On the Calculation of Steady Laminar Boundary Layer Flows with Continuous Suction

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In Memoriam HERBERT BILHARZ (1910–1956)

1. Introduction. We consider the plane and steady laminar boundary layer flow of an incompressible viscous fluid around a cylinder in the general case of arbitrary outer velocity distribution $U(x)$ and arbitrary suction velocity distribution $v_0(x)$ along the wall. (x denotes the distance along the wall from the forward stagnation point or, in case of a flat plate or cuspidal leading edge, from the leading edge; for $v_0(x) < 0$ we have suction, but $v_0(x) > 0$, blowing out, will also be permitted.)

It is the aim of this paper to give a formally exact solution of the boundary layer equations for this very general problem in the form of an infinite series and, furthermore, to show how a method for the numerical evaluation of such flows can be derived from this series. The main practical result will be that the application of this method to any special problem will prove to be extremely simple. The amount of work involved in calculating the boundary layer flow for any given $U(x)$ and $v_0(x)$ will, in fact, be comparable with that of the non-rigorous and unreliable parameter methods of the momentum integral type. The reason for this simplicity of application is the fact that the method allows the main part of the work to be carried out once and for all by preparing tables of certain universal functions. As soon as these are available the application of the method to any special problem will consist only in the introduction of the given data of the special case into given formulæ and the application of the tables. It is necessary to assume that certain rather general series expansions for $U(x)$ and $v_0(x)$ are possible, and the coefficients of these series are, together with the kinematic viscosity ν of the fluid, the quantities considered to be given in any special case of application.

Till now, exact solutions of the problem of boundary layer flow with suction have been found only for a few special cases, special with respect to the given outer velocity $U(x)$ and suction velocity $v_0(x)$; *cf. e.g.* [1] or [2]. Thus, for outer velocities of the form $U(x) = \text{const.} \cdot x^m$ and suction velocities prescribed accord-

ingly in the form $v_0(x) = \text{const.} \cdot x^{\frac{1}{2}(m-1)}$ the problem can be reduced to that of the ordinary FALKNER-SKAN-HARTREE differential equation. Beside these *similar* solutions the non-similar case of the flow along a semi-infinite flat plate with homogeneous suction ($U(x)$ and $v_0(x)$ constant) should be mentioned here, this special case having been given a rigorous treatment by R. IGLISCH [3]. Apart from these rigorous results considerable efforts have been made in applying non-rigorous parameter methods to calculate boundary layer flows with suction approximately in more general cases of interest. Such methods are unsatisfactory aside from the mathematical viewpoint. They are not only unreliable but also necessarily limited in accuracy. High accuracy, however, is often required, *e.g.* for the investigation of stability of such flows.

Numerical difference procedures (*cf.* [4], [5]), applied till now with good success to flows along impermeable walls, can also readily be applied to cases of continuous suction, but the amount of numerical work involved is considerable. They have, in fact, recently been applied in connection with the still more complicated problem of boundary layer flows with discontinuous suction distribution by W. RHEINBOLDT in his Freiburg thesis [6]. In the immediate neighborhood of a boundary layer cross-section where $v_0(x)$ is discontinuous (*e.g.* where a piece of impermeable wall is followed by a piece of porous wall or *vice versa*) RHEINBOLDT applies an interesting series method, and further downstream from this sudden change of conditions the difference procedure can again be applied.

In order that boundary layer theory remain applicable it is necessary to restrict investigations to relatively weak suction in the sense that only those particles within the boundary layer that are near to the wall are sucked away through the wall. It is well known from experiment that the effect of such weak suction is considerable and of special practical interest. In the following we may therefore assume that v_0 is small in magnitude compared to U .

The method that will be developed in this paper is an extension of a new series method recently published by the author [7] for the calculation of boundary layers along impermeable walls. This series represents a formally exact solution for any given outer velocity $U(x)$. Two main features of this series solution shall be mentioned here: (1) By a suitable choice of variables it has been possible to fulfill exactly the outer boundary condition for any $U(x)$ with the zero order term of the series; thus, the higher order terms give corrections only in the inner part of the boundary layer, and the zero order term alone already gives a good approximation from the forward point $x = 0$ surprisingly far downstream. (2) The coefficient functions of the series can be represented as linear combinations of universal functions for which tables of high accuracy are available for orders 0 through 5 for flows around cylinders with rounded noses as well as around cylinders with forward cuspidal leading edge (including flat plates with arbitrary outer pressure distribution). The work involved in applying the series method (within the fifth order approximation attainable) is thus reduced to a few simple numerical operations.

The extension of the new series method to problems of boundary layer flows with suction will then be of practical value only if it is possible to preserve the properties mentioned above. It is the special object of this paper to show that this is possible. Of course, it must be expected that the number of universal functions will be larger than in the case of no suction. Along with the arbitrary outer velocity the suction velocity now enters into the problem as a further arbitrary function.

The success of the new series method in many applications (*cf.* [7, 9, 10]), especially in cases of flat plate flows, seems to justify the tabulation of the universal functions for the treatment of flows with suction. It is hoped that such tables will soon be available. The universal functions of the impermeable wall are, without any change, also universal functions for the case of suction through the wall. Therefore only the additional universal functions due to suction must still be evaluated.

2. Transformation of the boundary layer equations. For the plane steady flow of an incompressible viscous fluid in a boundary layer we have

$$(1) \quad \begin{aligned} uu_x + vu_y &= U(x)U'(x) + \nu u_{yy}, \\ u_x + v_y &= 0 \end{aligned}$$

with the boundary conditions

$$(2) \quad \begin{aligned} u &= 0, \quad v = v_0(x) \quad \text{for } y = 0, \\ u &\rightarrow U(x) \quad \text{for } y\sqrt{\text{Re}} \rightarrow \infty. \end{aligned}$$

Here x denotes, as already stated above, the wall curve length measured from the forward singular point of the flow at which the fluid is parted when meeting the surface of the cylinder. $x = 0$ is usually a stagnation point (cases of cylinders with rounded noses or non-zero forward vertex angle of contour) but can also be a forward cuspidal point or the forward leading edge of a flat plate. y is the distance from the wall along the normal. u and v are the x and y components of the flow velocity in the boundary layer, $U(x)$ the given outer velocity, $v_0(x)$ the distribution of suction velocity along the wall, ν the kinematic viscosity of the fluid, and Re denotes a characteristic REYNOLDS number of the problem in question. The subscripts denote partial differentiation and U' is dU/dx .

