On the Solutions of Nonlinear Differential Equations, II

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In this paper we shall study the behavior of the solutions of the differential equation

$$(1) \quad x'' + p(t)x - 2q(t)x^3 = 0$$

where the nonlinear restoring force is also dependent on the time t . The free vibration of a soft spring which changes with time leads to a differential equation of the form (1). In another paper [1] we study a differential equation which may be considered to correspond to the free vibration of a hard spring.

In all the subsequent discussions we assume that the coefficients $p(t)$ and $q(t)$ in the equation (1) are real Lebesgue-measurable functions on $0 \leq t < \infty$. It is further assumed that $p(t)$ and $q(t)$ are positive for $t \geq 0$ and possess positive lower and upper bounds. Hence they are Lebesgue-integrable on $[0, a]$ for every positive a . All solutions considered are real and different from the identically zero solution.

Under our assumptions on $p(t)$ and $q(t)$ the solutions of (1) exist uniquely and at least locally and can be uniquely continued as far as they and their derivatives remain finite. (See, for instance, Chap. I and II in [2].)

It is instructive to observe first some properties of the solutions of (1) when $p(t)$ and $q(t)$ are positive constants. The following remarks serve to explain our approach in the general case. Let A and B be positive constants. Consider the solution $x(t)$ of

$$(2) \quad x'' + Ax - 2Bx^3 = 0, \quad x(0) = 0, \quad x'(0) = E > 0.$$

Clearly $x(t)$ also satisfies

$$(3) \quad x'^2 = Bx^4 - Ax^2 + E^2.$$

Let $E_0 > 0$ and $A^2 - 4BE_0^2 = 0$. For any E , $0 < E < E_0$, the solution $x(t)$ of (2) is periodic. Denoting the period of such a periodic solution by $4T$ and its amplitude by M , one can see by direct calculation that T and M increase as E increases, $0 < E < E_0$. As $E \rightarrow E_0 -$, $T \rightarrow \infty$. For $E \geq E_0$ no solution of (2)

is periodic. For $E = E_0$ the solution $x(t)$ of (2) is bounded on $0 \leq t < \infty$ and has no positive zero. For $E > E_0$ the solution of (2) is unbounded on $0 \leq t < \infty$ (as far as it exists) and has no positive zero. For the solution $x(t)$ of (2), $E = E_0$, the orbit $(x(t), x'(t))$ approaches the point $((A/2B)^{\frac{1}{2}}, 0)$ as $t \rightarrow \infty$, which is a saddle point for (2) considered as a system of equations.

Denote by $x(t) = x(t, E)$ the solution of (1) which satisfies the conditions

$$(1^*) \quad x(0) = 0, \quad x'(0) = E, \quad E > 0.$$

The following result is basic in our later discussions.

(I) *Let $0 < E_1 < E_2$. Then $x(t, E_1) < x(t, E_2)$ for all positive t as far as $x(t, E_1)$ remains non-negative.*

Proof. Set $x_i(t) = x(t, E_i)$, $i = 1, 2$. Using (1), we have

$$(4) \quad (x_1 x_2' - x_1' x_2)' = 2q x_1 x_2 (x_2^2 - x_1^2)$$

and, integrating from 0 to t ,

$$(5) \quad x_1 x_2' - x_1' x_2 = 2 \int_0^t q x_1 x_2 (x_2^2 - x_1^2) dt.$$

Suppose the conclusion is false. Since $x_2 - x_1$ vanishes at $t = 0$ and is positive for small positive t , there exists a positive t_1 such that $x_2 - x_1 = 0$ at t_1 , $x_2 - x_1 > 0$ and $x_1 > 0$ on $0 < t < t_1$. We have also $x_2' - x_1' < 0$ at t_1 , for the other possibility, $x_2' - x_1' = 0$ at t_1 , implies that x_1 and x_2 are identical. With $t = t_1$ the right hand side of (5) is positive and the left hand side non-positive. The contradiction proves I.

