Relations Between Properties of Solutions of Partial Differential Equations and the Coefficients of their Power Series Development

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1. INTRODUCTION

1.1. Problems considered in this paper. The theory of solutions of linear partial differential equations of elliptic type and with analytic coefficients,

$$(1.1) \quad \Delta v + \alpha(x, y)v_x + \beta(x, y)v_y + \gamma(x, y)v = 0,$$

can be based upon the theory of functions of a complex variable, as S. BERGMAN has recently shown in a number of important papers, *cf.* [1–6] of the bibliography at the end of this paper. Obviously this relation is a generalization of the well known correspondence between analytic and harmonic functions.

The present paper is concerned with two closely related problems. In Section 2 we shall obtain relations between properties of solutions of (1.1) and those of the coefficients of the corresponding power series development. By means of appropriate Bergman operators (definition below) theorems of that kind can be derived from well known similar theorems on analytic functions $f(z)$, $z = x + iy$. We restrict ourselves to some interesting problems which may arise in this connection. We first consider singularities of solutions of (1.1) and representations of these solutions in terms of particular solutions which are “most simple” in a sense defined in Section 2.1. Section 2.2 is devoted to solutions of (1.1) which satisfy also ordinary differential equations. This property is very advantageous since it permits the investigation of these solutions by means of the theory of ordinary differential equations. We shall see that the results thus obtained depend only on a certain subset of the coefficients in the power series development of the solutions. This fact leads to the important question as to what extent information on the behavior of a solution can be obtained from other subsets of those coefficients. In Section 3 we shall find relations between such subsets; if the coefficients of (1.1) are entire functions these relations either are completely independent of α , β , and γ or depend on these func-

tions in a trivial manner. For details of the problem and the results see Section 3.1.

Various similar results can be obtained by extending our approach to partial differential equations in three and more variables, *cf.* BERGMAN [7, 8]. In this case, in particular, one can generalize the theory of fundamental solutions, as developed by J. HADAMARD [9], T. Y. THOMAS & E. W. TITT [10], F. JOHN [11], and others.

1.2. Transformation of the partial differential equation (1.1). Since $\alpha(x, y)$, $\beta(x, y)$, and $\gamma(x, y)$ are assumed to be analytic functions we may continue them to complex values by introducing complex variables

$$z_1 = x + iX, \quad z_2 = y + iY.$$

Setting

$$z = z_1 + iz_2, \quad z^* = z_1 - iz_2$$

and using

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial z_1} - i \frac{\partial}{\partial z_2} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial z_1} + i \frac{\partial}{\partial z_2} \right)$$

we obtain from (1.1)

$$(1.2) \quad L(u) \equiv u_{zz^*} + a(z, z^*)u_z + b(z, z^*)u_{z^*} + c(z, z^*)u = 0$$

where

$$u(z, z^*) \equiv v(z_1, z_2), \quad a = \frac{1}{4}(\alpha + i\beta), \quad b = \frac{1}{4}(\alpha - i\beta), \quad c = \frac{1}{4}\gamma.$$

If z_1 and z_2 are real, the variables z and z^* are conjugate; otherwise they are independent of each other.

We assume that α , β , and γ are entire functions of z_1, z_2 . Then a , b , and c are entire functions of z, z^* .

In general, (1.2) is equivalent to a system of two partial differential equations for the real and imaginary part of the complex solution $u(z, z^*)$. If and only if

$$(1.3) \quad b = \bar{a}, \quad c = \bar{c}$$

those two equations become independent of each other and are of the same form. In this important case

$$(1.4) \quad U(z, z^*) = \frac{1}{2} [u(z, z^*) + \bar{u}(z^*, z)]$$

is also a solution of the differential equation under consideration; for real values of z_1, z_2 the function $U(z, z^*)$ is real. (Capital letters denote functions which are real for real values of the arguments.)

1.3. Bergman operators. Bergman operators [1-6] transform analytic functions $g(z)$, $z = x + iy$, into solutions $u(z, z^*)$ of (1.2). There exist infinitely

many such operators. The choice of the operator depends on the purpose. For our problem *Bergman operators of the first kind* are especially advantageous since they preserve many properties of the analytic functions involved. There are different representations of these operators; we choose the representation which is most suitable for our purpose.

Definition 1. An operator

$$(1.5) \quad \begin{aligned} u(z, z^*) &= p(g) \\ &\equiv h(z, z^*) \left\{ g(z) + \sum_{n=1}^{\infty} q_n(z, z^*) \int_0^z \int_0^{z_1} \cdots \int_0^{z_{n-1}} g(z_n) dz_n \cdots dz_1 \right\} \end{aligned}$$

where

$$(1.6) \quad h(z, z^*) = \exp \left\{ - \int_0^{z^*} a(z, t) dt \right\}$$

and

$$(1.7) \quad q_n(z, 0) = 0, \quad n = 1, 2, \cdots,$$

is called a *Bergman operator of the first kind*. $g(z)$ is an analytic function of $z = x + iy$ regular at the origin. $g(z)$ is called the p -associated function of $u(z, z^*)$ with respect to the "point of reference" z_0 of the operator. (In (1.5) $z_0 = 0$; we shall use only this special point of reference.)

The functions $q_n(z, z^*)$ depend on the coefficients of (1.2) but not on the choice of $g(z)$; they are entire if the coefficients are entire functions. The operator $p(g)$ has been introduced by BERGMAN [3]; he proved that functions q_n with the above properties exist and that $u(z, z^*)$, cf. (1.5), is defined in every domain $B_4 = B_2 \times (|z^*| < \infty)$ of zz^* -space where B_2 is any simply connected domain in the z -plane which contains the origin and in which $g(z)$ is regular.

Instead of (1.5) we may also write

$$(1.5') \quad u(z, z^*) = h(z, z^*) \left\{ g(z) + \sum_{n=1}^{\infty} \frac{q_n(z, z^*)}{\Gamma(n)} \int_0^z (z - t)^{n-1} g(t) dt \right\}$$

as can be seen by repeated integration by parts.

From (1.4) and (1.5) we obtain for the real solution

$$(1.8) \quad U(z, z^*) = \frac{1}{2}[p(g) + \bar{p}(\bar{g})]$$

where

$$(1.9) \quad \bar{p}(\bar{g}) = \bar{h}(z^*, z) \left\{ \bar{g}(z^*) + \sum_{n=1}^{\infty} \bar{q}_n(z^*, z) \int_0^{z^*} \int_0^{z_1} \cdots \int_0^{z_{n-1}} \bar{g}(z_n) dz_n \cdots dz_1 \right\}$$

and

$$(1.10) \quad \bar{h}(z^*, z) = \exp \left\{ - \int_0^z \bar{a}(z^*, t) dt \right\}.$$

2. SOME PROPERTIES OF SOLUTIONS OF THE PARTIAL DIFFERENTIAL EQUATION (1.2) GIVEN BY THEIR SERIES DEVELOPMENTS

2.1. Singularities and decomposition of solutions of (1.2). The investigation of the nature and location of singularities of solutions of (1.2) is of basic importance. We shall obtain some relations between the singularities of $u(z, z^*)$ and $U(z, z^*)$, cf. (1.4), and the coefficients of the corresponding series developments

$$(2.1) \quad u(z, z^*) = \sum_{m, n=0}^{\infty} u_{mn} z^m z^{*n}$$

and

$$(2.2) \quad U(z, z^*) = \sum_{m, n=0}^{\infty} U_{mn} z^m z^{*n}.$$

(Sometimes, for the sake of clarity, we write $u_{m,n}$ instead of u_{mn} .)