We now introduce the same independent and dependent variables as in our former treatment of flows without suction [7]. We repeat the transformation here in order to take into account the changes due to suction. The reasons for our choice of variables are fully explained in our former paper [7]. As independent variables we take the dimensionless quantities

$$(3) \quad \xi = \frac{1}{\nu} \int_0^x U(x) dx, \quad \eta = \frac{U(x)y/\nu}{\left(\frac{2}{\nu} \int_0^x U(x) dx\right)^{\frac{1}{2}}}.$$

Let $\psi(x, y)$ denote the stream function of the boundary layer flow ($u = \psi_y$, $v = -\psi_x$). As a new dependent variable we then choose the dimensionless quantity

$$(4) \quad F(\xi, \eta) = \frac{\psi(x, y)}{\left(2\nu \int_0^x U(x) dx\right)^{\frac{1}{2}}}.$$

Thus

$$(5) \quad \psi(x, y) = \nu \sqrt{2\xi} F(\xi, \eta), \quad \frac{u(x, y)}{U(x)} = F_\eta(\xi, \eta),$$

$$\frac{v(x, y)}{U(x)} = -\frac{1}{\sqrt{2\xi}} \{F(\xi, \eta) + 2\xi F_\xi(\xi, \eta) + (\beta(\xi) - 1)\eta F_\eta(\xi, \eta)\}.$$

Here

$$(6) \quad \beta(\xi) = \frac{2U'(x) \int_0^x U(x) dx}{U^2(x)}.$$

In [7] we have referred to this function as the *principal function* of the boundary layer.

By the introduction of the stream function the second equation (1) is fulfilled (equation of continuity), and with the variables (3), (4) the first equation (1) becomes

$$(7) \quad F_{\eta\eta\eta} + FF_{\eta\eta} + \beta(\xi)\{1 - F_\eta^2\} = 2\xi\{F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta}\}.$$

The boundary conditions (2) are equivalent to

$$(8) \quad F_\eta(\xi, 0) = 0, \quad F(\xi, 0) + 2\xi F_\xi(\xi, 0) = \gamma(\xi)$$

where, in the following,

$$(9) \quad \gamma(\xi) = -\frac{v_0(x)}{U(x)} \cdot \left(\frac{2}{\nu} \int_0^x U(x) dx\right)^{\frac{1}{2}}$$

will be called the *suction function*.* We have

$$(10) \quad \frac{v_0(x)}{U(x)} = -\frac{\gamma(\xi)}{\sqrt{2\xi}}.$$

If our assumption $|v_0(x)/U(x)| \ll 1$ is to be fulfilled for all values of x along the wall, it is readily seen that, according to (10), this implies $\gamma(\xi) = O(\xi^{\frac{1}{2}})$ for $\xi \rightarrow 0$.

For the purpose of checking the accuracy of numerical evaluations the first and sometimes also the second condition of compatibility at the wall (in German: *Grenzschichtbindungen*) are applied, the fulfillment of these conditions

* $\gamma(\xi) > 0$ for suction, $\gamma(\xi) < 0$ for blowing out.

being a necessary requirement for an exact solution. We note here that, in our new variables, these conditions are

$$(11a) \quad F_{\eta\eta\eta}(\xi, 0) + \gamma(\xi)F_{\eta\eta}(\xi, 0) = -\beta(\xi)$$

and

$$(11b) \quad F_{\eta\eta\eta\eta}(\xi, 0) + \gamma(\xi)F_{\eta\eta\eta}(\xi, 0) = 0.$$

In order to extend the method of [7] to our present problem we assume that the principal function and the suction function allow expansions in power series with respect to ξ convergent in the interval of interest, $0 \leq \xi \leq \xi_0$ ($0 \leq x \leq x_0$ with $\xi_0 = \xi(x_0)$). For these series we write

$$(12) \quad \beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k$$

and, for reasons of convenience at a later stage,

$$(13) \quad \gamma(\xi) = \sum_{k=0}^{\infty} (2k+1)\gamma_k \xi^k.$$

These two assumptions are equivalent to restrictions on the classes of outer velocities $U(x)$ and of suction velocity distributions $v_0(x)$. From the practical viewpoint of applications these restrictions prove to be sufficiently weak still to allow consideration of very general classes of functions $U(x)$ and $v_0(x)$.

The restriction on the class of outer velocities $U(x)$ due to assumption (12) has already been investigated in [7]. It was found that (12) is equivalent to assuming, for $U(x)$, a convergent series of the form

$$(14) \quad U(x) = x^m \{s_0 + s_1 x^{m+1} + s_2 x^{2(m+1)} + \dots\}$$

where

$$m = \frac{\beta_0}{2 - \beta_0} \quad \left(\beta_0 = \frac{2m}{m+1} \right) \quad \text{and} \quad s_0 \neq 0.$$

For the case $m = 0$ ($\beta_0 = 0$) of flows around contours with forward cuspidal point $x = 0$ (e.g. flows along flat plates), therefore, any arbitrary velocity distribution $U(x)$ allowing a power series expansion $U(x) = \sum_{k=0}^{\infty} s_k x^k$ with $s_0 \neq 0$, convergent in the interval of interest, is permissible. For the case $m = 1$ ($\beta_0 = 1$), the second case of major importance, the case of rounded nose $x = 0$, the restriction (14) is $U(x) = \sum_{k=0}^{\infty} s_k x^{2k+1}$ with $s_0 \neq 0$; any odd function of x for which this power series is convergent is permissible, leading to all flows of practical interest around contours with rounded noses and symmetrical with respect to the direction of the oncoming stream. The restricting condition of symmetry found here can easily be overcome by a simple generalization of the assumption (12) already studied and applied in [7]. (It is only necessary to assume, instead of (12), that $\beta(\xi) = \sum_{k=0}^{\infty} \beta_{\frac{1}{2}k} \xi^{\frac{1}{2}k}$. For $m = 1$ the equivalent class of permissible outer velocities is then $U(x) = \sum_{k=1}^{\infty} u_k x^k$, with arbi-

trary $u_k(u_1 \neq 0)$. Only for reasons of simplicity in the following representation do we confine ourselves to the assumption (12). There is no difficulty in carrying out the extension to the weaker condition just mentioned.) For general values of m in $0 \leq m \leq 1$ ($0 \leq \beta_0 \leq 1$) the outer velocities $U(x)$ of the form (14) are found along relatively arbitrary contours with forward vertex at $x = 0$ (vertex angle $\pi\beta_0$).

In the special case of outer velocity $U(x) = s_0 x^m$ the principal function reduces to $\beta(\xi) = \beta_0 = \text{const.}$ The solutions of the boundary layer problems for this class of outer velocities and impermeable walls are the well known *similar* solutions with $F = F_0(\eta)$ a function of η alone. More generally, allowing suction or blowing out through the wall under this class of outer velocities, similar solutions are found if and only if the suction velocity distribution is chosen accordingly as $v_0(x) = \text{const.} \cdot x^{\frac{1}{2}(m-1)}$ (cf. No. 3).

The relations between the coefficients β_k of (12) and the coefficients s_k of (14) for the general case can easily be calculated. They have been given [7] up to a sufficiently high order for practical applications.

We must now discuss in the same manner the class of suction velocity distributions $v_0(x)$ permissible under the assumption (13) (in connection with (12)).

3. Relations between the suction function coefficients and the coefficients of the prescribed outer and suction velocities. Restricting ourselves now to prescribed outer velocity distributions of the class (14) in accordance with the assumed class of principal functions (12), we find from definition (3)

$$(15) \quad \xi = \frac{x^{m+1}}{\nu(m+1)} \sum_{k=0}^{\infty} \frac{s_k}{k+1} x^{k(m+1)}.$$

Thus, the suction functions of class (13) can be expanded in power series with respect to x^{m+1} :

$$(16) \quad \gamma(\xi) = \sum_{k=0}^{\infty} (2k+1) \gamma_k \xi^k = \sum_{k=0}^{\infty} c_k x^{k(m+1)}.$$

The relations between the coefficients γ_k and c_k , starting with

$$(17) \quad c_0 = \gamma_0, \quad c_1 = \frac{3\gamma_1 s_0}{\nu(m+1)},$$

are given in Appendix 1 up to the order $k = 5$ although they will not be of importance for applications but only of interest for checking the further results stated in that appendix.

