Secondary Conditions for Linear Differential Operators of the Second Order

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1. Introduction. The differential equation

$$-y'' + [q(x) - \lambda]y = f(x), \quad a < x < b,$$

plays an important role in classical physics, quantum-mechanics, and in modern electromagnetic theory. In the early Sturm-Liouville theory the function $q(x)$ is restricted to be real and continuous on the closed interval $[a, b]$. Under these circumstances, it is not difficult to show the existence of discrete real eigenvalues and real eigenfunctions corresponding to a pair of real homogeneous boundary conditions, one at each end of the interval $[a, b]$. For each eigenvalue the corresponding eigenfunction is a solution of the homogeneous differential equation. For other values of the complex parameter λ there is a unique square-integrable solution to the non-homogeneous equation satisfying the boundary value problem. In other words, the boundary conditions give rise to a *well-set* problem. Later, as new physical problems were brought under study, the same differential equation was encountered again but in its real-singular form. The singularity arose either because the interval $[a, b]$ became infinite, or because the real function $q(x)$ possessed singularities at one or both ends of the interval. Questions as to what constituted a well-set problem were answered in part by the Weyl *limit-point, limit-circle* theory. This theory allows one to partition the set of functions that are real and continuous on the open interval (a, b) into four classes according as limit-point or limit-circle situations prevail at the ends of the interval. On the one hand, if the function $q(x)$ is of the limit-point type at an end point then the problem is automatically well-set with respect to this end of the interval and no further condition need be nor can be imposed there. In a sense, the boundary condition is implicit in the nature of $q(x)$. On the other hand, if one has a limit-circle at an end point of the interval $[a, b]$ then some further condition must be imposed in order to have a well-set problem. The Weyl theory gives one class of boundary conditions that serve this purpose. As it turns out, there are other conditions that serve the same general purpose.

With this brief background we turn to a two-fold extension of the theory mentioned above. In the first place, the continuous function $q(x)$ is allowed to be complex-valued as well as singular at the end points of the finite or infinite interval $[a, b]$. Furthermore, no additional assumptions regarding integrability, differentiability, or boundedness of the imaginary part of $q(x)$ are to be made.⁽¹⁾ Differential equations exhibiting this complex character (or ones easily transformed into this form) come chiefly from electromagnetic theory as a result of separating variables—in particular, radiation in a spherically stratified medium leads to such an equation. Considerations such as this have indicated a need to extend nearly all parts of the classical theory to embrace the situation where $q(x)$ is non-real. One of the purposes of this paper is to fulfill this need with respect to the particular subject of boundary conditions and certain portions of closely related spectral theory. In the second place, the usual notion of a boundary value problem is extended to include a wider class of problems that depend upon what are henceforth called *secondary conditions*—a generalization that appertains to the real as well as to the complex case. The advantage of this second extension lies in the fact that it enables one to delineate clearly the totality of well-set linear problems, not only for the ambiguous limit-circle case attendant in the real-singular situation, but also in the additional ambiguous cases that develop in consequence of the complex nature of $q(x)$.

The secondary conditions arise in quite a natural way when one attacks the general subject of boundary conditions from an operator-theoretic standpoint. They are defined in the following way. Let D^* stand for the largest linear manifold of $L^2[a, b]$ for which $Ky = -y'' + qy$ is defined and in $L^2[a, b]$, i.e., K is an operator with domain D^* and range in $L^2[a, b]$. Let P stand for any property that elements of D^* can have, and let D_P be the set of elements of D^* having property P . The property will be said to determine a secondary condition for

⁽¹⁾ By way of acknowledgment it should be mentioned that others have treated one or another aspect of the spectral theory associated with a $q(x)$ that is complex-valued, but otherwise severely restricted. However, the overlap in subject matter is slight and appertains chiefly to the incidental topic of location of the spectrum. Notable among these authors, and in chronological order of publication are:

R. S. PHILLIPS, N. Y. U. Research Report No. E. M.-42 (April 1952)

M. S. NAIMARK, Progress in Math. Sc., 8, No. 4, pp. 174–175 (1953)

J. SCHWARTZ, Pacific Journal of Math., 4, No. 2, (Sept. 1954).

The first of these treats the special situation where $I(q)$ is bounded and is aimed at determining the behavior of the spectrum and resolvent when a closed operator associated with a real $q(x)$ undergoes a small complex perturbation. Also subject to the same restriction, a start is made here on an integral representation theory for square integrable functions in which wave packets play the role of eigenfunctions. The second paper referred to (NAIMARK) further restricts $q(x)$ to a variety of severe integrability and differentiability conditions and obtains strong results regarding the nature of the spectrum and under very special circumstances an eigenfunction expansion. The third paper mentioned (SCHWARTZ), like the first (PHILLIPS), treats only bounded perturbations of real differential operators (with additional restrictions imposed in case the function $q(x)$ is singular at the end points) and succeeds in disclosing the spectral nature of these perturbed operators.

the differential equation if for each non-zero $f \in L^2[a, b]$, and for at least one λ , there exists one and only one $y \in D_p$ satisfying $L_p y - y = f$ (L_p is the contraction of K having domain D_p). In the sequel we will impose several restrictions. First of all, we will consider only linear secondary conditions, *i.e.*, those properties P for which D_p is a linear set. Second, we will require the inverse of $L_p - \lambda I$ to be a bounded operator. With these two hypotheses, the problem becomes that of determining all contractions of K having non-empty resolvent sets $\rho(L_p)$. For reasons that become apparent later, the imaginary part of $q(x)$ is restricted to be of one sign. The imaginary part of λ is restricted to be of the opposite sign. Later it will be shown that the restriction upon λ can be relaxed in certain of the cases, but that it seems to be fundamental in the remaining cases.

The first step consists of showing that there are always two linearly independent solutions, $\psi_0(x, \lambda)$ and $\psi_1(x, \lambda)$, of the homogeneous equation

$$-y'' + (q - \lambda)y = 0$$

having the property that $\psi_0(x, \lambda) \in L^2[a, r]$ and $\psi_1(x, \lambda) \in L^2[r, b]$ for arbitrary $r \in (a, b)$ and λ such that $I(\lambda) > 0$. These two solutions can be represented in the form

$$\psi_0(x, \lambda) = m(\lambda)\phi(x, \lambda) + \theta(x, \lambda)$$

and

$$\psi_1(x, \lambda) = M(\lambda)\phi(x, \lambda) + \theta(x, \lambda)$$

where ϕ and θ are linearly independent solutions of the homogeneous equation, continuous functions of x on the open interval (a, b) for each λ , and entire functions of λ for each x in (a, b) . Once the existence of the solutions ϕ and θ has been established, the existence of the required solutions ψ_0 and ψ_1 is confirmed by recourse to a limit-point, limit-circle argument similar to that employed in the classical situation where $I(q) = 0$. Here, however, the limit-point, limit-circle geometry differs from that existent in the classical situation in two important respects. In the first place, the geometric arguments do not hold for $I(\lambda) < 0$. In the second place, in contrast to the older theory with $I(q) = 0$, it is possible to have two linearly independent square integrable solutions at an end point, but to have a limit-point case. Examples demonstrating the existence of Case II are given in the next section of the paper.

At either end point of the interval (a, b) one of the three cases occurs:

Case I.	A limit-point case but only one square-integrable solution.
Case II.	A limit-point case and two square-integrable solutions.
Case III.	A limit-circle case and two square-integrable solutions.

There are six distinct cases in all.

Case (I, I).	Case I at a and Case I at b .
Case (I, II).	Case I at a and Case II at b .
Case (I, III).	Case I at a and Case III at b .
Case (II, II).	Case II at a and Case II at b .
Case (II, III).	Case II at a and Case III at b .
Case (III, III).	Case III at a and Case III at b .

As in the classical theory the case depends only upon $q(x)$, *i.e.*, the case is independent of λ .

The nature of the functions $M(\lambda)$ (and $m(\lambda)$) appearing in the previous definition of ψ_1 and (and ψ_0) have an important bearing upon the sequel. Their definition depends upon the limit-circle, limit-point geometry. For each λ , $I(\lambda) > 0$, and for each b' , $r < b' < b$, there exists a circle $C_{b'}(\lambda)$ lying in the complex l -plane. These circles are nested in such a fashion that $C_{b''}(\lambda) \subset C_{b'}(\lambda)$ if and only if $b'' \geq b'$. Since the circles are nested, a point $M(\lambda)$ can be chosen such that $M(\lambda) \notin C_{b'}(\lambda)$ for all $b' > r$. In the limit-point cases $M(\lambda)$ is uniquely determined and is analytic for $I(\lambda) > 0$. In the limit-circle case there is a choice of functions $M(\lambda)$, some of which are analytic for $I(\lambda) > 0$. The following two theorems summarize the results obtained from the study of these functions.

Theorem. *In each case there exists at least one analytic M -function, defined for $I(\lambda) > 0$. In Case III an analytic M -function can be found such that $M(\lambda_0) = M_0$ where M_0 is an arbitrary point of the limit-circle $C_b(\lambda_0)$; moreover, this function $M(\lambda)$ can be so chosen as to take its values on the boundaries of the circles $C_b(\lambda)$.*

Theorem. *In Case II $M(\lambda)$ is meromorphic. In Case III any analytic M -function is meromorphic. In either Case II or Case III $M(\lambda)$ has a pole at λ_1 if and only if $1 + (\lambda_1 - \lambda) \int_r^b \psi(x, \lambda) \phi(x, \lambda) dx = 0$ for all λ . The order of the pole at λ_1 is one greater than the order of the zero of the entire function $\int_r^b \phi^2(x, \lambda) dx$ at λ_1 . Also, if λ_1 and λ_2 are both poles of $M(\lambda)$ then $\int_r^b \phi(x, \lambda_1) \phi(x, \lambda_2) dx = 0$.*

This last theorem is particularly important in that it permits an extension of the theory to the lower half λ -plane in Cases II and III despite the fact that the limit-point, limit-circle type of argument cannot be employed there. Furthermore, the poles of $M(\lambda)$ (and of $m(\lambda)$) are closely associated with the point spectrum of certain of the contractions of the operator K introduced earlier. (Note: The above remarks regarding the function $M(\lambda)$ corresponding to the right hand end point b of the interval (a, b) hold equally well, after obvious modification, for the function $m(\lambda)$ corresponding to the left hand end point a .)

Another preliminary step toward the objective of determining arbitrary contractions L_r of the operator K consists in the study of a special class of contractions. These special contractions are designated base contractions and are denoted by the symbol L_0 . Once their properties have been established, the base contractions are used in the representation of arbitrary contractions $L_r \subset K$.

It is convenient to have the following definitions listed here although they appear elsewhere in the paper. Following the definitions the main results are summarized.

Definition. $X = L^2[a, b]$.

Definition. $D^* = \{f|f \in X, f \text{ and } f' \text{ absolutely continuous, } -f'' + qf \in X\}$.

Definition. $Kf = -f'' + qf, f \in D^*$.

Definition. $D_0 = \{f|f \in D^*, W_b[\psi_1, f] = W_a[\psi_0, f] = 0\}$.

(The limits $\lim_{x \rightarrow a} W_x[f, g]$ and $\lim_{x \rightarrow b} W_x[f, g]$ exist for all f and g in D^* , hence D_0 is well defined.)

Definition. $L_0f = Kf$ for $f \in D_0$.

(D_0 is the base domain—there is a plurality of base domains in Cases (II, II), (II, III), and (III, III).)

Definition.

$$R_0(\lambda)f = \frac{1}{w(\lambda)} \left[\psi_1(x, \lambda) \int_a^x \psi_0(\tau, \lambda) f(\tau) d\tau + \psi_0(x, \lambda) \int_x^b \psi_1(\tau, \lambda) f(\tau) d\tau \right]$$

where $w(\lambda) = M(\lambda) - m(\lambda) = W_x[\psi_1(x, \lambda), \psi_0(x, \lambda)]$.

Theorem. $R_0(\lambda)f \in X$, and for $I(\lambda) > 0$, $\|R_0(\lambda)f\| \leq v^{-1} \|f\|$ where $v = I(\lambda)$.

Theorem. $R_0(\lambda)$ is a linear bounded operator defined for all of X and for $I(\lambda) > 0$, and has the properties:

- (i) $R_0(\lambda)$ is the resolvent of L_0 .
- (ii) $\rho(L_0)$ contains the upper half λ -plane.
- (iii) $\|R_0(\lambda)\| \leq v^{-1}$ and $\int_a^b |R_0(\lambda)f|^2 (v - q_2) dx < \infty$.
- (iv) $R_0(\lambda)X = D_0$ and $U(\lambda)D_0 = X$ where $U(\lambda) = L_0 - \lambda I$.

The definition of D_0 given above is independent of λ , $I(\lambda) > 0$. In case (I, I) $D_0 = D^*$. In cases (I, II) and (II, II)

$$D_0 = \left\{ f|f \in D_1, \int_a^b |f|^2 (-q_2) dx < \infty \right\}$$

is equivalent to the former definition where $q_2 = I(q)$.

In Cases (II, II), (II, III), and (III, III) the restriction $I(\lambda) > 0$ can be removed. In these cases the spectrum $\sigma(L_0)$ is a pure point spectrum, consisting of discrete points in the λ -plane. This follows from the fact that $R_0(\lambda)$ is completely continuous and a theorem of PHILLIPS [3].

Theorem. In Cases (II, II), (II, III), and (III, III) the spectrum of L_0 , $\sigma(L_0)$, is a pure point spectrum consisting of discrete points in the lower half λ -plane. Furthermore, $\lambda_1 \in \sigma(L_0)$ if and only if λ_1 is a pole of $M(\lambda)$ and $m(\lambda)$, or a zero of $w(\lambda)$.

Having established the properties of the base contractions (L_0) and base domains (D_0), all contractions whose resolvent sets $\rho(L_r)$ contain at least one point of the upper half λ -plane are obtained. An indication of the procedure, the necessary definitions, and the main results are contained in the paragraphs below.

The following theorem is essential to the proofs of the sequel.

Theorem. $F(\lambda) = \psi_0(x_1\lambda)$ is a holomorphic vector-valued function from the upper half λ -plane into D^* .

The Cases (I, II) and (I, III) are treated first. Although the computations involved are simpler than in the remaining cases they serve as a convenient model from which the treatment of Cases (II, II), (II, III), and (III, III) can be generalized readily.

Definitions.

$$(a) \quad A[f, \lambda] = w(\lambda)^{-1}w_b[f, \psi_1(x, \lambda)].$$

$$B[f, \lambda] = w(\lambda)^{-1}w_a[\psi_0(x, \lambda), f].$$

(Note: $B[f, \lambda] = 0$ for all $f \in D^*$ in Cases (I, III) and (I, II).)

$$(b) \quad C(f) = \int_a^b f f_0 dx \text{ where } f_0 \text{ is an arbitrary element of } X.$$

For an arbitrary λ_0 , $I(\lambda_0) > 0$, and an arbitrary $f_0 \in X$ introduce the following definitions:

$$(c) \quad g(\lambda) = A[\psi_0(x, \lambda), \lambda_0] + C[U(\lambda_0)\psi_0(x, \lambda)], \quad I(\lambda) > 0.$$

(Note: $g(\lambda)$ is analytic for $I(\lambda) > 0$ and $g(\lambda_0) = -1$.)

$$(d) \quad C(f, \lambda) = -C[U(\lambda_0)R_0(\lambda)f]g(\lambda)^{-1} \text{ for } g(\lambda) \neq 0 \text{ and } I(\lambda) > 0.$$

(Note: $C(f, \lambda_0) = C(f)$.)

$$(e) \quad R_1(\lambda)f = R_0(\lambda)f + C(f, \lambda)\psi_0(x, \lambda).$$

$$(f) \quad D_1 = \{f | f \in D^*, A[f, \lambda] + C[U(\lambda)f, \lambda] = 0\}.$$

(Note: Definition (f) is independent of λ , i.e.,

$$A[f, \lambda] + C[U(\lambda)f, \lambda] = -g(\lambda)^{-1}\{A[f, \lambda_0] + C[U(\lambda_0)f]\}$$

for $g(\lambda) \neq 0$ and $I(\lambda) > 0$.)

For each initial pair (f_0, λ_0) one obtains a domain D_1 , an operator $R_1(\lambda)$, an analytic function $g(\lambda)$, and a functional $C(f)$.

Theorem. Let $L_1f = Kf$ for $f \in D_1$. Then the resolvent of L_1 is $R_1(\lambda)$, $R_1(\lambda)X = D_1$, and $U(\lambda)D_1 = X$. $\rho(L_1)$ contains all of the upper half λ -plane except for the zeros of the analytic function $g(\lambda)$. These zeros are in the point spectrum of L_1 . Thus for each pair (f_0, λ_0) there exists a contraction $L_1 \subset K$ whose resolvent set $\rho(L_1)$ contains the whole upper half λ -plane except possibly for a discrete set.

