On the Solutions of Nonlinear Differential Equations, I

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1. Introduction. In this paper we shall study the behavior of the solutions of the differential equation

$$(1) \quad x'' + p(t)x + 2q(t)x^3 = 0$$

with a time-dependent nonlinear restoring force. Our interests are the oscillatory properties, boundedness, and asymptotic behavior of the solutions as well as the existence of periodic solutions. In the subsequent discussions, the coefficients $p(t)$ and $q(t)$ in the equation (1) are assumed (*) to be real-valued functions defined and Lebesgue-measurable on $0 \leq t < \infty$ and Lebesgue-integrable on $0 \leq t \leq a$ for every positive a . It is further assumed that $p(t)$ and $q(t)$ are positive for all t and possess positive lower and upper bounds. In all subsequent statements, the above basic hypotheses shall be referred to as (*).

By a solution of (1) on a t -interval I we mean a function $x(t)$ which is defined and absolutely continuous together with its derivative $x'(t)$ on I and satisfying (1) almost everywhere on I , *i.e.*, in the sense of CARATHEODORY [6]. The usual method of successive approximation (Picard-Lindelof existence theorem) proves that the solutions of (1) exist uniquely and at least locally [6]. In this paper all solutions considered are real and different from the identically zero solution.

Our main results are stated in Theorems 1, 5, 6, 7, 8. The main part of Theorem 1 states that the first positive zero of the solution $x(t)$ of (1), $x(0) = 0$, $x'(0) = E > 0$, tends to zero as $E \rightarrow \infty$ and has a positive limit as $E \rightarrow 0$. An application of Theorem 1 and its basic idea yields the results in Theorems 5 and 6 which assert the existence of various periodic solutions and describe their oscillatory behavior under the assumption that $p(t)$ and $q(t)$, besides satisfying (*), are even periodic functions with a common period. In Theorem 7 we prove that every solution of the equation (1) and its derivative is bounded on $[0, \infty)$ provided that $p(t)$ and $q(t)$ are functions of bounded variation on $[0, \infty)$ and satisfy (*). In Theorem 8, under stronger conditions than in Theorem 7, we give an asymptotic representation of a solution of the equation (1), and of its derivative, by a

periodic function which satisfies a differential equation of the form (1) with constant coefficients. In Section 6 it is indicated that these results hold for a more general differential equation (43).

The literature of the problems considered in this paper in connection with the equation (1) is not extensive. For the same problems or related problems of similar equations, the readers are referred to the report of BELLMAN [3] for the literature before 1948. For recent results we refer the readers to the papers [1] and [2] by ATKINSON, [4] and [5] by BUTLEWSKI, and [8] and [9] by TAAM.

The free vibration, without damping, of some hard spring which changes with time leads to a differential equation of the form (1). In another paper [7] we study a differential equation of the same form (1) with the third term negative which may be considered to correspond to the free vibration, without damping, of some soft spring.

2. Preliminary observations. When $p(t)$ and $q(t)$ are positive constants, the nature of the solutions of the equation (1) can be understood by directly integrating the equation. We shall make such preliminary observations in this section and state the results in Lemma 1 for later use.

Let A and B be positive constants. Consider the solution $x(t)$ of

$$(2) \quad x'' + Ax + 2Bx^3 = 0, \quad x(0) = 0, \quad x'(0) = E > 0.$$

Multiplying (2) by $2x'$ and integrating from 0 to t , one obtains

$$(3) \quad x'^2 + Ax^2 + Bx^4 = E^2.$$

Clearly $x(t)$ is periodic. Let M be the amplitude of $x(t)$ and T one-fourth of its period. We have $E^2 = AM^2 + BM^4$ and

$$(4) \quad \begin{aligned} T &= \int_0^M [A(M^2 - x^2) + B(M^4 - x^4)]^{-\frac{1}{2}} dx \\ &= \int_0^1 (1 - y^2)^{-\frac{1}{2}} [A + BM^2(1 + y^2)]^{-\frac{1}{2}} dy, \end{aligned}$$

where $x = My$.

Lemma 1.

- (a) For fixed positive constants A and B , T decreases as E increases on $0 < E < \infty$ and $T \rightarrow 0$ as $E \rightarrow \infty$, $T \rightarrow \frac{1}{2}\pi A^{-\frac{1}{2}}$ as $E \rightarrow 0$.
- (b) For a fixed positive E , T decreases as both A and B increase, A and B being positive.

Proof. This lemma clearly follows from (4) and

$$A + BM^2(1 + y^2) = \frac{1}{2}[A(1 - y^2) + (A^2 + 4BE^2)^{\frac{1}{2}}(1 + y^2)].$$

Part (a) of Lemma 1 brings out a very important difference in the oscillation of linear homogeneous differential equations of second order and the equation (2). In the linear case, the zeros of a solution $x(t)$ are determined by the initial

condition $x(0) = 0$ and are independent of $x'(0) = E$, while in the case of equation (2), this initial value $x'(0) = E$ plays an important role in the distribution of zeros of a solution. On the other hand, part (b) of this lemma points out a similarity in the two cases, namely, T decreases as the coefficients increase.