Now, by definition (9) of the suction function, taking (16) and again (14) into account, we easily find that the class of permissible suction velocity distributions is given by all convergent series of the form

$$(18) \quad v_0(x) = x^{\frac{1}{2}(m-1)} \sum_{k=0}^{\infty} \sigma_k x^{k(m+1)}.$$

The coefficients σ_k ($k = 0, 1, 2, \dots$) can easily be found by comparing coefficients in (9) or, and that is the same, in

$$(19) \quad -\sum_{k=0}^{\infty} c_k u^k \cdot \sum_{k=0}^{\infty} s_k u^k = \sqrt{\frac{2}{\nu(m+1)}} \sqrt{\sum_{k=0}^{\infty} \frac{s_k u^k}{k+1}} \cdot \sum_{k=0}^{\infty} \sigma_k u^k$$

where, for abbreviation, we have set $u = x^{m+1}$. Thus, for given s_k and γ_k (given $U(x)$ and $\gamma(\xi)$) the coefficients σ_k of $v_0(x)$ can be calculated. The results for $k = 0, 1, \dots, 5$ are given in Appendix 1. As $U(x)$ and $v_0(x)$ usually will be prescribed we also give, in Appendix 1, the corresponding formulæ for the coefficients γ_k of the suction function $\gamma(\xi)$ for given coefficients s_k of the outer velocity $U(x)$ and given coefficients σ_k of the suction velocity $v_0(x)$.

It will be of interest to note here the first two of these relations:

$$(20) \quad \sigma_0 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot \gamma_0 s_0, \quad \sigma_1 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot \left(\frac{3}{4} \gamma_0 s_1 + \frac{3\gamma_1 s_0^2}{\nu(m+1)} \right).$$

We have $s_0 \neq 0$. We see that $\sigma_0 = 0$ if $\gamma_0 = 0$.

For $m = 0$ ($\beta_0 = 0$), the case of forward cuspidal point $x = 0$, we find, according to (18), arbitrary suction velocity distributions of the form

$$(18_0) \quad m = 0: \quad v_0(x) = \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \sigma_k x^k.$$

$v_0(x)$ is finite (zero) in $x = 0$ if and only if $\sigma_0 = 0$ ($\gamma_0 = 0$).

For $m = 1$ ($\beta_0 = 1$), the case of contours with rounded noses $x = 0$ and symmetrical with respect to the oncoming flow, we have arbitrary suction velocity distributions of the form

$$(18_1) \quad m = 1: \quad v_0(x) = \sum_{k=0}^{\infty} \sigma_k x^{2k}.$$

(As already mentioned above, admission of more general principal functions, as introduced in [7], would also permit the inclusion of arbitrary non-symmetrical distributions.)

4. The series solution of the boundary layer problem. We now return to the differential equation (7) and the boundary conditions (8) of the boundary layer flow:

$$(7) \quad F_{\eta\eta\eta} + FF_{\eta\eta} + \beta(\xi)\{1 - F^2\} = 2\xi\{F_{\eta}F_{\xi\eta} - F_{\xi}F_{\eta\eta}\},$$

$$(8) \quad F(\xi, 0) + 2\xi F_{\xi}(\xi, 0) = \gamma(\xi), \quad F_{\eta}(\xi, 0) = 0, \quad F_{\eta}(\xi, \infty) = 1$$

with principal and suction functions of the form

$$(12) \quad \beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k,$$

$$(13) \quad \gamma(\xi) = \sum_{k=0}^{\infty} (2k+1) \gamma_k \xi^k.$$

Accordingly, for the solution of (7), (8) we assume a convergent series of the form

$$(21) \quad F(\xi, \eta) = \sum_{k=0}^{\infty} F_k(\eta) \xi^k.$$

Quite as in [7] we find from (7):

$$(22) \quad F_0''' + F_0 F_0'' + \beta_0(1 - F_0'^2) = 0,$$

$$(23) \quad F_k''' + F_0 F_k'' - 2(\beta_0 + k) F_0' F_k' + (2k + 1) F_0'' F_k = R_{k-1} \quad (k = 1, 2, 3, \dots)$$

with

$$(24) \quad \begin{aligned} R_{k-1} = & \beta_k(F_0'^2 - 1) + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} \beta_i F_i' F_{k-i-j}' + F_0' \sum_{j=1}^{k-1} \beta_j F_{k-j}' \\ & + \sum_{j=1}^{k-1} \{\beta_0 + 2(k-j)\} F_j' F_{k-j}' - \sum_{j=1}^{k-1} (1 + 2j) F_j F_{k-j}''. \end{aligned}$$

The boundary conditions are

$$(25) \quad F_0(0) = \gamma_0, \quad F_0'(0) = 0, \quad F_0'(\infty) = 1;$$

$$(26) \quad F_k(0) = \gamma_k, \quad F_k'(0) = 0, \quad F_k'(\infty) = 0 \quad (k = 1, 2, 3, \dots).$$

Formally, the only change, compared with the case of the impermeable wall, is that the prescribed values of $F_k(0)$, $k = 0, 1, 2, \dots$, are not necessarily zero.

Writing, for abbreviation,

$$(27) \quad L_k[y] = y''' + F_0 y'' - 2(\beta_0 + k) F_0' y' + (2k + 1) F_0'' y$$

the first five equations (23) are

$$(28) \quad \begin{aligned} L_1[F_1] &= \beta_1(F_0'^2 - 1) \\ L_2[F_2] &= \beta_2(F_0'^2 - 1) + 2\beta_1 F_0' F_1' + (\beta_0 + 2) F_1'^2 - 3F_1 F_1'' \\ L_3[F_3] &= \beta_3(F_0'^2 - 1) + 2\beta_2 F_0' F_1' + \beta_1(F_1'^2 + 2F_0' F_2') \\ &\quad + 2(\beta_0 + 3) F_1' F_2' - 3F_1 F_2'' - 5F_1' F_2' \\ L_4[F_4] &= \beta_4(F_0'^2 - 1) + 2\beta_3 F_0' F_1' + \beta_2(F_1'^2 + 2F_0' F_2') \\ &\quad + 2\beta_1(F_0' F_3' + F_1' F_2') + (\beta_0 + 4)(F_2'^2 + 2F_1' F_3') \\ &\quad - 3F_1 F_3'' - 5F_2 F_2'' - 7F_3 F_1'' \\ L_5[F_5] &= \beta_5(F_0'^2 - 1) + 2\beta_4 F_0' F_1' + \beta_3(F_1'^2 + 2F_0' F_2') \\ &\quad + 2\beta_2(F_0' F_3' + F_1' F_2') + \beta_1(2F_0' F_4' + 2F_1' F_3' + F_2'^2) \\ &\quad + 2(\beta_0 + 5)(F_1' F_4' + F_2' F_3') - 3F_1 F_4'' \\ &\quad - 5F_2 F_3'' - 7F_3 F_2'' - 9F_4 F_1''. \end{aligned}$$

It will be noticed that whereas the differential equation for F_0 is non-linear—the well known FALKNER-SKAN-HARTREE equation (22)—the equations for $F_1, F_2, F_3, \dots$ are linear. In the following this will be of importance.