Corollary. For a fixed λ_0 distinct choices of functionals $C(f)$ yield distinct contractions L_1 . Furthermore, all contractions $L_\tau \subset K$ whose resolvent sets contain a single point of the upper half λ -plane are of the kind L_1 , corresponding to some pair (f_0, λ_0) .

Corollary. Let $L_1 \subset K$ be determined by f_0 and λ_0 . Then, for arbitrary $\lambda_0 \in \rho(L_1)$ and $I(\lambda_1) > 0$, L_1 is also determined by the pair (f_1, λ_1) where $f_1 = g(\lambda_1)^{-1}U(\lambda_0)R_0(\lambda_1)f_0$.

The theory below pertains to the remaining cases and is very similar to the preceding. Where formerly we had a single functional $C(f)$ and a single analytic function $g(\lambda)$ now we have two arbitrary functionals $C_0(f)$ and $C_1(f)$ and two functions $g_0(\lambda)$ and $g_1(\lambda)$. The domains D_1 are similar; however, now they correspond to triples (f_0, f_1, λ_0) instead of to pairs (f_0, λ_0) . Any contraction $L_\tau \subset K$ with a non-empty resolvent set $\rho(L_\tau)$ is generated from the base contractions (L_0) in a similar, but somewhat more complicated, fashion to that employed in the previous section. After making analogous definitions of D_1 and $R_1(\lambda)$ and after introducing a new analytic function $\Delta(\lambda)$ that depends upon $g_0(\lambda)$ and $g_1(\lambda)$ one obtains theorems and corollaries closely resembling those pertaining to Cases (I, II) and (I, III).

Theorem. Let $L_1 f = Kf$ for $f \in D_1$. Then $R_1(\lambda)$ is the resolvent of L_1 , $R_1(\lambda)X = D_1$, and $U(\lambda)D_1 = X$. $\sigma(L_1)$ is a discrete point spectrum and the points of $\sigma(L_1)$ are either in $\sigma(L_0)$ or they are zeros of the analytic function $\Delta(\lambda)$.

Corollary. For a fixed λ_0 distinct choices of functionals C_0 and C_1 yield distinct contractions $L_1 \subset K$.

Corollary. If $L_\tau \subset K$ and $\rho(L_\tau) = 0$ then $L_\tau = L_1$ for some choice of λ_0, C_0 , and C_1 .

The concluding portion of the paper is devoted to a classification of the domains D_τ into boundary condition domains and global domains.

If one looks at the definition of D_1 , one sees that membership in D_1 ordinarily depends upon the behavior of an element throughout the whole interval (a, b) . In one case, at least, this was not true; for the case where $f_0 = 0$ ($D_1 = D_0$), membership in D_0 depended only upon behavior at the end point $x = b$. In other words D_0 was a boundary condition domain.

Definition. $H = \{f | f \in D^*, f = 0 \text{ in a neighborhood of } a \text{ and of } b\}$.

Definition. D_τ is a boundary condition domain if $H \subset D_\tau$.

Definition. D_τ is a global domain if it is not a boundary condition domain.

Theorem. (Cases (I, II) and (I, III).) Let D_τ be determined by λ_0 and f_0 . D_τ is a boundary condition domain if and only if there exists a number α such that $f_0 = \alpha\psi_0(x, \lambda_0)$.

Theorem. (Cases (II, II), (II, III), and (III, III).) Let D_τ be determined by λ_0 , f_0 and f_1 . D_τ is a boundary condition domain if and only if there exist complex numbers α, β, γ , and δ such that $f_0 = \alpha\psi_0(x, \lambda_0) + \beta\psi_1(x, \lambda_0)$ and $f_1 = \gamma\psi_0(x, \lambda_0) + \delta\psi_1(x, \lambda_0)$.

The theorem is easy to prove if the functional-determining f -functions appearing in the statement of the above two theorems are restricted to lie in D^* . To show that there are no f -functions outside of D^* that furnish boundary condition domains takes somewhat more proving.

One immediate conclusion from the above is that boundary conditions constitute a small minority of suitable secondary conditions for the differential equation. The majority are global domains.

The theory that ensues when $q(x)$ is complex-valued differs in several respects from the classical theory where $q(x)$ is real-valued. In addition to the points of difference arising out of the limit-point, limit-circle geometry, the operator K , whose domain is defined precisely by

$$D^* = \{f|f \in L^2[a, b], f \text{ and } f' \text{ absolutely continuous, } -f'' + qf \in L^2[a, b]\},$$

is no longer symmetric, nor is it equal to its complex conjugate operator, $\overline{K}f = -f'' + \bar{q}f$, whose domain is $\hat{D}^* = \{f|\bar{f} \in D^*\}$. Furthermore, in consequence of the formula

$$W_x[f, g] - W_r[f, g] = \int_r^x [f(g'' - qg) - g(f'' - qf)]d\tau, \quad a < r < x < b,$$

where $W_x[f, g]$ stands for $fg' - f'g$, $\lim W_x[f, g]$ as $x \rightarrow a$ and $\lim W_x[f, g]$ as $x \rightarrow b$ both exist for functions in D^* . This fact enables one to define two operators, L and $\bar{L}$, contractions of K and $\bar{K}$, respectively, whose domains are

$$D = \{f|f \in D^*, W_b[f, g] - W_a[f, g] = 0 \text{ for all } g \in D^*\}$$

and $\hat{D} = \{f|\bar{f} \in D\}$. By a simple modification of proofs conducted by M. H. STONE [5] it is easy to show that the adjoints of the operators $K, \bar{K}, L$, and $\bar{L}$ are $K^* = \bar{L}, \bar{K}^* = L, L^* = \bar{K}$, and $\bar{L}^* = K$. These relations are in contrast to the relations $K^* = L, L^* = K, \bar{K} = K$, and $\bar{L} = L$ that hold in case $q(x)$ is real-valued. However, in our present circumstance where $q(x)$ is complex-valued, it still follows from the above relations, with the aid of the general Hilbert space theory [5], that the four operators above are closed and that the domains D and $\hat{D}$ are dense in $L^2[a, b]$. This last turns out to be of some importance in the sequel.

2. The Limit-Point, Limit-Circle Geometry and the Base Solutions. Let $[r, b)$ be a half-open interval where r is finite and b is arbitrary. Assume that $q(x)$ is continuous for x in $[r, b)$. Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be solutions of $-y'' + [q(x) - \lambda]y = 0$ having the properties $\phi(r, \lambda) = \sin \alpha$, $\phi'(r, \lambda) = -\cos \alpha$, $\theta(r, \lambda) = \cos \alpha$, and $\theta'(r, \lambda) = \sin \alpha$. The existence of solutions with the required properties can be confirmed by following an argument similar to that employed

by TITCHMARSH [6] for the case where $q(x)$ and α are real. The proof is not reproduced here.

Let λ and b' be fixed. Assume for the moment that neither $\phi(b', \lambda) = 0$ nor $\phi'(b', \lambda) = 0$. (Note: Since ϕ and ϕ' are entire functions of λ this assumption holds automatically for fixed b' for all λ 's, except possibly a discrete set.) For the fixed choice of λ and b'

$$(1) \quad l(b', \lambda, z) = -\frac{\theta(b', \lambda)z + \theta'(b', \lambda)}{(b', \lambda)z + \phi'(b', \lambda)}$$

defines a linear-fractional transformation from the complex z -plane to the complex l -plane. Since ϕ and θ are linearly independent the transformation is proper and, as $\phi(b', \lambda) \neq 0$ is assumed, the transformation does not degenerate into a linear one. The inverse transformation exists and is given by

$$(2) \quad z(b', \lambda, l) = -\frac{\phi'(b', \lambda)l + \theta'(b', \lambda)}{\phi(b', \lambda)l + \theta(b', \lambda)}.$$

Letting $I(\lambda) = v$, $I(q) = q_2$, and $I(\alpha) = \alpha_2$, with the aid of the formula

$$(3) \quad W_x[f, g] - W_v[f, g] = \int_x^v [f''g - fg''] dt,$$

the imaginary part of the critical point of the transformation (1) can be shown to be

$$(4) \quad I\left[-\frac{\phi'}{\phi}\right] = \frac{1}{|\phi|^2} \left[-\frac{1}{2} \sinh 2\alpha_2 + \int_r^{b'} (v - q_2) |\phi|^2 dx \right].$$

In case $q_2 \leq 0$, $\alpha_2 \leq 0$, and $v > 0$ this critical point is seen to lie in the upper half z -plane, and the lower half z -plane maps onto a circle $C_{b'}(\lambda)$ in the l -plane. From this last follows the observation that a point l is in $C_{b'}(\lambda)$ if and only if $I[z(b', \lambda, l)] < 0$ and is on the boundary of $C_{b'}(\lambda)$ if and only if $I[z(b', \lambda, l)] = 0$. Employing (2) and (3) the last statement can be replaced by the statement l is in or on $C_{b'}(\lambda)$ according as the following inequality or equality holds:

$$(5) \quad 2 \int_r^{b'} |\phi l + \phi|^2 (v - q_2) dx \leq (1 + |l|^2) \sinh 2\alpha_2 - 2I(l) \cosh 2\alpha_2.$$

Since inverse points are transformed into inverse points under linear-fractional transformations, the center of $C_{b'}(\lambda)$ is $p_{b'}(\lambda) = l[b', \lambda, -\bar{\phi}'(b', \lambda)/\bar{\phi}(b', \lambda)]$ or

$$(6) \quad p_{b'}(\lambda) = -\frac{W_{b'}[\bar{\phi}, \bar{\theta}]}{W_{b'}[\bar{\phi}, \phi]}.$$

Also, one notices that $-\theta'(b', \lambda)/\phi(b', \lambda)$ is on the boundary of $C_{b'}(\lambda)$ as it is the l -plane image of $z = 0$. Thus the radius of $C_{b'}(\lambda)$ is $r_{b'}(\lambda) = 1/W_{b'}[\bar{\phi}, \phi]$ or

$$(7) \quad r_{b'}(\lambda) = \left[-\sinh 2\alpha_2 + 2 \int_r^{b'} (v - q_2) |\phi|^2 dx \right]^{-1}.$$

The inverse transformation (2) takes the circle $C_{b'}(\lambda)$ into the real axis in the z -plane. Therefore, its critical point $-\theta(b', \lambda)/\phi(b', \lambda)$ must be on the boundary of the circle $C_{b'}(\lambda)$. It follows that

$$\left| \frac{-\theta'(b', \lambda)}{\phi'(b', \lambda)} - \frac{-\theta(b', \lambda)}{\phi(b', \lambda)} \right| \leq 2r_{b'}(\lambda)$$

and, with the aid of (7), that

$$(8) \quad -\sinh 2\alpha_2 + 2 \int_r^{b'} (v - q_2) |\phi|^2 dx \leq 2 |\phi(b', \lambda)| |\phi'(b', \lambda)|.$$

So far we have proceeded under the restriction that neither $\phi(b', \lambda)$ nor $\phi'(b', \lambda)$ is equal to zero. Now we remove this restriction but still require $q_2 \leq 0$, $\alpha_2 \leq 0$, and $v > 0$. If for arbitrary b' , $r < b' < b$, there exists a λ_0 such that either $\phi(b', \lambda_0)$ or $\phi'(b', \lambda_0)$ is equal to zero, then there exists a neighborhood of λ_0 , call it $N(\lambda_0)$, such that, say, $\phi(b', \lambda_0) \neq 0$ as long as $\lambda \neq \lambda_0$ and $\lambda \in N(\lambda_0)$. However, if one lets λ tend to λ_0 , the right side of (8) tends to zero, contradicting the fact that the left side of (8) is strictly positive as long as the restrictions upon the imaginary parts of q , α , and λ are in force. A similar argument shows that $\phi'(b', \lambda) \neq 0$.

Since (5) is now known to hold for all b' such that $r \leq b' < b$, it follows that the circle $C_{b'}(\lambda)$ tends to either a limit-circle or a limit-point as b' tends to b . In either case there is point interior to all of the circles $C_{b'}(\lambda)$. Letting $M(\lambda)$ stand for such a point, after replacing l by $M(\lambda)$ inequality (5) is seen to hold as $b' \rightarrow b$, i.e., $\int_r^b |\phi M + \theta|^2 (v - q_2) dx < \infty$. This result is important in the subsequent work and appears below as a theorem.

Theorem 1. *Let $[r, b]$ be a closed interval where r is finite and b arbitrary. Let $q(x)$ be continuous, $r \leq x < b$. Let $I(q) \leq 0$, $I(\alpha) \leq 0$, and $I(\lambda) > 0$. Let $M(\lambda)$ be any point interior to all $C_{b'}(\lambda)$; then $\psi(x, \lambda) = \phi(x, \lambda)M(\lambda) + \theta(x, \lambda)$ is a solution of $-y'' + (q(x) - \lambda)y = 0$ having the property that $\int_r^b |\psi(x, \lambda)|^2 (v - q_2) dx < \infty$.*

We see that in terms of the geometry of the l -plane we can have either a limit-circle or a limit-point situation at b . If a limit-circle situation prevails at b then from (7) $\int_r^b |\phi(x, \lambda)|^2 (v - q_2) dx < \infty$. In this case there are two linearly independent solutions of $-y'' + (q - \lambda)y = 0$, each square-integrable in $[r, b]$ with weight $v - q_2$; they are ϕ and ψ . If a limit-point situation is in force at b there is only one point $M(\lambda)$ interior to all $C_{b'}(\lambda)$, thus only one solution square-integrable in $[r, b]$ with weight $(v - q_2)$. A special subcase of the limit-point case occurs when $\int_r^b |\phi|^2 dx < \infty$ but $\int_r^b |\phi|^2 (v - q_2) dx = \infty$. From the foregoing, then, there are three rather different circumstances at the end point b :

Case I. There exists one and only one solution $\psi(x, \lambda)$ in $L^2[r, b]$, a limit-point case.

Case II. There exist two linearly independent solutions in $L^2[r, b]$, but only one solution $\psi(x, \lambda)$ square integrable in $[r, b]$ with weight $(v - q_2)$. This is a limit-point case too.

Case III. There exists two solutions square integrable in $[r, b]$ with weight $(v - q_2)$. This is the limit-circle case.

As has been noted above, Case II has no analog in the classical Weyl theory. In order to demonstrate the existence of this case two examples are given, the first appropriate to the infinite interval $(1, \infty)$ and the second to the finite interval $(0, 1)$. The first example is obtained by setting $q(x) = -x^6 - \frac{3}{2}ix^2$ and the second by setting $q(x) = \frac{5}{16}x^{-2} - ix^{-1}$. In the first example it is easy to show by employing the so-called normal solutions that there are two independent solutions of the form

$$y_1 = x^{-2\frac{1}{2}}e^{\frac{1}{2}ix^4}v_1(x) \quad \text{and} \quad y_2 = x^{-\frac{3}{2}}e^{-\frac{1}{2}ix^4}v_2(x)$$

where $v_1(x)$ and $v_2(x)$ are bounded for large x . In the second example recourse to the Frobenius theory enables one to show the existence to two independent solutions having the behavior $y_1 \sim x^{\frac{1}{2}}$ and $y_2 \sim x^{-\frac{1}{2}}$ as x tends to zero. From these observations the existence of Case II is apparent.

3. The Nature of the Functions $M(\lambda)$. The functions $M(\lambda)$ defined in Theorem 1 warrant further investigation. As a preliminary to their study we show that the case (I, II, or III) is independent of λ for $I(\lambda) > 0$. Our next regard is for questions of analyticity. In Case I $M(\lambda)$ is uniquely defined and analytic for $I(\lambda) > 0$. In Case II $M(\lambda)$ is uniquely defined and meromorphic. In Case III there exists a class of functions of the kind $M(\lambda)$ that are meromorphic; moreover, for any limit-circle $C(\lambda_0)$ and an arbitrary point lying in or on $C(\lambda_0)$, there exists a function $M(\lambda)$ of the meromorphic class that satisfies $M(\lambda_0) = m_0$, and, if desired, the function $M(\lambda)$ can be chosen with the additional property that it takes its values on the circles $C(\lambda)$ for each λ , $I(\lambda) > 0$. The proofs of these statements appear below.

For a fixed α , the solutions ϕ and θ can be expressed as linear combinations of the solutions ϕ and θ corresponding to any other suitable choice of α . From this it follows that the case is independent of α . On the other hand, if α is fixed the case is independent of λ . This last is fully substantiated by the proof of the following theorem.

