Extended Compatibility Conditions for the Study of Surfaces of Discontinuity in Continuum Mechanics

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1. Introduction. Suppose that the dependent variables of a dynamical problem, *e.g.* the pressure p , the density ρ , *etc.*, are continuous across a moving surface $\Sigma(t)$ but that discontinuities in the coordinate and time derivatives of these variables can occur over the surface. If we denote the discontinuities in the derivatives by means of a bracket $[\ ]$ as is customary, then, using the pressure for illustration, it is well known that we must have relations of the form

$$(1) \quad [p, i] = \xi \nu_i ; \quad [\partial p / \partial t] = -\xi G$$

over $\Sigma(t)$, where G denotes the velocity of this surface in the direction of the unit normal ν and ξ is some function defined over the surface. When the problem involves a system of differential equations of the first order, the above relations, together with the system of differential equations, yield the condition for $\Sigma(t)$ to be a characteristic surface provided the function ξ associated with the pressure p in the relations (1), or a corresponding function connected with one of the other dependent variables, does not vanish so that discontinuities in the derivatives necessarily occur over $\Sigma(t)$. Actually the use of relations of the general type (1) in the determination of characteristic surfaces for systems of differential equations of the first or higher order avoids some of the details of calculation which arise in the common Cauchy procedure.

Conditions analogous to (1) have been derived by HADAMARD¹ on the basis of the Lagrangian parameters rather than the space coordinates as Eulerian variables which we prefer. He has also considered an extension of such relations giving the discontinuities in the second and higher derivatives of quantities under the assumption that only the quantities themselves are continuous over $\Sigma(t)$. However these latter relations, like the former, involve the Lagrangian parameters and are, moreover, in a form which does not appear suitable for application.

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¹J. HADAMARD, *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique*, Paris, 1903.

In this paper² we have constructed explicit formulæ for the discontinuities in the first and second derivatives of functions without the usual assumption that the functions are continuous over the surface $\Sigma(t)$. The use of these formulæ, which are of the nature of compatibility conditions for the existence of discontinuities, should enable us to obtain some new and interesting results in the general theory of surfaces of discontinuity in continuum mechanics. Since the formulæ in question involve both the first and second fundamental forms of the surface $\Sigma(t)$ one must naturally expect that certain of the results obtainable will depend on the geometrical nature of the surface of discontinuity. Moreover it may be observed that the application of these extended compatibility conditions will enable us to remove, in a simple manner, the restriction inherent in the theory of the characteristic surface, namely, that we must limit ourselves to the consideration of discontinuities in the derivatives of highest order in the system of differential equations under discussion.

2. Basic formulæ from the theory of surfaces. Let

$$(2) \quad x_i = \phi_i(u^1, u^2, t), \quad i = 1, 2, 3,$$

represent the moving surface $\Sigma(t)$. Here u^1, u^2 are curvilinear coordinates of the surface Σ , t denotes the time and x_1, x_2, x_3 are the coordinates of an allowable rectangular system for the dynamical problem. It is assumed that the functions ϕ_i are continuously differentiable and that the surface $\Sigma(t)$, defined by (2), possesses a tangent plane at each point. Beyond this all necessary continuity and differentiability requirements will be assumed without special mention. With regard to the position of the indices on the quantities in the equations (2), and more generally on other quantities which enter the discussion, we shall use the lower or subscript position for indices associated with the rectangular x coordinate system for which there is no distinction between covariance and contravariance under orthogonal transformations. We must, however, distinguish between covariant and contravariant indices when the indices are associated with the curvilinear or surface coordinates of $\Sigma(t)$ since these coordinates are subject to arbitrary differentiable transformations.

The coefficients of the first fundamental form of the surface $\Sigma(t)$ are given by

$$(3) \quad g_{\alpha\beta} = \phi_{i,\alpha} \phi_{i,\beta},$$

where $\alpha, \beta = 1, 2$ and the "comma" in the right member of these equations denotes partial differentiation with respect to the curvilinear coordinates. Latin indices have the range 1, 2, 3 and Greek indices the range 1, 2 in (3) and throughout the remainder of this article. In equations (3) and in the following discussion we have adopted the usual convention that an index which occurs twice in a term is to be summed over the proper range of values of this index.

