there is a unique real root $A(s)$. Furthermore $A(s)$ is a real analytic function on $-\infty < s < \infty$.

Let $\xi(t) = x(t) - A$. Then one computes

$$\ddot{\xi} + (b + c)\dot{\xi} + (a + k + 3\beta A^2)\xi + 3\beta A\xi^2 + \beta\xi^3 = 0.$$

The corresponding differential system is

$$\begin{aligned} \dot{\xi} &= \eta \\ \dot{\eta} &= -(b + c)\eta - (a + k + 3\beta A^2)\xi - 3\beta A\xi^2 - \beta\xi^3. \end{aligned}$$

We must show that each solution $\xi(t)$, $\eta(t)$ approaches the origin as $t \rightarrow +\infty$.

Define a function of the phase (ξ, η) by

$$E(\xi, \eta) = \frac{1}{2}\eta^2 + \frac{1}{2}(a + k + 3\beta A^2)\xi^2 + \beta A\xi^3 + \frac{1}{4}\beta\xi^4.$$

Then the radial derivative of $E(\xi, \eta)$ is

$$\nabla E \cdot \mathbf{r} = \eta^2 + (a + k + 3\beta A^2)\xi^2 + 3\beta A\xi^3 + \beta\xi^4.$$

Now $(a + k + 3\beta A^2) + 3\beta A\xi + \beta\xi^2 > 0$, for $\xi \neq 0$, since the discriminant $9\beta^2 A^2 - 4\beta(a + k + 3\beta A^2) < 0$. Thus the loci of $E(\xi, \eta) = \text{constant}$ are simple closed differentiable curves encircling the origin in the phase plane.

Along a solution curve $\xi(t)$, $\eta(t)$ one computes

$$\frac{dE}{dt} = \frac{\partial E}{\partial \xi} \eta + \frac{\partial E}{\partial \eta} [-(b + c)\eta - (a + k + 3\beta A^2)\xi - 3\beta A\xi^2 - \beta\xi^3],$$

$\dot{E}(t) = -(b + c)\eta^2 \leq 0$. Therefore $\lim_{t \rightarrow +\infty} E(\xi(t), \eta(t)) = E^* \geq 0$. If $E^* = 0$, the theorem is proved.

Suppose $E^* > 0$. Then the solution $\xi(t), \eta(t)$ must spiral in towards the closed curve $E(\xi, \eta) = E^*$. But $\dot{E} < -\mu < 0$ outside a narrow strip about the ξ -axis. Moreover, the solution requires a certain minimal time interval T^* to form a half loop around $E(\xi, \eta) = E^*$, during which time $\dot{E} < -\mu < 0$. Since the solution curve penetrates the level $E(\xi, \eta) = E^* + \frac{1}{2}\mu T^*$, it must penetrate the level $E(\xi, \eta) = E^*$. Therefore, by contradiction, we see $E^* = 0$. *Q.E.D.*

3. Attraction to transform function.

Definition 2. Consider

$$(1) \quad \dot{x}^i + f^i(x) = h^i(z, \dot{z}) \quad (i = 1, 2, \dots, n)$$

as above. This system has attraction to the transform $A(z, \dot{z})$ (a continuous vector function of the vectors $z, \dot{z}$), for input bound $B_0 > 0$, in case

(i) for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ and an input class I_ϵ consisting of all vectors in $C^{(2)}$ satisfying bounds $|z(t)| < B_0$, $|\dot{z}(t)| < B_1(\epsilon)$, $|\ddot{z}(t)| < B_2(\epsilon)$, $t \geq 0$, and

(ii) for each $z(t) \in I_\epsilon$ and each corresponding output $x(t)$ one has $|x(t) - A(\dot{z}(t), z(t))| < \epsilon$ for $t \geq 0$, provided $|x(0) - A(z(0), \dot{z}(0))| < \delta$.

Here $|x|$ means the maximum of the absolute values of the components of the vector x , and we also use a similar notation for the norm of a matrix. We also note that Definition (2) refers only to a local form of stability.