Theorem 2. *The case depends only upon the coefficient function $q(x)$. If $\int_r^b |\phi(x, \lambda_0)|^2 dx < \infty$ and $\int_r^b |\theta(x, \lambda_0)|^2 dx < \infty$ then the same result holds at all other values of λ . If $\int_r^b |\phi(x, \lambda_0)|^2 (v_0 - q_2) dx < \infty$ and $\int_r^b |\theta(x, \lambda_0)|^2 (v_0 - q_2) dx < \infty$ then the same result holds at all other values of λ .*

In order to prove Theorem 2 the formula

$$(9) \quad \begin{aligned} \phi(x, \lambda) = \phi(x, \lambda_0) + (\lambda_0 - \lambda) & \left[\theta(x, \lambda_0) \int_r^x \phi(t, \lambda_0) \phi(t, \lambda) dt \right. \\ & \left. - \phi(x, \lambda_0) \int_r^x \theta(t, \lambda_0) \phi(t, \lambda) dt \right], \end{aligned}$$

obtained by variation of parameters, is employed. From (9)

$$\begin{aligned}
 |\phi(x, \lambda)|^2 &\leq 3 |\phi(x, \lambda_0)|^2 \\
 (10) \quad &+ 3 |\lambda - \lambda_0|^2 |\theta(x, \lambda_0)|^2 \int_r^b |\phi(t, \lambda_0)|^2 dt \int_r^{b'} |\phi(t, \lambda)|^2 dt \\
 &+ 3 |\lambda - \lambda_0|^2 |\phi(x, \lambda_0)|^2 \int_r^b |\theta(t, \lambda_0)|^2 dt \int_r^{b'} |\phi(t, \lambda)|^2 dt
 \end{aligned}$$

where $r \leq x \leq b' \leq b$. Letting $b'' < b$, integration of (10) from b'' to b' gives

$$(11) \quad \int_{b''}^{b'} |\phi(t, \lambda)|^2 dt \leq K_0 + K(b'') \int_r^{b'} |\phi(t, \lambda)|^2 dt$$

where $K_0 = 3 \int_r^b |\phi(t, \lambda_0)|^2 dt$ and

$$\begin{aligned}
 K(b'') &= 3 |\lambda - \lambda_0|^2 \left[\int_{b''}^b |\theta(t, \lambda_0)|^2 dt \int_r^b |\phi(t, \lambda_0)|^2 dt \right. \\
 &\quad \left. + \int_{b''}^b |\phi(t, \lambda_0)|^2 dt \int_r^b |\theta(t, \lambda_0)|^2 dt \right].
 \end{aligned}$$

After choosing b'' so large that $K(b'') < \frac{1}{2}$ it follows that

$$\int_{b''}^{b'} |\phi(t, \lambda)|^2 dt < 2K_0 + 2 \int_r^{b''} |\phi(t, \lambda)|^2 dt.$$

The right side of this last inequality is independent of b' ; thus by letting b' tend to b and holding b'' fixed one obtains $\int_r^b |\phi(x, \lambda)|^2 dx < \infty$.

A similar proof is valid for $\theta(x, \lambda)$. The first part of Theorem 2 is proved.

The hypotheses of the second part of Theorem 2 imply the hypotheses of the first part; therefore, in the right side of (10) the result of the first part of the theorem can be employed to justify replacing b' by b . After this has been done, multiplication of both sides of (10) by $-q_2$ followed by integration from b' to b completes the proof of the second part of Theorem 2.

Theorem 2 shows that the case depends only upon the coefficient function $q(x)$. The next step will consist in showing that in every case there are analytic M -functions.

Let z_0 be any complex number lying on or below the real axis in the z -plane. Let b_0 be arbitrary, $r < b_0 < b$. Consider the functions $l(b', \lambda, z_0)$ defined by formula (1) for $b_0 < b' < b$. Since $I(z_0) \leq 0$, one recalls from the remarks following (4) that $l(b', \lambda, z_0)$ is analytic for $I(\lambda) > 0$. Consulting formulæ (6) and (7) one observes that $r_{b_0}(\lambda)$ and $p_{b_0}(\lambda)$ are continuous functions of λ for $I(\lambda) > 0$, hence uniformly bounded in every compact subset of the upper half λ -plane. Since $|l(b', \lambda, z_0)| < r_{b_0}(\lambda) + |p_{b_0}(\lambda)|$ for $b_0 < b' < b$, the set of functions $l(b', \lambda, z_0)$ is uniformly bounded with respect to b' in every compact subset of the upper half λ -plane. As a result, for any compact set G of the upper half λ -plane there exists a sequence $b_i \rightarrow b$ such that $l(b_i, \lambda, z_0)$ converges to a limit $M(\lambda)$. By

VITALI'S theorem $M(\lambda)$ is analytic in G , and because of the arbitrary nature of G $M(\lambda)$ is analytic throughout the whole upper half λ -plane.

In the preceding argument one notices that if $I(z_0) = 0$ then $l(b', \lambda, z_0)$ is on the boundary of $C_b(\lambda)$ for each λ , hence $M(\lambda)$ is on the boundary of $C_b(\lambda)$. One should also observe that z_0 can be replaced in the argument by $f(b', \lambda)$ so long as each $f(b', \lambda)$ is analytic for $I(\lambda) > 0$ and $I[f(b', \lambda)] \leq 0$ for each b' .

In case III $M(\lambda)$ is not uniquely defined. On the contrary, there exists a whole class of analytic M -functions obtained by the limiting process described above, corresponding to different choices of $f(b', \lambda)$. For example, if M_0 is a point on or in the limit-circle $C_b(\lambda_0)$ then the choice $f(b', \lambda) = z(b', \lambda_0, M_0)$ (see (2)) yields an analytic M -function with the property $M(\lambda_0) = M_0$; thus there are at least as many distinct choices of analytic functions $M(\lambda)$ as there are points on an arbitrary limit-circle.

Concerning the question of prescribing values of the M -functions, a more complete result is given below. For arbitrary λ_0 , $I(\lambda_0) > 0$, choose a point M_0 on the boundary of $C_b(\lambda_0)$. On the boundary of each $C_{b'}(\lambda_0)$ select a point M' with the additional property that $M' \rightarrow M_0$ as $b' \rightarrow b$ and $M' \neq -\theta(b', \lambda_0)/\phi(b', \lambda_0)$. Now let $f(b', \lambda) = z(b', \lambda_0, M')$. The restriction on M' assures the analyticity of f . Furthermore, since M is on the boundary of $C_{b'}(\lambda_0)$ it follows that $I[f(b', \lambda)] = 0$, and, therefore, the functions $l(b', \lambda, f(b', \lambda))$ take their values on the boundary of $C_b(\lambda)$ and are analytic for $I(\lambda) > 0$. Repeating the earlier limiting process, there exists a subsequence of this last set of functions that converges to an analytic M -function, and, in view of the special nature of the members of the limiting sequence, this limit function $M(\lambda)$ takes its values on the boundary of $C_b(\lambda)$.

We have proved the following theorem:

Theorem 3. *In every case there exists an M -function that is analytic for $I(\lambda) > 0$. In Case III an analytic M -function can be found such that $M(\lambda) = M_0$, where M_0 is an arbitrary point of an arbitrary limit-circle $C_b(\lambda_0)$; moreover, an M -function can be found with the additional property that $M(\lambda)$ always lies on the boundary of the limit-circle $C_b(\lambda)$ for $I(\lambda) > 0$.*

In Case III any analytic M -function is obtained by the limiting process described during the proof of Theorem 3. One confirms this by choosing $f(b', \lambda) = z(b', \lambda, M(\lambda))$ where the right member is defined by (2).

In Cases II and III the fact that the base solutions ϕ and θ are square-integrable on the interval from r to b makes available more detailed information regarding the M -functions. In this regard we have Theorem 4 below.

Theorem 4. *In Case II $M(\lambda)$ is meromorphic. In Case III any analytic M -function is meromorphic. In either case $M(\lambda)$ has a pole at λ_1 if and only if $1 + (\lambda_1 - \lambda) \int_r^b \psi(x, \lambda) \phi(x, \lambda_1) dx = 0$ for all λ . The order of the pole is one greater than that of the zero of the entire function $\int_r^b \phi^2(x, \lambda) dx$ at λ_1 . If λ_1 and λ_2 are both poles of $M(\lambda)$ then $\int_r^b \phi(x, \lambda_1) \phi(x, \lambda_2) dx = 0$.*

In order to prove Theorem 4 we require the following lemmas.

Lemma 1. *In all three cases $W_b[\psi(x, \lambda), \psi(x, \lambda')] = 0$ for every analytic M -function as long as $I(\lambda) > 0$ and $I(\lambda') > 0$.*

Later it will be shown that in Cases II and III this result holds for all pairs λ, λ' even if one or both are poles of $M(\lambda)$. For the case at hand the reader can easily fashion a proof of this lemma of his own, modeled after the one given by TITCHMARSH [6] for the case where $I(q) = 0$.

Just as in the classical case with $I(q) = 0$, one obtains the following lemma with the aid of Lemma 1 and formula (3).

Lemma 2. *$M(\lambda) = M(\lambda') + (\lambda - \lambda') \int_r^b \psi(x, \lambda) \psi(x, \lambda') dx$ for $I(\lambda) > 0$ and $I(\lambda') > 0$ in every case.*

Lemma 3. *In Cases II and III $\int_r^b |\phi(x, \lambda)|^2 dx$ and $\int_r^b |\theta(x, \lambda)|^2 dx$ are bounded with respect to λ in every compact subset of the λ -plane.*

Let G be an arbitrary compact set in the λ -plane. In either Case II or Case III b' can be replaced by b in (10). Integrating both sides of the resulting expression from r to b one obtains, for arbitrary λ and λ' , the inequality

$$(12) \quad \int_r^b |\phi(x, \lambda)|^2 dx \leq 3 \int_r^b |\phi(x, \lambda')|^2 dx + K(\lambda') |\lambda - \lambda'| \int_r^b |\phi(x, \lambda)|^2 dx$$

where $K(\lambda') = 6(\int_r^b |\phi(x, \lambda')|^2 dx)(\int_r^b |\theta(x, \lambda')|^2 dx)$. About each λ' in G construct a neighborhood $N(\lambda')$ such that $K(\lambda') |\lambda - \lambda'| < \frac{1}{2}$ for $\lambda \in N(\lambda')$. Since G is compact, a finite sub-covering $N(\lambda_1), N(\lambda_2), \dots, N(\lambda_n)$ covers G . Let K_0 be the maximum of the numbers $6 \int_r^b |\phi(x, \lambda_j)|^2 dx, j = 1, 2, \dots, n$. Then $\int_r^b |\phi(x, \lambda)|^2 dx < K_0$ follows from (12). A similar argument applies for $\theta(x, \lambda)$. *Q.e.d.*

Lemma 4. *In cases II and III the functions $\int_r^b \phi^2(x, \lambda) dx$, $\int_r^b \phi(x, \lambda) \theta(x, \lambda) dx$, and $\int_r^b \theta^2(x, \lambda) dx$ are entire functions of λ ; moreover, $\phi(x, \lambda)$ and $\theta(x, \lambda)$ are entire functions (in the strong sense) from the λ -plane into $L^2[r, b]$.*

The proof of Lemma 4 follows from Lemma 3. By applying Lemma 3 to the right side of inequality (10) it is clear that $|\phi(x, \lambda)|$ is uniformly bounded for λ in an arbitrary compact set G and for all x in (r, b) . Hence, by the definition of the integral, there exists a uniformly bounded sequence of entire functions $F_n(\lambda)$ that tends pointwise to $\int_r^b \phi^2(x, \lambda) dx$ in G . Hence by VITALI's theorem $\int_r^b \phi^2(x, \lambda) dx$ is analytic in G . Furthermore, $\int_r^b \phi^2(x, \lambda) dx \rightarrow \int_r^b \phi^2(x, \lambda) dx$. Since the set of analytic functions $\int_r^b \phi^2(x, \lambda) dx$ is uniformly bounded in G (Lemma 3), $\int_r^b \phi^2(x, \lambda) dx$ is analytic in G . Similar arguments hold for $\int_r^b \phi(x, \lambda) \theta(x, \lambda) dx$ and $\int_r^b \theta^2(x, \lambda) dx$. The same result also holds for $\int_r^b f_n(x) \phi(x, \lambda) dx$ where $f_n(x)$ is an arbitrary continuous function of $L^2[r, b]$. By choosing a sequence of continuous functions f_n that are dense in $L^2[r, b]$, Lemma 3 attests to the fact that the sequences of analytic functions $\int_r^b f_n(x) \phi(x, \lambda) dx$ and $\int_r^b f_n(x) \theta(x, \lambda) dx$ tend uniformly to $\int_r^b f(x) \phi(x, \lambda) dx$ and $\int_r^b f(x) \theta(x, \lambda) dx$ in G . Thus these last two limit functions are also analytic in G , which fully proves Lemma 4.

By multiplying both sides of inequality (10) by $-q_2(x)$ one obtains that in

Case II $\int_r^b (v - q_2) |\phi^2(x, \lambda)| dx$ and $\int_r^b (v - q_2) |\theta(x, \lambda)|^2 dx$ are bounded in every compact subset G of the upper half λ -plane.

In Cases II and III the fact that ϕ and θ are square-integrable on the interval $[r, b]$ enables one to solve the equation of Lemma 2 for $M(\lambda)$. Doing so one obtains

$$(13) \quad M(\lambda) = \frac{M(\lambda') + (\lambda' - \lambda) \int_r^b \theta(x, \lambda) \psi(x, \lambda') dx}{1 + (\lambda - \lambda') \int_r^b \phi(x, \lambda) \psi(x, \lambda) dx}.$$

While Lemma 2 asserts that this expression holds for all pairs λ and λ' with $I(\lambda) > 0$ and $I(\lambda') > 0$, it is also true, in view of Lemma 4, that the right side of (13) is defined and analytic for *all* λ provided $I(\lambda') > 0$ and provided λ is not a zero of the denominator. We take (13) as our extended definition of $M(\lambda)$ (for arbitrary fixed λ' , $I(\lambda') > 0$) and notice that it is analytic except for poles occurring at the zeros of the entire function $1 + (\lambda - \lambda') \int_r^b \phi(x, \lambda) \psi(x, \lambda') dx$. The extended definition (13) agrees with the previous definition of $M(\lambda)$ for $I(\lambda) > 0$; thus $M(\lambda)$ is meromorphic.

In expression (13) λ' was an arbitrary complex number of the upper half λ -plane. Thus if λ_1 is a pole of $M(\lambda)$ the denominator of (13) must be zero for all λ' , $I(\lambda') > 0$. Since the denominator is a meromorphic function of λ' , this denominator must be zero for *all* λ' . That is to say, if λ_1 is a pole of $M(\lambda)$ then $1 + (\lambda_1 - \lambda') \int_r^b \phi(x, \lambda_1) \psi(x, \lambda') dx = 0$ for *all* λ' . On the other hand, if $1 + (\lambda_1 - \lambda') \int_r^b \phi(x, \lambda_1) \psi(x, \lambda') dx = 0$ for some λ_1 and all λ' , then by a straightforward computation it follows from this last equality that

$$M(\lambda') = - \frac{1 + (\lambda_1 - \lambda') \int_r^b \phi(x, \lambda_1) \theta(x, \lambda') dx}{(\lambda_1 - \lambda') \int_r^b \phi(x, \lambda_1) \phi(x, \lambda') dx}$$

and that $M(\lambda')$ has a pole at λ_1 . We see then that $M(\lambda)$ has a pole at λ_1 if and only if $1 + (\lambda_1 - \lambda) \int_r^b \phi(x, \lambda_1) \psi(x, \lambda) dx = 0$ for *all* λ .

By letting $\lambda \rightarrow \lambda_1$ one acquires the additional fact from this last expression that if $M(\lambda)$ has pole of order greater than one then $\int_r^b \phi^2(x, \lambda_1) dx = 0$, and that if λ_1 and λ_2 are both poles of $M(\lambda)$ then $\int_r^b \phi(x, \lambda_1) \phi(x, \lambda_2) dx = 0$. Also, from the definition (13) and from the ensuing observations, it is easy to see that Lemma 1 holds for *all* pairs λ, λ' (even in the limit if one or both are poles of $M(\lambda)$).

This completes the proof of Theorem 4.

4. Extension of the Theory to the Interval (a, b) . In the foregoing we considered the situation where the function $q(x)$ had a singularity at the right hand end point of the interval (r, b) where r was an arbitrary point of the larger interval (a, b) . In this section we extend the preceding results to include the circumstance when $q(x)$ has a singularity at $x = a$ as well as at $x = b$.