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From the covariant components $g_{\alpha\beta}$ of the metric tensor we can construct the contravariant components $g^{\alpha\beta}$ of this tensor in the usual manner. An essential fact to be observed is that the quantities $\phi_{i,\alpha}$ or $x_{i,\alpha}$ are the components of a covariant vector, for i fixed, under transformations of the curvilinear coordinates. On the other hand these quantities have the vector transformation, when α is fixed, under orthogonal transformations of the x coordinates. Moreover, this latter or space vector is tangent to the surface Σ and has the direction of the corresponding u^α curve. Hence we can write

$$(4) \quad \nu_i \nu_i = 1; \quad x_{i,\alpha} \nu_i = 0.$$

The first of these equations expresses the fact that the normal vector ν is a unit vector and the second equation gives the condition that the above mentioned vectors with components $x_{i,\alpha}$ are tangent to the surface Σ . The relations (4) will have frequent application in the following derivations.

Two basic formulæ from the theory of surfaces will be especially useful in our work. These are

$$(5) \quad x_{i,\alpha\beta} = b_{\alpha\beta} \nu_i; \quad \nu_{i,\alpha} = -g^{\beta\gamma} b_{\beta\alpha} x_{i,\gamma},$$

where the $x_{i,\alpha\beta}$ are the components of the second covariant derivative of the quantities x_i based on the above metric tensor of the surface and the $b_{\alpha\beta}$ are the components of the second fundamental form of this surface. The quantities $x_{i,\alpha\beta}$ are symmetric in the indices α, β and hence the $b_{\alpha\beta}$ are likewise symmetric in the indices α, β from the first set of equations (5). In addition the surface scalar Ω defined by

$$(6) \quad 2\Omega = g^{\alpha\beta} b_{\alpha\beta}$$

will appear in certain of the following formulæ. The scalar Ω is the mean curvature of the surface Σ .

3. Geometrical conditions of the first order. Let us now consider the “jumps” or discontinuities of a function $Z(x, t)$ and its derivatives which can arise when we traverse the moving surface $\Sigma(t)$. The function Z may, in particular, represent the pressure p , the density ρ or any one of the velocity components v_i of a dynamical problem. For definiteness we define the discontinuity in Z by writing $[Z] = (Z)_2 - (Z)_1$ where the subscripts 1 and 2 refer to the sides 1 and 2 of the surface $\Sigma(t)$ and we shall suppose that the unit normal vector ν to $\Sigma(t)$ is directed from the side 1 to the side 2. The discontinuities in the derivatives of Z are of course defined in a corresponding manner.

We shall find that the discontinuity $[Z]$ and the combinations $[Z_{,i}] \nu_i$ and $[Z_{,ij}] \nu_i \nu_j$ occur more or less frequently in the following formulæ. Hence we shall use the abbreviations

$$(7) \quad [Z] = A; \quad [Z_{,i}] \nu_i = B; \quad [Z_{,ij}] \nu_i \nu_j = C.$$

From the above definitions and with due regard to the necessary assumptions

concerning the existence of derivatives of Z on the surface and in space, as well as the limiting values of Z and its derivatives as one approaches the surface $\Sigma(t)$ from the side 1 and from the side 2, it is easily seen that we have

$$(8) \quad A_{,\alpha} = [Z_{,\alpha}] = [Z_{,i}]x_{i,\alpha},$$

in which the index α denotes partial differentiation with respect to the curvilinear coordinate u^α of the surface $\Sigma(t)$. For our immediate purpose it will now be helpful to make the following substitutions

$$w_{i\alpha} = x_{i,\alpha}; \quad w_{i3} = \nu_i; \quad Q_\alpha = A_{,\alpha}; \quad Q_3 = B.$$