We next prove a general stability theorem which later will be applied to an analysis of the spring servomechanism described earlier.

Theorem 2. Consider

$$(3) \quad \dot{x}^i = G_i^i x^i + F^i(t) + H^i(x, t) \quad (i = 1, 2, \dots, n)$$

where

(i) G_i^i is a real $n \times n$ constant matrix each eigenvalue of which has a negative real part,

(ii) $F^i(t)$, $H^i(x, t)$, $(\partial H^i / \partial x^j)(x, t)$ are real continuous functions for all real vectors x and times $-\infty < t < \infty$,

$$(iii) \quad H^i(0, t) \equiv 0 \quad \text{and} \quad \left| \frac{\partial H^i}{\partial x^j}(x, t) \right| < \frac{1}{4n \int_0^\infty |e^{s\alpha}| ds}$$

for sufficiently small $|x|$.

Then for each $\epsilon > 0$ there exist numbers $\delta(\epsilon) > 0$ and $M(\epsilon) > 0$ such that if $|F(t)| \leq M$, then $|x(t)| < \epsilon$ for $t \geq 0$, provided $|x(0)| < \delta$.

Proof. The proof follows three lemmas.

Lemma 1. Let $x_H(t)$ be the solution of the homogeneous equation $\dot{x}^i = \alpha_i^j x^j$ with $x_H(0) = x_0$. Given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $|x_H(t)| < \frac{1}{2}\epsilon$ for $t \geq 0$, provided $|x_H(0)| < \delta(\epsilon)$.

Proof. Since each eigenvalue of the matrix α has a negative real part, this lemma is a well known elementary fact.

Lemma 2. Let $G(t) = e^{t\alpha}$ be the Green's matrix for the homogeneous equation $\dot{x}^i = \alpha_i^j x^j$. Then there is a unique continuous vector function $x_P(t)$, for $t \geq 0$, satisfying the integral equation

$$(4) \quad x_P(t) = \int_0^t G(t-s)[F(s) + H(x_P(s) + x_H(s), s)] ds.$$

Furthermore $x_P(t)$ is in $C^{(1)}$ and the sum $x(t) = x_H(t) + x_P(t)$ is the unique solution of the differential equation (3) with $x_P(0) = x_H(0) = x_0$.

Proof. Let $x_P(t)$ be a continuous solution of the integral equation (4). Then

$$\dot{x}_P(t) = G(0)[F(t) + H(x(t), t)] + \alpha x_P(t)$$

and $G(0)$ is the identity matrix. Thus $x_P(t) \in C^{(1)}$ on $t \geq 0$ and is the unique solution of the differential equation (3) with the initial condition $x_P(0) = 0$.

Since the difference $x_P(t) = x(t) - x_H(t)$, where $x_H(t)$ is the solution of $\dot{x}^i = \alpha_i^j x^j$ with $x_H(0) = x_0$ and $x(t)$ is the solution of (3) with $x(0) = x_0$, satisfies the integral equation (4), the lemma is proved.

Lemma 3. Given $\epsilon > 0$, there exists $M(\epsilon) > 0$ such that if $|F(t)| \leq M$ for $t \geq 0$, then the solution of the integral equation (4) satisfies $|x_P(t)| \leq \frac{1}{2}\epsilon$.

Proof. Since each eigenvalue of α has a negative real part, one can show that $\int_0^\infty |G(s)| ds < \infty$.

Consider the successive approximation vectors $x_P^{(0)}(t), x_P^{(1)}(t), \dots, x_P^{(m)}(t), \dots$, continuous on $t \geq 0$ and defined by

$$x_P^{(0)}(t) = \frac{1}{2}\epsilon$$

$$x_P^{(m)}(t) = \int_0^t G(t-s)[F(s) + H(x_H(s) + x_P^{(m-1)}(s), s)] ds.$$

Here $x_H(t) < \frac{1}{2}\epsilon$ for all $t \geq 0$, as in Lemma 1.