5. Introduction of *universal* functions. It is now our object to reduce the problem of the determination of the coefficient functions $F_k(\eta)$ to that of the determination of a set of auxiliary functions which, if possible, do not depend on the prescribed quantities β_i and γ_i of any special problem in question and therefore can be tabulated once and for all.

Already in the case of the impermeable wall investigated in [7] a complete reduction in this sense proved to be impossible, and for good reasons. It must be expected that these sets of functions will depend on the type of prescribed flow singularity at $x = 0$ and will *e.g.* not be the same if this singularity is the leading edge of a flat plate or the rounded nose of a cylinder. In other words, it must be expected that it will not be possible to throw out the dependency on the parameter β_0 determining the vertex angle $\pi\beta_0$ at $x = 0$. If, however, we consider only contours with rounded noses or only contours with forward cuspidal point, then it may be hoped (and, in fact, we have shown in [7]) that a reduction to *universal* functions in the sense that these functions do not depend on the further data $\beta_1, \beta_2, \beta_3, \dots$ of any special problem under consideration will be possible. The set of universal functions for contours with forward cuspidal points ($\beta_0 = 0$) and the set of universal functions for contours with rounded noses ($\beta_0 = 1$) have been tabulated up to the order 5 inclusively. (Tabulations for other forward vertex angles $\pi\beta_0$ are of less interest and have therefore not yet been carried out.)

The mathematical reason for the situation just described is the fact that the zero order function F_0 , depending on β_0 , is the solution of a non-linear differential equation whereas the coefficient function F_k of the k -th order term ($k \geq 1$), depending on the β_j for $j = 0, 1, 2, \dots$, is determined by a linear differential equation.

In our more general case of boundary layers with suction we see from (25) that the zero order function F_0 now also depends on γ_0 . Along with β_0 now also γ_0 determines the behavior of the flow at its singularity $x = 0$. Therefore it can be hoped that for each fixed pair of values β_0, γ_0 the coefficient functions $F_1, F_2, F_3, \dots$ can be reduced to linear combinations of *universal* functions (meaning functions independent of $\beta_1, \beta_2, \beta_3, \dots$ and independent of $\gamma_1, \gamma_2, \gamma_3, \dots$). That this is possible will be demonstrated in the following. Let us first consider a little further the rôle of γ_0 .

As has already been stated in connection with (10), the value of $v_0(x)/U(x)$ remains finite for $x \rightarrow 0$ if and only if $\gamma(\xi) = O(\xi^{\frac{1}{2}})$ for $\xi \rightarrow 0$. Considering our assumption (13) for $\gamma(\xi)$, this necessary and sufficient condition is equivalent to $\gamma_0 = 0$. Therefore, for $\gamma_0 \neq 0$ the suction velocity $v_0(x)$ is not small in magnitude compared to the outer velocity $U(x)$ in the immediate vicinity of $x = 0$. The assumption of *weak suction* can be strictly fulfilled for all values of x including $x = 0$ only if $\gamma_0 = 0$. Then $v_0(0)/U(0) = 0$.

For a further investigation of the rôle of γ_0 in the vicinity of the forward point $x = 0$ we first consider the special class of outer velocities of the form $U(x) = s_0 x^m$ leading to $\beta(\xi) = \beta_0$ and to the well known *similar* solutions for impermeable walls. In the case of continuous suction distribution along the walls similar solutions remain possible if and only if $\gamma(\xi) = \gamma_0 = \text{const.}$ Then again $F(\xi, \eta) = F_0(\eta)$, a function of η only. The only difference is that, instead of $F_0(0) = 0$ for impermeable walls, we now have $F_0(0) = \gamma_0$. The suction velocity distribution is

$$(29) \quad v_0(x) = \sigma_0 x^{\frac{1}{2}(m-1)} \quad \text{with} \quad \sigma_0 = -\gamma_0 \sqrt{\frac{1}{2}\nu(m+1)s_0}.$$

For $\gamma_0 \neq 0$ ($\sigma_0 \neq 0$) and for the values of m of interest here ($0 \leq m \leq 1$), therefore, not only is it impossible to fulfill the assumption of weak suction but $v_0(x)$ remains finite only in the case $m = 1$. Although, therefore, the case $\gamma_0 \neq 0$ is rather unrealistic for $m < 1$, at least in an arbitrarily small neighborhood of $x = 0$, the similar solutions with suction velocity (29) have been studied in the literature, assuming that little harm is done by the singularity at $x = 0$ to the flow further downstream, and mainly for the reason that these solutions were, till now, practically the only exact solutions known.

Now, in the general case of the classes of widely arbitrary outer velocities $U(x)$ and suction velocities $v_0(x)$ considered in the present paper, we have, for sufficiently small $|x|$, approximately $U(x) \approx s_0 x^m$ and, for $\gamma_0 \neq 0$, approximately $v_0(x) \approx \sigma_0 x^{\frac{1}{2}(m-1)}$ and, in the same approximation, the flow is *similar* in the neighborhood of $x = 0$. We conclude that also in the general case the assumption $\gamma_0 \neq 0$ must be treated with care. It will be of main interest to study suction functions with $\gamma_0 = 0$. Then $\sigma_0 = 0$. Assuming that $\gamma_1 \neq 0$ we have, near the forward point $x = 0$, approximately $v_0(x) \approx \sigma_1 x^{\frac{1}{2}(3m+1)}$ and $v_0(x)/U(x) \approx (\sigma_1/s_0)x^{\frac{1}{2}(m+1)}$, both expressions retaining finite values at $x = 0$ for all values of m of interest in this paper. ($\sigma_1 = -3\gamma_1(s_0^3/2\nu(m+1))^{\frac{1}{2}}$.)

We shall now demonstrate, quite generally, the reduction of the coefficient functions F_k ($k = 1, 2, 3, \dots$) to linear combinations of *universal* functions in the sense explained above.