Consider again the solution $x = x(t, E)$ of (1) satisfying (1^*) . Multiplying (1) by $2x'$ and integrating from 0 to t , we obtain

$$(6) \quad (x')^2 + 2 \int_0^t p x x' dt - 4 \int_0^t q x^3 x' dt = E^2.$$

Write

$$(2^*) \quad A = \text{lub } p(t), \quad a = \text{glb } p(t), \quad b = \text{lub } q(t), \quad B = \text{glb } q(t)$$

for $0 \leq t < \infty$. (We observe that a, b, A , and B are by assumption positive.) For every positive E , $0 < E^2 < a^2/4b$, the solution $x(t, E)$ has a first positive zero. To see this, define two functions $p^*(t)$ and $q^*(t)$ on a closed interval I from 0 to some positive t where the solution $x(t, E)$ is increasing such that $p^*(0) = p(0)$, $q^*(0) = q(0)$ and

$$(7) \quad 2 \int_0^t p x x' dt = p^*(t) x^2, \quad 4 \int_0^t q x^3 x' dt = q^*(t) x^4$$

for positive t in I . Clearly $p^*(t)$ and $q^*(t)$ are continuous functions for positive t in I and $A \geq p^*(t) \geq a$, $b \geq q^*(t) \geq B$ for t in I . Rewrite (6) as

$$(8) \quad (x')^2 = q^* x^4 - p^* x^2 + E^2$$

for t in I . On the interval I , $x^2 \leq [a - (a^2 - 4bE^2)^{1/2}]/2b$ for the solution $x(t, E)$, $0 < E^2 < a^2/4b$, otherwise, from (8), $(x')^2$ is negative at some point in I . It follows from (1) that $x'' \leq (2bx^2 - a)x$ almost everywhere in I and $2bx^2 - a$ has a negative upper bound in I for the solution $x(t, E)$, $0 < E^2 < a^2/4b$. Clearly every solution $x(t, E)$, $0 < E^2 < a^2/4b$, has a first positive zero $T(E)$, say, and just one maximum in the interval $(0, T(E))$.

(II) *There exists a positive constant E_0 such that, for $0 < E < E_0$, every solution $x = x(t, E)$ of (1) satisfying (1*) has a first positive zero $T(E)$ and a positive absolute maximum $M(E)$ in the interval $(0, T(E))$. $M(E)$ and $T(E)$ are strictly increasing functions of E on $0 < E < E_0$ and tend to a positive limit and infinity, respectively, as $E \rightarrow E_0$ from the left. As $E \rightarrow 0$ from the right, $M(E)$ and $T(E)$ tend to zero and a positive limit, respectively. The point t where $x(t, E)$ first attains its absolute maximum $M(E)$ also increases with E .*

Proof. In (8), $(x')^2 > 0$ for $E^2 > A^2/4B$. This means that every solution $x(t, E)$, $E^2 > A^2/4B$, increases with t and hence does not vanish for positive t . Let S be the set of positive E such that the solution $x(t, E)$ has a positive zero. S contains all E such that $0 < E^2 < a^2/4b$ and is bounded above. Let E_0 be the least upper bound of S . In view of the result (I), S must contain all E in $0 < E < E_0$ and $M(E)$ and $T(E)$ are strictly increasing with E . For $0 < E_1 < E_2 < E_0$, the identity (5) and its consequence that $x(t, E_2)/x(t, E_1)$ is an increasing function of t as far as both solutions remain positive show that $x(t, E_1)$ must attain its absolute maximum $M(E_1)$ before $x(t, E_2)$ does. From equation (1), as $x'' > 0$ for large x , the set of $M(E)$, $0 < E < E_0$, is bounded and hence $M(E)$ tends to a positive limit as $E \rightarrow E_0^-$. For a positive E , $0 < E^2 < a^2/4b$, $x'' < 0$ in $(0, T(E))$ and hence, from (8) with $x' = 0$, we can see that $M(E) \rightarrow 0$ as $E \rightarrow 0$ from the right. Since $T(E)$ is increasing with E , $T(E)$ either tends to infinity or to a finite positive limit t_0 as $E \rightarrow E_0$ from the left. Suppose $T(E) \rightarrow t_0$ as $E \rightarrow E_0$ from the left. Consider the solution $x(t, E_0)$ and let I_0 be a closed interval from 0 to some positive t over which $x(t, E_0)$ exists. The solutions $x(t, E)$ and their derivatives $x'(t, E)$ are continuous functions in (t, E) for t in I_0 and E near E_0 (see, for instance, Theorem 7.1, p. 22, of [2] which also holds under our assumptions on p and q). The solutions $x(t, E)$ are uniformly bounded for $0 \leq t \leq T(E)$ and $0 < E < E_0$ and, upon integration of (1) once, under the condition $T(E) < t_0$, the derivatives $x'(t, E)$ are also uniformly bounded for $0 \leq t \leq T(E)$ and $0 < E < E_0$. It follows that $x(t, E_0)$ and $x'(t, E_0)$ must be uniformly bounded over any interval of existence in $0 \leq t < t_0$ and hence can be continued uniquely beyond t_0 . Since $x(t_0, E) < 0$ for E very close to E_0 , $E < E_0$, and $x(t_0, E_0) \geq 0$, it follows by continuity that $x(t_0, E_0) = 0$. Hence, for E close enough to E_0 , $E > E_0$, the solution $x(t, E)$ also exists beyond t_0 such that, at t_0 , x^2 is less than $a/2b$ and $x' < 0$, and so has a first positive zero greater than t_0 . This is impossible, as E_0 is the least upper bound of the set S . The contradiction proves that $T(E)$ tends to infinity as $E \rightarrow E_0$ from the left.