Compute

$$|x_P^{(1)}(t)| \leq \max_{0 \leq s \leq t} n |F(s) + H(x_H(s) + \frac{1}{2}\epsilon, s)| \int_0^t |G(t-s)| ds.$$

Restrict $\epsilon > 0$ to be so small that

$$|H(x, s)| < \frac{\epsilon}{4nq} \quad \text{when} \quad |x| < \epsilon,$$

and take $M = \epsilon/4nq$, where $q = \int_0^\infty |G(s)| ds$. Then $x_P^{(1)}(t) \leq \frac{1}{2}\epsilon$ for all $t \geq 0$.

We proceed to complete the induction. Compute

$$|x_P^{(m)}(t)| \leq \max_{0 \leq s \leq t} n |F(s) + H(x_H(s) + x_P^{(m-1)}(s), s)| \int_0^t |G(t-s)| ds.$$

Then $x_P^{(m)}(t) \leq \frac{1}{2}\epsilon$ for all $t \geq 0$, just as above.

Now examine the convergence of the sequence $x_P^{(m)}(t)$ on some compact interval $0 \leq t \leq T$. Compute

$$x_P^{(m+1)}(t) - x_P^{(m)}(t) = \int_0^t G(t-s)[H(x_H(s) + x_P^{(m)}(s), s) - H(x_H(s) + x_P^{(m-1)}(s), s)] ds$$

and

$$\max_{0 \leq t \leq T} |x_P^{(m+1)}(t) - x_P^{(m)}(t)| \leq \max_{\substack{0 \leq x \leq \epsilon \\ 0 \leq t \leq T}} \left| \frac{\partial H^i}{\partial x^i} \right| nq.$$

Again restrict $\epsilon > 0$ so that $\max_{0 \leq t \leq T} |x_P^{(m+1)}(t) - x_P^{(m)}(t)| \leq \frac{1}{4}$ and $\lim_{m \rightarrow \infty} x_P^{(m)}(t) = x_P(t)$ uniformly on $0 \leq t \leq T$. Thus $\lim_{m \rightarrow \infty} x_P^{(m)}(t) = x_P(t)$ for each $t \geq 0$ and $x_P(t)$ is continuous and satisfies the integral equation (4). Moreover $|x_P(t)| \leq \frac{1}{2}\epsilon$.

Proof of theorem. Given $\epsilon > 0$, take $\delta(\epsilon) > 0$ as in Lemma 1 and $M(\epsilon) > 0$ as in Lemma 3. By Lemma 2, if $x(t) = x_H(t) + x_P(t)$ is a solution of the differential equation (3) initiating with $|x(0)| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq 0$. *Q.E.D.*

We remark that condition (iii) in the hypothesis of Theorem 2 is automatically satisfied if $H^i(x, t) = \sum_{l=2}^{\infty} h_l^i(t)x^l$ where the $|h_l^i(t)|$ are uniformly bounded on $t \geq 0$ for all $i = 1, 2, \dots, n$ and $l = 2, 3, 4, \dots$. In this case it is easy to obtain the same conclusion with somewhat more general hypotheses. For example, one could replace the constant matrix $\mathfrak{A}$ by the continuous matrix $\mathcal{P}(t) + \mathfrak{B}(t)$ where $\mathcal{P}(t)$ is continuous and periodic with characteristic exponents each of which has negative real part, and $\lim_{t \rightarrow +\infty} |\mathfrak{B}(t)| = 0$.

Let us return to the differential equation

$$(2) \quad \ddot{x} + c\dot{x} + kx + \beta x^3 = b(\dot{z} - \dot{x}) + a(z - x)$$

which describes certain servomechanisms such as the double spring apparatus mentioned earlier. Suppose one fixes a bound B_0 for the inputs, $|z(t)| < B_0$, and a tolerance $\epsilon > 0$. Then one hopes that if $z(t)$ varies slowly, that is if $|\dot{z}(t)| < B_1(\epsilon)$ and $|\ddot{z}(t)| < B_2(\epsilon)$, there is a position of creeping equilibrium. One seeks transform functions $(A(z), A'(z)\dot{z})$ such that for each admissible input $z(t)$ and corresponding output $x(t)$, one has $|x(t) - A(z(t))| < \epsilon$, $|\dot{x}(t) - A'(z(t))\dot{z}(t)| < \epsilon$, provided $|x(0) - A(0)| < \delta$, $|\dot{x}(0) - A'(z(0))\dot{z}(0)| < \delta$.