Writing

$$(30) \quad F_1 = \beta_1 f_1 + \gamma_1 g_1$$

where f_1 and g_1 are functions of η , the boundary value problem (28) for the function F_1 immediately leads to the two *universal* boundary value problems

$$(31) \quad \begin{aligned} L_1[f_1] &= F_0'^2 - 1, & f_1(0) &= f_1'(0) = f_1'(\infty) = 0 \\ L_1[g_1] &= 0, & g_1(0) &= 1, \quad g_1'(0) = g_1'(\infty) = 0. \end{aligned}$$

With (30) the differential equation (28) for F_2 becomes

$$(32) \quad \begin{aligned} L_2[F_2] &= \beta_2(F_0'^2 - 1) + \beta_1^2[2F_0'f_1' + (\beta_0 + 2)f_1'^2 - 3f_1f_1''] \\ &+ \beta_1\gamma_1[2F_0'g_1' + (\beta_0 + 2)2f_1'g_1' - 3(f_1g_1'' + f_1'g_1')] \\ &+ \gamma_1^2[(\beta_0 + 2)g_1'^2 - 3g_1g_1'']. \end{aligned}$$

If we now write

$$(33) \quad F_2 = \beta_2 f_2 + \gamma_2 g_2 + \beta_1^2 f_{11} + \gamma_1^2 g_{11} + \beta_1 \gamma_1 h_{1,1}$$

and introduce this into (32), comparison of the coefficients of β_1 , β_2 , γ_1 , γ_2 leads us to

$$\begin{aligned} L_2[f_2] &= F_0'^2 - 1 \\ L_2[g_2] &= 0 \\ L_2[f_{11}] &= 2F_0'f_1' + (\beta_0 + 2)f_1'^2 - 3f_1f_1'' \\ L_2[g_{11}] &= (\beta_0 + 2)g_1'^2 - 3g_1g_1'' \\ L_2[h_{1,1}] &= 2F_0'g_1' + 2(\beta_0 + 2)f_1'g_1' - 3(f_1g_1'' + f_1'g_1') \end{aligned}$$

with

$$(34) \quad \begin{aligned} f_2(0) &= 0, & f_2'(0) &= 0, & f_2'(\infty) &= 0; \\ g_2(0) &= 1, & g_2'(0) &= 0, & g_2'(\infty) &= 0; \\ f_{11}(0) &= 0, & f_{11}'(0) &= 0, & f_{11}'(\infty) &= 0; \\ g_{11}(0) &= 0, & g_{11}'(0) &= 0, & g_{11}'(\infty) &= 0; \\ h_{1,1}(0) &= 0, & h_{1,1}'(0) &= 0, & h_{1,1}'(\infty) &= 0. \end{aligned}$$

As seen from the boundary value problems the auxiliary functions $f_{..}$, $g_{..}$, $h_{..}$ are universal functions in the sense stated above. This procedure can be continued for the higher order terms.

Quite generally, the inhomogeneous differential equations have homogeneous boundary conditions. Additionally, for every order $k > 0$ there is one homogeneous differential equation with inhomogeneous boundary condition for a function g_k introduced solely in order to fulfill the suction boundary condition $F_k(0) = \gamma_k$.

The symbols chosen for the universal functions may, at first sight, seem complicated. For practical applications, however, they will prove to be useful as they directly indicate which coefficients in the β_k 's and γ_k 's belong to them in the linear combinations leading to the F_k 's. This will become still more obvious when we write down the linear combinations for higher orders, *cf.* Appendix 2.

The functions $f_{..}$ are those that have already appeared in [7] for the boundary layer along impermeable walls. They, therefore, are already numerically available for the orders 0 through 5 for the cases $\beta_0 = 0$ and $\beta_0 = 1$. The functions $g_{..}$ depend only on the suction function $\gamma(\xi)$, just as the functions $f_{..}$ depend only on the principal function $\beta(\xi)$. The functions $h_{..}$ depend on $\beta(\xi)$ and $\gamma(\xi)$. In their subscripts those ahead of the comma indicate the factors β_k , those behind the comma indicate the factors γ_k belonging to them in the linear combinations.

As was to be expected from the beginning, the number of universal functions for problems with suction is considerably higher than without suction. If the

series is truncated after the n^{th} order term $F_n(\eta)\xi^n$ we have, including the zero order term $F_0(\eta)$, the following number of universal functions (total number for a truncated n^{th} order approximation):

n	No Suction	Suction
0	1	1
1	2	3
2	4	8
3	7	18
4	12	38
5	19	71

With suction, already for $n = 4$ we have twice as many (38) universal functions as formerly for $n = 5$ without suction (19), the order up to which, inclusively, tabulations were carried out in [7]. Even with the aid of electronic computation it will probably be prohibitive to go beyond a 4th order approximation. The 38 boundary value problems governing the universal functions of the orders 0 through 4 are given in Appendix 2. For any given pair of values β_0 , γ_0 this system must be solved recursively.

6. Concluding remarks. The good success of the new series method in the case of impermeable walls (*cf.* [7, 8, 9, 10]) has encouraged us to extend the method to problems of boundary layer flows with suction. Although we were able to show that this is possible in principle, the numerical task of tabulation of the universal functions at least for the two cases of main interest, $\beta_0 = 0$ (forward cuspidal points, including flat plates with arbitrary pressure distribution) and $\beta_0 = 1$ (contours with rounded noses) has still to be completed. This is planned for the two classes of problems mentioned and, for reasons stated above, for $\gamma_0 = 0$. For each of these two classes and for the orders 0 through 4 the tabulation of 38 universal functions is necessary. Of these, the twelve universal functions $f_{..}$ already appearing in [7] are known, [8].

As soon as the tables are available, applications will be possible with very little numerical work necessary in each special problem. For the tables with $\beta_0 = 0$, $\gamma_0 = 0$ immediate applications to problems with any given outer and suction velocities representable in the form

$$(35) \quad U(x) = \sum_{k=0}^{\infty} u_k x^k (u_0 \neq 0) \quad \text{and} \quad v_0(x) = \sqrt{x} \sum_{k=0}^{\infty} v_k x^k$$

with arbitrary coefficients u_k and v_k will be possible. Similarly, the tables with $\beta_0 = 1$, $\gamma_0 = 0$ will allow application to problems with arbitrary velocity distributions

$$(36) \quad U(x) = \sum_{k=0}^{\infty} u_k x^{2k+1} (u_0 \neq 0) \quad \text{and} \quad v_0(x) = \sum_{k=1}^{\infty} v_k x^{2k}.$$

The extension of tabulation work to further classes of problems included in the very general distributions (14), (18) would mean only a repetition of the same

computation program for other values β_0 , γ_0 . Their completion will depend mainly on existing practical interest.

REFERENCES

- [1] H. SCHLICHTING: *Grenzschicht-Theorie*. Verlag G. Braun, Karlsruhe, 1951.
- [2] H. SCHLICHTING: *Boundary layer theory*. Translated by J. KESTIN. Pergamon Press, London, and Verlag G. Braun, Karlsruhe, 1955.
- [3] R. IGLISCH: *Exakte Berechnung der laminaren Reibungsschicht an der längs angeströmten ebenen Platte mit homogener Absaugung*. Schriften der dt. Akad. d. Luftfahrtforschung 8B, Heft 1 (1944). English translation as NACA Techn. Mem. 1205 (1949).
- [4] H. GÖRTLER: Ein Differenzenverfahren zur Berechnung laminarer Grenzschichten. *Ing.-Archiv* 16, 173–187 (1948). English translation: Reports and Translations M.A.P. Völkenrode (1946).
- [5] H. WITTING: Verbesserung des Differenzenverfahrens von H. GÖRTLER zur Berechnung laminarer Grenzschichten. Thesis, University of Freiburg i.Br., 1953. *Z.A.M.P.* 4, 376–397 (1953).
- [6] W. RHEINBOLDT: Zur Berechnung stationärer Grenzschichten bei kontinuierlicher Absaugung mit unstetig veränderlicher Absaugengeschwindigkeit. Thesis, University of Freiburg i.Br. 1955. *Journ. Rat. Mech. Analysis* 5, 539–604 (1956).
- [7] H. GÖRTLER: A new series for the calculation of steady laminar boundary layer flows. *Journ. Math. Mech.* 6, 1–66 (1957).
- [8] H. GÖRTLER: *Zahlentafeln universeller Funktionen zur neuen Reihe für die Berechnung laminarer Grenzschichten*. Bericht Nr. 34 der Deutschen Versuchsanstalt für Luftfahrt, 1957.
- [9] H. GÖRTLER & H. WITTING: Zu den TANISCHEN Grenzschichten. *Österr. Ing.-Archiv*. (1957) To appear soon.
- [10] H. GÖRTLER & H. WITTING: Einige laminare Grenzschichtströmungen, berechnet mittels einer neuen Reihenmethode. To appear in *Z.A.M.P.*