Moreover we shall consider the decomposition of a solution of (1.2) into a sum of "most simple" particular solutions which we define as follows.

Definition 2. A solution of (1.2) for which in (1.5) and (1.8)

$$(2.3) \quad g(z) = k_s z^s \quad \text{or} \quad g(z) = k_s (z - \zeta)^{-s}, \quad \zeta \neq 0, \quad s = 0, 1, \dots,$$

is called an elementary solution.

Such a decomposition is of great interest since for an elementary solution the integrals in (1.5) and (1.9) can be expressed in terms of elementary functions and thus yield a detailed knowledge of the singularities.

Theorem 1. A solution $U(z, z^*)$ of (1.2) has at most finitely many singularity planes (which are pole planes of order $p \leq q$ and (for all $z^* \neq 0$) logarithmic branch planes), and the representation of $U(z, z^*)$ as a sum of elementary solutions contains at most finitely many terms each of which has just one singularity plane, if and only if at most finitely many of the determinants

$$(2.4) \quad D_r = |D_{r1} D_{r2} \cdots D_{r,q+1}|, \quad r = 0, 1, \dots,$$

with column vectors

$$(2.5) \quad D_{rj} = [U_{r+j-1,0} \ U_{r+j,0} \ \cdots \ U_{r+j+q-1,0}], \quad j = 1, 2, \dots, q+1,$$

are different from zero.

Proof. From (1.6) we have $h(z, 0) = 1$, hence from (1.8)

$$(2.6) \quad U(z, 0) = \frac{1}{2}[g(z) + \bar{h}(0, z)\bar{g}(0)].$$

Since $a(z, z^*)$ is entire, $h(z, z^*)$ is also entire. $U(z, 0)$ and $g(z)$ thus have the same singularities. Now

$$(2.7) \quad g(z) = \sum_{m=0}^{\infty} g_m z^m$$

is a rational function whose denominator has the degree $p \leq q$ if and only if the determinants

$$(2.8) \quad G_m^{(q)} = \begin{vmatrix} g_m & g_{m+1} & \cdots & g_{m+q+1} \\ g_{m+1} & g_{m+2} & \cdots & g_{m+q+2} \\ . & . & \cdots & . \\ g_{m+q+1} & g_{m+q+2} & \cdots & g_{m+2q+2} \end{vmatrix}, \quad m = 0, 1, \dots,$$

vanish with the exception of at most finitely many of them, cf. É. BOREL [12]. From (2.2), (2.6)–(2.8) we obtain (2.4). $U(z, z^*)$, considered as a function of z in the characteristic plane $z^* = 0$ of zz^* -space, thus possesses only finitely many poles $z = a_\beta$, $\beta = 1, 2, \dots, b$. These singularities are the intersections of singularity planes of zz^* -space with the plane $z^* = 0$. We represent the rational function $g(z)$ in the form

$$(2.9) \quad g(z) = r(z) + q(z), \quad r(z) = \sum_{\alpha=0}^m G_\alpha z^\alpha, \quad q(z) = \sum_{\beta=1}^b \sum_{\delta=1}^d G_{\beta\delta} (z - a_\beta)^{-s_\beta\delta}$$

and introduce (2.9) into (1.5). Then the integrals corresponding to the terms of $r(z)$ are regular. Let $g_{sc}(z) = k(z - c)^{-s}$, $k = \text{const.}$, $c \neq 0$, $s > 0$, denote an arbitrary term of $q(z)$. Then, if $s > n$,

$$(2.10) \quad \begin{aligned} J_n(g_{sc}) &\equiv \int_0^s \int_0^{s_1} \cdots \int_0^{s_{n-1}} (t - c)^{-s} dt dz_{n-1} \cdots dz_1 \\ &= \frac{(z - c)^{n-s}}{(1-s)(2-s) \cdots (n-s)} - \sum_{\beta=0}^{n-1} \frac{(-c)^{n-s-\beta}}{(1-s)(2-s) \cdots (n-s-\beta)} \frac{z^\beta}{\beta!}. \end{aligned}$$

In (1.5), for every order s of a pole of $g(z)$ there exist values $n \geq s$. For these values of n the integral J_n yields functions with logarithmic singularities. The singularity planes $z = a_\beta$ of $U(z, z^*)$ thus are not only pole planes but (for $z^* \neq 0$) also logarithmic branch planes. This completes the proof.

It is remarkable that the above conditions are independent of the special form of the entire functions a , b , and c in (1.2).

A similar result is valid for complex solutions $u(z, z^*)$, cf. (1.4), (1.5); we only have to replace in (2.4) each coefficient U_{mn} by the corresponding coefficient u_{mn} , cf. (2.1), (2.2). The conditions thus obtained are at the same time necessary and sufficient in order that $u(z, z^*)$ can be represented as a finite sum of elementary solutions.

However, Theorem 1 does not answer the question whether a real solution $U(z, z^*)$ can be represented as a finite sum of elementary solutions. The conditions for the existence of such a finite decomposition are somewhat more complicated since they necessarily depend on the coefficient $a(z, z^*)$ of (1.2), but only on the subsequence (a_{m0}) in the corresponding development

$$(2.11) \quad a(z, z^*) = \sum_{m,n=0}^{\infty} a_{mn} z^m z^{*n}.$$

Theorem 2. *A real solution $U(z, z^*)$ of (1.2), cf. (1.4), can be represented as a finite sum of elementary solutions and has at most finitely many singularities (which are all of the type indicated in Theorem 1) if and only if all but a finite number of the determinants*

$$(2.12) \quad E_r = |E_{r_1} E_{r_2} \cdots E_{r_{q+1}}|, \quad r = 0, 1, \cdots,$$

vanish. E_r has the column vectors

$$(2.13) \quad E_{rj} = [V_{r+j-1} \ V_{r+j} \ \cdots \ V_{r+j+q-1}], \quad j = 1, 2, \cdots, q+1,$$

where

$$(2.14) \quad V_m = 2U_{m0} - U_{00}H_m,$$

cf. (2.2), and the determinant

$$(2.15) \quad H_m = (-1)^m |H_{m1} H_{m2} \cdots H_{mm}|$$

has the order m and the column vectors

$$(2.16) \quad H_{mj} = [c_{j-1} \ c_{j-2} \ \cdots \ c_0 \ 1 \ 0 \ 0 \ \cdots], \quad j = 1, 2, \cdots, m,$$

$$c_0 = \bar{a}_{00}, \quad c_t = (j-1)(j-2) \cdots (j-t)\bar{a}_{0t}, \quad t = 1, 2, \cdots,$$

cf. (2.11); in (2.16) the number of zeros decreases with increasing j ; in $H_{m,m-1}$ there are no more zeros left, in H_{mm} the element c_0 is the last one.

