On Criteria for Linear Stability

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Communicated by L. CESARI

1. In the Appendix of a paper which appeared on pages 553–559 of volume 68 (1946) of the American Journal of Mathematics and which will be referred to as [*], as by-products of *non-linear* considerations I obtained a few simple results concerning asymptotic limitations of the solutions of homogeneous *linear* systems. Subsequently, the results and methods of [*] were applied by several authors to questions pertaining to problems of linear stability. A study of these questions will be found in a recent paper of R. CONTI (Rivista di Mat. della Univ. di Parma 6 (1955) pp. 3–35), which lists various references (but misses [*] and, correspondingly, does not mention that, for instance, the papers of T. WAŻEWSKI (1948), Z. BUTLEWSKI (1948) and A. ROSENBLATT (1954), quoted under [20], [5] and [17] in the paper, contain rediscoveries, or weakened forms, of the results of the paper [*] of 1946). It appears however that the ultimate consequences of the method of [*], which seem to be essential in various applications, have not been developed yet.

The object of the first part of this paper is to fill in this gap by presenting, on the one hand, a systematization of the results which can be obtained by the method of [*] and, on the other hand, the derivation of certain stability theorems which then result if recourse is had to Lagrange's perturbation principle.

The balance of the paper deals with a certain refinement of the question of the preservation of stability under "small" perturbations of the coefficient matrix. A known criterion of this type is what is quoted as (iii) in Section 6 below. But there is another criterion, (iv) in Section 6, which, in contrast to (iii), deals with the same question when the two systems compared are connected by an affine equivalence of the coefficient matrices. The deduction of this comparison theorem, (iv), will depend on a combination of (iii) with an abstract formulation of a method of proof due to L. CESARI (Ann. Scuola Norm. Pisa, ser. 2, 9 (1940) pp. 163–186).

2. For large positive t , let $A = A(t)$ be an n by n matrix which is a continuous function of t , and consider only those solutions $x = x(t)$ of

$$(1) \quad \frac{dx}{dt} = A(t)x$$

which are distinct from the trivial solution ($\equiv 0$). Then, if $r(t)$ denotes the length of the vector $x(t)$,

$$(2) \quad r = r(t) = |x(t)| > 0 \quad \text{for all } t.$$

In fact, if $x(t) = 0$ for some $t = t_0$, then, since the trivial solution is one solution which fulfills the initial condition $x(t_0) = 0$, and since $x(t_0)$ determines $x(t)$ for all t , Lipschitz's uniqueness condition being satisfied by (1), it follows that $x(t)$ must be the trivial solution. The latter will not of course be admitted when such questions are asked as whether or not (1) has a solution $x(t)$ satisfying

$$(3) \quad \liminf |x(t)| = 0, \text{ where } t \rightarrow \infty.$$

It will be convenient to assume that the elements of $A(t)$ (and the components of every $x(t)$) are real-valued. This involves no loss of generality, since n can be replaced by $2n$. Denote by A' the transposed matrix of A (at a fixed t) and put $A^\circ = \frac{1}{2}(A + A')$. Then, at every t , the matrix

$$(4) \quad A^\circ \text{ is symmetric}$$

and, since $A - A^\circ = \frac{1}{2}(A - A')$, the matrix

$$(5) \quad A - A^\circ \text{ is skew-symmetric.}$$

If C is any (real) matrix, let $C(x, y)$ denote its bilinear form, and $\text{tr } C$ the sum of the diagonal elements of C . Then it is clear that both expressions $C(x, x)$, $\text{tr } C$ are additive in C , and that both of them are 0 if C is skew-symmetric. It follows therefore from (4) and (5) that

$$(6) \quad A(x, x) = A^\circ(x, x)$$

is an identity in x , and that

$$(7) \quad \text{tr } A = \text{tr } A^\circ.$$

Let $\lambda = \lambda(t)$ denote the least, and $\mu = \mu(t)$ the greatest, eigenvalue of the symmetric matrix $A^\circ = A^\circ(t)$. Then it follows from (6) that, for any x ,

$$(8) \quad |x|^2 \lambda(t) \leq A(x, x) \leq |x|^2 \mu(t), \quad \text{where } A = A(t).$$

Similarly, since $\text{tr } C$ is known to be the sum of the n eigenvalues of any C , it follows from (7) that

$$(9) \quad n\lambda(t) \leq \text{tr } A(t) \leq n\mu(t).$$

If $x = x(t)$ is a solution of (1), then, since the scalar product $2x \cdot dx/dt$ is identical with $d(r^2)/dt$ by virtue of (2), it is seen from (1) and (6) that

$$(10) \quad \frac{d(r^2)}{dt} = 2A^\circ(x, x), \quad \text{where } A^\circ = A^\circ(t).$$

But if (10) is divided by r^2 , it follows from (2) and (8) that

$$(11) \quad \lambda(t) \leq \frac{d}{dt} \log |x(t)| \leq \mu(t)$$

holds for every t and for every (non-trivial) solution $x(t)$ of (1). The results of [*] (cf. p. 559) were concluded from the second of the inequalities (11) (rediscovered by WAŻEWSKI, and, under stricter restrictions, by others; cf. Section 1 above and R. BELLMAN's recent work, referred to by CONTI).