Then the equations (8) and the second equation (7) can be combined to give the single set of relations

$$(9) \quad [Z_{,i}]w_{ij} = Q_j.$$

To solve the equations (9), in a simple formal manner, for the quantities $[Z_{,i}]$ we first observe the validity of the matrix relation

$$(10) \quad ||w_{ij}w_{ik}|| = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

which follows readily from (3) and (4). Constructing the determinants of the above matrices we have $|w_{ij}|^2 = |g_{\alpha\beta}|$. Hence $|w_{ij}| \neq 0$ since $|g_{\alpha\beta}| > 0$ as is well known from the theory of surfaces. Hence quantities y_{ik} are determined uniquely as a solution of the equations

$$(11) \quad w_{ik}y_{ik} = \delta_{ij}.$$

We can also avail ourselves of the quantities G_{ij} defined as the elements of the matrix in the right member of (10). Then, denoting by G^{ij} the normalized cofactors of the elements of this matrix, we can easily deduce the equivalence of the following two sets of relations

$$(12) \quad w_{ij}w_{ik} = G_{jk}; \quad y_{ki} = G^{im}w_{km},$$

where

$$G^{\alpha\beta} = g^{\alpha\beta}; \quad G^{\alpha 3} = G^{3\alpha} = 0; \quad G^{33} = 1.$$

Hence, making use of (11), it follows from (9) and the second set of relations (12) that

$$(13) \quad [Z_{,i}]w_{ij}y_{ki} = [Z_{,k}] = G^{im}Q_jw_{km}.$$

Expanding the right member of (13) and taking account of the above special values of the G^{im} we now obtain the required solution of (9) in the form

$$(14) \quad [Z_{,i}] = B\nu_i + g^{\alpha\beta}A_{,\alpha}x_{i,\beta}$$

by which the discontinuities $[Z, \cdot]$ are expressed in terms of the two quantities A and B defined on the surface $\Sigma(t)$. In case the function Z is continuous across $\Sigma(t)$, so that $A = 0$ by (7), the relations (14) reduce to relations of the restricted type considered at the beginning of §1.

Relations (14) will be called *geometrical conditions of compatibility of the first order*.³ In the next section we shall derive the corresponding *kinematical conditions of compatibility of the first order* which involve the time derivative of the function Z . Conditions of compatibility of the second order will be constructed in §5 and §6.

4. The δ time derivative and the kinematical conditions of the first order. Consider two positions $\Sigma(t)$ and $\Sigma(t + \Delta t)$ of the discontinuity surface Σ . Erect the normal ν to the surface $\Sigma(t)$ at a point P and suppose that this normal intersects the surface $\Sigma(t + \Delta t)$ at the point P' . Let P have coordinates x_i and P' the coordinates $x_i + \Delta x_i$ so that Δx_i represents the coordinate differences of these two points. Then, denoting by ΔA the difference of the values of the quantity A at the points P and P' , we can write

$$(15) \quad \frac{\Delta A}{\Delta t} = \frac{(Z'_2 - Z_1) - (Z_2 - Z_1)}{\Delta t} = \frac{(Z'_2 - Z_2)}{\Delta t} - \frac{(Z'_1 - Z_1)}{\Delta t},$$

where the subscripts 1 and 2 refer to evaluation on the sides 1 and 2 of the surface and the prime indicates evaluation at the point P' .

Let us now define the δ time derivatives of the quantities A , $(Z'_1 - Z_1)$ and $(Z'_2 - Z_2)$ which appear in (15) as follows

$$(16) \quad \begin{cases} \frac{\delta A}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t}, \\ \frac{\delta Z_1}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{(Z'_1 - Z_1)}{\Delta t}, \\ \frac{\delta Z_2}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{(Z'_2 - Z_2)}{\Delta t}. \end{cases}$$

Hence from (15) and (16) we have

$$(17) \quad \frac{\delta A}{\delta t} = \frac{\delta Z_2}{\delta t} - \frac{\delta Z_1}{\delta t} = \left[\frac{\delta Z}{\delta t} \right].$$

It is clear that the process of δ time differentiation is, quite generally, applicable to any function defined over the moving surface $\Sigma(t)$. The δ time derivatives of other functions will enter into the formulæ of the following sections.

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³We use the designation *geometrical conditions* for those compatibility relations which do not depend on the actual motion of the surface Σ . Conditions which involve time derivatives and which depend on the motion of Σ are called *kinematical conditions of compatibility*.