APPENDIX 1†

Formulæ for the coefficients of the suction function $\gamma(\xi)$ for given outer velocity distribution $U(x)$ and suction velocity distribution $v_0(x)$.

a. **Auxiliary coefficients c_k defined in Eq. (16).** The coefficients c_k are not needed for the application of the method. They are useful for checking the further formulæ of this appendix as derived from equations (16) and (19).

As an abbreviation we introduce

$$\frac{(2k+1)\gamma_k}{\nu^k(m+1)^k} = \Gamma_k.$$

$$c_0 = \Gamma_0$$

$$c_1 = s_0 \Gamma_1$$

$$c_2 = \frac{s_1}{2} \Gamma_1 + s_0^2 \Gamma_2$$

†I am much indebted to Dr. THEO GEIS for the careful calculation of all formulæ in Appendix 1 and Appendix 2.

$$c_3 = \frac{s_2}{3} \Gamma_1 + s_0 s_1 \Gamma_2 + s_0^3 \Gamma_3$$

$$c_4 = \frac{s_3}{4} \Gamma_1 + (\frac{2}{3} s_0 s_2 + \frac{1}{4} s_1^2) \Gamma_2 + \frac{3}{2} s_0^2 s_1 \Gamma_3 + s_0^4 \Gamma_4$$

$$c_5 = \frac{s_4}{5} \Gamma_1 + (\frac{1}{3} s_1 s_2 + \frac{1}{2} s_0 s_3) \Gamma_2 + (s_0^2 s_2 + \frac{3}{4} s_0 s_1^2) \Gamma_3 + 2 s_0^3 s_1 \Gamma_4 + s_0^5 \Gamma_5 .$$

b. Coefficients σ_k for prescribed coefficients s_k and Γ_k .

$$\sigma_0 = -\sqrt{\frac{\nu(m+1)}{2s_0}} s_0 \Gamma_0$$

$$\sigma_1 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot \left\{ \frac{3}{4} s_1 \Gamma_0 + s_0^2 \Gamma_1 \right\}$$

$$\sigma_2 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot \left\{ \left(\frac{5}{6} s_2 - \frac{5}{32} \frac{s_1^2}{s_0} \right) \Gamma_0 + \frac{5}{4} s_0 s_1 \Gamma_1 + s_0^3 \Gamma_2 \right\}$$

$$\begin{aligned} \sigma_3 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot & \left\{ \left(\frac{7}{8} s_3 - \frac{7}{24} \frac{s_1 s_2}{s_0} + \frac{7}{128} \frac{s_1^3}{s_0^2} \right) \Gamma_0 \right. \\ & + \left(\frac{7}{6} s_0 s_2 + \frac{7}{32} s_1^2 \right) \Gamma_1 + \frac{7}{4} s_0^2 s_1 \Gamma_2 + s_0^4 \Gamma_3 \left. \right\} \\ \sigma_4 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot & \left\{ \left(\frac{9}{10} s_4 - \frac{1}{8} \frac{s_2^2}{s_0} - \frac{9}{32} \frac{s_1 s_3}{s_0} + \frac{9}{64} \frac{s_1^2 s_2}{s_0^2} - \frac{45}{2048} \frac{s_1^4}{s_0^3} \right) \Gamma_0 \right. \\ & + \left(\frac{9}{8} s_0 s_3 + \frac{3}{8} s_1 s_2 - \frac{3}{128} \frac{s_1^3}{s_0} \right) \Gamma_1 \\ & + \left(\frac{27}{32} s_0 s_1^2 + \frac{3}{2} s_0^2 s_2 \right) \Gamma_2 + \frac{9}{4} s_0^3 s_1 \Gamma_3 + s_0^5 \Gamma_4 \left. \right\} \end{aligned}$$

$$\begin{aligned} \sigma_5 = -\sqrt{\frac{\nu(m+1)}{2s_0}} \cdot & \left\{ \left(\frac{11}{12} s_5 - \frac{11}{48} \frac{s_2 s_3}{s_0} - \frac{11}{40} \frac{s_1 s_4}{s_0} + \frac{11}{96} \frac{s_1 s_2^2}{s_0^2} \right. \right. \\ & + \frac{33}{256} \frac{s_1^2 s_3}{s_0^2} - \frac{55}{768} \frac{s_1^3 s_2}{s_0^3} + \frac{77}{8192} \frac{s_1^5}{s_0^4} \left. \right) \Gamma_0 \\ & + \left(\frac{11}{10} s_0 s_4 + \frac{11}{32} s_1 s_3 + \frac{11}{72} s_2^2 - \frac{11}{192} \frac{s_1^2 s_2}{s_0} + \frac{11}{2048} \frac{s_1^4}{s_0^2} \right) \Gamma_1 \\ & + \left(\frac{11}{8} s_0^2 s_3 + \frac{11}{8} s_0 s_1 s_2 + \frac{11}{128} s_1^3 \right) \Gamma_2 \\ & + \left(\frac{11}{6} s_0^3 s_2 + \frac{55}{32} s_0^2 s_1^2 \right) \Gamma_3 + \frac{11}{4} s_0^4 s_1 \Gamma_4 + s_0^6 \Gamma_5 \left. \right\} . \end{aligned}$$

c. Coefficients Γ_k for prescribed coefficients s_k and σ_k .