3. A solution $x(t)$ of (1) is called stable if

$$(12) \quad \limsup |x(t)| < \infty,$$

where $t \rightarrow \infty$. If both (12) and

$$(13) \quad \liminf |x(t)| > 0$$

hold, then $x(t)$ will be called strictly stable. If every (non-trivial) solution of (1) is strictly stable, then (1) itself is called strictly stable. Correspondingly, (1) is called stable if all solutions of (1) are stable.

(i) If $\mu = \mu(t)$ denotes the greatest eigenvalue of the symmetric matrix $A^\circ = A^\circ(t)$, then

$$(14) \quad -\infty \leq \limsup_{t \rightarrow \infty} \int_{t_0}^t \mu(s) ds < \infty$$

is sufficient in order that (1) be stable. If, in addition to (14),

$$(15) \quad -\infty < \liminf_{t \rightarrow \infty} \int_{t_0}^t \lambda(s) ds \leq \infty$$

holds, where $\lambda(t)$ denotes the least eigenvalue of $A^\circ(t)$, then (1) is strictly stable.

The first of these assertions (i) is identical with the theorem italicized in [*], p. 558. The second assertion of (i) can be refined to the statement that, whether (1) is or is not stable, (15) alone (without regard to (14)) is sufficient in order that (13) should hold for all (non-trivial) solutions $x(t)$ of (1). All of this is an immediate consequence of (11).

The point is that if $A(t)$ is written in the form $H(t) + S(t)$, where $H = A^\circ$ is symmetric and $S = A - A^\circ$ is skew-symmetric (cf. (4) and (5)), then *only* $H(t)$ enters in (i), even though the assertions of (i) refer to (1), rather than just to $x' = H(t)x$, the particular case $S(t) \equiv 0$ of (1). There is a similar, though different, instance of the elimination of the skew-symmetric part in case of linear differential systems of second order.¹

Conditions, distinct from those of (i), for the stability or the strict stability of (1) will be given in Section 6 below. There the consideration will involve the

¹ A. WINTNER, Amer. Journ. of Math. 71 (1949) p. 365.

Jacobi-Liouville identity

$$(16) \quad \det X(t) = \beta \exp \int_{t_0}^t \operatorname{tr} A(s) ds, \quad \beta = \det X(t_0) \neq 0,$$

where $X(t)$ denotes any fundamental matrix of (1), that is, any matrix $X = (x_1, \dots, x_n)$ the columns of which are n , linearly independent, solutions $x_i = x_i(t)$ of (1).

3 bis. The first assertion of (i) contains the following corollary (in which y, f, g are scalars, and f, g are real-valued, continuous functions for large positive t):

If $f(t) \equiv 1$ and $g(t) \geq 0$ or, more generally, if

$$\int_{t_0}^{\infty} |f(t) - 1| dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} |g^-(t)| dt < \infty$$

(where $g^- = g - g^+$ if $g^+ = \max(0, g) \geq 0$), then all solutions $y(t)$ of

$$y'' + g(t)y' + f(t)y = 0,$$

along with their derivatives $y'(t)$, stay bounded as $t \rightarrow \infty$.

What is not of a standard type in this lemma is *the unilateral character* of the assumption $g(t) \geq 0$ or of its integral refinement. If $g(t) \geq 0$ is strengthened to $g(t) \geq \text{const.} > 0$, (and if $f(t) \equiv 1$), then it is known² that all solutions $y(t)$ are not only bounded, but tend to 0 with the rapidity of e^{-ct} , where $c > 0$, as $t \rightarrow \infty$.