If $b + c > 0$, $a + k > 0$, and $\beta > 0$ is sufficiently small (compared to B_0), then the system (2) does have the property of attraction to a transform. We note that $B_0 > 0$ can be fixed at an arbitrarily large value provided $\beta > 0$ is then restricted appropriately, $\beta < \beta(B_0)$. After the proof of these statements

in the next theorem, we shall give explicit estimates for $\beta(B_0)$, $B_1(\epsilon)$, $B_2(\epsilon)$, $\delta(\epsilon)$ for given B_0 , ϵ and coefficients a , b , c , k , β .

Theorem 3. *The differential equation*

$$(2) \quad \ddot{x} + c\dot{x} + kx + \beta x^3 = b(\dot{z} - \dot{x}) + a(z - x),$$

where $b + c > 0$, $a + k > 0$, and $0 < \beta < \beta(B_0)$, has attraction to the transform $(A(z), A'(z)\dot{z})$, where $A(z)$ is the unique real root of the cubic equation $\beta A^3 + (a + k)A - az = 0$.

Proof. Consider the polynomial equation $P(A) = \beta A^3 + (a + k)A - az = 0$. Since $P'(A) = 3\beta A^2 + (a + k) > 0$, there is a unique real root $A(z)$ of $P(A) = 0$ and, further, $A(z)$ is a real analytic function on $-\infty < z < \infty$.

Let $\xi(t) = x(t) - A(t)$, where $A(t) = A(z(t))$ for an input $z(t)$ in $C^{(2)}$ with $|z(t)| < B_0$, $|\dot{z}(t)| < B_1(\epsilon)$, $|\ddot{z}(t)| < B_2(\epsilon)$ for $t \geq 0$. The bounds $B_1(\epsilon)$, $B_2(\epsilon)$ and the number $\delta(\epsilon) > 0$ must be determined for each $\epsilon > 0$ so that $|\xi(t)| < \epsilon$, $|\dot{\xi}(t)| < \epsilon$, provided $|\xi(0)| < \delta$, $|\dot{\xi}(0)| < \delta$.

For the function $\xi(t)$ we have the differential equation

$$(5) \quad \ddot{\xi} + (b + c)\dot{\xi} + (a + k)\xi = F(t) + H(\xi, t)$$

where $F(t) = -\ddot{A}(t) - (b + c)\dot{A}(t) + b\dot{z}(t)$ and $H(\xi, t) = -\beta\xi^3 - 3\beta A(t)\xi^2 - 3\beta A(t)^2\xi$. When this equation (5) is considered as a differential system, with matrix

$$\alpha = \begin{pmatrix} 0 & 1 \\ -(a + k) & -(b + c) \end{pmatrix},$$

all the hypotheses of Theorem 2 hold provided $|F(t)|$ and $|\partial H(\xi, t)/\partial \xi|$ are small enough for small $|\xi|$.

For a fixed $B_0 > 0$ and $\epsilon > 0$ one can choose $\beta > 0$, $B_1(\epsilon) > 0$, $B_2(\epsilon) > 0$ so small that $|F(t)| < M(\epsilon)$ and $|\partial H/\partial \xi| < 1/8q$ for sufficiently small $|\xi|$. Then by Theorem 2 one concludes that $|x(t) - A(t)| < \epsilon$ and $|\dot{x}(t) - \dot{A}(t)| < \epsilon$ for all $t \geq 0$, provided they are less than $\delta(\epsilon) > 0$ at $t = 0$. *Q.E.D.*

Finally we indicate satisfactory values for the bounds $\beta(B_0)$, $B_1(\epsilon)$, $B_2(\epsilon)$, $\delta(\epsilon)$ for a given pair $\epsilon > 0$, $B_0 > 0$ in order to assure attraction to the transform $(A(z), A'(z)\dot{z})$ for

$$(2) \quad \ddot{x} + c\dot{x} + kx + \beta x^3 = b(\dot{z} - \dot{x}) + a(z - x).$$

As stated earlier $b + c > 0$, $a + k > 0$, $\beta > 0$ and we restrict $\epsilon < 1$.