As $E \rightarrow 0$ from the right, $T(E)$ tends to a positive limit T_w which is the first

positive zero of the solution $w(t)$ of

$$(9) \quad w'' + p(t)w = 0, \quad w(0) = 0, \quad w'(0) \neq 0.$$

The proof of this is similar to that of a result in [1] (see the last part of the proof of Theorem 1 in [1]) and so is omitted here.

(III). For the same E_0 of (II), the solution $x(t, E_0)$ of (1) satisfying (1*) for $E = E_0$ exists and is bounded for all positive t and has no positive zero. For $E_0 < E < \infty$, $x(t, E)$ tends to infinity as t tends to some finite positive number $I(E)$. $I(E)$ does not increase with E . For every positive real number r , there exists an E , $E_0 < E < \infty$, such that $I(E) = r$.

Proof. The proof of (II) shows clearly that $x(t, E_0)$ has no positive zero. We have also seen in the proof of (II) that the solutions $x(t, E)$ are uniformly bounded for $0 \leq t \leq T(E)$ and $0 < E < E_0$. The derivatives $x'(t, E)$ of all such solutions are also uniformly bounded for $0 \leq t \leq T(E)$ and $0 < E < E_0$. To see this let $x = x(t, E)$ be one such solution and take a point t_1 , $0 \leq t_1 \leq T(E)$, at which $x' \neq 0$, let t_2 be a point in $0 \leq t \leq T(E)$ which is nearest to t_1 and at which $x' = 0$, then x' does not change sign between t_1 and t_2 . Multiplying (1) by $2x'$ and integrating from t_1 to t_2 , we can express x' at t_1 by

$$(10) \quad (x')^2 = q^*(x^4 - M^4) - p^*(x^2 - M^2)$$

where $M = x(t_2)$ and p^* and q^* possess the same bounds as those defined in (7). It follows from the uniform boundedness of p^* , q^* , x , and M that $x'(t, E)$ is also uniformly bounded for $0 \leq t \leq T(E)$ and $0 < E < E_0$. By virtue of the fact that $x(t, E)$ and $x'(t, E)$ are continuous in (t, E) , $x(t, E_0)$ and $x'(t, E_0)$ are uniformly bounded from 0 to any positive t over which they exist and hence they can be uniquely continued to all positive t .