3. Oscillatory properties. Every solution $x(t)$ of (1) is oscillatory in the sense that $x(t)$ has infinitely many zeros on $0 \leq t < \infty$. This can be seen by applying Sturm's comparison theorem [6] or a similar argument to the equation (1) written in the form

$$(1') \quad x'' + [p(t) + 2q(t)x^2]x = 0$$

and

$$x'' + p_1x = 0,$$

where p_1 is a positive constant not greater than $p(t)$ on $0 \leq t < \infty$, and the fact that every solution of the second equation is oscillatory.

In this section we investigate the oscillatory properties of the solutions of the equation (1). Consider the solution $x(t)$ of the equation

$$(1^*) \quad x'' + px + 2qx^3 = 0, \quad x(0) = 0, \quad x'(0) = E > 0.$$

Let p_1 and q_1 be the greatest lower bounds of $p(t)$ and $q(t)$, respectively, on $0 \leq t < \infty$ and p_2 and q_2 their least upper bounds. We note that p_1 , p_2 , q_1 , and q_2 are all finite and positive (see Section 1). Let M_1 and M_2 be, respectively, the positive roots of the equations

$$(5) \quad \begin{aligned} p_2M_1^2 + q_2M_1^4 &= E^2, \\ p_1M_2^2 + q_1M_2^4 &= E^2 \end{aligned}$$

and set

$$(6) \quad \begin{aligned} E_1^2 &= p_1M_1^2 + q_1M_1^4, \\ E_2^2 &= p_2M_2^2 + q_2M_2^4. \end{aligned}$$

Clearly $M_2 \geq M_1$ and $E_2 \geq E_1$. Let $u(t)$ and $v(t)$ be the solutions of the equations

$$(7) \quad u'' + p_1u + 2q_1u^3 = 0, \quad u(0) = 0, \quad u'(0) = E_1,$$

and

$$(8) \quad v'' + p_2v + 2q_2v^3 = 0, \quad v(0) = 0, \quad v'(0) = E_2,$$

respectively. If

T_u , T_x , T_v are the first positive zeros of $u(t)$, $x(t)$, $v(t)$, respectively,
 $T_{u'}$, $T_{x'}$, $T_{v'}$ are the first positive zeros of $u'(t)$, $x'(t)$, $v'(t)$, respectively,
 M_u , M_x , M_v are the first maxima of $u(t)$, $x(t)$, $v(t)$, respectively,
 T_w is the first positive zero of the solution $w(t)$ of

$$w'' + p(t)w = 0, \quad w(0) = 0, \quad w'(0) \neq 0,$$

then

Theorem 1.

- (a) $T_{\bullet} \leq T_x \leq T_u$, $T_{\bullet'} \leq T_{x'} \leq T_{u'}$, $M_u \leq M_x \leq M_{\bullet}$;
- (b) $T_x \rightarrow 0$, $M_x \rightarrow \infty$ as $E \rightarrow \infty$;
- (c) $T_x \rightarrow T_w$, $M_x \rightarrow 0$ as $E \rightarrow 0$;
- (d) $0 < T_x < T_w$.

Proof. Since $x'' < 0$ on $0 < t < T_x$, $x(t)$ has one and only one maximum M_x on $0 < t < T_x$. Multiplying the equation (1*) by $2x'$ and integrating from t to $T_{x'}$, $0 \leq t \leq T_{x'}$, we obtain

$$(9) \quad (x')^2 = 2 \int_t^{T_{x'}} p x x' dt + 4 \int_t^{T_{x'}} q x^3 x' dt.$$

Setting

$$\begin{aligned} 2 \int_t^{T_{x'}} p x x' dt &= \int_t^{T_{x'}} p d(x^2) = p^*(M_x^2 - x^2), \\ 4 \int_t^{T_{x'}} q x^3 x' dt &= \int_t^{T_{x'}} q d(x^4) = q^*(M_x^4 - x^4) \end{aligned}$$

for $0 \leq t \leq T_{x'}$, the relation (9) becomes

$$(10) \quad (x')^2 = p^*(M_x^2 - x^2) + q^*(M_x^4 - x^4).$$

Clearly $p^*(t)$ and $q^*(t)$ are defined and continuous on $0 \leq t < T_{x'}$ and, setting $p^* = p$, $q^* = q$ at $T_{x'}$,

$$0 < p_1 \leq p^*(t) \leq p_2 , \quad 0 < q_1 \leq q^*(t) \leq q_2$$

on $0 \leq t \leq T_{x'}$. Hence

$$(10') \quad E^2 = p^*(0)M_x^2 + q^*(0)M_x^4$$

and, using (5), $M_u = M_1 \leq M_x \leq M_2 = M_{\bullet}$. From (10), we have

$$(11) \quad T_{x'} = \int_0^{M_{\bullet}} [p^*(M_x^2 - x^2) + q^*(M_x^4 - x^4)]^{-\frac{1}{2}} dx,$$