Proof. We start from (2.6). We write

$$(2.17) \quad \bar{h}(0, z) \equiv \exp \{K(z)\} \equiv \bar{h}(0, 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \bar{h}(0, z)}{\partial z^m} \Big|_{z=0} z^m \equiv \sum_{m=0}^{\infty} H_m z^m.$$

We prove by induction that

$$(2.18) \quad \frac{\partial^m \bar{h}(0, z)}{\partial z^m} = F_m \bar{h}(0, z), \quad m = 1, 2, \cdots,$$

where

$$F_m = |F_{m1} F_{m2} \cdots F_{mm}|, \quad m = 1, 2, \cdots,$$

is a determinant of order m and with column vectors

$$(2.19) \quad F_{mj} = \left[\binom{j-1}{j-1} K^{(j)} \quad \binom{j-1}{j-2} K^{(j-1)} \cdots \binom{j-1}{0} K' \quad -1 \ 0 \ 0 \ \cdots \right],$$

$$j = 1, 2, \cdots, m,$$

where $K^{(r)} \equiv d^r K / dz^r$ and the number of zeros decreases with increasing j ; $F_{m,m-1}$ contains no more zeros, and $\binom{m-1}{0} K'$ is the last element of F_{mm} . Indeed, (2.18) is true for $m = 1$. From

$$\frac{\partial^{m-1} \bar{h}(0, z)}{\partial z^{m-1}} = F_{m-1} \bar{h}$$

we find

$$\frac{\partial^m \bar{h}(0, z)}{\partial z^m} = (F'_{m-1} - F_{m-1} K') \bar{h} \equiv F_m \bar{h},$$

and the terms F'_{m-1} and F_{m-1} correspond to the two determinants in the development of F_m by its first row. From (1.6) and (2.11) we have

$$\left. \frac{d^s K(z)}{dz^s} \right|_{z=0} = -(s-1)! \bar{a}_{0, s-1}.$$

From this and (2.18) we obtain (2.15). From (1.6), (2.6), and (2.7) we find

$$U(0, 0) = \frac{1}{2}(g_0 + \bar{g}_0).$$

We may assume g_0 to be real, *i.e.* we neglect the imaginary part of g_0 which corresponds to just one (regular) elementary solution in the decomposition of $U(z, z^*)$. Then $g_0 = U(0, 0)$. Hence

$$(2.20) \quad g(z) = 2U(z, 0) - \bar{h}(0, z)U(0, 0),$$

cf. (2.6). The coefficient of each power of z must be the same on both sides; we thus find (*cf.* (2.2) and (2.7))

$$(2.21) \quad g_m = 2U_{m0} - H_m U_{00}.$$

If we apply to (2.21) the theorem of BOREL used in the proof of Theorem 1 the statement of Theorem 2 follows.

2.2. Solutions of (1.2) which satisfy ordinary differential equations. Solutions of (1.2) which satisfy ordinary differential equations with algebraic coefficients are of particular interest, *cf.* the introduction to this paper. Such solutions can be obtained by means of *Bergman operators of the third kind*, *i.e.* operators of the form

$$(2.22) \quad u(z, z^*) = P(f) \equiv \int_{-1}^1 E(z, z^*, t) f(\tfrac{1}{2}z(1-t^2))(1-t^2)^{-\frac{1}{2}} dt$$

where the “generating function” $E(z, z^*, t)$ satisfies a linear ordinary differential equation in z and also a linear partial differential equation in t and z , *cf.* BERGMAN [3], KREYSZIG [14–16]; the “ P -associated function” $f(z)$ of $u(z, z^*)$ is an analytic function of a complex variable, regular at the origin. By inserting (2.22) into (1.2) we see that E satisfies

$$(2.23) \quad (1-t^2)(E_{z^*t} + aE_t) - t^{-1}(E_{z^*} + aE) + 2ztL(E) = 0.$$

It is remarkable that the problem of determining such solutions can be also considered from our present point of view, *i.e.* as a coefficient problem.

Theorem 3. *If the coefficients u_{mn} of the development (2.1) are such that, for a certain fixed value $z^* = z_0^*$, at most finitely many of the determinants*

$$(2.24) \quad Q_r = |Q_{r1} \ Q_{r2} \ \cdots \ Q_{rr}|, \quad r = 1, 2, \dots,$$

with column vectors

$$Q_{ri} = \left[(j-1)! \sum_{n=0}^{\infty} u_{i-1,n} z_0^{*n}, j! \sum_{n=0}^{\infty} u_{i,n} z_0^{*n}, \dots, (j+r-2)! \sum_{n=0}^{\infty} u_{i+r-2,n} z_0^{*n} \right]$$

are different from zero, then the corresponding solution $u(z, z^*)$ of (1.2) satisfies a linear ordinary differential equation whose coefficients are independent of z (but may depend on z_0^*).

Proof. The function

$$G(z) \equiv u(z, z_0^*) = \sum_{m=0}^{\infty} (m!)^{-1} u_m z^m$$

satisfies an ordinary linear homogeneous differential equation (whose coefficients are independent of z) if and only if all but a finite number of the determinants

$$D_w = \begin{vmatrix} u_0 & u_1 & \cdots & u_w \\ u_1 & u_2 & \cdots & u_{w+1} \\ \cdot & \cdot & \cdots & \cdot \\ u_w & u_{w+1} & \cdots & u_{2w} \end{vmatrix}, \quad w = 0, 1, \dots,$$

vanish, cf. E. BEKE [17]. Since we may write (2.1) in the form

$$u(z, z^*) = \sum_{m=0}^{\infty} (m!)^{-1} u_m(z^*) z^m, \quad u_m(z^*) = m! \sum_{n=0}^{\infty} u_{mn} z^{*n}, \quad m = 0, 1, \dots,$$

we thus obtain the statement of Theorem 3.

3. RELATIONS BETWEEN SUBSETS OF THE COEFFICIENTS OF THE SERIES DEVELOPMENTS OF COMPLEX AND REAL SOLUTIONS OF (1.2)

3.1. Introduction to Section 3.

3.1a. Characterization of the problem. We again consider solutions $u(z, z^*)$ and $U(z, z^*)$ of (1.2), cf. (1.4), given by their series developments

$$(3.1) \quad (a) \quad u(z, z^*) = \sum_{m,n=0}^{\infty} u_{mn} z^m z^{*n}, \quad (b) \quad U(z, z^*) = \sum_{m,n=0}^{\infty} U_{mn} z^m z^{*n}.$$

Information about singularities and other properties of the solutions can be obtained from

$$(3.2) \quad (a) \quad u(z, 0) = u_0(z) = \sum_{m=0}^{\infty} u_{m0} z^m, \quad (b) \quad U(z, 0) = U_0(z) = \sum_{m=0}^{\infty} U_{m0} z^m,$$

i.e. if the sequence (u_{m0}) , or (U_{m0}) , respectively, of the coefficients in (3.1) is known. Cf. the preceding section.

It is obvious that information on the behavior of $u(z, z^*)$ or $U(z, z^*)$ should be obtained also from the other sequences (u_{mn}) or (U_{mn}) , $n > 0$ and fixed. The basic question thus arises as to what extent such possibilities exist. This problem will be considered in what follows, starting with complex solutions $u(z, z^*)$.