Next we show that no solution $x(t, E)$, $E_0 < E < \infty$, is bounded on its maximal interval of existence. Suppose that there exists $E_1 > E_0$ such that $x(t, E_1)$ is bounded on its maximal interval of existence which must then be the interval $0 \leq t < \infty$. Let $x_1 = x(t, E_1)$, $x_0 = x(t, E_0)$. We have (see (5))

$$(11) \quad x_0 x'_1 - x'_0 x_1 = 2 \int_0^t q x_0 x_1 (x_1^2 - x_0^2) dt.$$

Since $x_0 > 0$ for $t > 0$, clearly

$$(12) \quad \frac{d}{dt} \left(\frac{x_1}{x_0} \right) = \frac{2}{x_0^2} \int_0^t q x_0 x_1 (x_1^2 - x_0^2) dt \quad (t > 0).$$

From (I), $0 < x_0 < x_1$ for $t > 0$, and hence the integral in (12) is positive and increasing with t . In view of the boundedness of x_0^2 , we have

$$(13) \quad \frac{d}{dt} \left(\frac{x_1}{x_0} \right) \geq \epsilon > 0$$

for some constant ϵ and for large t . Clearly (13) implies x_1/x_0 tends to infinity

with t . As $x'' < 0$ for small positive x (see equation (1)) and x_0 has no positive zero, x_0 has a positive lower bound for all t greater than any fixed positive number. Hence x_1 tends to infinity with t . The contradiction proves that all solutions $x(t, E)$, $E_0 < E < \infty$, are unbounded. Consider $x_1 = x(t, E_1)$, $E_0 < E_1$. Since x_1 is not bounded on its maximal interval of existence, there exists a positive t_0 such that

$$(14) \quad x_1^2 > A/B, \quad x_1' > 0, \quad x_1'' = 2qx_1^3 - px_1 > Bx_1^3 > 0$$

hold at t_0 and hence hold for all $t > t_0$ as far as x_1 exists. (The last inequality only holds almost everywhere.) The last inequality in (14) yields

$$(15) \quad 2(x_1')^2 \geq B(x_1^4 - C)$$

for $t \geq t_0$, where C is some constant such that $x_1^4 > C$ for $t \geq t_0$. From (15) we have

$$(16) \quad \frac{dx_1}{(x_1^4 - C)^{\frac{1}{2}}} \geq \left(\frac{B}{2}\right)^{\frac{1}{2}} dt.$$

Since x_1 increases with t for $t \geq t_0$ and the integral of the left side of (16) from $x_1(t_0)$ to ∞ converges, this proves that x_1 tends to ∞ as t tends to some finite positive number $I(E_1)$, say. That $I(E_1)$ does not increase with E_1 follows from the result (I).

Integrating the relation (8) one finds that $I(E) \rightarrow 0$ as $E \rightarrow \infty$. From the fact that $x(t, E)$ is a continuous function of (t, E) and the fact that $x(t, E_0)$ is bounded for all positive t , $I(E) \rightarrow \infty$ as $E \rightarrow E_0$ from above. Let r be a given positive number. Suppose that there exists no E , $E_0 < E < \infty$, such that $I(E) = r$. Divide the open connected set $E_0 < E < \infty$ into two sets S_1 and S_2 such that E is in S_1 if $I(E) < r$ and E is in S_2 if $I(E) > r$; S_1 and S_2 are not empty. Let E' be in S_2 . Since $x(t, E')$ and $x'(t, E')$ exist for some t beyond r , all solutions $x(t, E)$ exist for some t beyond r provided E is sufficiently close to E' ; hence S_2 is open. Let E^* be in S_1 . Then S_1 contains every $E > E^*$ as $I(E)$ does not increase with E . Let t' be arbitrarily close to $I(E^*)$, $t' < I(E^*)$. Then $x(t', E)$ and $x'(t', E)$ can be arbitrarily large if E is sufficiently close to E^* , $E < E^*$. Integration of (16) from t' to $I(E)$ shows that $I(E)$ can be arbitrarily close to $I(E^*)$ if E is sufficiently close to E^* , $E < E^*$. This shows that S_1 is also open. But this is impossible. Hence $I(E) = r$ for some $E > E_0$. Clearly $I(E)$ is a continuous function of E , $E_0 < E < \infty$.