$$\begin{aligned}
 \Gamma_0 &= -\sqrt{\frac{2s_0}{\nu(m+1)}} \cdot \frac{\sigma_0}{s_0} \\
 \Gamma_1 &= -\sqrt{\frac{2s_0}{\nu(m+1)}} \cdot \frac{1}{s_0^{\frac{1}{2}}} \cdot \left\{ -\frac{3}{4} \frac{s_1}{s_0} \sigma_0 + \sigma_1 \right\} \\
 \Gamma_2 &= -\sqrt{\frac{2s_0}{\nu(m+1)}} \cdot \frac{1}{s_0^{\frac{3}{2}}} \cdot \left\{ \left(\frac{35}{32} \frac{s_1^2}{s_0^2} - \frac{5}{6} \frac{s_2}{s_0} \right) \sigma_0 - \frac{5}{4} \frac{s_1}{s_0} \sigma_1 + \sigma_2 \right\} \\
 \Gamma_3 &= -\sqrt{\frac{2s_0}{\nu(m+1)}} \cdot \frac{1}{s_0^4} \cdot \left\{ \left(-\frac{7}{8} \frac{s_3}{s_0} + \frac{21}{8} \frac{s_1 s_2}{s_0^2} - \frac{231}{128} \frac{s_1^3}{s_0^3} \right) \sigma_0 \right. \\
 &\quad \left. + \left(-\frac{7}{6} \frac{s_2}{s_0} + \frac{63}{32} \frac{s_1^2}{s_0^2} \right) \sigma_1 - \frac{7}{4} \frac{s_1}{s_0} \sigma_2 + \sigma_3 \right\} \\
 \Gamma_{4,4} &= -\sqrt{\frac{2s_0}{\nu(m+1)}} \cdot \frac{1}{s_0^5} \cdot \left\{ \left(-\frac{9}{10} \frac{s_4}{s_0} + \frac{99}{32} \frac{s_1 s_3}{s_0^2} + \frac{11}{8} \frac{s_2^2}{s_0^2} \right. \right. \\
 &\quad \left. \left. - \frac{429}{64} \frac{s_1^2 s_2}{s_0^3} + \frac{6435}{2048} \frac{s_1^4}{s_0^4} \right) \sigma_0 + \left(-\frac{9}{8} \frac{s_3}{s_0} + \frac{33}{8} \frac{s_1 s_2}{s_0^2} - \frac{429}{128} \frac{s_1^3}{s_0^3} \right) \sigma_1 \right. \\
 &\quad \left. + \left(-\frac{3}{2} \frac{s_2}{s_0} + \frac{99}{32} \frac{s_1^2}{s_0^2} \right) \sigma_2 - \frac{9}{4} \frac{s_1}{s_0} \sigma_3 + \sigma_4 \right\} \\
 \Gamma_5 &= -\sqrt{\frac{2s_0}{\nu(m+1)}} \cdot \frac{1}{s_0^6} \cdot \left\{ \left(-\frac{11}{12} \frac{s_5}{s_0} + \frac{143}{40} \frac{s_1 s_4}{s_0^2} + \frac{143}{48} \frac{s_2 s_3}{s_0^2} \right. \right. \\
 &\quad \left. \left. - \frac{2145}{256} \frac{s_1^2 s_3}{s_0^3} - \frac{715}{96} \frac{s_1 s_2^2}{s_0^3} + \frac{12155}{768} \frac{s_1^3 s_2}{s_0^4} - \frac{46189}{8192} \frac{s_1^5}{s_0^5} \right) \sigma_0 \right. \\
 &\quad \left. + \left(-\frac{11}{10} \frac{s_4}{s_0} + \frac{143}{32} \frac{s_1 s_3}{s_0^2} + \frac{143}{72} \frac{s_2^2}{s_0^2} \right. \right. \\
 &\quad \left. \left. - \frac{715}{64} \frac{s_1^2 s_2}{s_0^3} + \frac{12155}{2048} \frac{s_1^4}{s_0^4} \right) \sigma_1 \right. \\
 &\quad \left. + \left(-\frac{11}{8} \frac{s_3}{s_0} + \frac{143}{24} \frac{s_1 s_2}{s_0^2} - \frac{715}{128} \frac{s_1^3}{s_0^3} \right) \sigma_2 \right. \\
 &\quad \left. + \left(-\frac{11}{6} \frac{s_2}{s_0} + \frac{143}{32} \frac{s_1^2}{s_0^2} \right) \sigma_3 - \frac{11}{4} \frac{s_1}{s_0} \sigma_4 + \sigma_5 \right\}.
 \end{aligned}$$

APPENDIX 2

Differential equations for the universal functions of orders 0 through 4.

The universal functions are introduced by the equations

$$F_1 = \beta_1 f_1 + \gamma_1 g_1 ;$$

$$F_2 = \beta_2 f_2 + \gamma_2 g_2 + \beta_1^2 f_{11} + \gamma_1^2 g_{11} + \beta_1 \gamma_1 h_{1,1} ;$$

$$F_3 = \beta_3 f_3 + \gamma_3 g_3 + \beta_1 \beta_2 f_{12} + \gamma_1 \gamma_2 g_{12} + \beta_1 \gamma_2 h_{1,2} + \beta_2 \gamma_1 h_{2,1} \\ + \beta_1^3 f_{111} + \gamma_1^3 g_{111} + \beta_1^2 \gamma_1 h_{11,1} + \beta_1 \gamma_1^2 h_{1,11} ;$$

$$F_4 = \beta_4 f_4 + \gamma_4 g_4 + \beta_2^2 f_{22} + \gamma_2^2 g_{22} + \beta_2 \gamma_2 h_{2,2} \\ + \beta_1 \beta_3 f_{13} + \gamma_1 \gamma_3 g_{13} + \beta_1 \gamma_3 h_{1,3} + \beta_3 \gamma_1 h_{3,1} \\ + \beta_1^2 \beta_2 f_{112} + \gamma_1^2 \gamma_2 g_{112} + \beta_1^2 \gamma_2 h_{11,2} + \beta_1 \beta_2 \gamma_1 h_{12,1} + \beta_1 \gamma_1 \gamma_2 h_{1,12} + \beta_2 \gamma_1^2 h_{2,11} \\ + \beta_1^4 f_{1111} + \gamma_1^4 g_{1111} + \beta_1^3 \gamma_1 h_{111,1} + \beta_1^2 \gamma_1^2 h_{11,11} + \beta_1 \gamma_1^3 h_{1,111} .$$

The boundary conditions are

for $y = g_k(\eta)$, $k = 1, 2, 3, 4, \dots$:

$$y(0) = 1, \quad y'(0) = 0, \quad y'(\infty) = 0;$$

for all other universal functions $f_{..}$, $g_{..}$ and $h_{..}$:

$$y(0) = 0, \quad y'(0) = 0, \quad y'(\infty) = 0.$$

The function $F_0(\eta)$ is the solution of the boundary value problem

$$F_0''' + F_0 F_0'' + \beta_0(1 - F_0^2) = 0;$$

$$F_0(0) = \gamma_0, \quad F_0'(0) = 0, \quad F_0'(\infty) = 0.$$

Using the differential operator L_k defined by (27), the differential equations for the universal functions of orders 0 to 4 are:

$$k = 1 \quad L_1[f_1] = F_0'^2 - 1$$

$$L_1[g_1] = 0$$

$$k = 2 \quad L_2[f_2] = F_0'^2 - 1$$

$$L_2[g_2] = 0$$

$$L_2[f_{11}] = 2F_0'f_1' + (\beta_0 + 2)f_1'^2 - 3f_1f_1''$$

$$L_2[g_{11}] = (\beta_0 + 2)g_1'^2 - 3g_1g_1''$$

$$L_2[h_{1,1}] = 2F_0'g_1' + 2(\beta_0 + 2)f_1'g_1' - 3(f_1g_1'' + f_1'g_1')$$

$$k = 3 \quad L_3[f_3] = F_0'^2 - 1$$

$$L_3[g_3] = 0$$

$$L_3[f_{12}] = 2F_0'(f_1' + f_2') + 2(\beta_0 + 3)f_1'f_2' - 3f_1f_2'' - 5f_1'f_2'$$