where p^* and q^* are expressed as functions of x . Changing the variable $x = M_x z$, (11) yields

$$\begin{aligned} (12) \quad T_{x'} &= \int_0^1 \{(1 - z^2)[p^* + q^* M_x^2(1 + z^2)]\}^{-\frac{1}{2}} dz \\ &\leq \int_0^1 \{(1 - z^2)[p_1 + q_1 M_1^2(1 + z^2)]\}^{-\frac{1}{2}} dz = T_{u'}. \end{aligned}$$

Similarly

$$(13) \quad T_{x'} \geq \int_0^1 \{(1 - z^2)[p_2 + q_2 M_2^2(1 + z^2)]\}^{-\frac{1}{2}} dz = T_{v'}.$$

A relation similar to (10) also holds on $T_{x'} \leq t \leq T_x$ and similar arguments lead to the inequalities

$$(14) \quad T_v - T_{v'} \leq T_x - T_{x'} < T_u - T_{u'},$$

which implies $T_v \leq T_x \leq T_u$. Clearly, as $E \rightarrow \infty$, we have $M_1 = M_u \rightarrow \infty$ and $M_2 = M_v \rightarrow \infty$ (from (5)), and so $T_u = 2T_{u'} \rightarrow 0$ which implies $T_x \rightarrow 0$. On the other hand, as $E \rightarrow 0$, M_u , M_x , and $M_v \rightarrow 0$. Let T_v be the first positive zero of the solution y of

$$(15) \quad y'' + (p + 2qM_x^2)y = 0, \quad y(0) = 0, \quad y'(0) \neq 0.$$

Applying Sturm's comparison theorem to (1'), (15), and the equation satisfied by w , the inequalities

$$(16) \quad T_v < T_x < T_w$$

hold. Since the solution $y(t)$ of (15) is a continuous function of M_x for $-\infty < M_x < \infty$, it is easy to see that T_v is also a continuous function of M_x with $T_v = T_w$ when $M_x = 0$. It follows, from (16), that $T_x \rightarrow T_w$ as $E \rightarrow 0$. This completes the proof.

Theorem 1 gives a method of finding upper and lower bounds of T_x , $T_{x'}$, and M_x .

If $p(t)$ and $q(t)$ are non-decreasing or non-increasing we compare the solutions of (1*) with the solutions of

$$(17) \quad y'' + p_0 y + 2q_0 y^3 = 0, \quad y(0) = 0, \quad y'(0) = E_0 > 0,$$

where p_0 and q_0 are positive constants. Let T_x , $T_{x'}$, M_x , T_v , $T_{v'}$, M_v have the same meanings as before for the solutions x and y , respectively.

Theorem 2. *Let $x(t)$ be the solution of (1*) and $y(t)$ the solution of (17). If $p(t)$ and $q(t)$ are, besides satisfying (*), non-decreasing (non-increasing) and not less (not greater) than p_0 and q_0 respectively, and if $E \geq E_0$ ($E \leq E_0$), then T_x is not greater (not less) than T_v .*

Proof. Clearly the relations (9)-(11) and the first half of (12) still hold. Consider the case that $p(t)$ and $q(t)$ are non-decreasing. From the definition of $p^*(t)$ and $q^*(t)$ clearly we have $p(t) \leq p^*(t)$ and $q(t) \leq q^*(t)$ on $0 \leq t \leq T_{x'}$. Since $p^*(t)$ and $q^*(t)$ are absolutely continuous on $0 \leq t < T_{x'}$, it is easy to verify that the derivatives of $p^*(t)$ and $q^*(t)$ are non-negative on $0 \leq t < T_{x'}$, whenever they exist. Hence $p^*(t)$ and $q^*(t)$ are non-decreasing on $0 \leq t < T_{x'}$. From the first half of (12) we have

$$(18) \quad T_{x'} \leq \int_0^1 \{(1 - z^2)[p^*(0) + q^*(0)M_x^2(1 + z^2)]\}^{-\frac{1}{2}} dz.$$

In view of the relation (10'), the expression on the right hand of (18) is equal to $T_{v'}$ of the solution $g(t)$ of

$$(19) \quad g'' + p^*(0)g + 2q^*(0)g^3 = 0, \quad g(0) = 0, \quad g'(0) = E.$$

Since $p^*(0) \geq p_0$, $q^*(0) \geq q_0$, $E \geq E_0$, an appeal to Lemma 1 gives $T_{v'} \leq T_v$, and hence $T_{x'} \leq T_{v'}$. For $T_{x'} \leq t \leq T_x$, define $p^*(t)$ and $q^*(t)$ by the same equations as before. Then, proceeding in the same way as before, we can prove that $p^*(t)$ and $q^*(t)$ are non-decreasing on $T_{x'} \leq t \leq T_x$ and bounded below by $p^*(0)$ and $q^*(0)$, respectively. Then the equation (10) holds on $T_{x'} \leq t \leq T_x$. Repeating the argument, one obtains $T_x - T_{x'} \leq T_{v'} \leq T_v$, and hence $T_x \leq 2T_{v'} = T_v$. The case of non-increasing $p(t)$ and $q(t)$ can be proved in a similar way.