Consider the solution $x(t, k, E)$ of (1) which satisfies

$$(1^{**}) \quad x(0) = k, \quad x'(0) = E.$$

The result (I) holds for $x_1 = x(t, k, E_1)$ and $x_2 = x(t, k, E_2)$ provided $0 < E_1 < E_2$, $k \geq 0$. For a positive k , $0 < k^2 < a/2b$, we can modify the previous arguments to show that there exists a positive $E(k) > 0$ such that, for all non-negative t , $x(t, k, E(k))$ exists, remains bounded, and does not possess positive zeros. In the following theorem we consider two such solutions $x_i = x_i(t, k_i, E(k_i))$,

$0 \leq k_i < (a/2b)^{\frac{1}{2}}, i = 1, 2$. We observe that each of these solutions must remain greater than some positive number for all large positive t .

(IV). Let $0 \leq k_1 < k_2 < (a/2b)^{\frac{1}{2}}$. Then $x_1 < x_2$ for all $t \geq 0$. Furthermore both $x_2 - x_1 \rightarrow 0$ and $x'_2 - x'_1 \rightarrow 0$ as $t \rightarrow \infty$.

Proof. In the same manner as the derivation of (12), we have for $t > 0$

$$(17) \quad \left(\frac{x_2}{x_1}\right)' = \frac{2 \int_0^t q x_1 x_2 (x_2^2 - x_1^2) dt + C}{x_1^2},$$

where $C = k_1 E(k_2) - k_2 E(k_1)$. The following argument shows that the constant C is negative: The supposition $C \geq 0$ and the fact that $x_2 > x_1 > 0$ for small t imply that the integral in (17) is positive for positive t and increasing, hence x_2/x_1 is unbounded as t tends to infinity, contradicting the facts that x_2 is bounded and x_1 has a positive lower bound for all large positive t .

Hence $x_2/x_1 > 1$ and is decreasing for small positive t . Suppose $x_2/x_1 = 1$ for some positive t and denote by t' the smallest of such t ; then the right-hand expression in (17) must be negative at t' , for the other possibility is zero which implies $x_1 = x_2, x'_1 = x'_2$ at t' and hence $x_1 \equiv x_2$ for all t . Hence the supposition $x_2/x_1 = 1$ at t' implies that x_2/x_1 is decreasing for $t \geq t'$ and that the right-hand side of (17) is less than some negative number for all $t \geq t'$. But then x_2 must vanish for some positive t , contradicting the assumption on x_2 . This proves that $x_2/x_1 > 1$ for all positive t .

The argument showing the constant C negative also proves that the right-hand side of (17) remains negative for all positive t and then x_2/x_1 is decreasing and tends to a limit 1 as t tends to infinity. Clearly this is equivalent to $x_2 - x_1 \rightarrow 0$ as $t \rightarrow \infty$. As the right-hand side of (17) tends to zero as t tends to infinity, the left-hand side yields the result $x'_2 - x'_1 \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof of (IV).

The following four theorems are concerned with the existence of periodic solutions and their oscillatory behavior. The methods and proofs are similar to those in [1]. Hence we describe the proofs briefly and omit some of the details. In the following theorem T_ω is the first positive zero of the solution of (9).

(V) Let $p(t)$ and $q(t)$ be even (positive) periodic functions with a common least period L . If n is any positive integer such that nL is greater than $2T_\omega$, then there exists a periodic oscillatory solution of (1) with least period nL and having exactly three zeros, namely, 0, $nL/2$, and nL , in the closed interval $[0, nL]$.

Proof. Consider the solution $x(t, E)$ of (1) satisfying the initial condition (1*) and denote by $T(E)$ the mapping from the initial condition (1*) to the first positive zero of $x(t, E)$. $T(E)$ is defined for all E , $0 < E < E_0$, where E_0 is the constant described in the result (II). As $E \rightarrow 0+$, $T(E) \rightarrow T_\omega$ and as $E \rightarrow E_0-$, $T(E) \rightarrow \infty$. For all E , $0 < E < E_0$, $T_\omega < T(E) < \infty$. Since $x(t, E)$ and $x'(t, E)$ are continuous in t and E and, for $0 < E < E_0$, $x(t, E) = 0$,

$x'(t, E) \neq 0$ at $t = T(E)$, $T(E)$ is continuous on $0 < E < E_0$. Since $T_w < \frac{1}{2}nL < \infty$, there exists a positive number E^* , $0 < E^* < E_0$, such that $T(E^*) = \frac{1}{2}nL$, i.e., $\frac{1}{2}nL$ is the first positive zero of the solution $x(t, E^*)$. Now some routine computation shows that $x(t, E^*) = -x(-t, E^*)$, $x(t, E^*) = -x(nL - t, E^*)$ and hence $x(t, E^*)$ is periodic with least period nL and has exactly three zeros, namely, 0 , $\frac{1}{2}nL$, and nL , in the closed interval $[0, nL]$.