Let α be any real number. The solution appropriate to the interval $[r, b)$ that was previously designated $\psi(x, \lambda)$ is now designated

$$(14) \quad \psi_1(x, \lambda) = M(\lambda)\phi(x, \lambda) + \theta(x, \lambda)$$

where $M(\lambda)$ is an analytic M -function. As before, let

$$\begin{aligned} \phi(r, \lambda) &= \sin \alpha, & \phi'(r, \lambda) &= -\cos \alpha, \\ \theta(r, \lambda) &= \cos \alpha, & \theta'(r, \lambda) &= \sin \alpha. \end{aligned}$$

Let

$$\Phi(x, \lambda) = -\phi(-x, \lambda), \quad \Theta(x, \lambda) = \theta(-x, \lambda).$$

Let $Q(x) = q(-x)$ for $-b \leq x \leq -a$. Let $\gamma = -\alpha$. Consider the differential equation

$$(15) \quad -y'' + [Q - \lambda]y = 0$$

Applying the previous theory to this differential equation and the interval $[-r, -a]$, there exists a solution $\Psi(x, \lambda)$ with the property

$$\int_{-r}^{-a} |\Psi(x, \lambda)|^2 (v - Q_2) dx < \infty, \quad W_{-a}[\Psi(x, \lambda), \Psi(x, \lambda')] = 0,$$

$$\Psi(x, \lambda) = \Phi(x, \lambda)M_0(\lambda) + \Theta(x, \lambda),$$

where $M_0(\lambda)$ is analytic for $I(\lambda) > 0$. Letting $M(\lambda) = -M_0(\lambda)$ and $\psi_0(x, \lambda) = \Psi(-x, \lambda)$, we have at once

$$\psi_0(x, \lambda) = M(\lambda)\phi(x, \lambda) + \theta(x, \lambda), \quad \int_a^r |\psi_0|^2 (v - q_2) dx < \infty,$$

and $M(\lambda)$ is analytic for $I(\lambda) > 0$. Moreover, $W_a[\psi_0(x, \lambda), \psi_0(x, \lambda')] = 0$ for all pairs of complex numbers λ and λ' such that $I(\lambda) > 0$ and $I(\lambda') > 0$.

So far we have shown the existence of two solutions

$$\psi_0(x, \lambda) = m(\lambda)\phi(x, \lambda) + \theta(x, \lambda) \quad \text{and} \quad \psi_1(x, \lambda) = M(\lambda)\phi(x, \lambda) + \theta(x, \lambda)$$

both defined on (a, b) , and having the properties $\int_a^r |\psi_0(x, \lambda)|^2 (v - q_2) dx < \infty$ and $\int_r^b |\psi_1(x, \lambda)|^2 (v - q_2) dx < \infty$ for $I(\lambda) > 0$. Our next objective is to show that these two solutions are linearly independent for $I(\lambda) > 0$. Applying inequality (5) to ψ_0 and ψ_1 we have on the one hand

$$\int_a^r |m(\lambda)\phi(x, \lambda) + \theta(x, \lambda)|^2 (v - q_2) dx \leq I[m(\lambda)]$$

and on the other hand

$$\int_r^b |M(\lambda)\phi(x, \lambda) + \theta(x, \lambda)|^2 (v - q_2) dx \leq -I[M(\lambda)],$$

from which it follows that $I(M) > 0$ and $I(m) < 0$ for $I(\lambda) > 0$; thus ψ_0 and ψ_1 are linearly independent for $I(\lambda) > 0$.

With the last result this section can be summarized by the following theorem.

Theorem 5. *Let $[a, b]$ be an arbitrary closed interval of the real line. Let $q(x)$ be continuous and $I(q) = q_2 \leq 0$ in the open interval (a, b) . Let α be real and λ complex with $I(\lambda) = v > 0$. Then there exist functions $m(\lambda)$ and $M(\lambda)$ and linearly independent solutions*

$$\psi_0(x, \lambda) = m(\lambda)\phi(x, \lambda) + \theta(x, \lambda) \quad \text{and} \quad \psi_1(x, \lambda) = M(\lambda)\phi(x, \lambda) + \theta(x, \lambda)$$

of $-y'' + [q(x) - \lambda]y = 0$ having the following properties:

- (a) $M(\lambda)$ and $m(\lambda)$ are analytic for $I(\lambda) > 0$ (Theorems 3 and 4 hold for m as well as M).
- (b) $\phi(r, \lambda) = \sin \alpha$, $\phi'(r, \lambda) = -\cos \alpha$, $\theta(r, \lambda) = \cos \alpha$, and $\theta'(r, \lambda) = \sin \alpha$ for arbitrary $r \in (a, b)$.

$$(c) \quad \int_a^r |\psi_0(x, \lambda)|^2 (v - q_2) dx < \infty \quad \text{and} \quad \int_r^b |\psi_1(x, \lambda)|^2 (v - q_2) dx < \infty.$$

- (d) $W_a[\psi_0(x, \lambda), \psi_0(x, \lambda')] = W_b[\psi_1(x, \lambda), \psi_1(x, \lambda')] = 0$ for $I(\lambda) > 0$ and $I(\lambda') > 0$.

Moreover, the solutions ϕ and θ are continuous functions of x in (a, b) and entire functions of λ .

One should notice that a limit-circle, limit-point geometry exists at each end of the interval (a, b) for $I(\lambda) > 0$, thus giving, in all, nine cases. Since we have no occasion to distinguish between (I, II) and (II, I), (I, III) and (III, I), or (II, III) and (III, II), for our purposes there are but six distinct cases to be considered in the sequel; they are (I, I), (I, II), (I, III), (II, II), (II, III), and (III, III).

5. The Base Contractions of K . In the introduction we defined the domain

$$(16) \quad D^* = \{f | f \in L^2[a, b], f \text{ and } f' \text{ absolutely continuous, } -f'' + qf \in L^2[a, b]\}$$

and the operator

$$(17) \quad Kf = -f'' + qf,$$

for $f \in D^*$. D^* is the largest set in $L^2[a, b]$ for which the differential expression on the right side of (17) is defined and in $L^2[a, b]$. Our immediate aim is to show that the contraction $L_0 \subset K$ whose domain is

$$(18) \quad D_0 = \{f | f \in D^*, W_a[\psi_0(x, \lambda_0), f] = W_b[\psi_1(x, \lambda_0), f] = 0, I(\lambda_0) > 0\}$$

has a resolvent set $\rho(L_0)$ containing all the λ -plane lying above the real axis, and that the definition of D_0 is independent of λ_0 . (Note: The solutions ψ_0 and ψ_1 appearing in (18) are those of Theorem 5.) Since $\psi_0 \in L^2[a, r]$ and $\psi_1 \in L^2[r, b]$ it follows from the formula

$$(19) \quad W_z[\psi_i(x, \lambda), f] - W_r[\psi_i(x, \lambda), f] = \int_r^x \psi_i[f'' - (q - \lambda)f] dx, \quad i = 0, 1,$$

that $\lim W_x[\psi_0(x, \lambda), f]$ exists as $x \rightarrow a$ and $\lim W_x[\psi_1(x, \lambda), f]$ exists as $x \rightarrow b$ for all $f \in D^*$. This last observation assures us that (18) is a proper definition. Furthermore, $D_0 \neq \emptyset$ because D_0 contains the set of functions H_0 contained in D^* that, along with the first derivatives, are zero in a neighborhood of both a and b .

The base solutions ψ_0 and ψ_1 are utilized to construct a bounded right inverse $R_0(\lambda)$ with domain $L^2[a, b]$ for the operator $K - \lambda I$ for each λ with $I(\lambda) > 0$. Next it is shown that $R_0(\lambda)$ is a left inverse for the operator

$$(20) \quad U(\lambda) = L_0 - \lambda I$$

as long as $I(\lambda) > 0$. Later, in three of the six distinct cases, the resolvent set will be shown to contain all of the finite λ -plane except for a discrete set.

For arbitrary λ , $I(\lambda) > 0$, define

$$(21) \quad G(x, t, \lambda) = \psi_1(x, \lambda)\psi_0(t, \lambda), \quad a < t \leq x < b,$$

$$(22) \quad G(x, t, \lambda) = \psi_1(t, \lambda)\psi_0(x, \lambda), \quad a < x \leq t < b,$$

$$(23) \quad w(\lambda) = M(\lambda) - m(\lambda).$$

For arbitrary $f \in L^2[a, b]$, define

$$(24) \quad \Phi(x, \lambda) = \int_a^b G(x, t, \lambda)f(t) dt,$$

or alternatively

$$(25) \quad \Phi(x, \lambda) = \frac{1}{w(\lambda)} \left[\psi_1(x, \lambda) \int_a^x \psi_0(t, \lambda)f(t) dt + \psi_0(x, \lambda) \int_x^b \psi_1(t, \lambda)f(t) dt \right].$$

Each of the integrals in (25) exists by virtue of Theorem 5. The derivative of Φ with respect to x exists *everywhere* and is equal to

$$(26) \quad \Phi'(x, \lambda) = \frac{1}{w(\lambda)} \left[\psi_1'(x, \lambda) \int_a^x \psi_0(t, \lambda)f(t) dt + \psi_0'(x, \lambda) \int_x^b \psi_1(t, \lambda)f(t) dt \right].$$

The second derivative exists almost everywhere and is equal to

$$(27) \quad \begin{aligned} \Phi''(x, \lambda) = \frac{1}{w(\lambda)} \left[\psi_1''(x, \lambda) \int_a^x \psi_0(t, \lambda)f(t) dt \right. \\ \left. + \psi_0''(x, \lambda) \int_x^b \psi_1(t, \lambda)f(t) dt \right] - f(x) \end{aligned}$$

hence

$$(28) \quad -\Phi'' + [q(x) - \lambda]\Phi = f.$$

It is also true that $\Phi \in L^2[a, b]$ for all $f \in L^2[a, b]$ provided $I(\lambda) > 0$. The proof is a generalization of the one given by TITCHMARSH [6] for the case with $q(x)$ real and appears below as Theorem 6.

Theorem 6. *If $f \in L^2[a, b]$ then in every case $\Phi(x, \lambda) \in L^2[a, b]$ for $I(\lambda) > 0$; moreover, $\|\Phi\| \leq v^{-1} \|f\|$ where $v = I(\lambda)$.*

For $a < a' < b' < b$ we have, on the one hand, from (20), that

$$\int_{a'}^{b'} [\bar{\Phi}(-\Phi'' + q\Phi) - \Phi(-\bar{\Phi}'' + \bar{q}\bar{\Phi})] dx = 2iv \int_{a'}^{b'} |\Phi|^2 dx - 2i \int_{a'}^{b'} I[f\bar{\Phi}] dx$$

and, on the other hand, by direct computation, that the integral on the left is equal to

$$2i \int_{a'}^{b'} q_2 |\Phi|^2 dx + W_{b'}[\Phi, \bar{\Phi}] - W_{a'}[\Phi, \bar{\Phi}],$$

hence

$$(29) \quad 2i \int_{a'}^{b'} (v - q_2) |\Phi|^2 dx = 2i \int_{a'}^{b'} I[f\bar{\Phi}] dx + W_{b'}[\Phi, \bar{\Phi}] - W_{a'}[\Phi, \bar{\Phi}].$$

Define $f_{a', b'} = f$ for $a' < x < b'$ and $f_{a', b'} = 0$ elsewhere in (a, b) . Let $\Phi_{a', b'} = \int_{a'}^x G(x, t, \lambda) f_{a', b'} dt$. In particular

$$\Phi_{a', b'}(b') = \frac{1}{w(\lambda)} \psi_1(b', \lambda) \int_{a'}^{b'} \psi_0(t, \lambda) f(t) dt,$$

$$\Phi'_{a', b'}(b') = \frac{1}{w(\lambda)} \psi_1(b', \lambda) \int_{a'}^{b'} \psi_0(t, \lambda) f(t) dt,$$

$$\Phi_{a', b'}(a') = \frac{1}{w(\lambda)} \psi_0(a', \lambda) \int_{a'}^{b'} \psi_1(t, \lambda) f(t) dt,$$

$$\Phi'_{a', b'}(a') = \frac{1}{w(\lambda)} \psi'_0(a', \lambda) \int_{a'}^{b'} \psi_1(t, \lambda) f(t) dt.$$

Consequently,

$$W_{b'}[\Phi_{a', b'}, \bar{\Phi}_{a', b'}] = |\lambda|^{-2} W_{b'}[\psi_1, \bar{\psi}_1] \left| \int_{a'}^{b'} \psi_0 f dt \right|^2,$$

$$W_{a'}[\Phi_{a', b'}, \bar{\Phi}_{a', b'}] = |\lambda|^{-2} W_{a'}[\psi_0, \bar{\psi}_0] \left| \int_{a'}^{b'} \psi_1 f dt \right|^2.$$

Substituting into (29) the relations

$$W_{b'}[\psi_1, \bar{\psi}_1] = 2iI[M] + 2i \int_r^{b'} (v - q_2) |\psi_1|^2 dx,$$

$$W_{a'}[\psi_0, \bar{\psi}_0] = 2iI[m] - 2i \int_{a'}^r (v - q_2) |\psi_0|^2 dx$$

proved earlier one obtains in this special case

$$\begin{aligned}
 \int_{a'}^{b'} (v - q_2) |\Phi_{a'b'}|^2 dx &= \int_{a'}^{b'} I[f\bar{\Phi}_{a'b'}] dx \\
 (30) \quad &+ \frac{1}{|w|^2} \left| \int_{a'}^{b'} \psi_0 f dx \right|^2 \{I[M] + \int_{a'}^{b'} (v - q_2) |\psi_1|^2 dx\} \\
 &+ \frac{1}{|w|^2} \left| \int_{a'}^{b'} \psi_1 f dx \right|^2 \{-I[M] + \int_{a'}^{b'} (v - q_2) |\psi_0|^2 dx\}.
 \end{aligned}$$

From inequality (5) the last two terms on the right side of (30) are negative and the two following inequalities ensue

$$(31) \quad \int_{a'}^{b'} (v - q_2) |\Phi_{a'b'}|^2 dx \leq \left(\int_{a'}^{b'} |f|^2 dx \right)^{\frac{1}{2}} \left(\int_{a'}^{b'} |\Phi_{a'b'}|^2 dx \right)^{\frac{1}{2}},$$

$$(32) \quad \left(\int_{a'}^{b'} |\Phi_{a'b'}|^2 dx \right)^{\frac{1}{2}} \leq \frac{1}{v} \left(\int_{a'}^{b'} |f|^2 dx \right)^{\frac{1}{2}}.$$

From (31) and (32) one obtains

$$(33) \quad \int_{a'}^{b'} (v - q_2) |\Phi_{a'b'}|^2 dx \leq \frac{1}{v} \int_{a'}^{b'} |f|^2 dx.$$

By definition $|\Phi_{a'b'}|^2$ tends to $|\Phi|^2$ (point-wise) as a' tends to a and b' tends to b ; therefore, by (33) and FATOUE's theorem it follows that

$$\int_a^b (v - q_2) |\Phi|^2 dx \leq \frac{1}{v} \int_a^b |f|^2 dx.$$

This last completes the proof of Theorem 6.

Defining the operator $R_0(\lambda)f$ by the relation $R_0(\lambda)f = \Phi(x, \lambda)$ one can paraphrase Theorem 6 as follows.

Corollary. $R_0(\lambda)$ is a linear bounded operator defined on all of $X = L^2[a, b]$ and has the following properties for $I(\lambda) > 0$:

- (i) $R_0(\lambda)X \subset D^*$.
- (ii) $\|R_0(\lambda)\| = v^{-1}$.
- (iii) $U(\lambda)R_0(\lambda) = I$ where $U(\lambda) = K - \lambda I$.
- (iv) $\int_a^b |R_0(\lambda)f|^2 (v - q_2) dx < \infty$ for all $f \in X$.

Let us make the temporary definition

$$(34) \quad D_0 = \{f|f \in D^*, W_a[f, \psi_0] = W_b[f, \psi_1] = 0 \text{ for all } \lambda, I(\lambda) > 0\}.$$

The proof that definition (34) is independent of the choice of λ so long as $I(\lambda) > 0$ will be deferred for the moment; however, at the present time, we will show that the contraction $L_0 \subset K$ whose domain is $D_0 \subset D^*$ has the property $R(L_0, \lambda) = R_0(\lambda)$ for all $\lambda, I(\lambda) > 0$. ($R_0(\lambda)$ is the resolvent of L_0 and the resolvent set $\rho(L_0)$ contains the upper half λ -plane.) In order to see this, one may observe that since

$R_0(\lambda)U(\lambda)f \in D^*$ for all $f \in D^*$, $R_0(\lambda)U(\lambda)f - f$ is a solution of $-y'' + (q - \lambda)y = 0$ for all $f \in D^*$ and all λ , $I(\lambda) > 0$. This solution is continuous in (a, b) hence there exist numbers dependent upon λ and f such that

$$(35) \quad R_0(\lambda)U(\lambda)f - f = A[f, \lambda]\psi_0(x, \lambda) + B[f, \lambda]\psi_1(x, \lambda).$$

Returning to the definition (27) and taking the Wronskian of both sides of (35) with respect to ψ_0 and ψ_1 , one is entitled to pass to the limits a and b respectively. The existence of these limits is assured by the properties of ψ_0 and ψ_1 (Theorem 5) and the text following (19). The result of a simple computation shows that the following lemma is true.