From the definition of the velocity G of the moving surface $\Sigma(t)$ we have⁴

$$(18) \quad \frac{\delta x_i}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x_i}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta s}{\Delta t} \right) \nu_i = G \nu_i ,$$

where Δs denotes the distance of the point P' from the point P . Hence from the equation

$$\frac{Z'_1 - Z_1}{\Delta t} = (Z_{,i})_1 \frac{\Delta x_i}{\Delta t} + \left(\frac{\partial Z}{\partial t} \right)_1 ,$$

relative to the side 1 of $\Sigma(t)$ and the corresponding equation for the side 2 of this surface it follows that

$$(19) \quad \frac{\delta Z_1}{\delta t} = (Z_{,i})_1 G \nu_i + \left(\frac{\partial Z}{\partial t} \right)_1 ,$$

$$(20) \quad \frac{\delta Z_2}{\delta t} = (Z_{,i})_2 G \nu_i + \left(\frac{\partial Z}{\partial t} \right)_2 .$$

Hence from (17), (19) and (20) we obtain

$$\frac{\delta A}{\delta t} = [Z_{,i}] G \nu_i + \left[\frac{\partial Z}{\partial t} \right] .$$

The above relation, or

$$(21) \quad \left[\frac{\partial Z}{\partial t} \right] = -BG + \frac{\delta A}{\delta t} ,$$

is called the *kinematical condition of compatibility of the first order*. If Z is continuous across the surface $\Sigma(t)$ the condition (21) reduces to the special compatibility condition of the type represented by the second equation (1).

5. Geometrical conditions of the second order. Replacing the function Z in (14) by one of its derivatives $Z_{,i}$ we obtain a relation of the form

$$(22) \quad [Z_{,ij}] = B_i \nu_j + g^{\alpha\beta} A_{i,\alpha} x_{j,\beta} ,$$

where, in accordance with (7), the quantities A_i and B_i are given by

$$(23) \quad A_i = [Z_{,i}]; \quad B_i = [Z_{,ii}] \nu_i .$$

But the expression in the right member of (22) must be symmetric in the indices i and j since the left member of (22) has this property. Hence

$$(24) \quad B_i \nu_j + g^{\alpha\beta} A_{i,\alpha} x_{j,\beta} = B_j \nu_i + g^{\alpha\beta} A_{j,\alpha} x_{i,\beta} .$$

Multiplying (22) by ν_i and ν_j and summing on the repeated indices i and j we find that $B_i \nu_i = C$ as defined by (7). Next multiply (24) by ν_i and sum on j to obtain

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⁴See, T. Y. THOMAS, Singular surfaces and flow lines in the theory of Plasticity, *Jour. Rat. Mech. and Anal.*, **2**, 1953, p. 341.

$$(25) \quad B_i = C\nu_i + g_{\alpha\beta}(A_{i,\alpha}\nu_i)x_{i,\beta}.$$

But

$$A_{i,\alpha}\nu_i = [Z_{,i}]_{,\alpha}\nu_i = ([Z_{,i}]\nu_i)_{,\alpha} - [Z_{,i}]\nu_{i,\alpha},$$

or

$$A_{i,\alpha}\nu_i = B_{,\alpha} + [Z_{,i}]g^{\mu\nu}b_{\mu\alpha}x_{i,\nu},$$

when use is made of the second equation (5) and the second equation (7). Hence, replacing the quantity $[Z_{,i}]$ by its value as given by (14), we find that the resulting expression can be reduced to

$$A_{i,\alpha}\nu_i = B_{,\alpha} + g^{\mu\nu}b_{\mu\alpha}A_{,\nu}.$$

When this substitution is made in the right member of (25) we then have

$$(26) \quad B_i = C\nu_i + g^{\alpha\beta}(B_{,\alpha} + g^{\mu\nu}b_{\mu\alpha}A_{,\nu})x_{i,\beta}.$$

It remains to consider the quantity $A_{i,\alpha}$ in the right member of (22). But

$$A_{i,\alpha} = [Z_{,i}]_{,\alpha} = (B\nu_i + g^{\mu\nu}A_{,\mu}x_{i,\nu})_{,\alpha}$$

when we make the substitution (14). Now the indicated surface differentiation, with respect to the coordinates u^α , of the parenthesis expression in the right member of these relations may be taken to be either partial or covariant differentiation. Assuming covariant differentiation we have

$$A_{i,\alpha} = B_{,\alpha}\nu_i + B\nu_{i,\alpha} + g^{\mu\nu}A_{,\mu\alpha}x_{i,\nu} + g^{\mu\nu}A_{,\mu}x_{i,\nu\alpha},$$

where the quantities $A_{,\mu\alpha}$ and $x_{i,\nu\alpha}$ are the components of second covariant derivatives. Hence, using (5), the expression for $A_{i,\alpha}$ becomes

$$(27) \quad A_{i,\alpha} = B_{,\alpha}\nu_i - Bg^{\beta\gamma}b_{\alpha\beta}x_{i,\gamma} + g^{\sigma\tau}A_{,\alpha\sigma}x_{i,\tau} + g^{\sigma\tau}A_{,\sigma}b_{\alpha\tau}\nu_i.$$