For n even, it is an easy consequence of (I) that $x'(t, E^*)$ vanishes at $\frac{1}{4}nL$ and $\frac{3}{4}nL$.

(VI). Let $p(t)$ and $q(t)$ be even (positive) periodic functions with a common least period L and let m be any positive integer. If n is a positive integer such that $2m\pi < nLa^{\frac{1}{2}}$, then there exists a periodic oscillatory solution of (1) with period nL and having exactly $2m + 1$ zeros, including 0 , $\frac{1}{2}nL$, and nL , in the closed interval $[0, nL]$.

The constant a in this theorem is defined in (2*).

Proof. Consider the solution $x = x(t, E)$ of (1) satisfying the condition (1*) and denote by $Z(E)$ the mapping from the initial condition $(0, E)$ to the m th positive zero of x whenever the latter exists. By assumption $m\pi a^{-\frac{1}{2}} < \frac{1}{2}nL$. We first show that $Z(E)$ is defined and $Z(E) < \frac{1}{2}nL$ for small positive E . Let ϵ be any positive number such that $\epsilon < a$ and $m\pi(a - \epsilon)^{-\frac{1}{2}} < \frac{1}{2}nL$. Since $x(t, E) \equiv 0$ for $E = 0$, there exists $\delta > 0$ such that the solution $x(t, E)$ exists on $[0, \frac{1}{2}nL]$ and $2qx^2 < \epsilon$ there for all E , $0 < E < \delta$. (See Theorem 7.1, p. 22, [2].) Write the terms $px - 2qx^3$ in (1) as $(p - 2qx^2)x$ and apply Sturm's comparison theorem [2] to the solution $x(t, E)$ of (1) satisfying (1*), $0 < E < \delta$, and the solution $y(t)$ of

$$(18) \quad y'' + (a - \epsilon)y = 0, \quad y(0) = 0, \quad y'(0) \neq 0,$$

obtaining $Z(E) < m\pi(a - \epsilon)^{-\frac{1}{2}}$ and hence

$$(19) \quad Z(E) < \frac{1}{2}nL, \quad 0 < E < \delta.$$

Now we consider the maximal interval S of positive numbers E , including the open interval $(0, \delta)$, for which $Z(E)$ is defined. We now prove that the set $Z(E)$, E being in S , is unbounded. Suppose the contrary is true. Then there is a positive number Z^* such that $0 < Z(E) < Z^*$ for all E in S . S is bounded above by E_0 of (II) and hence there exists a least upper bound E' . In view of the continuity of $x(t, E)$ and $x'(t, E)$ in (t, E) for E near E' and for all t where they exist, $x(t, E')$ must exist as far as to have m positive zeros, that is, $Z(E')$ is defined and E' is in S . But then, for the same reason, for E slightly greater than E' , $x(t, E)$ must also exist as far as to have m positive zeros. This contradicts the fact that E' is the least upper bound of S and hence the set $Z(E)$, E being in S , is unbounded. Again by virtue of the continuity of $x(t, E)$ and $x'(t, E)$ in (t, E) and the fact that, for E in S , $x(t, E) = 0$ and $x'(t, E) \neq 0$ at $Z(E)$, $Z(E)$ is continuous on S . Since $Z(E)$ is unbounded on S and $Z(E) < \frac{1}{2}nL$ for small positive E , there

exists a positive E^* such that $Z(E^*) = \frac{1}{2}nL$. Now it can be easily verified that $x(t, E^*)$ is a solution with the properties described in the theorem.

It is not known to the authors whether these periodic solutions are stable.