Consider another equation

$$(20) \quad z'' + fz + 2gz^3 = 0, \quad z(0) = 0, \quad z'(0) = e > 0.$$

Theorem 3. *Let the pairs $f(t)$, $g(t)$ and $p(t)$, $q(t)$ satisfy (*). Let $f(t)$ and $g(t)$ be non-decreasing and $p(t)$ and $q(t)$ non-increasing and $f(t) \geq p(t)$, $g(t) \geq q(t)$. If $e \geq E > 0$, then the first positive zero T_z of the solution $z(t)$ of (20) is not greater than the first positive zero T_x of the solution $x(t)$ of (1*).*

Proof. This theorem is an immediate consequence of Theorem 2. To see this let y be the solution of (17) whose constant coefficients p_0 and q_0 are chosen to satisfy $f(0) \geq p_0 \geq p(0)$, $g(0) \geq q_0 \geq q(0)$, and $e \geq E_0 \geq E > 0$. Then, applying Theorem 2, we have $T_z \leq T_y \leq T_x$.

Theorem 4. *Let T_k be the distance between the k th and $(k+1)$ th positive zero of a solution $x(t)$ of the equation (1). If $p(t)$ and $q(t)$ are, besides satisfying (*), non-decreasing (non-increasing), then the sequence $\{T_k\}$ is non-increasing (non-decreasing).*

Proof. Since every solution $x(t)$ of the equation (1) is oscillatory, let a be the k th positive zero of $x(t)$ and let $E = x'(a)$. Without loss of generality, we assume that $E > 0$. Clearly $z(t) = x(t+a)$ is the solution of

$$(21) \quad z'' + p(t+a)z + 2q(t+a)z^3 = 0, \quad z(0) = 0, \quad z'(0) = E,$$

and $y(t) = -x(-t+a)$ is the solution of

$$(22) \quad y'' + p(-t+a)y + 2q(-t+a)y^3 = 0, \quad y(0) = 0, \quad y'(0) = E.$$

The first positive zero of $z(t)$ is T_k and the first positive zero of $y(t)$ is T_{k-1} . If $p(t)$ and $q(t)$ are non-decreasing, then the coefficients of (21) are non-decreasing and the coefficients of (22) are non-increasing. From Theorem 3 $T_k \leq T_{k-1}$. The other case holds for similar reasons. This proves the theorem.

Under the conditions of Theorem 4, it is also known that the maxima of $|x(t)|$ and those of $|x'(t)|$ also form monotone sequences (see [8]).

4. Periodic solutions. In this section we shall apply the results of Theorem 1 to investigate the conditions of existence of periodic solutions of the equation (1) and their oscillatory behavior.

In any closed interval $[0, a]$ where a solution $x(t)$ of the equation (1) exists, there are at most a finite number of zeros of $x(t)$, because if the contrary were true, the zeros would have an accumulation point in $[0, a]$ at which $x(t)$ and $x'(t)$ would also vanish; this in turn would imply that $x(t)$ vanishes identically on $[0, a]$, contradicting the assumption $x(t) \not\equiv 0$. According to the usual Picard-Lindelof existence theorem with some modification, every solution of (1) exists uniquely and at least locally. According to Theorem 1, every solution $x(t)$ of (1*) is bounded by M_* as far as it exists in any interval containing no positive zeros, hence it can be uniquely continued until it has a positive zero, for instance, by applying repeatedly a finite number of times the Picard-Lindelof existence theorem. But then every solution of (1*) can be uniquely continued so as to have as many positive zeros as is desirable. Hence for any solution of (1*), we can say that it has a k th positive zero for any positive integer k .

Consider the solution $x = x(t, E)$ of (1*) and denote the mapping from $x'(0) = E$ to the first positive zero of x by $T(E)$. For each E , $0 < E < \infty$, the solution x of (1*) exists over some interval containing a first positive zero, hence $T(E)$ is defined for all positive E . We now show that $T(E)$ is continuous for $E > 0$. Let $E_0 > 0$. Then $x(t, E_0) = 0$ and $x'(t, E_0) \neq 0$ at $T(E_0)$. Since $x(t, E)$ and $x'(t, E)$ are continuous in (t, E) for $0 \leq t \leq T(E_0) + \epsilon$ and $|E - E_0| \leq \delta$ for small positive ϵ and δ (see, for instance, Theorem 7.1 in [6], p. 22, which also holds under our assumptions on the coefficients $p(t)$ and $q(t)$) the usual proof of the existence theorem for implicit functions also proves that $x(t, E) = 0$ has a continuous solution $t = Z(E)$ in the neighborhood of $(T(E_0), E_0)$ and $Z(E_0) = T(E_0)$. Clearly $Z(E)$ must be the first positive zero of $x(t, E)$ and $Z(E) = T(E)$, otherwise there would exist some E such that the solution $x(t, E)$ possesses a double zero between 0 and $T(E_0) + \epsilon$, contradicting the assumption that $x \not\equiv 0$. This proves that the mapping $T(E)$ is continuous for positive E .