If x denotes the vector with the $n = 2$ components y, y' , then the differential equation of second order can be written in the form (1), with

$$A = \begin{bmatrix} 0 & 1 \\ -f & -g \end{bmatrix}; \quad \text{hence} \quad 2A^\circ = \begin{bmatrix} 0 & 1-f \\ 1-f & -2g \end{bmatrix},$$

by the definition of A° ; cf. (4) and (5). Suppose first that $f(t) \equiv 1$. Then A° is the diagonal matrix having the eigenvalues 0, $-g$, and so the definition of μ in (i) shows that $\mu = g^-$. Hence (14) will be satisfied if $g^-(t)$ is (absolutely) integrable. Thus (i) contains the truth of the last italicized lemma for the case $f(t) \equiv 1$. Finally, if $f(t) \equiv 1$ is generalized to the absolute integrability of $f(t) - 1$, then the assertion follows by first using its truth for the case $f(t) \equiv 1$ and then applying (iii bis) below.

4. It is clear from the principle of superposition that (1) is stable if and only if all n^2 scalars which constitute $X(t)$ are bounded functions of t (for some and/or every choice of the fundamental matrix X); that, if no restriction is placed on (1) and if an $X(t)$ is given, then

$$(17) \quad x(t) = X(t)c$$

² A. WINTNER, Journ. of Math. and Mech. 5 (1957), pp. 109-117.

is a solution of (1) for every choice of the constant vector c ; that every solution $x(t)$ of (1) determines a unique c for which (17) holds; and that $c = 0$ belongs to the trivial solution of (1).

Since (16) implies that $\det X \neq 0$ for every t , there exists³ a *unique* pair of matrices, say $P = P(t)$ and $R = R(t)$, which, like $X = X(t)$, are continuously differentiable and have the following properties: P is symmetric and positive definite, R is orthogonal (*i.e.*, $R' = R^{-1}$), and

$$(18) \quad X = RP$$

(which, since $\det P > 0$, implies that R is a rotation or a reflection for every t according as $\det X > 0$ or $\det X < 0$ throughout).

Since it is clear from (18) that $X'X = P^2$, and since the definition of X , given at the end of Section 3, shows that $X'X$ is identical with the matrix of the scalar products $x_i \cdot x_k$, it follows that

$$(19) \quad P^2 = (x_i \cdot x_k)$$

for every t . But $P = P(t)$ is the positive definite square root of (19). Hence the algebraic construction leading to this square root makes it clear that $P = P(t)$ will tend, as $t \rightarrow \infty$, to a limit matrix if and only if the same is true of (19) itself. On the other hand, $x_i + x_k$ and $x_i - x_k$ are solutions of (1), since x_i and x_k are. Hence, if $x_i \cdot x_k = |x_i|^2$ or $4x_i \cdot x_k = |x_i + x_k|^2 - |x_i - x_k|^2$ is used according as $i = k$ or $i \neq k$, then, since $x_1, \dots, x_n$ is a complete set of linearly independent solutions of (1), it follows that $P(t)$ tends, as $t \rightarrow \infty$, to a limit, $P(\infty)$, if and only if there exists a (finite, non-negative) $\lim |x(t)|$ for every solution $x(t)$ of (1).

Clearly, this is a sufficient, but not a necessary, condition for the stability of (1). In view of the diagonal elements of the matrix (19), what is necessary and sufficient for the stability of (1) is that the second factor, $P = P(t)$, of (18) be bounded as $t \rightarrow \infty$, which requires less than the existence of $P(\infty)$.

Sufficient for the existence of $P(\infty)$ is that $A^\circ(t)$ be non-positive definite at every t (*cf.* [*], p. 557). In fact, the latter condition means that $\mu(t)$, the greatest eigenvalue of $A^\circ(t)$, is nowhere positive. Consequently, (11) implies that $\log |x(t)|$, hence $|x(t)|$ itself, is a non-increasing function of t , if $x(t)$ is any solution, and so every $\lim |x(t)|$ exists (with the understanding that it can be 0 or a positive number).

Another sufficient condition for the existence of $P(\infty)$ is the (possibly just conditional) convergence of both integrals⁴

$$(20) \quad \int_{t_0}^{\infty} \lambda(t) dt, \quad \int_{t_0}^{\infty} \mu(t) dt, \quad \text{where} \quad \int_{t_0}^{\infty} = \lim_{T \rightarrow \infty} \int_{t_0}^T.$$

But then $\lim |x(t)|$ not only exists, but is even distinct from 0, for every non-trivial solution $x(t)$ of (1). In fact, it is clear from (11) that the existence of both

³ *Cf.* A. WINTNER, *Annali di Mat.*, ser. 4, **13** (1934) pp. 105–112 (footnote 11 on p. 108).