Consider the solution $x(t, k, 0)$ of (1) which satisfies the condition (1**) for $k > 0$, $E = 0$. The same proof as in (I) also proves that, for $0 < k_1 < k_2$, $x(t, k_1, 0) < x(t, k_2, 0)$ for all positive t as far as $x(t, k_1, 0)$ remains non-negative. Modifying the methods in (II) and (III), one can show that there exists a positive constant k_0 such that, for $0 < k < k_0$, $x(t, k, 0)$ has a first positive zero $t(k)$, say. $t(k)$ is strictly increasing on $0 < k < k_0$ and tends to infinity as k tends to $k_0 -$. And $t(k)$ tends to t_w from the right as k tends to $0+$, t_w being the first positive zero of the solution $w(t)$ of

$$(20) \quad w'' + p(t)w = 0, \quad w(0) \neq 0, \quad w'(0) = 0.$$

For all positive t , the solution $x(t, k_0, 0)$ exists, remains bounded, and has no zero. Applying the methods in (V) and (VI) to the solution $x(t, k, 0)$, $0 < k < k_0$, one can establish the following Theorems (VII) and (VIII).

(VII). *Let $p(t)$ and $q(t)$ be even (positive) periodic functions with a common least period L . If n is any positive integer such that nL is greater than $2t_w$, then there exists a periodic oscillatory solution x of (1) of the form $x(t, k, 0)$ with least period $2nL$ such that x has exactly two zeros, namely $\frac{1}{2}nL$ and $\frac{3}{2}nL$, and x' has zeros at 0 , nL , and $2nL$, in the closed interval $[0, 2nL]$.*

This theorem also can be obtained from Theorem (V) by taking n even and by the change of independent variable $t = t' + \frac{1}{2}nL$.

(VIII). *Let $p(t)$ and $q(t)$ be even (positive) periodic functions with a common least period L and let m be any positive integer. If n is a positive integer such that $(2m - 1)\pi < nLa^{\frac{1}{2}}$, then there exists a periodic oscillatory solution x of (1) of the form $x(t, k, 0)$ with period $2nL$ such that x has exactly $2(2m - 1)$ zeros, including $\frac{1}{2}nL$ and $\frac{3}{2}nL$, and x' has zeros at 0 , nL , and $2nL$, in the closed interval $[0, 2nL]$. For the definition of a see (2*).*

In the next theorem we compare the distances between two successive zeros of solutions of two different differential equations

$$(21, i) \quad x'' + p_i(t)x - 2q_i(t)x^3 = 0, \quad x(0) = 0, \quad x'(0) = E_i > 0, \quad (i = 1, 2),$$

where p_i and q_i satisfy the same conditions as p and q . The result resembles the Sturm's comparison theorem.

(IX). *Let x_i , $i = 1, 2$, be the solution of (21, i) with a first positive zero T_i and let M_i be the (absolute) maximum of x_i on $[0, T_i]$. If $p_1 \geq p_2 > 0$, $q_2 \geq q_1 > 0$ on $[0, T_1]$ and if $E_2 \geq E_1 > 0$, then $x_2 \geq x_1$ on $[0, T_1]$ and $T_2 \geq T_1$, $M_2 \geq M_1$. The equality $T_2 = T_1$ holds if and only if $E_2 = E_1$ and $p_2 = p_1$, $q_2 = q_1$ (almost everywhere) on $[0, T_1]$.*

The proof given below also proves that x_1 has a first positive zero T_1 if x_2 is known to have a first positive zero T_2 .

Proof of (IX). Using (21, 1) and (21, 2), we have

$$(22) \quad (x_1'x_2 - x_2'x_1)' = x_1x_2(p_2 - p_1) + 2x_1x_2(q_1x_1^2 - q_2x_2^2).$$

Then, integrating (22) from 0 to t ,

$$(23) \quad x_1'x_2 - x_2'x_1 = \int_0^t x_1x_2(p_2 - p_1) dt + 2 \int_0^t x_1x_2(q_1x_1^2 - q_2x_2^2) dt.$$

Case 1. $E_2 > E_1 > 0$. For small positive t , $x_2 > x_1 > 0$. Hence the right-hand side of (23) is negative and so $(x_1/x_2)' < 0$ for small positive t . Then, using (23) again, $(x_1/x_2)'$ must remain negative as far as both x_1 and x_2 are positive. This means that $x_2 > x_1$ as far as both remain positive and $T_2 \geq T_1$, $M_2 > M_1$.