Lemma. $W_a[R_0(\lambda)f, \psi_0] = W_b[R_0(\lambda)f, \psi_1] = 0$ for all λ , $I(\lambda) > 0$ and all $f \in X$, i.e., $R_0(\lambda)X \subset D_0$.

Applying this lemma with f replaced by $U(\lambda)f$ one obtains formulæ for the coefficients of ψ_0 and ψ_1 appearing in (35). The formulæ are

$$(36) \quad A[f, \lambda] = \frac{1}{w(\lambda)} W_b[f, \psi_1], \quad B[f, \lambda] = \frac{1}{w(\lambda)} W_a[\psi_0, f].$$

(One should recall from Theorem 5 that $w(\lambda) = M(\lambda) - m(\lambda)$ is never zero for $I(\lambda) > 0$.)

From (36), definition (34), and the corollary to Theorem 6 the following theorem ensues.

Theorem 7. For $I(\lambda) > 0$ $R_0(\lambda)$ has the following properties:

- (i) $U(\lambda)R_0(\lambda)f = f$ for all $f \in X$.
- (ii) $R_0(\lambda)U(\lambda)f = f$ for all $f \in D_0$.
- (iii) $\|R_0(\lambda)\| \leq v^{-1}$.
- (iv) $\int_a^b |R_0(\lambda)f|^2 (v - q_2) dx < \infty$ for all $f \in X$.

Employing the lemma above and Theorem 7 one obtains the following corollary.

Corollary. $R_0(\lambda)$ is the resolvent of L_0 and the resolvent set $\rho(L_0)$ contains the upper half λ -plane.

Our next concern is for a better characterization of the domain D_0 , and particularly for showing that the definition (34) is independent of λ , $I(\lambda) > 0$. We proceed by cases.

In Case (I, I) there are no solutions, other than zero, of $-y'' + (q - \lambda)y = 0$ in $L^2[a, b]$; therefore, consulting (35), one observes that $A[f, \lambda] = B[f, \lambda] = 0$ for all $f \in D^*$ and λ such that $I(\lambda) > 0$. As a result $D_0 = D^*$ in Case (I, I). In the remaining cases no such simple situation prevails; a closer scrutiny is required.

In Case (I, II) there is only one solution of $-y'' + (q - \lambda)y = 0$ in $L^2[a, b]$ and it is ψ_0 . It follows from (35) that $B[f, \lambda] = 0$ for all $f \in D^*$ and $I(\lambda) > 0$. From (iv) of Theorem 7, $R_0(\lambda)U_0(\lambda)f \in L^2[-q_2; a, b]$ for $f \in D^*$ and $I(\lambda) > 0$. Let $f \in D^* \cap L^2[-q_2; a, b]$. Then from (35) $A[f, \lambda] = 0$ since there are no non-

trivial solutions of $-y'' + (q - \lambda)y = 0$ in $L^2[-q_2; a, b]$ in Case (I, II). Thus, $D^* \cap L^2[-q_2; a, b] \subset D_0$. On the other hand, if $f \in D_0$ then $f = R_0(\lambda)U(\lambda)f$ and $f \in D^* \cap L^2[-q_2; a, b]$; consequently $D_0 = \{f | f \in D^*, \int_a^b -q_2 |f|^2 dx < \infty\}$. This last definition is independent of λ .

An equivalent definition can be obtained as follows. Let $f \in D^*$ have the property that $A[f, \lambda_0] = 0$ for some λ_0 , $I(\lambda_0) > 0$; then from (35) $f \in L^2[-q_2; a, b]$, and from the above argument $f \in D_0$. Employing definition (34) $A[f, \lambda] = 0$ for all λ , $I(\lambda) > 0$. With this last one can make the definition $D_0 = \{f | f \in D^*, A[f, \lambda_0] = B[f, \lambda_0] = 0 \text{ for one } \lambda_0, I(\lambda_0) > 0\}$.

In Case (II, II), by pursuing an argument too similar to the one given above to repeat here, one acquires the same two equivalent definitions of D_0 .

In Case (I, III) $B[f, \lambda] = 0$ for all $f \in D^*$ as ψ_0 is the only solution of $-y'' + (q - \lambda)y = 0$ that is in $L^2[a, b]$. Now suppose $f \in D^*$ has the additional property that $A[f, \lambda_0] = 0$ for some fixed λ_0 , $I(\lambda_0) > 0$. From (35) $f = R_0(\lambda_0)U(\lambda_0)f$ and from (27)

$$W_x[f, \psi_1(x, \lambda)] = \frac{1}{w(\lambda_0)} \left[W_x[\psi_0(x, \lambda_0), \psi_1(x, \lambda)] \int_a^x U(\lambda_0) f \psi_0(x, \lambda) dx \right. \\ \left. + W_x[\psi_0(x, \lambda_0), \psi_0(x, \lambda)] \int_x^b U(\lambda_0) \psi_1(x, \lambda) dx \right].$$

From (d), Theorem 5, $W_x[\psi_1(x, \lambda_0), \psi_1(x, \lambda)] \rightarrow 0$ as $x \rightarrow b$. Since $\psi_0 \in D^*$, $\lim W_x[\psi_0(x, \lambda_0), \psi_1(x, \lambda)]$ exists as $x \rightarrow b$ and $\lim W_x[f, \psi_1(x, \lambda)] \rightarrow 0$ as $x \rightarrow b$, i.e., $A[f, \lambda] = 0$.

In short, if $A[f, \lambda_0] = 0$ for $f \in D^*$ and $I(\lambda_0) > 0$, then $A[f, \lambda] = 0$ for all λ , $I(\lambda) > 0$, and $D_0 = \{f | f \in D^*, A[f, \lambda_0] = B[f, \lambda_0] = 0 \text{ for arbitrary } \lambda_0, I(\lambda_0) > 0\}$.

In the two remaining cases, (II, III) and (III, III), the argument just applied to the right end points and the definition of D_0 just given are obtained. This definition is seen to be independent of λ , $I(\lambda) > 0$.

In all six cases the definition

$$(37) D_0 = \{f | f \in D^*, A[f, \lambda_0] = B[f, \lambda_0] = 0 \text{ for one } \lambda_0, I(\lambda_0) > 0\}$$

is equivalent to (34). This enables us to prove that $R_0(\lambda)X = D_0$ and $U(\lambda)D_0 = X$ for arbitrary λ , $I(\lambda) > 0$. The proof can be accomplished by a line of reasoning similar to that employed during the discussion of Case (I, III) and is omitted.

The above results are summarized by the following theorem.

Theorem 8. $R_0(\lambda)$ is the resolvent of $L_0 \subset K$ with domain $D_0 \subset D^*$; moreover, $R_0(\lambda)X = D_0$, $U(\lambda)D_0 = X$, $\|R_0(\lambda)\| = v^{-1}$, and $\rho(L_0)$ contains all complex numbers λ such that $I(\lambda) > 0$.

6. Extension of the Theory to the Entire λ -Plane. In this section we show that in cases (II, II), (II, III), and (III, III) we can avoid the restriction that λ lies in the upper half complex plane. We have already seen (see Theorem 4) that in these cases $m(\lambda)$ and $M(\lambda)$ are meromorphic, thus $\psi_0(x, \lambda)$ and $\psi_1(x, \lambda)$

are defined and finite throughout the whole λ -plane except at the poles of these functions. Moreover, in these three cases $\psi_0(x, \lambda)$ and $\psi_1(x, \lambda)$ are in $L^2[a, b]$. From this it follows that for each f in $L^2[a, b]$

$$(38) \quad \Phi(x, \lambda) = \frac{1}{w(\lambda)} \left[\psi_1(x, \lambda) \int_a^x \psi_0(t, \lambda) f(t) dt + \psi_0(x, \lambda) \int_x^b \psi_1(t, \lambda) f(t) dt \right]$$

is a well defined function of x even if $I(\lambda) > 0$, provided λ is neither a pole of $m(\lambda)$ nor of $M(\lambda)$ nor a zero of $w(\lambda)$. Furthermore, for each $x \in (a, b)$, $\Phi(x, \lambda)$ is a solution of $-y'' + (q - \lambda)y = f$. As we cannot vouch for the existence of a limit-point, limit-circle geometry for any λ such that $I(\lambda) < 0$, we cannot use the arguments of the earlier sections of this chapter to show that $\Phi(x, \lambda f)$ is in $L^2(a, b)$; however, by following a different line we prove the following theorem.

Theorem 9. *In cases (II, II), (II, III), and (III, III) the spectrum $\sigma(L_0)$ of L_0 is a pure point spectrum consisting of discrete points lying in the lower half λ -plane. Moreover, $\lambda \in \sigma(L_0)$ if and only if λ is a pole of both $M(\lambda)$ and $m(\lambda)$ or a zero of $w(\lambda)$. Also*

$$(39) \quad R(L_0, \lambda)f = \Phi(x, \lambda, f) \quad \text{for all } f \in X, \quad \lambda \in \rho(L_0).$$

Before proceeding we prove the following lemma.

Lemma 1. *In cases (II, II), (II, III), and (III, III), for fixed λ_0 , $I(\lambda_0) > 0$, $R_0(\lambda_0)$ is a completely continuous operator; i.e., $R_0(\lambda_0)$ maps bounded sets in X into sets with compact closure. Moreover, $\sigma(L_0)$ is a pure point spectrum and is a discrete set lying in the lower half λ -plane.*

The proof of complete continuity for the special case where the interval (a, b) is of finite length can be accomplished by reference to the literature [1].

The fact that in our theory the interval (a, b) may be infinite will present no obstacle if the reader attempts to carry out his own proof; such proofs depend only upon the fact that the two solutions, ψ_0 and ψ_1 , are square-integrable in the interval (a, b) .

In the general Banach space theory it is shown [5] that if a closed operator has a completely continuous resolvent for one point of the resolvent set then the spectrum is discrete and is a pure point spectrum. We have shown that L_0 meets these hypotheses and we conclude that $R(L_0, \lambda)$ exists for all values of λ except for a discrete set $\sigma(L_0)$. From the analysis of the earlier sections we know that $\sigma(L_0)$ lies beneath the real axis in the λ -plane. Moreover, for $\lambda \in \sigma(L_0)$ we know that there exists a solution $\Psi(x, \lambda)$ of $-y'' + [q(x) - \lambda]y = 0$ such that $\Psi(x, \lambda) \in D_0$.

We require a second lemma.

Lemma 2. *In cases (II, II), (II, III), and (III, III) $\psi_0(x, \lambda)$ and $\psi_1(x, \lambda)$ are meromorphic vector-valued functions from the λ -plane into D^* . They are analytic except at the poles of M or m . In particular, ψ_0 and ψ_1 are holomorphic in the upper*

half λ -plane, i.e., (ψ_0, f) and (ψ_1, f) are holomorphic in the Cauchy sense for every $f \in X$ [2].

We show that, for any $f_0 \in X$, $\int_a^b \psi_0(x, \lambda) f_0(x) dx$ and $\int_a^b \psi_1(x, \lambda) f(x) dx$ are analytic functions of λ except for poles. In Lemma 3 of Theorem 4 (Section 3) we showed that in cases II and III $\int_r^b |\phi(x, \lambda)|^2 dx$ and $\int_r^b |\theta(x, \lambda)|^2 dx$ are bounded in every compact region of the λ -plane. Applying the same lemma to the interval (a, r) we obtain that $\int_a^r |\phi(x, \lambda)|^2 dx$ and $\int_a^r |\theta(x, \lambda)|^2 dx$ are bounded in every compact region of the λ -plane. Thus, $\int_a^b |\phi(x, \lambda)|^2 dx$ and $\int_a^b |\theta(x, \lambda)|^2 dx$ are bounded in every compact region of the λ -plane. Moreover, from the definitions of ψ_0 and ψ_1 we have that

$$\int_a^b |\psi_0(x, \lambda)|^2 dx \leq 2 |m(\lambda)|^2 \int_a^b |\phi(x, \lambda)|^2 dx + 2 \int_a^b |\phi(x, \lambda)|^2 dx$$

and

$$\int_a^b |\psi_1(x, \lambda)|^2 dx \leq 2 |M(\lambda)|^2 \int_a^b |\phi(x, \lambda)|^2 dx + \int_a^b |\theta(x, \lambda)|^2 dx.$$

Since $m(\lambda)$ and $M(\lambda)$ are meromorphic functions of λ from the above, $\int_a^b |\psi_0(x, \lambda)|^2 dx$ and $\int_a^b |\psi_1(x, \lambda)|^2 dx$ are bounded functions of λ in every compact region of the λ -plane that excludes the poles of $M(\lambda)$ and $m(\lambda)$.

We observe, without repeating an argument given several times before, that, for arbitrary $f_0 \in X$, $\int_{a'}^{b'} f_0 \psi_0(x, \lambda) dx$ and $\int_{a'}^{b'} f_0 \psi_1(x, \lambda) dx$ are meromorphic functions of λ where $a < a' < b' < b$. Furthermore, since $|\int_{a'}^{b'} f_0 \psi_0(x, \lambda) dx| \leq \|\psi_0(x, \lambda)\| \|f_0\|$ and $|\int_{a'}^{b'} f_0 \psi_1(x, \lambda) dx| \leq \|\psi_1(x, \lambda)\| \|f_0\|$, the left members of these inequalities are bounded uniformly with respect to a' , b' , and λ in an arbitrary compact region of the λ -plane that excludes the poles of m and M . Thus in every such compact region of the λ -plane $\int_a^b f_0 \psi_0(x, \lambda) dx$ and $\int_a^b f_0 \psi_1(x, \lambda) dx$ are the point-wise limits of uniformly bounded sequences of analytic functions and are, therefore, analytic in such a region. This proves Lemma 2.

Lemma 3. $f \in D_0$ if and only if $W_a[f, \psi_0(x, \lambda)] = W_b[f, \psi_1(x, \lambda)] = 0$ for all λ . Moreover, if λ_1 is a pole of $m(\lambda)$ or of $M(\lambda)$ then the lemma holds in the limit as $\lambda \rightarrow \lambda_1$.

This lemma is already known to hold for $I(\lambda) > 0$. For $f \in D^*$ we have

$$(40) \quad \begin{aligned} W_b[f, \psi_1(x, \lambda)] &= W_r[f, \psi_1(x, \lambda)] + \int_r^b [U(\lambda)f] \psi_1(x, \lambda) dx, \\ W_a[f, \psi_0(x, \lambda)] &= W_r[f, \psi_0(x, \lambda)] - \int_a^r [U(\lambda)f] \psi_0(x, \lambda) dx. \end{aligned}$$

By inspection, the first terms of the right members of (40) are analytic except at the poles of $m(\lambda)$ and of $M(\lambda)$. By Lemma 2, the second terms are analytic except at the poles of $m(\lambda)$ or of $M(\lambda)$. Thus we have that $W_a[f, \psi_0(x, \lambda)]$ and $W_b[f, \psi_1(x, \lambda)]$ are analytic except at the poles of $m(\lambda)$ and $M(\lambda)$ for $f \in D^*$.

In particular, if $f \in D_0$ $W_b[f, \psi_1(x, \lambda)] = 0$ and $W_a[f, \psi_1(x, \lambda)] = 0$ for $I(\lambda) > 0$; therefore, from the foregoing $W_b[f, \psi_1(x, \lambda)] = 0$ and $W_a[f, \psi_0(x, \lambda)] = 0$ for all λ , even in the limit, as λ tends to a pole of $m(\lambda)$ or of $M(\lambda)$. Thus if $f \in D_0$ we have $W_b[f, \psi_1(x, \lambda)] = W_a[f, \psi_0(x, \lambda)] = 0$ for all λ .

On the other hand, if $W_b[f, \psi_1(x, \lambda)] = 0$ and $W_a[f, \psi_0(x, \lambda)] = 0$ for all λ , then in particular $W_b[f, \psi_1(x, \lambda)] = 0$ and $W_a[f, \psi_0(x, \lambda)] = 0$ in the upper half plane and we have, by the earlier theory, $f \in D_0$. This last completes the proof of Lemma 3.

Lemma 4. For $f \in X$, $R(L_0, \lambda)f = \Phi(x, \lambda, f)$ provided that λ is not one of the following:

- (a) a pole of $m(\lambda)$,
- (b) a pole of $M(\lambda)$,
- (c) a zero of $w(\lambda) = M(\lambda) - m(\lambda) = W_x[\psi_1, \psi_0]$, or
- (d) a point of $\sigma(L_0)$.