In the following theorem let T_* have the same meaning as in Theorem 1.

Theorem 5. *Let $p(t)$ and $q(t)$ be, besides satisfying (*), even periodic functions with a least common period L . If kL is less than $2T_*$ and $n = 1, 2, \dots, k$, then there exists a periodic oscillatory solution of (1*) with least period nL and having exactly one zero, namely $\frac{1}{2}nL$, in the open interval $(0, nL)$.*

Proof. By assumption $0 < \frac{1}{2}nL < T_*$. Consider the solution $x = x(t, E)$ of (1*) and the mapping $T(E)$. By Theorem 1, $0 < T(E) < T_*$ for all positive E and $T(E) \rightarrow T_*$ as $E \rightarrow 0$, $T(E) \rightarrow 0$ as $E \rightarrow \infty$. Since $T(E)$ is continuous for positive E , there exists a positive E^* such that $T(E^*) = \frac{1}{2}nL$, i.e., $\frac{1}{2}nL$ is the first positive zero of $x(t, E^*)$. We now prove that $x(t, E^*)$ is periodic with least period nL . Since $p(t)$ and $q(t)$ are even and periodic, $p(t) = p(nL - t)$ and $q(t) = q(nL - t)$ for all t . It follows that $x(t, E^*)$ and $-x(nL - t, E^*)$ are solutions of (1) having the same value (namely 0) and derivative at $\frac{1}{2}nL$ and, by

the uniqueness of the solutions, $x(t, E^*) = -x(nL - t, E^*)$ for all t . Changing the variable, we have $x(t + nL, E^*) = -x(-t, E^*)$. On the other hand, $x(t, E^*) = -x(-t, E^*)$. This completes the proof.

Theorem 6. *Let $p(t)$ and $q(t)$ be, besides satisfying (*), even periodic functions with a least common period L and let n be any positive integer. If m is a positive integer greater than $(2\pi)^{-1}nL [\text{lub } p(t)]^{\frac{1}{2}}$, then there exists a periodic oscillatory solution of (1*) with period nL and having exactly $2m - 1$ zeros in the open interval $(0, nL)$.*

Proof. Consider the solution $x = x(t, E)$ of (1*) and denote the mapping from $x'(0) = E > 0$ to the m th positive zero of x by $T_m(E)$. Proceeding in the same way as for $T(E)$, $T_m(E)$ is defined and continuous for all positive E . Set $p_2 = \text{lub } p(t)$. By assumption $\frac{1}{2}nL < m\pi p_2^{-\frac{1}{2}}$. Let a be the m th positive zero of the solution w of $w'' + pw = 0$, $w(0) = 0$, $w'(0) \neq 0$. Writing (1) as (1') and using Sturm's comparison theorem, we have $T_m(E) < a$ for $E > 0$. Since $x(t, E) \equiv 0$ for $E = 0$, to each $\epsilon > 0$ there exists $\delta > 0$ such that the solution $x(t, E)$ of (1*) exists on $0 \leq t \leq a$ and $|2q(t)x^2| < \epsilon$ for $0 < E < \delta$. (See Theorem 7.1, p. 22 in [6]). Applying Sturm's comparison theorem again we have $m\pi(p_2 + \epsilon)^{-\frac{1}{2}} < T_m(E)$ for $0 < E < \delta$. This proves that $\frac{1}{2}nL < T_m(E)$ if E is positive and small enough. On the other hand, the proof in Theorem 1 (using (10) for the interval $T_x' \leq t \leq T_x$) shows that $(x')^2 \rightarrow \infty$ at T_x as $E \rightarrow \infty$. Hence by induction and Theorem 1, $T_m(E) \rightarrow 0$ as $E \rightarrow \infty$. This proves that there exists a positive E^* such that $T_m(E^*) = \frac{1}{2}nL$. The same argument used in the proof of Theorem 5 shows that $x(t, E^*)$ is a periodic solution with period nL and having exactly $2m - 1$ zeros in the open interval $(0, nL)$.

5. Boundedness and asymptotic behavior of the solutions. In this section we shall give a sufficient condition for the boundedness of the solutions of (1) and a sufficient condition under which we can represent a solution of (1) asymptotically by a periodic function which also satisfies a differential equation.