Case 2. $E_2 = E_1 > 0$. The solution $x_2 = x_2(t, E_2)$ is continuous in (t, E_2) . Since $x_2 > x_1 > 0$ in the open interval $(0, T_1)$ for E_2 slightly greater than E_1 , it follows that $x_2 \geq x_1$ on $[0, T_1]$ and $T_2 \geq T_1$, $M_2 \geq M_1$ for $E_2 = E_1$.

Suppose $T_2 = T_1$. Setting $t = T_1$ in (23), its left-hand side is 0 and the integrands in the right-hand side are non-positive and hence $p_2 = p_1$, $q_1 = q_2$ (almost everywhere), $x_1 \equiv x_2$ on $[0, T_1]$.

It is also easy to see that if $x_1 = x_2$ at a point t' inside $[0, T_1]$, then $x_1 \equiv x_2$, $p_1 = p_2$, and $q_1 = q_2$ (almost everywhere) on $[0, t']$, by virtue of the fact that x_1/x_2 does not increase on $[0, T_1]$ under the conditions considered.

If x_i takes the (absolute) maximum M_i at more than one point in $[0, T_i]$, let t_i be the smallest. Then $t \geq t_1$. For if $t_2 < t_1$, then $x_1/x_2 = x_1/M_2$ at t_2 is smaller than $x_1/x_2 = M_1/x_2$ at t_1 , contradicting the fact that x_1/x_2 is not increasing inside $[0, T_1]$.

Theorem (IX) applies equally well if the initial conditions of x_i are $x(0) = k > 0$, $x'(0) = E_i > 0$. Under these initial conditions the right-hand side of (23) acquires an additional term $k(E_1 - E_2)$ which is non-positive.

By considering the differential equation obtained from (1) by replacing p and q by their least upper bounds or greatest lower bounds, and as a consequence of Theorem (IX), upper bounds and lower bounds can be obtained for the solution of (1), satisfying the conditions (1*) over some interval from the origin, and for its first positive zero.

As another application of Theorem (IX) we prove the following theorem.

(X). Let $p(t)$ be a non-decreasing (non-increasing) function and $q(t)$ non-increasing (non-decreasing). Let x be the solution of (1) satisfying (1*) and having zeros $0 = T_0, T_1, T_2, \dots$ arranged in increasing magnitude. If M_i is the (absolute) maximum of $|x|$ on $[T_{i-1}, T_i]$ and V_i is the value of $|x'|$ at T_{i-1} , then the sequences $\{T_i - T_{i-1}\}$ and $\{M_i\}$ are non-increasing (non-decreasing) and $\{V_i\}$ non-decreasing (non-increasing).

In the case of non-decreasing $p(t)$ and non-increasing $q(t)$ the following proof

shows that if the solution x of (1) satisfying (1*) has a first positive zero it has an infinite number of positive zeros. In the opposite case x may have just one or a finite number of positive zeros.

Proof of (X). Setting $t = T_i + t'$ and $t = T_i - t'$ in the equation (1) we obtain two differential equations. Applying Theorem (IX) to these two equations we have the results for the sequences $\{T_i - T_{i-1}\}$ and $\{M_i\}$.

Evaluating the integrals in (6) by parts we get

$$(24) \quad (x')^2 + px^2 - qx^4 = E^2 + \int_0^t x^2 dp - \int_0^t x^4 dq.$$

Setting $t = T_i$ in (24) we have the result about the sequence $\{V_i\}$.

It is interesting to observe that, in the case of non-decreasing $p(t)$ and non-increasing $q(t)$, the constant E_0 of Theorem (II) separates all solutions of (1) satisfying (1*) into two groups such that the group corresponding to $0 < E < E_0$ contains the solutions each of which has an infinite number of zeros and the group corresponding to $E_0 \leq E < \infty$ contains all the solutions which do not vanish for positive t .

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