Making the substitutions (26) and (27) in the relation (22) and combining terms we now obtain

$$(28) \quad [Z_{,ij}] = C\nu_i\nu_j + g^{\alpha\beta}(B_{,\alpha} + g^{\sigma\tau}b_{\alpha\sigma}A_{,\tau})(\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) \\ + g^{\alpha\beta}g^{\sigma\tau}(A_{,\alpha\sigma} - Bb_{\alpha\sigma})x_{i,\beta}x_{j,\tau}.$$

These relations are the *geometrical conditions of compatibility of the second order*. We observe that the right member of (28) satisfies the required property of being symmetric in the indices i and j . If, in particular, the function Z is continuous across the surface $\Sigma(t)$ the relations (28) reduce to

$$(29) \quad [Z_{,ij}] = C\nu_i\nu_j + g^{\alpha\beta}B_{,\alpha}(\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) - Bg^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}x_{i,\beta}x_{j,\tau}.$$

If, furthermore, the first derivatives of the function Z with respect to the coordinates x are also continuous, the relations become merely

$$(30) \quad [Z_{,ij}] = C\nu_i\nu_j.$$

Certain compatibility expressions which can be derived from (28) will be found to be useful in the applications of this formula. Thus, if we contract the indices i and j in (28), we obtain

$$(31) \quad [Z_{,ii}] = C + g^{\alpha\beta} A_{,\alpha\beta} - 2B\Omega,$$

when use is made of (3) and (6).

6. Kinematical conditions of the second order. Let us now replace Z by the derivative $\partial Z/\partial t$ in the conditions (14) and (21). This leads to relations of the form

$$(32) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] = B' \nu_i + g^{\alpha\beta} A'_{,\alpha} x_{i,\beta},$$

$$(33) \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = -B'G + \frac{\delta A'}{\delta t},$$

where

$$(34) \quad A' = \left[\frac{\partial Z}{\partial t} \right]; \quad B' = \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] \nu_i.$$

We can derive an equivalent expression for the quantity B' by considering the δ time derivative of B . In fact we have

$$(35) \quad \frac{\delta B}{\delta t} = \frac{\delta}{\delta t} ([Z_{,i}] \nu_i) = \nu_i \frac{\delta [Z_{,i}]}{\delta t} + [Z_{,i}] \frac{\delta \nu_i}{\delta t}.$$

Now from the first equation (4) it follows that

$$(36) \quad \nu_i \frac{\delta \nu_i}{\delta t} = 0.$$

Hence the last term in (35) becomes

$$[Z_{,i}] \frac{\delta \nu_i}{\delta t} = (B \nu_i + g^{\alpha\beta} A_{,\alpha} x_{i,\beta}) \frac{\delta \nu_i}{\delta t} = g^{\alpha\beta} A_{,\alpha} x_{i,\beta} \frac{\delta \nu_i}{\delta t}.$$

Making this substitution and expanding the first term in the right member of (35) we have

$$\frac{\delta B}{\delta t} = [Z_{,ii}] G \nu_i \nu_i + \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] \nu_i + g^{\alpha\beta} A_{,\alpha} x_{i,\beta} \frac{\delta \nu_i}{\delta t},$$

or

$$(37) \quad B' = -CG + \frac{\delta B}{\delta t} - g^{\alpha\beta} A_{,\alpha} x_{i,\beta} \frac{\delta \nu_i}{\delta t},$$

from the third equation (7) and the second equation (34). When we make the substitution (37) the relations (32) and (33) now become

$$(38) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] = \left(-CG + \frac{\delta B}{\delta t} - g^{\alpha\beta} A_{,\alpha} x_{k,\beta} \frac{\delta \nu_k}{\delta t} \right) \nu_i + g^{\alpha\beta} A'_{,\alpha} x_{i,\beta} ,$$

$$(39) \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = CG^2 - G \frac{\delta B}{\delta t} + G g^{\alpha\beta} A_{,\alpha} x_{i,\beta} \frac{\delta \nu_i}{\delta t} + \frac{\delta A'}{\delta t} .$$