Theorem 7. *If $p(t)$ and $q(t)$ are, besides satisfying (*), functions of bounded variation on $[0, \infty)$, then every solution of the equation (1) and its derivative are bounded on $[0, \infty)$.*

Proof. Since $p(t)$ and $q(t)$ are functions of bounded variation on $[0, \infty)$, we can subdivide $[0, \infty)$ into n subintervals $I_1, I_2, \dots, I_n$ by the points $0 = a_0 < a_1 < a_2 < \dots < a_{n-1} < \infty$ in such way that the total variation of $p(t)$ and $q(t)$ over each subinterval I_i is not greater than $\frac{1}{2}p_1$ and $\frac{1}{2}q_1$, respectively. (p_1 and q_1 are the greatest lower bounds of $p(t)$ and $q(t)$ on $[0, \infty)$, respectively. They are positive by assumption.) Multiply (1) by $2x'$ and integrate the result from 0 to t , giving

$$(20) \quad (x')^2 + px^2 + qx^4 = c^2 + \int_0^t x^2 dp + \int_0^t x^4 dq, \quad c^2 > 0.$$

Let M_1 be the maximum of x^2 on a subinterval $[0, t_1]$ of I_1 and suppose that it occurs at the point t . Then, from (20),

$$(21) \quad \begin{aligned} p_1 M_1 + q_1 M_1^2 &\leq c^2 + M_1 \int_0^{a_1} |dp| + M_1^2 \int_0^{a_1} |dq| \\ &\leq c^2 + \frac{1}{2} p_1 M_1 + \frac{1}{2} q_1 M_1^2 \end{aligned}$$

and

$$(22) \quad p_1 M_1 + q_1 M_1^2 \leq 2c^2.$$

This proves that x^2 is bounded on I_1 by a positive number which can be expressed in terms of p_1 , q_1 , and c alone. Using (20), it is clear that $(x')^2$ is bounded on I_1 by the last expression in (21) and hence also by a positive number expressed in terms of p_1 , q_1 , and c alone. By induction x^2 and $(x')^2$ are bounded on $[0, \infty)$.

We have assumed that $p(t)$ and $q(t)$ do not have jumps greater than $\frac{1}{2}p_1$ and $\frac{1}{2}q_1$, respectively. If they do, we can enclose each of these points with a small interval so that, by continuity, x and x' change only slightly over these intervals, and so remain bounded on $[0, \infty]$.

In the proof above we have assumed the existence of the solution x on $[0, \infty)$. This assumption is justified by virtue of the fact that a solution x of (1) is uniformly bounded on every interval of existence by a positive number and hence can be extended to exist over $[0, \infty)$. (See, for instance, p. 13 in [6]).

In the following theorem we shall represent a solution of (1) by a function $\phi(\theta)$ which satisfies

$$(23) \quad \frac{d^2\phi}{d\theta^2} + \phi + 2\phi^3 = 0, \quad \phi(0) = 0, \quad \phi'(0) = E > 0.$$

Clearly $\phi(\theta)$ also satisfies

$$(24) \quad \left(\frac{d\phi}{d\theta}\right)^2 + \phi^2 + \phi^4 = E^2$$

and is a periodic function of θ . The least period of $\phi(\theta)$ can be determined by integrating (24).

Theorem 8. *Let $p(t)$ and $q(t)$ satisfy (*). Let A and B be positive constants and let $p(t) = A + f(t)$, $q(t) = B + g(t)$. If $f(t)$ and $g(t)$ are functions of bounded variation on $[0, \infty)$ and if $tf(t)$ and $tg(t)$ are Lebesgue-integrable on $[0, \infty)$, then every solution $x(t)$ of (1) and its derivative $x'(t)$ can be represented by*

$$(25) \quad x(t) = A^{\frac{1}{2}} B^{-\frac{1}{2}} \phi(A^{\frac{1}{2}} t + C) + e(t),$$

$$(26) \quad x'(t) = A B^{-\frac{1}{2}} \phi'(A^{\frac{1}{2}} t + C) + e'(t),$$

where C is a constant, $e(t)$ and $e'(t) \rightarrow 0$ as $t \rightarrow \infty$, and $\phi(\theta)$ is the periodic function defined by (23) for some fixed $E > 0$.

Proof. Let $x(t)$ be a solution of (1). By Theorem 7 and the remark after its proof, $x(t)$ and $x'(t)$ exist and are bounded on $0 \leq t < \infty$.

Consider (24) as the equation of a closed curve in the (ϕ, ϕ') plane. Since $x(t)$ and $x'(t)$ never vanish simultaneously we can determine a unique positive function $r(t)$ such that this point $(x/r, x'/rA^{\frac{1}{2}})$ lies on the curve (24) for each fixed t . It follows that for each t there exists a $\theta = \theta(t)$ such that

$$(27) \quad x(t) = r(t)\phi(\theta(t)), \quad x'(t) = A^{\frac{1}{2}}r(t)\phi'(\theta(t))$$

for all t on $0 \leq t < \infty$. Since $\phi(\theta)$ is periodic, the $\theta(t)$ in (27) is uniquely determined up to an integral multiple of the least period.