In cases (II, II), (II, III), and (III, III), according to (25) and Lemma 2, for fixed $f \in X$, $\Phi(x, \lambda, f)$ is a meromorphic vector-valued function from the λ -plane into X , analytic except possibly at the points mentioned in the statement of the current lemma. In particular $\Phi(x, \lambda, f)$ is analytic for $\lambda \in \rho(L_0)$. In the literature [4] it is shown that $R(L_0, \lambda)f$ is analytic on $\rho(L_0)$. Since it is known that $R(L_0, \lambda)f = \Phi(x, \lambda, f)$ for $I(\lambda) > 0$, it follows by analytic continuation that this equality holds for all $\lambda \in \rho(L_0)$ such that $w(\lambda) \neq 0$, $M(\lambda) \neq \infty$, and $m(\lambda) \neq \infty$. This proves the lemma.

Later we show that $\lambda \in \sigma(L_0)$ if and only if (a) λ_1 is a pole of both $M(\lambda)$ and $m(\lambda)$ or (b) $w(\lambda_1) = 0$. However, first we prove several other lemmas.

Lemma 5. If $\lambda \in \sigma(L_0)$ then λ_1 is either a pole of $m(\lambda)$, a pole of $M(\lambda)$, or a zero of $w(\lambda)$.

Assume that $\lambda_1 \in \sigma(L_0)$ and that λ_1 is neither a pole of $m(\lambda)$ nor a pole of $M(\lambda)$ nor a zero of $w(\lambda)$. From Lemma 1, there exists a solution $\Psi(x, \lambda_1)$ of $-y'' + (q - \lambda_1)y = 0$ with the property that $\Psi(x, \lambda_1) \in D_0$. By assumption, λ_1 is not a pole of $m(\lambda)$ nor of $M(\lambda)$ nor a zero of $w(\lambda)$, therefore $\psi_0(x, \lambda_1)$ and $\psi_1(x, \lambda_1)$ are linearly independent elements of D^* . We have $\Psi(x, \lambda_1) = \alpha\psi_0(x, \lambda_1) + \beta\psi_1(x, \lambda_1)$. Since $\Psi(x, \lambda_1) \in D_0$ we have, from Lemma 3, that

$$W_b[\Psi(x, \lambda_1), \psi_1(x, \lambda)] = 0 \quad \text{for all } \lambda.$$

Hence

$$W_b[\Psi(x, \lambda_1), \psi_1(x, \lambda)] = \alpha W_b[\psi_0(x, \lambda_1), \psi_1(x, \lambda)] + \beta W_b[\psi_1(x, \lambda_1), \psi_1(x, \lambda)] = 0.$$

From Lemma 1 of Theorem 4 $W_b[\psi_1(x, \lambda_1), \psi_1(x, \lambda)] = 0$ for all λ . Therefore, we have $\alpha W_b[\psi_0(x, \lambda_1), \psi_1(x, \lambda)] = 0$ for all λ . Furthermore,

$$\alpha W_b[\psi_0(x, \lambda_1), \psi_1(x, \lambda)] = \alpha W_b[\psi_1(x, \lambda_1) - w(\lambda_1)\phi(x, \lambda_1), \psi_1(x, \lambda)].$$

Again, as $W_b[\psi_1(x, \lambda_1), \psi_1(x, \lambda)] = 0$, we have $\alpha w(\lambda_1) W_b[\phi(x, \lambda_1), \psi_1(x, \lambda)] = 0$.

However,

$$W_b[\phi(x, \lambda_1), \psi_1(x, \lambda)] = 1 + (\lambda_1 - \lambda) \int_r^b \phi(x, \lambda_1) \psi_1(x, \lambda) dx.$$

From Theorem 4, we recall that if this expression is equal to zero for all λ then λ_1 is a pole of $M(\lambda)$ contrary to the hypotheses; therefore, as $w(\lambda_1) \neq 0$ and we have $\alpha = 0$ hence $\psi_1(x, \lambda_1) \in D_0$. As $\psi_1(x, \lambda_1) \in D_0$, $W_a[\psi_1(x, \lambda_1), \psi_0(x, \lambda)] = 0$ for all λ by Lemma 3. Also,

$$\begin{aligned} W_a[\psi_0(x, \lambda), \psi_1(x, \lambda_1)] &= W_a[\psi_0(x, \lambda), \psi_0(x, \lambda) + w(\lambda_1)\phi(x, \lambda_1)] \\ &= w(\lambda_1)W_a[\psi_0(x, \lambda), \phi(x, \lambda_1)], \end{aligned}$$

because $W_a[\psi_0(x, \lambda), \psi_0(x, \lambda_1)] = 0$ for all λ and λ_1 . By a second application of Theorem 4, this time to the interval (a, r) , we have that $W_a[\psi_0(x, \lambda), \phi(x, \lambda_1)] = 0$ for all λ only if λ_1 is a pole of $m(\lambda)$, which is contrary to the hypotheses. Thus we have that $w(\lambda_1) = 0$ which is again a contradiction. We have shown that if $\lambda \in \sigma(L_0)$ then λ_1 is either a pole of $M(\lambda)$ or of $m(\lambda)$ or a zero of $w(\lambda)$.

This proves the lemma.

Lemma 6. *If λ_1 is a pole of $M(\lambda)$ but not a pole of $m(\lambda)$, or vice versa, then $\lambda_1 \in \rho(L_0)$.*

Assume that λ_1 is a pole of $M(\lambda)$ but not a pole of $m(\lambda)$ and that $\lambda_1 \in \sigma(L_0)$. Since $\sigma(L_0)$ is a pure point spectrum there exist numbers α and β such that $\Psi = \alpha\phi(x, \lambda) + \beta\theta(x, \lambda) \in D_0$. By our criterion for membership in D_0 it follows that $W_a[\Psi, \psi_0(x, \lambda)] = 0$ and $W_b[\Psi, \psi_1(x, \lambda)] = 0$ for all λ . In other words,

$$(41) \quad \alpha W_a[\phi(x, \lambda_1), \psi_0(x, \lambda)] + \beta W_a[\theta(x, \lambda_1), \psi_0(x, \lambda)] = 0,$$

$$(42) \quad \alpha W_b[\phi(x, \lambda_1), \psi_1(x, \lambda)] + \beta W_b[\theta(x, \lambda_1), \psi_1(x, \lambda)] = 0.$$

Inasmuch as λ_1 is a pole of $M(\lambda)$, Theorem 4 asserts that

$$W_b[\phi(x, \lambda_1), \psi_1(x, \lambda)] = 1 + (\lambda_1 - \lambda) \int_r^b \phi(x, \lambda_1) \psi_1(x, \lambda) dx = 0$$

for all λ . This last equation and equation (42) yield $\beta = 0$ after letting $\lambda \rightarrow \lambda_1$; and in order to satisfy equation (41) non-trivially we must have

$$W_a[\phi(x, \lambda_1), \psi_0(x, \lambda)] = 0 \quad \text{for all } \lambda.$$

However, applying Theorem 4 to the interval (a, r) , $W_a[\phi(x, \lambda_1), \psi_0(x, \lambda)] = 0$ for all λ implies that λ_1 is a pole of $m(\lambda)$, contrary to our assumption.

In order to complete the proof of Theorem 9, three more lemmas are required.

Lemma 7. *If λ_1 is a pole of both $M(\lambda)$ and $m(\lambda)$ then $\lambda \in \sigma(L_0)$.*

We observe from the expressions employed in the previous proof that if λ_1 is

a pole of both $M(\lambda)$ and $m(\lambda)$ then

$$W_a[\phi(x, \lambda_1), \psi_0(x, \lambda)] = W_b[\phi(x, \lambda_1), \psi_1(x, \lambda)] = 0 \quad \text{for all } \lambda,$$

and in particular $\phi(x, \lambda) \in D_0$, i.e., $\lambda_1 \in \sigma(L_0)$.

Lemma 8. *If $w(\lambda) = 0$ then $\lambda_1 \in \sigma(L_0)$.*

We have, identically, that

$$\begin{aligned} W_b[\psi_0(x, \lambda_1), \psi_1(x, \lambda)] \\ = [\lambda(\lambda_1) - M(\lambda_1)]W_b[\phi(x, \lambda_1), \psi_1(x, \lambda)] + W_b[\psi_1(x, \lambda_1), \psi_1(x, \lambda)]. \end{aligned}$$

The first term of this last expression is equal to zero for all λ as $w(\lambda_1) = 0$. The second term is equal to zero for all pairs of complex numbers λ and λ_1 . Thus $W_b[\psi_0(x, \lambda_1), \psi_1(x, \lambda)] = 0$ for all λ . Furthermore, $W_a[\psi_0(x, \lambda_1), \psi_0(x, \lambda)] = 0$ for all pairs λ and λ_1 . These two equalities imply that $\psi_0(x, \lambda_1) \in D_0$ which completes the proof of the lemma.

So far we have proved $\lambda_1 \in \sigma(L_0)$ if and only if (a) $W(\lambda_1) = 0$ or (b) λ_1 is a pole of both $M(\lambda)$ and $m(\lambda)$. We conclude the section with a final lemma.

Lemma 9. *$R_0(\lambda_1) = R(L_0, \lambda_1)$ if λ_1 is a pole of $M(\lambda)$ (or $m(\lambda)$) alone.*

This follows from the observation, drawn from Lemma 6, that a pole of $M(\lambda)$ alone (or of $m(\lambda)$ alone) is in the resolvent set $\rho(L_0)$. This means that $R(L_0, \lambda_1)$ exists and is analytic in a neighborhood $N(\lambda_1)$ of λ_1 . In Lemma 4 we saw that $R(L_0, \lambda) = \Phi(x, \lambda, f)$ for $\lambda \in N(\lambda_1)$, $\lambda \neq \lambda_1$. From this we obtain $\lim_{\lambda \rightarrow \lambda_1} \Phi(x, \lambda, f) = R(L_0, \lambda_1)$ as $\lambda \rightarrow \lambda_1$.

We have completed the proof of Theorem 9.

7. Generation of Arbitrary Contractions L_r from the Base Contractions L_0 .
Cases (I, II) and (I, III). In this and the following section we obtain a characterization of an arbitrary $L_r \subset K$ whose resolvent set $\rho(L_r)$ contains at least one point λ_0 in the upper half λ -plane. It turns out that if $\rho(L_r)$ does contain a single point λ_0 , $I(\lambda_0) > 0$, then $\rho(L_r)$ contains all of the upper half plane except possibly a discrete set.

It will not be necessary to discuss the case (I, I) for, as we have seen in Section 5, $\rho(K)$ contains the upper half λ -plane, hence, for every contraction $L_r \subset K$, $\rho(L_r)$ contains no points of this half plane.

In the remaining five cases, the intersection of the set $\rho(K)$ with the upper half plane is empty and we will be able to show the existence of contractions L_r other than L_0 such that $\rho(L_r)$ contains all of the upper half λ -plane.

Of these remaining five cases, the cases (I, II) and (I, III) have two important features in common; they are the only two where a non-meromorphic $M(\lambda)$ is possible, and they are the only two where $\psi_0(x, \lambda)$ is the only solution of $U(\lambda)y = 0$ in X . In this section we restrict ourselves to these two cases. Because of the similarity between these two cases, we need make no distinction between them.

The proofs, formulæ, *etc.* will apply equally well to either and are to be so interpreted unless otherwise stated. Later the points of difference between these two cases will be discussed.

For each value of λ , $I(\lambda) > 0$, $\psi_0(x, \lambda)$ is an element of D^* . Thus $\psi_0(x, \lambda)$ defines a vector-valued function $F(\lambda)$ from the upper half λ -plane into D^* ; *i.e.*, the image of λ is $\psi_0(x, \lambda)$. Taking a fixed element $f \in X$ and defining $H(\lambda) = R_0(\lambda)f$, $H(\lambda)$ has the same domain of definition as that of $F(\lambda)$ and takes its values in D_0 .

Our immediate object is to show that $F(\lambda)$ is holomorphic for $I(\lambda) > 0$. That is to say, for every $f \in X$, $(F(\lambda), f)$ is a holomorphic complex-valued function of λ for $I(\lambda) > 0$. Our proof will depend upon the fact that $H(\lambda)$ is holomorphic.

In the literature [4] it is shown that

$$R_0(\lambda) - R_0(\lambda') = (\lambda' - \lambda)R_0(\lambda)R_0(\lambda')$$

for λ and λ' in $\rho(L_0)$. The equation (4.1.1) in [4] is known as the first resolvent equation and it implies that

$$\left\| \frac{R_0(\lambda)f - R_0(\lambda')f}{\lambda - \lambda'} R_0^2(\lambda)f \right\| \rightarrow 0 \quad \text{as } \lambda' \rightarrow \lambda,$$

i.e., the strong derivative of $H(\lambda)$ exists and is equal to $R^2(\lambda)f$. From this we have, *a fortiori*, that $d(H(\lambda), f_0)/d\lambda = (dH(\lambda)/d\lambda, f_0)$ for arbitrary $f_0 \in X$. That is

$$\frac{d}{d\lambda} \int_a^b (R_0(\lambda)f, f_0) dx = \int_a^b (R_0^2(\lambda)f, f_0) dx,$$

which shows that $H(\lambda)$ is holomorphic.

With regard to $F(\lambda)$ we have the following theorem.

Theorem 10. $F(\lambda) = \psi_0(x, \lambda)$ is a holomorphic vector-valued function from the upper half λ -plane ($I(\lambda) > 0$) into D^* .

It will be shown that, for arbitrary $f \in X$, $\int_a^b \psi_0(x, \lambda)f(x) dx$ is analytic for $I(\lambda) > 0$. The crux of the proof is to show that $\|\psi_0(x, \lambda)\|$ is bounded in every compact subset of the upper half λ -plane. Once this fact is established, Theorem 10 follows easily, for by following a familiar argument (Lemma 4, Section 3) one shows that $\int_{a'}^{b'} \psi_0(x, \lambda)f(x) dx$ is analytic for $a < a' < b' < b$, $I(\lambda) > 0$, and $f \in X$. If $\|\psi_0(x, \lambda)\|$ is bounded in any compact set, $I(\lambda) > 0$, it follows, for a fixed $f \in X$, that the set of functions $\int_{a'}^{b'} \psi_0(x, \lambda)f(x) dx$ is bounded with respect to a' , b' . By letting $a' \rightarrow a$, $b' \rightarrow b$ and applying VITALI's theorem one concludes the proof of Theorem 10.

As $\psi_0(x, \lambda) \in D^*$ we have

$$\psi_0(x, \lambda) = R_0(\lambda_0)U(\lambda_0)\psi_0(x, \lambda) + A[\psi_0(x, \lambda), \lambda_0]\psi_0(x, \lambda_0)$$

or

$$\psi_0(x, \lambda) = (\lambda - \lambda_0)R_0(\lambda_0)\psi_0(x, \lambda) - A[\psi_0(x, \lambda), \lambda_0]\psi_0(x, \lambda_0),$$

for $I(\lambda_0) > 0$ and $I(\lambda) > 0$. From this we have

$$||\psi_0(x, \lambda)|| \leq \frac{1}{v_0} |\lambda - \lambda_0| ||\psi_0(x, \lambda)|| + |A[\psi_0(x, \lambda), \lambda_0]| ||\psi_0(x, \lambda_0)||$$

and choosing λ such that $2|\lambda - \lambda_0| < v_0$ we obtain

$$||\psi_0(x, \lambda)|| < 2 |A[\psi_0(x, \lambda), \lambda_0]| ||\psi_0(x, \lambda_0)||.$$

By definition

$$\begin{aligned} A[\psi_0(x, \lambda), \lambda_0] &= \frac{W_b[\psi_0(x, \lambda), \psi_1(x, \lambda_0)]}{w(\lambda_0)} \\ &= \frac{W_b[\psi_1(x, \lambda) - w(\lambda)\phi(x, \lambda), \psi_1(x, \lambda_0)]}{w(\lambda_0)} \\ &= -\frac{W_b[\phi(x, \lambda), \psi_1(x, \lambda_0)]w(\lambda)}{w(\lambda_0)}, \end{aligned}$$

employing $W_b[\psi_1(x, \lambda), \psi_1(x, \lambda_0)] = 0$. Also

$$W_b[\phi(x, \lambda), \psi_1(x, \lambda_0)] = 1 + (\lambda - \lambda_0) \int_r^b \phi(x, \lambda) \psi_1(x, \lambda_0) dx.$$

Therefore, we have that

$$A[\psi_0(x, \lambda), \lambda_0] = -\left[1 + (\lambda - \lambda_0) \int_r^b \phi(x, \lambda) \psi_1(x, \lambda_0) dx\right] \frac{w(\lambda)}{w(\lambda_0)},$$

and that

$$\begin{aligned} |A[\psi_1(x, \lambda), \lambda_0]| \\ \leq \left[1 + |\lambda - \lambda_0| \left(\int_r^b |\phi(x, \lambda)|^2 dx\right)^{\frac{1}{2}} \left(\int_r^b |\psi_1(x, \lambda_0)|^2 dx\right)^{\frac{1}{2}}\right] \frac{|w(\lambda)|}{|w(\lambda_0)|}. \end{aligned}$$

$w(\lambda)$ is analytic for $I(\lambda) > 0$, hence bounded in every compact subset of the upper half λ -plane. In Lemma 3 of Theorem 4 we saw that $\int_r^b |\phi(x, \lambda)|^2 dx$ was bounded on any such set. Thus we have that $A[\psi_0(x, \lambda), \lambda_0]$ is bounded in every compact subset of the upper half λ -plane, and from inequality (43) that the same is true of $||\psi_0(x, \lambda)||$. *Q.e.d.*

Now we construct all possible contractions $L_r \subset K$ such that $\rho(L_r)$ contains at least one point λ_0 , $I(\lambda_0) > 0$.