The relations (38) and (39) are the *kinematical conditions of compatibility of the second order*. In case the normal directions to the surface $\Sigma(t)$ at $t = t_0$ remain normal to $\Sigma(t)$ for $t > t_0$, e.g. in the case of plane parallel waves or spherical waves having a common center, the derivatives $\delta \nu_i / \delta t$ vanish and the conditions (38) and (39) take the simpler form

$$(40) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] = \left(-CG + \frac{\delta B}{\delta t} \right) \nu_i + g^{\alpha\beta} A'_{,\alpha} x_{i,\beta} ,$$

$$(41) \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = CG^2 - G \frac{\delta B}{\delta t} + \frac{\delta A'}{\delta t} .$$

If the function Z is continuous over $\Sigma(t)$, then $A = 0$, and the expression (21) for A' reduces to $-BG$. Hence, making these substitutions in (38) and (39), we have

$$(42) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] = \left(-CG + \frac{\delta B}{\delta t} \right) \nu_i - g^{\alpha\beta} (BG)_{,\alpha} x_{i,\beta} ,$$

$$(43) \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = CG^2 - 2G \frac{\delta B}{\delta t} - B \frac{\delta G}{\delta t} .$$

In certain applications where this case is assumed it is found that $G = \text{const.}$ which results in a further simplification of the conditions (42) and (43). Finally, if both the function Z and its first coordinate derivatives $Z_{,i}$ are continuous across $\Sigma(t)$, the compatibility conditions reduce to

$$\left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] = -CG \nu_i; \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = CG^2 .$$

Remark. It is possible to eliminate the derivatives $A'_{,\alpha}$ and $\delta A' / \delta t$ from the relations (38) and (39) and thus to decrease the number of basic surface scalars needed to express the kinematical conditions of the second order. The derivatives $\delta \nu_k / \delta t$ can also be eliminated from these kinematical conditions.

To eliminate the quantity $A'_{,\alpha}$ from (38) we can proceed as follows. First, consider the fact that

$$(44) \quad A'_{,\alpha} = \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] x_{i,\alpha} ,$$

which follows directly from (38). Now from the definition of the δ time derivative we see that

$$(45) \quad \frac{\delta [Z_{,i}]}{\delta t} = [Z_{,ij}] G \nu_j + \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] .$$

Hence from (44) and (45) we have

$$(46) \quad A'_{,\alpha} = \frac{\delta[Z_{,i}]}{\delta t} x_{i,\alpha} = \frac{\delta}{\delta t} ([Z_{,i}]x_{i,\alpha}) - [Z_{,i}] \frac{\delta x_{i,\alpha}}{\delta t}.$$

But from (14) it follows that the parenthesis expression in the right member of (46) reduces to $A_{,\alpha}$. Hence (46) becomes

$$(47) \quad A'_{,\alpha} = \frac{\delta A_{,\alpha}}{\delta t} - [Z_{,i}] \frac{\delta x_{i,\alpha}}{\delta t}.$$

Performing the time differentiations indicated in the right member of (47) we have

$$(48) \quad \begin{cases} \frac{\delta A_{,\alpha}}{\delta t} = \frac{\partial^2 A}{\partial u^\alpha \partial u^\beta} \frac{\delta u^\beta}{\delta t} + \frac{\partial^2 A}{\partial u^\alpha \partial t}, \\ \frac{\delta x_{i,\alpha}}{\delta t} = \frac{\partial^2 x_i}{\partial u^\alpha \partial u^\beta} \frac{\delta u^\beta}{\delta t} + \frac{\partial^2 x_i}{\partial u^\alpha \partial t}. \end{cases}$$

Hence, in place of (47), we can write

$$(49) \quad A'_{,\alpha} = \left(\frac{\partial^2 A}{\partial u^\alpha \partial u^\beta} - [Z_{,i}] \frac{\partial^2 x_i}{\partial u^\alpha \partial u^\beta} \right) \frac{\delta u^\beta}{\delta t} + \frac{\partial^2 A}{\partial u^\alpha \partial t} - [Z_{,i}] \frac{\partial^2 x_i}{\partial u^\alpha \partial t}.$$