$$L_3[g_{12}] = 2(\beta_0 + 3)g_1'g_2' - 3g_1g_2'' - 5g_1'g_2'$$

$$L_3[h_{1,2}] = 2F_0'g_2' + 2(\beta_0 + 3)f_1'g_2' - 3f_1g_2'' - 5f_1'g_2'$$

$$L_3[h_{2,1}] = 2F_0'g_1' + 2(\beta_0 + 3)f_2'g_1' - 3f_2'g_1' - 5f_2g_1''$$

$$L_3[f_{111}] = 2F_0'f_{11}' + f_1'^2 + 2(\beta_0 + 3)f_1'f_{11}' - 3f_1f_{11}'' - 5f_1'f_{11}'$$

$$\begin{aligned}
L_3[g_{111}] &= 2(\beta_0 + 3)g'_1g'_{11} - 3g_1g''_{11} - 5g'_1g'_{11} \\
L_3[h_{11,1}] &= 2F'_0h'_{1,1} + 2f'_1g'_1 + 2(\beta_0 + 3)(f'_1h'_{1,1} + f'_{11}g'_1) \\
&\quad - 3(f_1h'_{1,1} + f'_{11}g_1) - 5(f'_1h'_{1,1} + f_{11}g'_1) \\
L_3[h_{1,11}] &= 2F'_0g'_{11} + g_1'^2 + 2(\beta_0 + 3)(f'_1g'_{11} + g'_1h'_{1,1}) \\
&\quad - 3(f_1g'_{11} + g_1h'_{1,1}) - 5(f'_1g_{11} + g'_1h'_{1,1})
\end{aligned}$$

$$\begin{aligned}
k = 4 \quad L_4[f_4] &= F_0'^2 - 1 \\
L_4[g_4] &= 0 \\
L_4[f_{22}] &= 2F'_0f_2' + (\beta_0 + 4)f_2'^2 - 5f_2g_2'' \\
L_4[g_{22}] &= (\beta_0 + 4)g_2'^2 - 5g_2g_2'' \\
L_4[h_{2,2}] &= 2F'_0g_2' + 2(\beta_0 + 4)f_2'g_2' - 5(f_2g_2'' + f_2'g_2) \\
L_4[f_{1,3}] &= 2F'_0(f_1' + f_3') + 2(\beta_0 + 4)f_1'f_3' - 3f_1f_3'' - 7f_1'f_3 \\
L_4[g_{1,3}] &= 2(\beta_0 + 4)g_1'g_3' - 3g_1g_3'' - 7g_1'g_3 \\
L_4[h_{1,3}] &= 2F'_0g_3' + 2(\beta_0 + 4)f_1'g_3' - 3f_1g_3'' - 7f_1'g_3 \\
L_4[h_{3,1}] &= 2F'_0g_1' + 2(\beta_0 + 4)f_3'g_1' - 3f_3'g_1 - 7f_3g_1'' \\
L_4[f_{112}] &= 2F'_0(f'_{11} + f'_{12}) + 2f_1'f_2' + f_1'^2 + 2(\beta_0 + 4)(f_1'f'_{12} + f'_{11}f_2) \\
&\quad - 3f_1f'_{12} - 5(f_{11}f_2'' + f'_{11}f_2) - 7f_1'f'_{12} \\
L_4[g_{112}] &= 2(\beta_0 + 4)(g_1'g'_{12} + g'_{11}g_2') - 3g_1g'_{12} - 5(g_{11}g_2'' + g'_{11}g_2) \\
&\quad - 7g_1'g'_{12} \\
L_4[h_{11,2}] &= 2F'_0h'_{1,2} + 2f_1'g_2' + 2(\beta_0 + 4)(f_1'h'_{1,2} + f'_{11}g_2') - 3f_1h'_{1,2} \\
&\quad - 5(f_{11}g_2'' + f'_{11}g_2) - 7f_1'h'_{1,2} \\
L_4[h_{12,1}] &= 2F'_0(h'_{1,1} + h'_{2,1}) + 2(f_1' + f_2')g_1' + 2(\beta_0 + 4)(f_1'h'_{2,1} \\
&\quad + f'_{12}g_1' + f_2'h'_{1,1}) - 3(f_1h'_{2,1} + f'_{12}g_1) - 5(f_2h'_{1,1} + f_2'h'_{1,1}) \\
&\quad - 7(f_1'h'_{2,1} + f_{12}g_1') \\
L_4[h_{1,12}] &= 2F'_0g'_{12} + 2g_1'g_2' + 2(\beta_0 + 4)(f_1'g'_{12} + g_1'h'_{1,2} + g_2'h'_{1,1}) \\
&\quad - 3(f_1g'_{12} + g_1h'_{1,2}) - 5(g_2h'_{1,1} + g_2'h'_{1,1}) - 7(f_1'g_{12} + g_1'h'_{1,2}) \\
L_4[h_{2,11}] &= 2F'_0g'_{11} + g_1'^2 + 2(\beta_0 + 4)(g_1'h'_{2,1} + f_2'g'_{11}) \\
&\quad - 3g_1h'_{2,1} - 5(f_2g'_{11} + f_2'g_{11}) - 7g_1'h'_{2,1} \\
L_4[f_{1111}] &= 2F'_0f'_{111} + 2f_1'f'_{11} + (\beta_0 + 4)(2f_1'f'_{111} + f_{11}'^2) \\
&\quad - 3f_1f'_{111} - 5f_{11}f'_{11} - 7f_1'f'_{111}
\end{aligned}$$

$$\begin{aligned}
L_4[g_{1111}] &= (\beta_0 + 4)(2g'_1g'_{111} + g'^2_{11}) - 3g_1g'_{11} - 5g_{11}g'_{11} - 7g'_{11}g_{111} \\
L_4[h_{111,1}] &= 2F'_0h'_{11,1} + 2(f'_1h'_{1,1} + f'_{11}g'_1) \\
&\quad + 2(\beta_0 + 4)(f'_1h'_{11,1} + f'_{111}g'_1 + f'_{11}h'_{1,1}) - 3(f_1h'_{11,1} + f'_{11}g_1) \\
&\quad - 5(f_{11}h'_{1,1} + f'_{11}h_{1,1}) - 7(f'_{11}h_{11,1} + f_{111}g'_{11}) \\
L_4[h_{11,11}] &= 2F'_0h'_{1,11} + 2(f'_1g'_{11} + g'_1h'_{1,1}) \\
&\quad + (\beta_0 + 4)(2f'_1h'_{1,11} + 2g'_1h'_{11,1} + 2f'_{11}g'_{11} + h'^2_{1,1}) \\
&\quad - 3(f_1h'_{1,11} + g_1h'_{11,1}) - 5(f_{11}g'_{11} + f'_{11}g_{11} + h_{1,1}h'_{1,1}) \\
&\quad - 7(f'_{11}h_{1,11} + g'_{11}h_{11,1}) \\
L_4[h_{1,111}] &= 2F'_0g'_{111} + 2g'_1g'_{11} + 2(\beta_0 + 4)(f'_1g'_{111} + g'_1h'_{1,11} + g'_{11}h'_{1,1}) \\
&\quad - 3(f_1g'_{111} + g_1h'_{1,11}) - 5(g_{11}h'_{1,1} + g'_{11}h_{1,1}) - 7(f'_{11}g_{111} + g'_{11}h_{1,11})
\end{aligned}$$

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