For this purpose we introduce the functions

$$(3.3) \quad u_n(z) = \sum_{m=0}^{\infty} u_{mn} z^m, \quad n = 1, 2, \dots$$

We shall obtain relations between $u_0(z)$ and $u_n(z)$, $n > 0$ and fixed, i.e. the desired connection between the sequences (u_{m0}) and (u_{mn}) , $n > 0$ and fixed. We shall prove the following basic fact which characterizes the whole situation:

While at every point where $u_n(z)$ is singular, $u_0(z)$ is also singular, the converse may not be true, i.e. a point at which $u_0(z)$ is singular may be a regular point of $u_n(z)$, $n > 0$.

We thus have two cases:

(I) $u_n(z)$ has a singularity at $z = a$ if $u_0(z)$ is singular at $z = a$.

(II) $u_n(z)$ is regular at a singular point of $u_0(z)$.

We shall obtain conditions under which one of the two cases is excluded. Cf. Lemma 7, Theorems 10, 11, Lemma 12, Theorems 13, 14.

These facts are of interest, e.g. for the determination of the domain of regularity of $u(z, z^*)$ which may be based on

Theorem 4. (S. BERGMAN [1].) *If the associated function $g(z)$, cf. (1.5), is regular in a simply-connected domain B_2 containing the origin, the corresponding solution $u(z, z^*)$ of (1.2) is regular in the product domain $B_4 = B_2 \times (|z^*| < \infty)$.*

In the above Case (II) we have two possibilities (cf. Section 3.2b):

(A) For every choice of $u_0(z)$ the function $u_n(z)$, $n > 0$, is entire.

(B) Certain singular points of some special functions $u_0(z)$ correspond to regular points of $u_n(z)$, $n > 0$ and fixed.

All relations between $u_0(z)$ and $u_n(z)$ will be obtained from a system (S_n) of ordinary linear differential equations involving the functions $u_0, u_1, \dots, u_n$, cf. (3.7). There are two different ways:

(1) The functions $u_1(z), \dots, u_{n-1}(z)$ and their derivatives may be eliminated from the system (S_n) . This way will lead, in general, to a linear differential equation of order $k \leq n$, involving $u_0(z)$, $u_n(z)$ and their derivatives. Cf. Section 3.2c.

(2) The system (S_n) may be considered in its original form. Cf. Sections 3.2a, 3.2d, 3.2e.

The method (1) is not quite satisfactory. It leads to complicated conditions under which certain basic relations between $u_0(z)$ and $u_n(z)$ hold. A more important objection is that there exist exceptional cases for which the method yields no result. Hence it is necessary to determine classes of partial differential equations (1.2) for which relatively simple relations between $u_0(z)$ and $u_n(z)$ hold. This will be done by method (2). Cf. Sections 3.2d, 3.2e.

In Section 3.3 we shall extend the results to the important case of real solutions $U(z, z^*)$.

In consequence of the subsequent results the conditions on (u_{m0}) or (U_{m0}) occurring in the theorems of Section 2 can be replaced by conditions on (u_{mn}) or (U_{mn}) , $n > 0$ and fixed. This leads to a natural generalization of these and similar theorems.

3.1b. The system (S_n) of ordinary differential equations relating the functions $u_0(z)$, $u_1(z)$, $\dots$, $u_n(z)$. The power series developments

$$(3.4) \quad \begin{aligned} a(z, z^*) &= \sum_{m,n=0}^{\infty} a_{mn} z^m z^{*n}, & b(z, z^*) &= \sum_{m,n=0}^{\infty} b_{mn} z^m z^{*n}, \\ c(z, z^*) &= \sum_{m,n=0}^{\infty} c_{mn} z^m z^{*n} \end{aligned}$$

of the analytic coefficients of (1.2) can be written as power series in z^* . Using (3.3) we have

$$(3.5) \quad \begin{aligned} u(z, z^*) &= \sum_{n=0}^{\infty} u_n(z) z^{*n}, & a(z, z^*) &= \sum_{n=0}^{\infty} a_n(z) z^{*n}, \\ b(z, z^*) &= \sum_{n=0}^{\infty} b_n(z) z^{*n}, & c(z, z^*) &= \sum_{n=0}^{\infty} c_n(z) z^{*n}, \end{aligned}$$

where the coefficients are of the form

$$(3.6) \quad \begin{aligned} u_n(z) &= \sum_{m=0}^{\infty} u_{mn} z^m, & a_n(z) &= \sum_{m=0}^{\infty} a_{mn} z^m, \\ b_n(z) &= \sum_{m=0}^{\infty} b_{mn} z^m, & c_n(z) &= \sum_{m=0}^{\infty} c_{mn} z^m. \end{aligned}$$

We insert (3.5) into (1.2). In order that the resulting equation be satisfied the coefficient of every power of z^* must vanish. We thus obtain the following system (S_n) of ordinary linear differential equations

$$(3.7) \quad u'_n + b_0 u_n = \frac{1}{n} G_n(u_0, u'_0, u_1, u'_1, \dots, u_{n-1}, u'_{n-1}), \quad n = 1, 2, \dots,$$

where

$$G_n = - \left\{ a_0 u'_{n-1} + c_0 u_{n-1} + \sum_{p=0}^{n-2} (a_{n-p-1} u'_p + (p+1) b_{n-p-1} u_{p+1} + c_{n-p-1} u_p) \right\}$$

and primes denote derivatives with respect to z .

3.2. Relations between subsequences (u_{m0}) and (u_{mn}) , $n > 0$ and fixed, of the coefficients of the series development of $u(z, z^*)$.

3.2a. Behavior of $u_0(z)$ at singular points of $u_n(z)$. Let $z = d$ be a singularity

of $u_n(z)$, $n > 0$ and fixed. According to (3.7) this singularity results from an expression which involves $u_0, u'_0, \dots, u_{n-1}, u'_{n-1}$. Since the coefficients a, b , and c of (1.2) and therefore also the functions $a_p(z), b_p(z), c_p(z)$, $p = 0, 1, \dots$, cf. (3.5), are entire, one (or several) of the functions $u_s(z)$, $s < n$, must be singular at $z = d$. By repeating this conclusion we obtain the simple but important theorem which follows:

Theorem 5. *If $u_n(z)$, $n > 0$, is singular at a point, say $z = d (\neq 0)$, the function $u_0(z)$ is also singular at this point.*

Hence

$$s(n) \leq s(0), \quad n = 1, 2, \dots,$$

where $s(p)$ is the number of singular points of $u_p(z)$ in a (fixed) domain B (each point counted just once).

Corollary 6. *If $g(z) \equiv u_0(z)$ is regular in a certain simply-connected and bounded domain B then $u_n(z)$, $n = 1, 2, \dots$, is also regular in B .*

While $u_0(z)$ must be singular at a singular point $z = d$ of $u_n(z)$, the function $u_p(z)$, $0 < p < n$, can be regular at $z = d$. Example:

$$(3.8) \quad u_{zz^*} + u_{z^*} + z^{*3}u = 0;$$

here singular points of $u_4(z)$ correspond to regular points of $u_1(z), u_2(z), u_3(z)$. Thus $s(p)$ is by no means a monotone function.