The positive function $r(t)$ in (27) can be determined by solving ϕ and ϕ' in (27) and substituting them in (24). An explicit expression for $r(t)$ would show that it is absolutely continuous on $0 \leq t < \infty$. Equating $\theta(0)$ to one of its possible values, $\theta(t)$ also can be so determined that it is absolutely continuous on $0 \leq t < \infty$, because at least one of the equations in (27) is solvable for $\theta = \theta(t)$ by virtue of the fact that ϕ' and ϕ'' never vanish simultaneously.

With the new notations, (20) becomes

$$(28) \quad (x')^2 + (A + f)x^2 + (B + g)x^4 = c^2 + \int_0^t x^2 df + \int_0^t x^4 dg, \quad c^2 > 0.$$

The two integrals in (28) are convergent as $t \rightarrow \infty$ by virtue of the boundedness of x and the bounded variation of f and g . We write (28) as

$$(29) \quad (x')^2 + (A + f)x^2 + (B + g)x^4 = E_1^2 - \int_t^\infty x^2 df - \int_t^\infty x^4 dg.$$

Clearly our hypotheses imply that f and g tend to zero when $t \rightarrow \infty$ and hence E_1^2 is non-negative. We show $E_1^2 > 0$. If $E_1 = 0$, let t_0 be so large that $f > -\frac{1}{2}A$, $g > -\frac{1}{2}B$, $V(f) < \frac{1}{2}A$, $V(g) < \frac{1}{2}B$ on $t_0 \leq t < \infty$ ($V(f)$ is the total variation of f over the interval indicated) and, since $E_1 = 0$ implies that $x \rightarrow 0$ as $t \rightarrow \infty$, let M be the maximum of x^2 on $t_0 \leq t < \infty$ which occurs at t_1 (say), then (29) with $t = t_1$ yields

$$AM + BM^2 < AM + BM^2 + E_1^2$$

and hence $E_1^2 > 0$. Rewrite (29) as

$$(30) \quad (x')^2 + Ax^2 + Bx^4 = E_1^2 + e$$

where

$$(31) \quad e(t) = -fx^2 - gx^4 - \int_t^\infty x^2 df - \int_t^\infty x^4 dg.$$

It is easy to verify that

$$(32) \quad e(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty, \quad e(t) \in L(0, \infty).$$

Setting $E^2 = E_1^2 BA^{-2}$, from (24) and (27) we find r satisfying

$$(33) \quad 2E_1^2 BA^{-1} r^2 = (x')^2 + Ax^2 + \{[(x')^2 + Ax^2]^2 + 4E_1^2 Bx^4\}^{\frac{1}{2}}.$$

Denote the sum of the first two terms on the right side of (33) by h and the third term by k and use (30), obtaining

$$(34) \quad h = E_1^2 - Bx^4 + e$$

and

$$(35) \quad \begin{aligned} k &= [(E_1^2 - Bx^4 + e)^2 + 4E_1^2 Bx^4]^{\frac{1}{2}} \\ &= [(E_1^2 + Bx^4)^2 + O(e)]^{\frac{1}{2}} = E_1^2 + Bx^4 + O(e). \end{aligned}$$

It follows that from (33), (34), and (35)

$$(36) \quad r^2 = AB^{-1} + O(e).$$

From (27) we have

$$(37) \quad \frac{x'(t)}{x(t)} = A^{\frac{1}{2}} \frac{\phi'(\theta)}{\phi(\theta)}$$

whenever $x \neq 0$ and, by differentiation with respect to t ,

$$(38) \quad \frac{x''}{x} - \left(\frac{x'}{x}\right)^2 = A^{\frac{1}{2}} \left[\frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2 \right] \theta'$$

whenever x'' exists. Simplify (38) with the aid of (1), (30), (23), and (24), obtaining

$$(39) \quad r^2 A^{\frac{1}{2}} (\phi^4 + E^2) \theta' = Bx^4 + E_1^2 + fx^2 + 2gx^4 + e$$

and hence

$$(40) \quad \theta' = \frac{Bx^4 + E_1^2}{(\phi^4 + E^2)r^2 A^{\frac{1}{2}}} + O(f, g, e),$$

where $O(f, g, e) = O(f) + O(g) + O(e)$. Using (27), (36), and $E^2 = E_1^2 BA^{-2}$ in (40) we obtain after simplification

$$(41) \quad \theta' = A^{\frac{1}{2}} + O(f, g, e).$$

Since θ is absolutely continuous for $t \geq 0$ and f, g, e are Lebesgue-integrable on $[0, \infty)$, the relation (41) holds almost everywhere on $[0, \infty)$ and hence

$$(42) \quad \begin{aligned} \theta &= A^{\frac{1}{2}} t + \theta(0) + \int_0^t O(f, g, e) dt \\ &= A^{\frac{1}{2}} t + C - \int_t^\infty O(f, g, e) dt \end{aligned}$$

for some constant C . Using the results (36) and (42), the relations in (27) clearly are equivalent to the expressions (25) and (26) for some functions $e(t)$ and $e'(t)$ ($e(t)$ in (25) is not the one defined by (31)) which tend to zero as $t \rightarrow \infty$ and $e'(t)$ is the derivative of $e(t)$. This completes the proof of the theorem.