To simplify the above expression for $A'_{,\alpha}$ let us consider the following formulæ for the components of the second covariant derivatives of A and x_i regarded as scalars on the surface Σ , namely

$$(50) \quad \begin{cases} A_{,\alpha\beta} = \frac{\partial^2 A}{\partial u^\alpha \partial u^\beta} - A_{,\sigma} \Gamma_{\alpha\beta}^\sigma, \\ x_{i,\alpha\beta} = \frac{\partial^2 x_i}{\partial u^\alpha \partial u^\beta} - x_{i,\sigma} \Gamma_{\alpha\beta}^\sigma, \end{cases}$$

where the Γ 's are Christoffel symbols based on the metric of the surface. Hence the parenthesis expression in the right member of (49) can be written

$$A_{,\alpha\beta} + A_{,\sigma} \Gamma_{\alpha\beta}^\sigma - [Z_{,i}] x_{i,\alpha\beta} - [Z_{,i}] x_{i,\sigma} \Gamma_{\alpha\beta}^\sigma.$$

But the second and last terms in this expression are seen to cancel when we make the substitution (14). Hence, using the first relation (5), it follows that (49) becomes

$$(51) \quad A'_{,\alpha} = (A_{,\alpha\beta} - B b_{\alpha\beta}) \frac{\delta u^\beta}{\delta t} + \frac{\partial^2 A}{\partial u^\alpha \partial t} - [Z_{,i}] \frac{\partial^2 x_i}{\partial u^\alpha \partial t}.$$

To derive a formula for the time derivatives $\delta u^\beta / \delta t$ in (51) let us observe that the equation (18) can be expressed as

$$(52) \quad x_{i,\alpha} \frac{\delta u^\alpha}{\delta t} + \frac{\partial x_i}{\partial t} = G \nu_i.$$

Multiplying both members of (52) by ν_i and again by $x_{i,\beta}$ and in each case summing on the repeated index i , we obtain

$$(53) \quad G = \nu_i \frac{\partial x_i}{\partial t}; \quad x_{i,\alpha} x_{i,\beta} \frac{\delta u^\alpha}{\delta t} + x_{i,\beta} \frac{\partial x_i}{\partial t} = 0.$$

The first relation (53) will not concern us at present. In the second relation (53) the coefficient of $\delta u^\alpha/\delta t$ is $g_{\alpha\beta}$ from (3) and hence this relation can be solved to obtain

$$(54) \quad \frac{\delta u^\beta}{\delta t} = -g^{\beta\gamma} x_{i,\gamma} \frac{\partial x_i}{\partial t}.$$

Let us next turn our attention to the derivation of a formula for the derivatives $\delta\nu_k/\delta t$ which occur in (38) and (39). We begin by differentiating the two relations (4) to obtain

$$(55) \quad \nu_i \frac{\delta\nu_i}{\delta t} = 0; \quad x_{i,\alpha} \frac{\delta\nu_i}{\delta t} = -\nu_i \frac{\delta x_{i,\alpha}}{\delta t}.$$

Now multiply both members of the second equation (55) by $g^{\alpha\beta} x_{i,\beta}$ and sum on the repeated indices. This gives

$$(56) \quad g^{\alpha\beta} x_{i,\alpha} x_{i,\beta} \frac{\delta\nu_i}{\delta t} = -\nu_i g^{\alpha\beta} x_{i,\beta} \frac{\delta x_{i,\alpha}}{\delta t}.$$

But it can be shown that

$$g^{\alpha\beta} x_{i,\alpha} x_{i,\beta} = \delta_{ii} - \nu_i \nu_i,$$

where δ_{ii} is the Kronecker δ . Making this replacement of the coefficient of $\delta\nu_i/\delta t$ in (56) and using the first equation (55) it follows that

$$(57) \quad \frac{\delta\nu_i}{\delta t} = -\nu_i g^{\alpha\beta} x_{i,\beta} \frac{\delta x_{i,\alpha}}{\delta t}.$$