From (3.7) we may easily derive the next lemma:

Lemma 7. *Suppose the developments (3.4) of the coefficients $a(z, z^*)$ and $c(z, z^*)$ of (1.2) do not involve constant terms or terms depending only on z . Let z^{**} be the smallest power of z^* occurring in these series. Then for any choice of $u_0(z)$ the functions $u_1(z), u_2(z), \dots, u_s(z)$ are entire.*

3.2b. Singular points of $u_0(z)$ at which $u_n(z)$, $n > 0$, is regular. While singular points of $u_n(z)$ are always singular points of $u_0(z)$ the converse may not be true, i.e. a singular point of $u_0(z)$ may be a regular point of $u_n(z)$. If this could not happen the determination of the domain of regularity and other properties of $u(z, z^*)$ from a given sequence (u_{mn}) , $n > 0$ and fixed, cf. (3.1), would be as easy as it is if the sequence (u_{m0}) is known. Hence that fact is of basic interest. There are two different cases:

(A) *For every choice of $u_0(z)$ the function $u_n(z)$, $n > 0$, is entire.*

(B) *Certain singular points of some special functions $u_0(z)$ correspond to regular points of $u_n(z)$, $n > 0$ and fixed.*

Case (A). In this case the functions $u_0(z)$ and $u_n(z)$ are completely independent. This case can occur if all the functions a_p, b_p, c_p in the corresponding equation (3.7) vanish identically. It is trivial that these conditions may be replaced by similar ones. (Example: u_1, u_2, u_3 in (3.8).)

In order that several subsequent functions $u_p(z)$, say $u_1, u_2, \dots, u_n$, be completely independent of u_0 for every choice of u_0 we must have

$$(3.9) \quad a_p(z) \equiv 0, \quad c_p(z) \equiv 0, \quad p = 0, 1, \dots, n-1,$$

as follows from (3.7). If these functions should also be completely independent of each other we must have

$$(3.10) \quad b_p(z) \equiv 0, \quad p = 1, 2, \dots, n-1.$$

Hence

$$u_{zz} + b(z)u_z = 0$$

is the only special form of (1.2) for which *all* the functions $u_0, u_1, \dots$ are completely independent of each other.

Case (B). Example: The solution of

$$u_{zz} + (z-d)^{k+1}u_z + u_z + (z-d)^k u = 0, \quad k > 0$$

for which $g(z) \equiv u_0(z) = (z-d)^{-k}$. Here $u_1(z)$ is regular at $z = d$ since the pole of $u_0(z)$ coincides with a zero of $a_0(z)$ and $c_0(z)$ of sufficiently large order.

In order that a singular point $z = d (\neq 0)$ of $u_0(z)$ be also a singular point of $u_1(z)$ it suffices that $a_0(d) \neq 0$, as follows from (3.7). Conversely, if $a_0(z) \equiv 0$ and $c_0(z) \equiv 0$ then, for any choice of $u_0(z)$, the function $u_1(z)$ is regular; in this case we cannot obtain information about the singularities of $u(z, z^*)$ from $u_1(z)$.

$u_1(z)$ is the only function whose singularities are related to those of $u_0(z)$ in such a simple manner. For $u_2(z), u_3(z), \dots$ the situation is more complicated. A singular point of $u_0(z)$ may be different from every zero of the functions $a_p(z), b_p(z), c_p(z), p = 1, 2, \dots$, and nevertheless be a regular point of $u_s(z), s = 2, 3, \dots$. Example:

$$(3.11) \quad u_{zz} + (1+z^*)u_z - z^*u_z + (1+z^*)u = 0;$$

here every singular point of $u_0(z)$ is also a singular point of $u_1(z)$ but a regular point of $u_2(z)$. We shall now consider the problem of relating singularities of $u_0(z)$ to those of $u_s(z), s = 2, 3, \dots$.

3.2c. Transformation of the system (S_n) into a single differential equation relating $u_0(z)$ and $u_n(z)$. Since we are interested in relating $u_0(z)$ and $u_n(z)$, $n > 0$ and fixed, it is natural to eliminate from the system (S_n) , given by (3.7), the functions $u_1, \dots, u_{n-1}$ and their derivatives. We may consider (S_n) as a system of n homogeneous equations in the "unknowns" $1, u_1(z), u_2(z), \dots, u_{n-1}(z)$ and their first derivatives; the coefficient of the "unknown" 1 consists of the terms containing $u_0(z), u_n(z)$ and their derivatives. By differentiating each equation of (S_n) a suitable number of times we shall obtain a system (S_n^*) whose number of "unknowns" is equal to the number of equations. Since (S_n^*) has a non-trivial solution, the determinant of the coefficients vanishes; we denote

this determinant by D_n . The equation $D_n = 0$ is a linear differential equation relating $u_0(z)$ and $u_n(z)$, $n > 0$ and fixed.

B. INGERSOLL [18] considered this method for the special case of (1.2) satisfying (1.3). We develop it for the general case of (1.2) and shall find a more explicit expression for the equation $D_n = 0$ than is given in [18].

There are two exceptional cases where the method of elimination does not yield a result:

- (a) *There may not exist a relation between $u_0(z)$ and $u_n(z)$.*
- (b) *There may exist such a relation but $D_n \equiv 0$.*

It is not necessary to pay particular attention to Case (b), since in this case the method of elimination can be applied to a system of fewer equations obtained from (S_n^*) .

If we differentiate each equation of (S_n) $n - 1$ times, cf. (3.7), we obtain a system (S_n^*) of n^2 equations in n^2 "unknowns". Let $\langle p, 1 \rangle$ denote the equation of (S_n) corresponding to $n = p$, and $\langle p, q \rangle$ the equation obtained by differentiating $\langle p, 1 \rangle$ $q - 1$ times. We may write (S_n^*) in the form

$$(3.12) \quad pu_p^{(q)} + \sum_{m=0}^{p-1} \left\{ \sum_{r=0}^{q-1} \binom{q-1}{r} (a_{p-m-1}^{(q-r-1)} u_m^{(r+1)} + (m+1) b_{p-m-1}^{(q-r-1)} u_{m+1}^{(r)} + c_{p-m-1}^{(q-r-1)} u_m^{(r)}) \right\} = 0, \quad p = 1, 2, \dots, n, \quad q = 1, 2, \dots, n.$$