⁴ A. WINTNER, *Quart. of Appl. Math.* **8** (1950) pp. 102–104.

limits (20), being equivalent to

$$\int_u^v \lambda(t) dt \rightarrow 0, \quad \int_u^v \mu(t) dt \rightarrow 0 \quad \text{as} \quad \min(u, v) \rightarrow 0,$$

implies that $\log |x(v)| - \log |x(u)| \rightarrow 0$ as $\min(u, v) \rightarrow 0$, and this means that $\log |x(t)|$ tends to a finite limit, hence $|x(t)|$ to a finite non-vanishing limit, as $t \rightarrow \infty$.

5. Another variant of (i) is contained in the first sentence of the following statement (ii):

(ii) *If $t \rightarrow \infty$, then*

$$(21) \quad \int_{t_0}^t \lambda(s) ds \rightarrow -\infty$$

is necessary, and

$$(22) \quad \int_{t_0}^t \mu(s) ds \rightarrow -\infty$$

is sufficient, in order that

$$(23) \quad x(t) \rightarrow 0$$

holds for every solution of (1) or, what is the same thing, in order that

$$(24) \quad P(t) \rightarrow 0$$

holds for a fundamental matrix (18), where $X = X(t)$. On the other hand, the condition

$$(25) \quad \int_{t_0}^t \operatorname{tr} A(s) ds \rightarrow -\infty,$$

(which in view of (8) is intermediate between (21) and (22)) is necessary and sufficient for

$$(26) \quad \det P(t) \rightarrow 0$$

(which is less than (24)).

First, the assertions of (ii) concerning (21), (22) and (23) follow from (11), since (23) is equivalent to $\log |x(t)| \rightarrow -\infty$ by virtue of (2). On the other hand, (19) shows that (24) is equivalent to (23), where $x = x_1, \dots, x_n$. Finally, since (18) implies that $\det X = \pm \det P$, it is seen from (16) that (25) is equivalent to (26).

Remark. The same formalism leads to a simple wording of the proof of a lemma of P. HARTMAN⁵ which, under the hypothesis that $\lim |x(t)|$ exists for every solution $x(t)$ of (1), states that (25) is necessary and sufficient for the

⁵ P. HARTMAN, Amer. Journ. of Math. **70** (1948) pp. 395-399.

existence of *some* non-trivial solution $x(t)$ satisfying (23). In fact, the italicized hypothesis is equivalent to the existence of $P(\infty)$. Since (25) is equivalent to (23), it follows that, under the italicized hypothesis, (25) is equivalent to $\det P(\infty) = 0$. But (17) shows that $|x(t)|^2 = x(t) \cdot x(t)$ is identical with the expression $c'X'(t)X(t)c$, which, since (18) implies that $X'X = P^2$, is identical with the quadratic form $Q(c, c)$ belonging to the positive definite matrix $Q = P^2$, where $P = P(t)$. Consequently, (23) will hold for some choice of the constant vector $c \neq 0$ in (17) if and only if there exists some $c \neq 0$ for which the quadratic form $Q_\infty(c, c)$, where Q_∞ denotes the square of the matrix $P(\infty)$, becomes 0. But since the form $Q_\infty(c, c)$ is non-negative definite, it will become 0 for some $c \neq 0$ if and only if $\det Q_\infty = 0$ or, since Q_∞ is the square of $P(\infty)$, if and only if $\det P(\infty) = 0$ and, therefore, if and only if (25) is satisfied.

6. The following comparison theorem, connecting an equation (1) with another such equation, say with

$$(28) \quad \frac{dx}{dt} = B(t)x,$$

was proved some time ago:⁶

(iii) Suppose that (1) is stable, that

$$(29) \quad -\infty < \liminf_{t \rightarrow \infty} \int_{t_0}^t \operatorname{tr} A(s) ds \leq \infty,$$

and that⁷

$$(30) \quad \int_{t_0}^{\infty} |A(t) - B(t)| dt < \infty.$$

Then the solutions of (1) and those of (28) are in asymptotic one-to-one correspondence.

By this is meant that if (I) denotes either (1) or (28), and (II) either (28) or (1), respectively, then there belongs to every solution $x = x_I(t)$ of (I) a unique solution $x = x_{II}(t)$ of (II) satisfying

$$(31) \quad x_I(t) - x_{II}(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

⁶ A. WINTNER, Amer. Journ. of Math. **68** (1946) pp. 199–201.

⁷ In view of well known algebraic facts, which concern themselves with certain inequalities valid for any constant matrix C , it is immaterial whether (for a fixed t) the norm $|C|$ of $C = A - B$ in (30) is meant to be

$$|C| = \sum_{i=1}^n \sum_{k=1}^n |c_{ik}|, \quad \text{where} \quad C = (c_{ik}),$$

or

$$|C| = \max |C(u, v)| \quad \text{for} \quad |u| = 1, \quad |v| = 1,$$

where $C(u, v)$ denotes the bilinear form belonging to C .