Now when we eliminate the time derivatives in the right member of (57) by means of the second relation (48) and then eliminate the quantities $\delta u^\beta/\delta t$, which are thereby introduced, by the use of (54) we obtain

$$(58) \quad \frac{\delta\nu_i}{\delta t} = \nu_i g^{\alpha\beta} g^{\sigma\tau} \frac{\partial^2 x_i}{\partial u^\alpha \partial u^\sigma} x_{i,\beta} x_{k,\tau} \frac{\partial x_k}{\partial t} - g^{\alpha\beta} x_{i,\beta} \nu_i \frac{\partial^2 x_i}{\partial u^\alpha \partial t}.$$

Next eliminate the second derivatives in the first term in the right member of (58) by means of the second relation (50). We then find that the resulting term in the symbols Γ vanishes on account of the second equation (4). Hence (58) becomes

$$(59) \quad \frac{\delta\nu_i}{\delta t} = \left(g^{\sigma\tau} b_{\alpha\sigma} x_{k,\tau} \frac{\partial x_k}{\partial t} - \nu_i \frac{\partial^2 x_i}{\partial u^\alpha \partial t} \right) g^{\alpha\beta} x_{i,\beta},$$

when use is made of the first relation (5). But

$$\nu_i \frac{\partial^2 x_i}{\partial u^\alpha \partial t} = \left(\nu_i \frac{\partial x_i}{\partial t} \right)_{,\alpha} - \nu_{i,\alpha} \frac{\partial x_i}{\partial t},$$

or

$$\nu_i \frac{\partial^2 x_i}{\partial u^\alpha \partial t} = G_{,\alpha} + g^{\sigma\tau} b_{\alpha\sigma} x_{k,\tau} \frac{\partial x_k}{\partial t},$$

on account of the first equation (53) and the second equation (5). Hence (59) reduces to the rather interesting relation

$$(60) \quad \frac{\delta \nu_i}{\delta t} = -g^{\alpha\beta} G_{,\alpha} x_{i,\beta}.$$

Now eliminate the derivatives $A'_{,\alpha}$ and $\delta \nu_k / \delta t$ from (38) by the substitution (51) and (60) and then remove the resulting derivatives $\delta u^\beta / \delta t$ by means of (54). After some reduction this leads to the formula

$$(61) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] = \left(-CG + \frac{\delta B}{\delta t} + g^{\alpha\beta} G_{,\alpha} A_{,\beta} \right) \nu_i - g^{\alpha\beta} g^{\sigma\tau} A_{,\alpha\sigma} x_{i,\beta} x_{k,\tau} \frac{\partial x_k}{\partial t} \\ - B g^{\alpha\beta} G_{,\alpha} x_{i,\beta} + g^{\alpha\beta} x_{i,\beta} \left(\frac{\partial^2 A}{\partial u^\alpha \partial t} - g^{\sigma\tau} A_{,\sigma} x_{k,\tau} \frac{\partial^2 x_k}{\partial u^\alpha \partial t} \right).$$

For the purpose of application it may be useful to note that (61) is equivalent to the following two relations

$$(62) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] \nu_i = -CG + \frac{\delta B}{\delta t} + g^{\alpha\beta} G_{,\alpha} A_{,\beta},$$

$$(63) \quad \left[\frac{\partial^2 Z}{\partial x_i \partial t} \right] x_{i,\alpha} = -g^{\sigma\tau} A_{,\alpha\sigma} x_{k,\tau} \frac{\partial x_k}{\partial t} - B G_{,\alpha} + \frac{\partial^2 A}{\partial u^\alpha \partial t} - g^{\sigma\tau} A_{,\sigma} x_{k,\tau} \frac{\partial^2 x_k}{\partial u^\alpha \partial t}.$$

Similarly we obtain from (39) that

$$(64) \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = CG^2 - G \frac{\delta B}{\delta t} - G g^{\alpha\beta} A_{,\alpha} G_{,\beta} + \frac{\delta}{\delta t} \left[\frac{\partial Z}{\partial t} \right],$$

or

$$(65) \quad \left[\frac{\partial^2 Z}{\partial t^2} \right] = CG^2 - 2G \frac{\delta B}{\delta t} - B \frac{\delta G}{\delta t} - G g^{\alpha\beta} A_{,\alpha} G_{,\beta} + \frac{\delta^2 A}{\delta t^2},$$

when we eliminate the quantity $[\partial Z / \partial t]$ in (64) by means of (21). With the relations (61) and (65) we have now attained our stated objective mentioned at the beginning of this remark.