Here $u_p^{(a)} \equiv d^a u_p / dz^a$, etc., and each quantity with a negative subscript or superscript is zero. We now arrange each equation so that the terms involving u_0 , u_n and their derivatives occur first. In $\langle p, 1 \rangle$ the terms containing u_0 and u'_0 are of the form

$$E_{p1} = a_{p-1} u'_0 + c_{p-1} u_0, \quad p = 1, 2, \dots, n,$$

cf. (3.7). From this we obtain the terms of $\langle p, q \rangle$ involving u_0 and its derivatives in the form

$$(3.13) \quad E_{pq} = \sum_{r=0}^{q-1} \binom{q-1}{r} (a_{p-1}^{(q-r-1)} u_0^{(r+1)} + c_{p-1}^{(q-r-1)} u_0^{(r)}) \equiv \sum_{k=0}^n s_{pq}(k) u_0^{(k)},$$

$$p, q = 1, 2, \dots, n,$$

where

$$(3.14) \quad s_{pq}(k) = \begin{cases} 0 & (k > q) \\ a_{p-1} & (k = q) \\ \binom{q-1}{k-1} a_{p-1}^{(q-k)} + \binom{q-1}{k} c_{p-1}^{(q-k-1)} & (q > k > 0) \\ c_{p-1}^{(q-1)} & (k = 0) \end{cases}$$

is the coefficient of $u_0^{(k)}$ in $\langle p, q \rangle$. The function u_n and its derivatives occur only

in $\langle n, 1 \rangle, \dots, \langle n, n \rangle$. In $\langle n, 1 \rangle$ the function F_1 involving u_n and u'_n has the form

$$F_1 = n(u'_n + b_0 u_n).$$

In $\langle n, q \rangle$ the function F_q involving u_n and its derivatives thus has the form

$$(3.15) \quad F_q = n \left\{ u_n^{(q)} + \sum_{r=0}^{q-1} \binom{q-1}{r} b_0^{(q-r-1)} u_n^{(r)} \right\} \equiv \sum_{k=0}^n t_q(k) u_n^{(k)},$$

$q = 1, 2, \dots, n,$

where

$$(3.16) \quad t_q(k) = \begin{cases} 0 & (k > q) \\ n & (k = q) \\ n \binom{q-1}{k} b_0^{(q-k-1)} & (k < q) \end{cases}$$

is the coefficient of $u_n^{(k)}$ in $\langle n, q \rangle$. From (3.12) we see that in $\langle p, q \rangle$ the function $u_j^{(k)}$ has the coefficient

$$(3.17) \quad r_{pq}(j, k) = \begin{cases} p & (j, k) = (p, q) \\ \binom{q-1}{k-1} a_{p-j-1}^{(q-k)} + \binom{q-1}{k} (j b_{p-j}^{(q-k-1)} + c_{p-j-1}^{(q-k-1)}) & \begin{cases} 0 < j \leq p < n, \\ 0 < k \leq q, \end{cases} \\ 0 & \text{otherwise,} \end{cases}$$

where each quantity with a negative subscript or superscript is zero.

We now consider the equation $D_n = 0$. If we develop D_n by its first column we obtain

$$(3.18) \quad \sum_{p, q=1}^n E_{pq} D_{n, pq} + \sum_{q=1}^n F_q D_{n, nq} = 0;$$

here $D_{n, pq}$ is the cofactor in D_n of the element in the first column corresponding to equation $\langle p, q \rangle$. From this and (3.13), (3.15) we have

$$(3.19) \quad \sum_{k=0}^n (J_k u_0^{(k)} + L_k u_n^{(k)}) = 0$$

where

$$(3.20) \quad J_k = \sum_{p, q=1}^n s_{pq}(k) D_{n, pq}, \quad L_k = \sum_{q=1}^n t_q(k) D_{n, nq}, \quad k = 0, 1, \dots, n.$$

If $L_k \equiv 0$, $k = 0, \dots, n$, then $J_k \equiv 0$, $k = 0, \dots, n$, must hold; otherwise u_0 would be the same for solutions $u(z, z^*)$ of (1.2) corresponding to any function $u_n(z)$ whatsoever. Hence $D_n \equiv 0$; we thus have one of the above cases (a) or (b). If $J_k \equiv 0$, $k = 0, \dots, n$, but $L_k \not\equiv 0$ for at least one value of k then (3.19)

reduces to a differential equation by which $u_n(z)$ is determined, regardless of the special form of $u_0(z)$; we thus have case (a). Excluding these cases, at least one function J_k and at least one function L_k do not vanish identically. If $k = s$ is the largest value for which $J_k \not\equiv 0$ then (3.19) has the order s ($\leq n$) and we may write

$$(3.21) \quad \sum_{k=0}^s \frac{J_k}{J_s} u_0^{(k)} + P(u_n, u_n', \dots, u_n^{(n)}) = 0, \quad P = \sum_{k=0}^n \frac{L_k}{J_s} u_n^{(k)}.$$

The general solution $u_0(z)$ of (3.21) is regular in the intersection of the domains of regularity of the coefficients and P . Since J_k and L_k are entire functions we thus obtain

Theorem 8. *Let there exist a number k for which $J_k \not\equiv 0$, cf. (3.20), and let $k = s$ be the largest number with this property. If $u_n(z)$ is regular at a point $z = d$ at which $J_s \not\equiv 0$ then the function $u_0(z)$ is regular at $z = d$.*

Hence the only regular points of $u_n(z)$ which may correspond to singular points of $u_0(z)$ are the zeros of J_s .

From Theorems 4 and 8 we find (cf. also INGERSOLL [18])

Theorem 9. *Let $u_n(z)$ be regular in a simply connected domain B'_2 containing the origin. Let J_k satisfy the condition stated in Theorem 8. Let z_a , $a = 1, 2, \dots$, be the zeros of J_s in B'_2 . Then $u(z, z^*)$ is regular in $B_4 = B_2 \times (|z^*| < \infty)$ where $B_2 = B'_2 - \{z_a\}$.*

3.2d. Singular points of $u_0(z)$ at which $u_n(z)$ is singular. The hypotheses in Theorem 8 are relatively complicated, even for small values of n . For interesting classes of partial differential equations (1.2) simpler theorems can be obtained from the system (S_n) in its original form (3.7). In Section 3.2a we have proven that every singular point of $u_n(z)$ is also a singular point of $u_0(z)$. We now consider the converse problem.

Theorem 10. *Let $u(z, z^*)$ be a solution of (1.2) for which $g(z) \equiv u_0(z)$ has a pole of order $r > 1$ at a point, say $z = d$ ($\neq 0$). Let*

$$P_1(z) = a_0(z), \quad P_{n+1}(z) = a_n(z) - \sum_{m=0}^{n-1} \frac{a_m}{n-m} P_{n-m}(z), \quad n = 1, 2, \dots,$$

cf. (3.6). Then, if

$$(3.22) \quad P_m(d) \neq 0, \quad m = 1, 2, \dots, n,$$

the function $u_n(z)$ is singular at $z = d$; it consists of a sum of terms one of which has a pole of order r at $z = d$ and the others may have poles of lower order or logarithmic singularities.