Let λ_0 be fixed but arbitrary with $I(\lambda_0) > 0$, and let C be an arbitrary linear bounded functional. Define

$$(44) \quad R_1(\lambda_0)f = R_0(\lambda_0)f + C(f)\psi_0(x, \lambda_0), \quad \text{for } f \in X,$$

and

$$(45) \quad D_1 = R_1(\lambda_0)X.$$

By inspection of (44) we see that $R_1(\lambda_0)f$ is a linear bounded transformation defined on all of X with range in D^* .

We also confirm directly that $R_1(\lambda_0)U(\lambda_0)f = f$ for $f \in D_1$. If $f \in D_1$, by definition (45), $f = R_0(\lambda_0)h + C(h)\psi_0(x, \lambda_0)$ for some $h \in X$. $U(\lambda_0)f = h$ and $R_1(\lambda_0)U(\lambda_0)f = R_1(\lambda_0)h = f$. Thus we have, so far, $R_1(\lambda_0)U(\lambda_0)f = f$ for $f \in D_1$ and $U(\lambda_0)R_1(\lambda_0)f = f$ for $f \in X$. Define

$$(46) \quad g(\lambda) = A[\psi_0(x, \lambda), \lambda_0] + C[U(\lambda_0)\psi_0(x, \lambda)], \quad I(\lambda) > 0,$$

$$(47) \quad C(f, \lambda) = -\frac{C[U(\lambda_0)R_0(\lambda)f]}{g(\lambda)} \quad \text{for } g(\lambda) \neq 0 \quad \text{and} \quad I(\lambda) > 0,$$

and

$$(48) \quad R_1(\lambda)f = R_0(\lambda)f + C(f, \lambda)\psi_0(x, \lambda).$$

We have shown earlier in this section that $A[\psi_0(x, \lambda), \lambda_0]$ and $C[U(\lambda_0)\psi_0(x, \lambda)] = (\lambda - \lambda_0) C[\psi_0(x, \lambda)]$ are analytic for $I(\lambda) > 0$; therefore, the zeros of $g(\lambda)$ are discrete. Except for these possible zeros $C(f, \lambda)$ is a linear bounded functional for each λ , $I(\lambda) > 0$. From this last, and from definition (48), it follows that $R_1(\lambda)$ is a linear bounded transformation defined on all of X with range in D^* provided $I(\lambda) > 0$ and $g(\lambda) \neq 0$.

We now prove that $R_1(\lambda)$ is the resolvent for that contraction $L_1 \subset K$ having domain D_1 as defined by (45).

Theorem 11. *Let $L_1f = Kf$ for $f \in D_1$. Then $R(L_1, \lambda) = R_1(\lambda)$, $R_1(\lambda)X = D_1$, $U(\lambda)D_1 = X$, and $\rho(L_1)$ contains all of the upper half λ -plane except for the zeros of $g(\lambda)$ and these zeros are in the point spectrum of L_1 . Thus for each λ_0 , $I(\lambda_0) > 0$, and each linear bounded functional C there exists a contraction of K whose resolvent set contains the upper half plane except possibly a discrete set.*

Lemma 1. $D_1 = \{f | f \in D^*, A[f, \lambda] + C[U(\lambda)f, \lambda] = 0\}$ where λ is arbitrary, $I(\lambda) > 0$, and $g(\lambda) \neq 0$.

We first prove that the lemma is true for $\lambda = \lambda_0$. If $f \in D^*$, then

$$\begin{aligned} R_1(\lambda_0)U(\lambda_0)Uf &= R_0(\lambda_0)U(\lambda_0)f + C[U(\lambda_0)f]\psi_0(x, \lambda_0) \\ &= f + \{A[f, \lambda_0] + C[U(\lambda_0)f]\}\psi_0(x, \lambda_0). \end{aligned}$$

From this last and (45) we obtain the lemma in the special case when $\lambda = \lambda_0$.

If $\lambda \neq \lambda_0$ we have, for any $f \in D^*$,

$$\begin{aligned} C[U(\lambda)f, \lambda] &= -\frac{C}{g(\lambda)} [U(\lambda_0)R_0(\lambda)U(\lambda)f] \\ &= -\frac{1}{g(\lambda)} \{C[U(\lambda_0)f] - A[f, \lambda]C[U(\lambda_0)\psi_0(x, \lambda)]\}. \end{aligned}$$

Therefore,

$$\begin{aligned} A[f, \lambda] + C[U(\lambda)f, \lambda] &= \frac{1}{g(\lambda)} \{A[f, \lambda]g(\lambda) - C[U(\lambda_0)f] - A[f, \lambda]C[U(\lambda)\psi_0(x, \lambda)]\}. \end{aligned}$$

Replacing $g(\lambda)$ in the first term by $A[\psi_0(x, \lambda), \lambda p] + C[U(\lambda_0)\psi_0(x, \lambda)]$, we have that

$$A[f, \lambda] + C[U(\lambda)f, \lambda] = \frac{1}{g(\lambda)} \{A[f, \lambda]A[\psi_0(x, \lambda), \lambda_0] - C[U(\lambda_0)f]\}.$$

As $f = R(\lambda_0)U(\lambda_0)f - A[f, \lambda_0]\psi_0(x, \lambda_0)$, by taking Wronskians of both sides of this equality and by the definition (36) of $A[f, \lambda]$ we have $A[f, \lambda] = -A[f, \lambda_0] \cdot A[\psi_0(x, \lambda_0), \lambda]$ from which

$$\begin{aligned} A[f, \lambda] + C[U(\lambda)f, \lambda] \\ = -\frac{1}{g(\lambda)} \{A[f, \lambda_0]A[\psi_0(x, \lambda), \lambda_0]A[\psi_0(x, \lambda_0), \lambda] + C[U(\lambda_0)f]\}. \end{aligned}$$

From the equality

$$\psi_0(x, \lambda) = R_0(\lambda)U(\lambda)\psi_0(x, \lambda) - A[\psi_0(x, \lambda), \lambda_0]\psi_0(x, \lambda_0);$$

then since $A[\psi_0(x, \lambda), \lambda] = -1$, after taking Wronskians of both sides of the above, $A[\psi_0(x, \lambda), \lambda_0] \cdot A[\psi_0(x, \lambda_0), \lambda] = 1$. It follows at once that

$$(49) \quad A[f, \lambda] + C[U(\lambda)f, \lambda] = -\frac{1}{g(\lambda)} \{A[f, \lambda_0] + C[U(\lambda_0)f]\}$$

for all $f \in D^*$, $I(\lambda) > 0$, $g(\lambda) \neq 0$.

From the fact that the lemma holds for $\lambda = \lambda_0$ we obtain the lemma for the other values of λ by (49). *Q.e.d.*

We have shown that the definition of D_1 is independent of λ .

With the aid of Lemma 1 the following lemma is immediate.

Lemma 2. $R_1(\lambda)X \subset D_1$ for $I(\lambda) > 0$ and $g(\lambda) \neq 0$.

$R_1(\lambda)f = R_0(\lambda)f + C(f, \lambda)\psi_0(x, \lambda)$ by definition (44). Thus $A[R_1(\lambda)f, \lambda] = C(f, \lambda)A[\psi_0(x, \lambda), \lambda] = -C(f, \lambda)$ as $R_0(\lambda)f \in D_0$ and $A[\psi_0(x, \lambda), \lambda] = -1$. Also, $C[U(\lambda)R_1(\lambda)f, \lambda] = C(f, \lambda)$. Therefore, $C[U(\lambda)R_1(\lambda)f] + A[R_1(\lambda)f, \lambda] = 0$ and from Lemma 1 we obtain Lemma 2.

A second consequence of Lemma 1 is the following lemma.

Lemma 3. $R_1(\lambda)U(\lambda)f = f$ for $f \in D_1$, $I(\lambda) > 0$, and $g(\lambda) \neq 0$.

For every $f \in D^*$ we have

$$\begin{aligned} R_1(\lambda)U(\lambda)f &= R_0(\lambda)U(\lambda)f + C[U(\lambda)f, \lambda]\psi_0(x, \lambda) \\ &= f + \{A[f, \lambda] + C[U(\lambda)f, \lambda]\}\psi_0(x, \lambda), \end{aligned}$$

i.e.,

$$(50) \quad R_1(\lambda)U(\lambda)f = f + \{A[f, \lambda] + C[U(\lambda)f, \lambda]\}\psi_0(x, \lambda),$$

for $f \in D^*$, $I(\lambda) > 0$, and $g(\lambda) \neq 0$. This last and Lemma 1 yield Lemma 3.

If $g(\lambda_1) = 0$ then definition (46) and Lemma 1 demonstrate that $\psi_0(x, \lambda_1) \subset D_1$

so that λ_1 is in the point spectrum of L_1 . Thus, with the aid of the three lemmas, it is easy to verify Theorem 11.

We next show that for a fixed λ_0 different choices of the linear bounded functional C yield distinct contractions of K . In order to see this let $C_0(f) = \int_a^b f f_0 dx$ and $C_1(f) = \int_a^b f f_1 dx$ and let λ_0 be fixed. Let $L_{1,0}$ and $L_{1,1}$ be the contractions associated with C_0 and C_1 respectively; let their domains be $D_{1,0}$ and $D_{1,1}$. If $f \in D_{1,0} \cap D_{1,1}$ then we have from Theorem 10 that $A[f, \lambda_0] + C_0[U(\lambda_0)f] = 0$ and $A[f, \lambda_0] + C_1[U(\lambda_0)f] = 0$, i.e., $\int_a^b [U(\lambda_0)f] (f_0 - f_1) dx = 0$. If $D_{1,0} = D_{1,1}$ we have $U(\lambda_0) D_{1,1} \perp (\tilde{f}_0 - \tilde{f}_1)$ and, as $U(\lambda_0) D_{1,1} = X$, $X \perp (\tilde{f}_0 - \tilde{f}_1)$. Therefore $f_0 = f_1$ as was claimed. In particular we obtain that $D_1 = D_0$ if and only if C is the zero functional.

Next it will be proved that given an $L_r \subset K$ and a point $\lambda_0 \in \rho(L_r)$, $I(\lambda_0) > 0$, then L_r can be constructed by the methods of Theorem 11. To prove this we notice that $R(L_r, \lambda_0)f - R_0(\lambda_0)f$ is in D^* for all f in X and $U(\lambda_0)[R(L_r, \lambda_0)f - R_0(\lambda_0)f] = 0$. Therefore, there exists a number $C(f)$ such that $R(L_r, \lambda_0)f - R_0(\lambda_0)f = C(f)\psi_0(x, \lambda_0)$. Furthermore, $C(f)$ is necessarily linear and bounded. We proceed, using $C(f)$ as the arbitrary linear bounded functional of Theorem 11, and obtain again the contraction $L_1 \subset K$ with domain D_1 . We have $R_1(\lambda_0)f = R(L_r, \lambda_0)f$ for all $f \in X$ by the definition of R_1 . Also $D_1 = R(\lambda)X$ for $I(\lambda) > 0$, $g(\lambda) \neq 0$ since $g(\lambda_0) = -1$, $g(\lambda) \equiv 0$ is not possible. Since the resolvent set is open, $R(L_r, \lambda)$ and $R_1(\lambda)$ exist in a neighborhood $N(\lambda_0)$ of λ_0 . By the definition of the resolvent, $R(L_r, \lambda_0)X = D_1$. Therefore we have $R(L_r, \lambda)f - R_1(\lambda)f = \gamma\psi_0(x, \lambda)$ and, as there are no solutions of $U(\lambda)y = 0$ in D_1 if $\lambda \in N(\lambda_0)$, $\gamma = 0$, $R(L_r, \lambda)f = R_1(\lambda)f$ for $\lambda \in N(\lambda_0)$. As $R_1(\lambda)f$ is holomorphic for all λ such that $I(\lambda) > 0$, $g(\lambda) \neq 0$, and since $R(L_r, \lambda)$ is holomorphic in $N(\lambda_0)$, it follows that this last equality holds for all λ , $I(\lambda) > 0$ and $g(\lambda) \neq 0$.

The above observations are summarized by the following corollary to Theorem 11.

Corollary 1. *For a fixed λ_0 distinct choices of functionals C yield distinct contractions $L_r \subset K$. Furthermore, all contractions $L_r \subset K$ with a resolvent set containing a point of the upper half plane are obtained by the methods delineated during the proof of Lemma 1 of Theorem 11.*

While it is true that a fixed λ_0 , $I(\lambda_0) > 0$, and a fixed $f_0 \in X$ fully determine a unique contraction $L_1 \subset K$, it is also true that for any other value $\lambda_1 \in \rho(L_1)$, $I(\lambda) > 0$, a unique element $f_1 \in X$ can be found such that the pair λ_1, f_1 also determines L_1 . In order to see that this is the case we consult definition (46) and Lemma 1. The first of these states that $g(\lambda) = A[\psi_0(x, \lambda), \lambda_0] + C[U(\lambda_0)\psi_0(x, \lambda)]$. The lemma states that $f \in D_1$ if and only if $A[f, \lambda_0] + C[U(\lambda_0)f_0] = 0$. Thus if $g(\lambda_1) = 0$ we have from the lemma that $\psi_0(x, \lambda_1) \in D_1$, contradicting the assumption that $\lambda_1 \in \rho(L_1)$. Therefore, if $I(\lambda_1) > 0$ and $\lambda_1 \in \rho(L_1)$ we have $g(\lambda_1) \neq 0$.

If $I(\lambda_1) > 0$ and $\lambda_1 \in \rho(L_1)$ then

$$R_1(\lambda_1)f = R_0(\lambda_1)f + C(f, \lambda_1)\psi_0(x, \lambda_1)$$

where

$$\begin{aligned} C(f, \lambda_1) &= \frac{1}{g(\lambda_1)} \int_a^b [U(\lambda_0)R_0(\lambda_1)f]f_0 \, dx \\ &= \frac{1}{g(\lambda_1)} \left\{ \int_a^b ff_0 \, dx + (\lambda_1 - \lambda_0) \int_a^b [R_0(\lambda_1)f] \cdot f_0 \, dx \right\}. \end{aligned}$$

Now, $\int_a^b f(R_0(\lambda)g) \, dx = \int_a^b gR_0(\lambda)f \, dx$ for f and g in X and $I(\lambda) > 0$, as one easily proves by first letting g be a function such that g and g' are both zero in a neighborhood of both a and b , and then, employing the fact that the set of such functions is dense in X , the proof follows for arbitrary $g \in X$. This last enables us to write

$$\begin{aligned} C(f, \lambda) &= \frac{1}{g(\lambda_1)} \int_a^b f_0f \, dx + (\lambda_1 - \lambda_0) \int_a^b fR_0(\lambda_1)f_0 \, dx \\ &= \frac{1}{g(\lambda_1)} \int_a^b f[U(\lambda_0)R_0(\lambda_1)f_0] \, dx \int_a^b f_0f_1 \, dx \end{aligned}$$

where $f_1 = g(\lambda_1)^{-1}U(\lambda_0)R_0(\lambda_1)f_0$. We know that $R(\lambda_1)X = D_1$. Suppose we consider the domain determined by λ_1 and f_1 . The resolvent for this domain, call it D_2 , is given by $R_2(\lambda_1)f = R_0(\lambda_1) + C_2(f)\psi_0(x, \lambda_1)$, where $C_2(f) = \int_a^b ff_1 \, dx = C(f, \lambda_1)$ and, of course, $R_2(\lambda_1)X = D_2$. However, by comparison of $R_2(\lambda_1)$ with $R_1(\lambda_1)$ we see that $R_1(\lambda_1)$ and $R_2(\lambda_1)$ are equal. Thus $D_1 = D_2$ and the pair λ_0, f_0 and the pair λ_1, f_1 determine the same contraction of K .