Hence (iii) contains the following corollary (which claims, of course, much less than (iii) itself):

(iii bis) *Under the assumptions (29) and (30), the stability of (1) is equivalent to the stability of (28).*

Corresponding to Riemann's concept of *Art* (Poincaré's *genre*) in the analytic case of a complex independent variable, let (28) be called *similar* to (1) if there exist a continuous matrix function $T = T(t)$ of non-vanishing determinant satisfying

$$(32) \quad B = T^{-1}AT$$

(as an identity in t). Clearly, all similar equations (1) form a class.

By adapting a formalism applied by L. CESARI (1940; cf. the reference in Section 1 above and pp. 6–12 of R. CONTI's paper (1955), mentioned in Section 1), the following conclusion will now be drawn from (iii):

(iv) *Suppose that (1) and (28) are similar, and that (32) holds for a continuous $T(t)$ for which*

$$(33) \quad T(\infty) = \lim_{t \rightarrow \infty} T(t) \text{ exists and } \det T(\infty) \neq 0$$

*and which is of finite total variation:*⁸

$$(34) \quad \int_{t_0}^{\infty} |dT(t)| < \infty.$$

Concerning $A(t)$, suppose that (1) is stable and that (29) is satisfied. Then the solutions of (28) are in asymptotic one-to-one correspondence with the solutions of (1).

By this is meant the correspondence explained in connection with (31). Hence it is clear that (iv) contains the following corollary:

(iv bis) *Suppose $T(t)$ satisfies (33) and (34), that (29) holds for $A(t)$, and that $B(t)$ is defined by (32). Then the stability of (28) is equivalent to the stability of (1).*

For the sake of simplicity, it will be assumed in the proof that $T(t)$ is continuously differentiable. Actually, this assumption, being purely local in nature, can afterwards be dispensed with, by applying appropriate approximations on finite t -intervals $t_0 \leq t \leq t_0 + m$, where $m = 1, 2, \dots$ and $m \rightarrow \infty$. Thus (34) will be used in the form

$$(35) \quad \int_{t_0}^{\infty} |T'(t)| dt < \infty$$

(with a continuous $T' = dT/dt$; from now on, a prime will denote differentiation and *not*, as above, transposition).

⁸ Needless to say, the first (but not the second) of the assumptions (33) is implied by (34).

7. If $A(t)$ is continuous and if $T(t)$ is continuously differentiable and of non-vanishing determinant, define $B(t)$ by (32), and $Z(t)$ by

$$(36) \quad Z = T^{-1}X,$$

where $X(t)$ is a fundamental matrix of (1), finally $C(t)$ by

$$(37) \quad C = B - T^{-1}T'.$$

Then $Z(t)$ is a fundamental matrix of

$$(38) \quad x' = C(t)x.$$

In view of the differentiation rule $(T^{-1})' = -T^{-1}T'T^{-1}$, the truth of this statement follows by direct substitutions.

Suppose that $T(t)$ and $A(t)$ satisfy the assumptions of (iv). Then $X(t)$ is bounded, as $t \rightarrow \infty$, since (1) is stable, and so (36) and (33) show that $Z(t)$ is bounded as $t \rightarrow \infty$, which means that (38) is stable. In addition,

$$(39) \quad -\infty < \liminf_{t \rightarrow \infty} \int_{t_0}^t \operatorname{tr} C(s) ds \leq \infty.$$

In fact, it is clear from (35) and (33) that

$$(40) \quad \int_{t_0}^{\infty} |T^{-1}T'| dt < \infty.$$

On the other hand, $\operatorname{tr} B = \operatorname{tr} A$, by (32), and so it is clear from (29) and (37) that (39) is certainly true if

$$\int_{t_0}^{\infty} |\operatorname{tr} (T^{-1}T')| dt < \infty.$$

But this is obvious from (40).

Accordingly, (38) is stable and its coefficient matrix satisfies (39). This means that both conditions imposed on A in (iii) are satisfied if the A of (iii) is identified with C . Hence, if (40) is identified with (30), it follows from (37) that (iii) leads to the following information: The solutions of (38) are in asymptotic one-to-one correspondence with those of (28).

In order to complete the proof of (iv), all that remains to be ascertained is that the solutions of (38) are in asymptotic one-to-one correspondence with those of (1). But this is clear from (33) and (36), since $Z(t)$ and $X(t)$ are fundamental matrices of (38) and (1) respectively (and since, (38) being stable, the function Z of t is bounded and so, in view of (36) and (33), the same is true of the function X).

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