Proof. Setting $n = 1$ we have from (3.7)

$$(3.23) \quad u'_1 + b_0 u_1 = -(a_0 u'_0 + c_0 u_0).$$

Since $a_0(z)$ and $c_0(z)$ are entire and $a_0(d) \neq 0$, cf. (3.22), the statement is true for $u_1(z)$. We assume the statement to be true for $u_{n-1}(z)$, i.e. we assume that $u_1, \dots, u_{n-1}$ have poles of r^{th} order (and possibly logarithmic singularities) at $z = d$. (3.7) yields

$$(3.24) \quad u_n(z) = \frac{1}{nR} \left\{ \int R \left(\sum_{m=0}^{n-1} (a_{n-m-1}u'_m + c_{n-m-1}u_m) + \sum_{m=0}^{n-2} (m+1)b_{n-m-1}u_{m+1} \right) dz + k_n \right\},$$

where

$$R = \exp \{ \int b_0 dz \}, \quad k_n = \text{const.}$$

We may consider only terms of the integrand with poles of order $r+1$, i.e. the function

$$(3.25) \quad h_n(z) = \sum_{m=0}^{n-1} a_{n-m-1}u'_m;$$

poles of lower order and logarithmic singularities may occur in the other terms, but these terms cannot compensate those in (3.25) since a_p, b_p, c_p are entire functions. We write (3.7) in the form

$$(3.26) \quad u'_m = -\frac{1}{m} \sum_{p=0}^{m-1} a_{m-p-1}u'_p + T_m(u_0, \dots, u_m)$$

and insert this expression for u'_m into (3.25); for the same reasons as above we may leave aside the function T_m since it involves only terms with poles of order $q \leq r$ or logarithmic singularities at $z = d$. We finally obtain in this way the function

$$(3.27) \quad H_n(z) = P_n(z)u'_0(z).$$

From this and (3.22) it follows that $u_n(z)$ has a pole of r^{th} order at $z = d$. Logarithmic singularities may occur from the terms which were left aside. This completes the proof.

It can easily be checked whether a given equation (1.2) satisfies the assumptions of Theorem 10. For example, if the coefficients of (1.2) depend on z and z^* and in particular are linear functions of z^* we have only the two conditions

$$a_0(d) \neq 0, \quad a_1(d) - a_0^2(d) \neq 0.$$

In a similar way we may prove

Theorem 11. *Let $u(z, z^*)$ be a solution of (1.2) for which $u_0(z)$ has a pole of r^{th} order at a point $z = d$ ($\neq 0$). If*

$$(3.28) \quad pb_1(d) + c_0(d) \neq 0, \quad a_p(d) \neq 0, \quad c_p(d) \neq 0, \quad p = 0, 1, \dots, r-1,$$

then $u_p(z)$, $p = 1, 2, \dots, r-1$, has at $z = d$ a pole of at least $(r-p)^{\text{th}}$ order, and $u_r(z)$ has a logarithmic singularity at this point.

3.2e. Relations between $u_0(z)$ and $u_n(z)$ if the coefficients of the partial differential equation (1.2) depend on only one variable. We now consider the important special case where the coefficients a , b , and c of (1.2) depend on only one of the variables z or z^* .

Lemma 12. *Let $u(z, z^*)$ be a solution of*

$$(3.29) \quad u_{zz^*} + a(z)u_z + b(z)u_{z^*} + c(z)u = 0.$$

Let $z = d (\neq 0)$ be an arbitrary point at which $a(z) \neq 0$. Then $u_n(z)$ has a singularity at $z = d$ if and only if $u_0(z)$ is singular at $z = d$. If moreover $c(d) \neq 0$ then a pole of order $r > 1$ of $u_0(z)$ at $z = d$ corresponds to a pole of the same order of $u_n(z)$, $n < r$, at $z = d$ and to a logarithmic singularity of $u_n(z)$, $n \geq r$. If $a(d) \neq 0$ and $c(z) \equiv 0$ then a pole of r^{th} order of $u_0(z)$ at $z = d$ corresponds to a pole of $u_n(z)$ of the same order and location.

Proof. We have $a_p = b_p = c_p = 0$, $p \geq 1$. Thus (3.7) takes the form

$$(3.30) \quad u'_n + b_0 u_n = -\frac{1}{n} (a_0 u'_{n-1} + c_0 u_{n-1}).$$

A pole of u_{n-1} of s^{th} order at $z = d (\neq 0)$ corresponds to a pole of u'_{n-1} of $(s+1)^{\text{th}}$ order at $z = d$. Since a_0 and c_0 are entire this pole may be cancelled if and only if it coincides with a zero of a_0 . If we exclude this possibility, $u_n(z)$ has a pole of s^{th} order at $z = d$ and possibly a logarithmic singularity which may come from the second term on the right hand side of (3.30). From this the statement follows by induction.

If we combine Lemma 12 and Theorem 4 we obtain

Theorem 13. *Let B_2 be a simply-connected domain containing the origin but no zero of $a(z)$. Then a solution $u(z, z^*)$ of (3.29) is regular in the domain $B_4 = B_2 \times (|z^*| < \infty)$ of zz^* -space if and only if the corresponding function $u_n(z)$ is regular in B_2 .*

We may similarly prove

Theorem 14. *Let $u(z, z^*)$ be a solution of*

$$(3.31) \quad u_{zz^*} + a(z^*)u_z + c(z^*)u = 0, \quad c(0) \neq 0.$$

If the corresponding function $u_0(z)$ has a pole of k^{th} order at a point $z = p (\neq 0)$ the function $u_n(z)$, $n = 1, 2, \dots$, is singular at $z = p$; in particular $u_m(z)$, $m \geq k$, has a logarithmic singularity at $z = p$.

3.3. Extension of the results of Section 3.2 to real solutions of (1.2). We shall now assume z_1 and z_2 to be real. Then z and z^* are conjugate, i.e. $z^* = \bar{z}$. In order to extend the results of Section 3.2 to the important case of real solutions $U(z, z^*)$ of (1.2), cf. (1.3), (1.4), we write (3.1) and (3.4) in the form

$$(3.32) \quad \begin{aligned} u(z, \bar{z}) &= \sum_{m=0}^{\infty} \tilde{u}_m(\bar{z}) z^m, & a(z, \bar{z}) &= \sum_{m=0}^{\infty} \tilde{a}_m(\bar{z}) z^m, \\ b(z, \bar{z}) &= \sum_{m=0}^{\infty} \tilde{b}_m(\bar{z}) z^m, & c(z, \bar{z}) &= \sum_{m=0}^{\infty} \tilde{c}_m(\bar{z}) z^m, \end{aligned}$$

where

$$(3.33) \quad \begin{aligned} \tilde{u}_m(\bar{z}) &= \sum_{n=0}^{\infty} u_{mn} \bar{z}^n, & \tilde{a}_m(\bar{z}) &= \sum_{n=0}^{\infty} a_{mn} \bar{z}^n, \\ \tilde{b}_m(\bar{z}) &= \sum_{n=0}^{\infty} b_{mn} \bar{z}^n, & \tilde{c}_m(\bar{z}) &= \sum_{n=0}^{\infty} c_{mn} \bar{z}^n. \end{aligned}$$

If we insert this into (1.2) we obtain

$$(3.34) \quad \tilde{u}'_m + \tilde{a}_0 \tilde{u}_m = m^{-1} \tilde{G}_m(\tilde{u}_0, \tilde{u}'_0, \dots, \tilde{u}_{m-1}, \tilde{u}'_{m-1}), \quad m = 1, 2, \dots,$$

where

$$\tilde{G}_m = - \left\{ \tilde{b}_0 \tilde{u}'_{m-1} + \tilde{c}_0 \tilde{u}_{m-1} + \sum_{r=0}^{m-2} [(r+1) \tilde{a}_{m-r-1} \tilde{u}_{r+1} + \tilde{b}_{m-r-1} \tilde{u}'_r + \tilde{c}_{m-r-1} \tilde{u}_r] \right\}$$

and primes denote derivatives with respect to $\bar{z}$.