First find $\beta(B_0)$ by the condition on

$$\frac{\partial H}{\partial \xi} = -3\beta\xi^2 - 6\beta A\xi - 3\beta A^2 = -3\beta(\xi + A)^2.$$

From $\beta A^3 + (a + k)A - az = 0$ one computes $|A| \leq |aB_0|/(a + k)$. The

Green's function for $\ddot{\xi} + (b + c)\dot{\xi} + (a + k)\xi = 0$ is

$$G(t) = \frac{1}{m_2 - m_1} [e^{m_2 t} - e^{m_1 t}]$$

if the roots m_1, m_2 of $m^2 + (b + c)m + (a + k) = 0$ are distinct, and

$$G(t) = te^{m t} \quad \text{if} \quad m_1 = m_2 = m.$$

Now

$$\int_0^t |G(t-s)| ds \leq q = \begin{cases} \frac{4}{(b+c)|m_2-m_1|} & \text{if } m_1 \neq m_2 \\ \frac{4}{(b+c)^2(1-e^{-1})} & \text{if } m_1 = m_2. \end{cases}$$

Thus one requires that $|\partial H / \partial \xi| = 3\beta |\xi + A|^2 < 1/8q$ and since $|\xi| < \epsilon < 1$ one finds

$$\beta \leq \beta(B_0) = \frac{1}{24q \left(1 + \frac{|a| B_0}{a+k}\right)^2}.$$

Once β has been restricted to the interval $0 < \beta < \beta(B_0)$, the differential equation (2) has attraction to the transform $(A(z), A'(z)\dot{z})$. One finds the bounds $B_1(\epsilon) > 0, B_2(\epsilon) > 0$ from the condition

$$|F(t)| = |-\ddot{A} - (b+c)\dot{A} + b\dot{z}| < M = \frac{\epsilon}{8q}.$$

This holds whenever

$$|a| B_2(a+k)^2 + 6\beta |a^3| B_0 B_1^2 + |a| B_1(b+c)(a+k)^2 + |b| B_1(a+k)^3 < \frac{\epsilon}{8q}.$$

Finally we must specify $\delta(\epsilon) > 0$ so that $|x(t) - A(t)| < \epsilon, |\dot{x}(t) - \dot{A}(t)| < \epsilon$ whenever these differences are initially less than $\delta(\epsilon)$. Examine the explicit solutions of the homogeneous equation $\ddot{\xi} + (b + c)\dot{\xi} + (a + k)\xi = 0$ and demand that $|\xi_H(t)| < \frac{1}{2}\epsilon, |\dot{\xi}_H(t)| < \frac{1}{2}\epsilon$ when $|\xi_H(0)| < \delta, |\dot{\xi}_H(0)| < \delta$. One computes

$$\delta = \begin{cases} \frac{1}{2} \epsilon \frac{|m_2 - m_1|}{2|m_1 m_2| + (|m_1| + |m_2|) + 2} & \text{if } m_1 \neq m_2 \\ \frac{1}{2} \epsilon \frac{1}{\left(1 + \frac{b+c}{2}\right)\left(3 + \frac{2}{e(b+c)}\right)} & \text{if } m_1 = m_2. \end{cases}$$

These are not the best possible estimates for $\beta(B_0), B_1(\epsilon), B_2(\epsilon), \delta(\epsilon)$ but they are easily and conveniently obtained.

Princeton University
Princeton, New Jersey