6. Remarks. Results similar to Theorem 1 and Theorems 5-8 can be easily established by the methods used in this paper for the differential equation

$$(43) \quad (mx')' + \sum_{k=1}^n kp_k x^{2k-1} = 0, \quad (n > 1)$$

where $m, p_1, p_2, \dots, p_n$ are real Lebesgue-measurable functions on $0 \leq t < \infty$, all having positive lower and upper bounds. To extend Theorems 1, 5, and 6, it is convenient to change the independent variable t by setting

$$(44) \quad t^* = \int_0^t \frac{dt}{m(t)}.$$

The new equation still has the form of (43), with m replaced by 1 and p_k by mp_k . If m and all p_k are even periodic functions with respect to t having a common period L , then all mp_k are also even and periodic with respect to t^* having a common period L^* determined by (44). When these conditions are satisfied, the equation (43) will possess various periodic solutions similar to those of the equation (1) as described in Theorems 5 and 6.

To extend Theorem 8, let $mp_k = A_k + f_k$, where A_k is a positive constant. If all f_k are functions of bounded variation on $[0, \infty)$ and if all tf_k are Lebesgue-integrable on $[0, \infty)$ and if we write

$$(45) \quad x(t) = r(t)\phi(\theta), \quad m(t)x'(t) = r(t)\phi'(\theta),$$

where $x(t)$ is any solution of (43) and $\phi(\theta)$ satisfies

$$\frac{d^2\phi}{d\theta^2} + \sum_{k=1}^n kA_k\phi^{2k-1} = 0, \quad \phi(0) = 0, \quad \phi'(0) = E > 0$$

for a suitable E , then one can prove that

$$r^2 = 1 + O(e_1), \quad \theta'(t) = \frac{1}{m} + O(e_2)$$

and both e_1 and e_2 tend to 0 as $t \rightarrow \infty$ and are Lebesgue-integrable on $0 \leq t < \infty$. We observe that the second relation in (45) only holds almost everywhere.

The restriction $n > 1$ in the equation (43) is necessary. For instance, Theorem 6 would be false if applied to the "linear" equation

$$(46) \quad w'' + \left(\frac{1}{16} + \delta \cos t\right)w = 0$$

where $\delta > 0$ is small with respect to $\frac{1}{1-\delta}$. The condition

$$m > (2\pi)^{-1}nL[\text{lub } p(t)]^{\frac{1}{2}}$$

for $m = 1$, $n = 1$, $L = 2\pi$, $\text{lub } p(t) = \frac{1}{1-\delta} + \delta \cong \frac{1}{1-\delta}$ is certainly satisfied. On the other hand an application of Sturm's comparison theorem shows that every solution $w(t)$ of (46) is oscillatory with the successive zeros separated by a distance of approximately 4π and hence equation (46) has no periodic solution of period 2π .

We observe that in both Theorems 5 and 6 the lower and upper bounds $0 < q_1 < q_2$ of $q(t)$ do not occur explicitly. However, the example just given shows that both theorems do not hold "at the limit" as $q_2 \rightarrow 0$.

Let $x(t, k, 0)$ be the solution of equation (1) satisfying the initial condition

$$(47) \quad x(0) = k, \quad x'(0) = 0.$$

One can show, by modifying the method in Section 3, that the m th positive zero, m being fixed, of $x(t, k, 0)$ tends to zero as $k \rightarrow \infty$ and has a certain positive limit as $k \rightarrow 0$. Using these facts and the method in Section 4 one can establish two more results similar to Theorems 5 and 6 concerning periodic solutions satisfying the initial condition (47).

With Fu denoting the expression or mapping $u'' + pu + 2qu^3$ and the condition (*) assumed, it is interesting to note that the boundary value problem

$$(48) \quad Fu = \lambda u, \quad u(0) = u(a) = 0, \quad (a > 0)$$

has infinitely many solutions. Using the methods developed in Sections 3 and 4, it is easy to see that any number λ less than $p_1 = \text{glb } p(t)$ is an eigenvalue of the boundary value problem and that, corresponding to each eigenvalue $\lambda < p_1$ and to each positive integer m greater than certain positive number m_0 , dependent on λ , a , and $p(t)$ (see Theorem 6), there is an eigenfunction $u(t)$ with exactly m zeros in the open interval $(0, a)$.

It is not known to the author whether any or all periodic solutions of (1), as described in Theorems 5 and 6, are stable in some sense.

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