From this we find (cf. also E. NETANYAHU [19])

Theorem 15. *Let $U(z, \bar{z})$ and $u(z, \bar{z})$ be two solutions of (1.2) related by (1.4). Then corresponding functions*

$$(3.35) \quad U_n(z) = \sum_{m=0}^{\infty} U_{nm} z^m, \quad n = 0, 1, \dots,$$

cf. (3.1b), and $u_n(z)$, cf. (3.6), have the same singularities.

Proof. From (1.5) and (2.7) we obtain

$$(3.36) \quad \tilde{u}_0(\bar{z}) = u(0, \bar{z}) = g_0 \exp \left\{ - \int a(0, \bar{z}) d\bar{z} \right\}.$$

Since $a(z, \bar{z})$ is entire, $\tilde{G}_1 = -(\tilde{b}_0 \tilde{u}'_0 + \tilde{c}_0 \tilde{u}_0)$, cf. (3.34), is also entire. We make the induction hypothesis that $\tilde{G}_m(\bar{z})$ is entire. Then the solution

$$\tilde{u}_m(\bar{z}) = \frac{1}{m} \exp \left\{ - \int \tilde{a}_0(\bar{z}) d\bar{z} \right\} \left[\int \tilde{G}_m(\bar{z}) \exp \left\{ \int \tilde{a}_0(\bar{z}) d\bar{z} \right\} d\bar{z} + k_m \right]$$

of (3.34) is also entire. Hence the functions $\tilde{u}_m(\bar{z})$ are entire. If we compare corresponding terms of the developments (3.1) we find

$$U_{mn} = \frac{1}{2} (u_{mn} + \bar{u}_{nm}).$$

From this it follows that

$$(3.37) \quad U_n = \frac{1}{2} (u_n + \bar{u}_n), \quad n = 0, 1, \dots$$

Since $\bar{u}_n$ is an entire function of z the statement thus follows.

In consequence of this theorem the results of Section 3.2 may be formulated as theorems about $U(z, \bar{z})$; this extension is immediate.

3.4. Partial differential equations (1.2) with singular coefficients. If (1.2) has entire coefficients the number of singular points of $u_n(z)$ in a domain B (each point counted just once) cannot exceed the number of singular points of $u_0(z)$ in B , cf. Section 3.2a. This is no longer true if (1.2) has coefficients with singularities, as can be seen from (3.7). To a function $u_0(z)$ regular in B there will then correspond, in general, a function $u_n(z)$ with singularities in B . Similarly while in the case of entire coefficients of (1.2) the functions $\tilde{u}_m(z^*)$, cf. (3.33), are entire, in the case of singular coefficients they may have singularities. The investigation of this problem may be based upon either the respective systems (3.7) and (3.34) or a single differential equation obtained from these systems (cf. Section 3.2c) provided such an equation exists. In this paper we confine ourselves to some remarks.

The fact that (1.2) was assumed to have entire coefficients was essential to the proofs in Section 3.2. From this we drew the conclusion that certain functions are regular or, what is more important, that certain singularities cannot be cancelled. Some singular points of $u_0(z)$, which in the case of entire coefficients are singular points of $u_n(z)$, may be regular points of $u_n(z)$ if (1.2) has singular coefficients. In consequence of this situation it is not permissible to split the present problem into two parts and consider the singularities of $u_n(z)$ resulting from singular points of $u_0(z)$ separately from those caused by the singularities of the coefficients of (1.2).

If $a(z, z^*)$, considered as a function of z^* , has a pole of order $r > 1$ at a point, say $z^* = d^*$, then $\tilde{u}_0(z^*)$ has an essential singularity at this point. If $b(z, z^*)$ and $c(z, z^*)$ are entire and $\tilde{u}_n(z^*)$ has a singularity at $z^* = d^*$ then $\tilde{u}_0(z^*)$ and $\tilde{a}_0(z^*)$ are singular at $z^* = d^*$. These and similar relations may be derived from (3.34) and (3.36).

Theorem 16. *Let the coefficients of (1.2) depend on z^* only. If $a(z^*)$ has a pole of order $r > 1$ at a point $z^* = p^*$ at which $b(z^*)$ and $c(z^*)$ are regular and $b(z^*)$ is different from zero then the functions $\tilde{u}_n(z^*)$, $n = 0, 1, \dots$, have essential singularities at $z^* = p^*$.*

Proof. From (3.36) we have

$$\tilde{u}_0'(z^*) = -\tilde{a}_0(z^*)\tilde{u}_0(z^*).$$

For $m = 1$ (3.34) thus takes the form

$$\tilde{u}_1' + \tilde{a}_0\tilde{u}_1 = W_1\tilde{u}_0, \quad W_1 = \tilde{a}_0\tilde{b}_0 - \tilde{c}_0.$$

Noting (3.36) we obtain the solution

$$\tilde{u}_1(z^*) = V_1(z^*)\tilde{u}_0(z^*), \quad V_1(z^*) = \int W_1(z^*) dz^* + k_1,$$

where k_1 is a constant. We may set

$$(3.38) \quad \tilde{u}_m(z^*) = V_m(z^*)\tilde{u}_0(z^*), \quad m = 2, 3, \dots$$

Since a , b , and c do not depend on z we have $\tilde{a}_p = \tilde{b}_p = \tilde{c}_p = 0$, $p = 1, 2, \dots$. Hence (3.34) reduces to

$$(3.39) \quad \tilde{u}'_m + \tilde{a}_0 \tilde{u}_m = W_m \tilde{u}_0, \quad W_m = m^{-1}[\tilde{b}_0(\tilde{a}_0 V_{m-1} - W_{m-1}) - \tilde{c}_0 V_{m-1}].$$

We prove by induction that $W_m(z^*)$ has a pole at $z^* = p^*$. Since $\tilde{c}_0(z^*)$ is regular at $z^* = p^*$ and $\tilde{b}_0(p^*) \neq 0$, the function $W_1(z^*)$ has a pole of order $s_1 = r$ at $z^* = p^*$. We now assume that $W_{m-1}(z^*)$ has a pole of order $s_{m-1} (> 1)$ at $z^* = p^*$. Then the function

$$V_{m-1}(z^*) = \int W_{m-1}(z^*) dz^* + k_{m-1}$$

has a pole of order $s_{m-1} - 1$ at $z^* = p^*$. The terms $\tilde{a}_0 V_{m-1}$ and $\tilde{c}_0 V_{m-1}$ in (3.39) thus have poles of different orders $s_{m-1} + r - 1$ and $s_{m-1} - 1$, respectively. Hence these poles cannot be cancelled. Since $\tilde{b}_0(p^*) \neq 0$ the functions $W_m(z^*)$ and $V_m(z^*)$ thus have poles at $z^* = p^*$. The pole of $\tilde{a}(z^*)$ at $z^* = p^*$ corresponds to an essential singularity of $\tilde{u}_0(z^*)$. From this and (3.38) the statement follows.

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