Furthermore, for fixed λ_1 , the function $f_1(x)$ is unique by virtue of Corollary 1. Also, by direct computation, if $f_0 \in D_0$ then $f_1 \in D_0$ and $U(\lambda_1)f_1 = g(\lambda_1)^{-1}U(\lambda_0)f_0$.

We summarize the above with a second corollary.

Corollary 2. *Let L_1 be determined by λ_0 and f_0 . Let $\lambda_1 \in \rho(L_1)$ and $I(\lambda_1) > 0$. Then L_1 is also determined by λ_1 and $f_1 = g(\lambda_1)^{-1}U(\lambda_0)R_0(\lambda_1)f_0$. Moreover, if $f_0 \in D_0$ then $f_1 \in D_0$ and $U(\lambda_1)f = g(\lambda_1)^{-1}U(\lambda_0)f_0$.*

We turn now to a parallel discussion of the remaining cases.

8. Generation of Arbitrary Contractions. Cases (II, II), (II, III), and (III, III).

The object of this section is to confirm that a theory similar to the one expounded in the previous section holds for the cases (II, II), (II, III), and (III, III). These three cases have two important features in common. In each of them ψ_0 and ψ_1 are in $L^2[a, b]$ and in each of them $w(\lambda)$ is a meromorphic function of λ (see Theorem 4).

We will be able to show that $\psi_0(x, \lambda)$ and $\psi_1(x, \lambda)$ can be regarded as meromorphic vector-valued functions from the λ -plane into D^* , and also that every contraction $L_r \subset K$ with $\rho(L_r) \neq 0$ has a pure point spectrum and that this spectrum is discrete.

The important difference between this and the preceding section is that we escape, in part, the restriction that λ lie in the upper half complex plane, and

that questions of analyticity are easier to answer. Otherwise, the content herein is similar to that of the previous section.

Let C_0 and C be arbitrary linear bounded functionals and let λ_0 be arbitrary, $\lambda_0 \in \rho(L_0)$. Define

$$(51) \quad R_1(\lambda_0)f = R_0(\lambda_0)f + C_0(f)\psi_0(x, \lambda_0) + C_1(f)\psi_1(x, \lambda_0)$$

and define

$$(52) \quad D_1 = R_1(\lambda_0)X.$$

It is clear that $R_1(\lambda_0)$ is a linear bounded transformation defined on all of X and has range in D^* , and that $U(\lambda_0)R_1(\lambda_0)f = f$ for $f \in X$ and $R_1(\lambda_0)U(\lambda_0)f = f$ for $f \in D_1$.

If we let $L_1f = Kf$ for $f \in D_1$ then all that has been shown so far is that $\lambda_0 \in \rho(L_1)$. As in Section 7, we now show that $\rho(L_1)$ contains all of the upper half λ -plane except possibly a discrete set. For convenience, definition (36) is repeated here

$$(53) \quad A[f, \lambda] = \frac{W_b[f, \psi_1(x, \lambda)]}{w(\lambda)}, \quad B[f, \lambda] = \frac{W_a[\psi_0(x, \lambda), f]}{w(\lambda)}.$$

Define

$$(54) \quad \begin{aligned} \Delta(\lambda) = & \{A[\psi_0(x, \lambda), \lambda_0] + C_0[U(\lambda_0)\psi_0(x, \lambda)]\} \\ & \cdot \{B[\psi_1(x, \lambda), \lambda_0] + C_1[U(\lambda_0), \psi_1(x, \lambda)]\} \\ & - C_0[U(\lambda_0)\psi_1(x, \lambda)]C_1[U(\lambda_0)\psi_0(x, \lambda)] \end{aligned}$$

for $\lambda \in \rho(L_0)$. From Lemma 2 of Theorem 9 and the formula

$$A[\psi_0(x, \lambda), \lambda_0] = - \left[1 + (\lambda - \lambda_0) \int_r^b \phi(x, \lambda) \psi_1(x, \lambda_0) dx \right] \frac{w(\lambda)}{w(x)}$$

we see that $\Delta(\lambda)$ is a meromorphic function of λ and that it is analytic in the upper half λ -plane. In particular, its zeros and poles constitute a discrete set. Define

$$(55) \quad g_0(\lambda) = A[\psi_0(x, \lambda), \lambda_0] + C_0[U(\lambda_0)\psi_0(x, \lambda)]$$

and

$$(56) \quad g_1(\lambda) = B[\psi_1(x, \lambda), \lambda_0] + C_1[U(\lambda_0)\psi_1(x, \lambda)].$$

The definition of g_0 here is identical with the definition (46); the definition of $g_1(\lambda)$ bears the same relation to ψ_1 , C_1 , and B that $g_0(\lambda)$ does to ψ_0 , C_0 , and A . With these last definitions we may restate (52) as

$$(57) \quad \Delta(\lambda) = g_0(\lambda) \cdot g_1(\lambda) - C_0[U(\lambda_0)\psi_1(x, \lambda)] \cdot C_1[U(\lambda_0)\psi_0(x, \lambda)].$$

We define two linear bounded functionals, $C_0(f, \lambda)$ and $C_1(f, \lambda)$, as the solu-

tions of the two linear equations written below

$$(58) \quad g_0(\lambda)C_0(f, \lambda) + C_0[U(\lambda_0)\psi_1(x, \lambda)]C_1(f, \lambda) = -C_0[U(\lambda_0)R_0(\lambda)f],$$

$$(59) \quad C_1[U(\lambda_0)\psi_0(x, \lambda)]C_0(f, \lambda) + g_1(\lambda)C_1(f, \lambda) = -C_1[U(\lambda_0)R_0(\lambda)f].$$

The determinant of coefficients is seen to be $\Delta(\lambda)$, hence $C_0(f, \lambda)$ and $C_1(f, \lambda)$ are well defined for all $f \in X$ provided that $\Delta(\lambda) \neq 0$ and $\lambda \in \rho(L_0)$. Since $\sigma(L_0)$ is discrete (see Theorem 9) and lies in the lower half λ -plane, the functionals $C_0(f, \lambda)$ and $C_1(f, \lambda)$ are well defined for all $f \in X$ and all λ except possibly for a discrete set. Moreover, the only points in the upper half λ -plane for which the definition fails are the zeros of $\Delta(\lambda)$. Excluding these exceptional values of λ we solve equations (58) and (59) for $C_0(f, \lambda)$ and $C_1(f, \lambda)$ obtaining

$$(60) \quad C_0(f, \lambda) = -\frac{1}{\Delta(\lambda)} \{g_1(\lambda)C_0[U(\lambda_0)R_0(\lambda)f] - C_0[U(\lambda_0)\psi_1(x, \lambda)]C_1[U(\lambda_0)R_0(\lambda)f]\}.$$

and

$$(61) \quad C_1(f, \lambda) = \frac{1}{\Delta(\lambda)} \{C_1[U(\lambda_0)\psi_0(x, \lambda)]C_0[U(\lambda_0)R_0(\lambda)f] - g_0(\lambda)C_1[U(\lambda_0)R_0(\lambda)f]\}.$$

Define

$$(62) \quad R_1(\lambda) = R_0(\lambda)f + C_0(f, \lambda)\psi_0(x, \lambda) + C_1(f, \lambda)\psi_1(x, \lambda)$$

for $f \in X$ and $\lambda \in \rho(L_0)$, $\Delta(\lambda) \neq 0$.

As $R_0(\lambda)$ is a linear bounded transformation and as $C_0(f, \lambda)$ and $C_1(f, \lambda)$ are linear bounded functionals, $R_1(\lambda)f$ is a linear bounded transformation defined on all of X and with range in D^* provided $\lambda \in \rho(L_0)$ and $\Delta(\lambda) \neq 0$.

Theorem 12. *Let $L_1f = Kf$ for $f \in D_1$. Then $R_1(\lambda) = R(L_1, \lambda)$, $R_1(\lambda)X = D_1$, and $U(\lambda)D_1 = X$. Furthermore, $\sigma(L_1)$, the spectrum of L_1 , is a discrete point spectrum and the points of $\sigma(L_1)$ are either in $\sigma(L_0)$ or they are zeros of $\Delta(\lambda)$.*

Lemma 1. $D_1 = \{f | f \in D^*, A[f, \lambda] + C_0[U(\lambda)f, \lambda] = 0, B[f, \lambda] + C_1[U(\lambda)f, \lambda] = 0 \text{ for } \lambda \in \rho(L_0), \Delta(\lambda) \neq 0\}.$

We first prove the lemma for the special case $\lambda = \lambda_0$. If $f \in D^*$ then by definition (62) applied to $U(\lambda_0)f$,

$$R_1(\lambda_0)U(\lambda_0)f = R_0(\lambda_0)U(\lambda_0)f + C_0[U(\lambda_0)f]\psi_0(x, \lambda_0) + C_1[U(\lambda_0)f]\psi_1(x, \lambda_0),$$

and as

$$R_0(\lambda_0)U(\lambda_0)f = f + A[f, \lambda_0]\psi_0(x, \lambda_0) + B[f, \lambda_0]\psi_1(x, \lambda_0)$$

we have

$$(63) \quad R_1(\lambda_0)U(\lambda_0)f = f + \{A[f, \lambda_0] + C_0[U(\lambda_0)f]\}\psi_0(x, \lambda_0) + \{B[f, \lambda_0] + C_1[U(\lambda_0)f]\}\psi_1(x, \lambda_0).$$

From this last we see that the lemma is true for the special case $\lambda = \lambda_0$. Consider the expression

$$(64) \quad \Delta(\lambda) \{A[f, \lambda] + C_0[U(\lambda)f, \lambda]\}$$

and

$$(65) \quad \Delta(\lambda) \{B[f, \lambda] + C_1[U(\lambda)f, \lambda]\}$$

where $f \in D^*$, $\Delta(\lambda) \neq 0$, and $\lambda \in \rho(L_0)$. After a tedious computation similar to the one used in proving Lemma 1 of Theorem 11, it can be shown that (64) and (65) are equal to

$$(66) \quad -g_1(\lambda) \{C_0[U(\lambda_0)f] + A[f, \lambda_0]\} + C_0[U(\lambda_0)]\psi_1(x, \lambda) \{C_1[U(\lambda_0)f] + B[f, \lambda_0]\}$$

and

$$(67) \quad -C_1[U(\lambda_0)\psi_0(x, \lambda)] \{C_0[U(\lambda_0)f] + A[f, \lambda_0]\} + g_0(\lambda) \{C_1[U(\lambda_0)f] + B[f, \lambda_0]\}$$

respectively.

Assume that $f \in D_1$. By the hypotheses $\Delta(\lambda) \neq 0$. Since we have shown that the lemma is true for the special case $\lambda = \lambda_0$ each of the four terms in (66) and (67) is equal to zero. Thus from (66) and (67) we have $A[f, \lambda] + C_0[U(\lambda)f, \lambda] = 0$ and $B[f, \lambda] + C_1[U(\lambda)f, \lambda] = 0$ for $\lambda \in \rho(L_0)$, $\Delta(\lambda) \neq 0$.

On the other hand, if $\Delta(\lambda) \neq 0$ and $\lambda \in \rho(L_0)$, and if $A[f, \lambda] + C_0[U(\lambda)f, \lambda] = 0$ and $B[f, \lambda] + C_1[U(\lambda)f, \lambda] = 0$ for $f \in D^*$, then (66) and (67) are equal to zero. Hence (64) and (65) are both equal to zero, i.e., $A[f, \lambda_0] + C_0[U(\lambda_0)f] = 0$ and $B[f, \lambda_0] + C_1[U(\lambda_0)f] = 0$. Since the lemma is true for the special case $\lambda = \lambda_0$, we have $f \in D_1$. This last completes the proof of Lemma 1.

Lemma 2. $R_1(\lambda)X \subset D_1$ for $\lambda \in \rho(L_0)$, $\Delta(\lambda) \neq 0$.

On account of Lemma 1 it is sufficient to show that $A[R_1(\lambda)f, \lambda] + C_0[U(\lambda)R_1(\lambda)f, \lambda] = 0$ and $B[R_1(\lambda)f, \lambda] + C_1[U(\lambda)R_1(\lambda)f, \lambda] = 0$ for $f \in X$. From definition (62) we have

$$R_1(\lambda)f = R_0(\lambda)f + C_0(f, \lambda)\psi_0(x, \lambda) + C_1(f, \lambda)\psi_1(x, \lambda).$$

From this we obtain $U(\lambda)R_1(\lambda)f = f$. And, as $R_0(\lambda)f \in D_0$, $A[\psi_0(x, \lambda), \lambda] = -1$ and $B[\psi_1(x, \lambda), \lambda] = -1$; we have $A[R_1(\lambda)f, \lambda] = -C_0(f, \lambda)$ and $B[R_1(\lambda)f, \lambda] = -C_1(f, \lambda)$. Q.e.d.

Lemma 3. If $\lambda \in \rho(L_0)$ and $\Delta(\lambda) \neq 0$ then $R_1(\lambda)U(\lambda)f = f$ for $f \in D_1$.

$$\begin{aligned} R_1(\lambda)U(\lambda)f &= f + \{A[f, \lambda] + C_0[U(\lambda)f, \lambda]\}\psi_0(x, \lambda) \\ &\quad + \{B[f, \lambda] + C_1[U(\lambda)f, \lambda]\}\psi_1(x, \lambda). \end{aligned}$$

From this and from Lemma 1 the result is immediate.

As we have seen already, $U(\lambda)R_1(\lambda)f = f$ for $f \in D^*$, $\lambda \in \rho(L_0)$, and $\Delta(\lambda) \neq 0$. This with Lemma 2 and Lemma 3 yields

Lemma 4. If $\Delta(\lambda) \neq 0$ and $\lambda \in \rho(L_0)$ then $U(\lambda)R_1(\lambda)f = f$ for $f \in X$.

Because $R_1(\lambda)$ is bounded we have that $R_1(\lambda) = R(L_1, \lambda)$ provided $\Delta(\lambda) \neq 0$ and $\lambda \in \rho(L_0)$.

In Section 6 we saw that $R_0(\lambda)$ is completely continuous. The operator $C_0(f)\psi_0(x, \lambda) + C_1(f)\psi_1(x, \lambda)$ is finite dimensional and is, therefore, completely continuous. Thus $R_1(\lambda_0)$ is the sum of two completely continuous operators, hence it is also completely continuous. As was pointed out in Section 6, this last implies that $\sigma(L_1)$ is a pure point spectrum and that it is a discrete set.

From the preceding paragraphs we know that if $\lambda_1 \in \sigma(L_1)$ then either $\lambda_1 \in \sigma(L_0)$ or $\Delta(\lambda_1) = 0$. In particular, the only points of $\sigma(L_1)$ that lie above the real axis in the λ -plane are zeros of $\Delta(\lambda)$. On the other hand, every zero of $\Delta(\lambda)$ that is not in $\sigma(L_0)$ is a point of $\sigma(L_1)$. In order to verify this we assume that $\Delta(\lambda_1) = 0$ and $\lambda_1 \in \rho(L_0)$. Since $\lambda_1 \in \rho(L_0)$, $\psi_0(x, \lambda_1)$ and $\psi_1(x, \lambda_1)$ are well defined elements of $L^2(a, b)$, the point λ_1 is in $\sigma(L_1)$ provided we can determine numbers α and β such that $\Psi(x, \lambda_1) = \alpha\psi_0(x, \lambda_1) + \beta\psi_1(x, \lambda_1) \in D_1$. According to Lemma 1, $\Psi(x, \lambda_1) \in D_1$ if $A[\Psi(x, \lambda_1), \lambda_0] + C_0[U(\lambda_0)\Psi(x, \lambda_1)] = 0$ and $B[\Psi(x, \lambda_1), \lambda_0] + C_1[U(\lambda_0)\Psi(x, \lambda_1)] = 0$. Since $A[\psi_1(x, \lambda_1), \lambda_0] = 0$ and $B[\psi_0(x, \lambda_1), \lambda_0] = 0$ these two equations become

$$g_0(\lambda_1)\alpha + C_0[U(\lambda_0)\Psi_1(x, \lambda_1)]\beta = 0$$

$$C_1[U(\lambda_0)\psi_0(x, \lambda_1)] + g(\lambda_1)\beta = 0.$$

The determinant of coefficients is $\Delta(\lambda_1) = 0$; hence the required numbers α and β can be found and $\lambda_1 \in \sigma(L_1)$. This last completes the proof of Theorem 12.

As in Cases (I, II) and (II, III) one can prove that for a fixed λ_0 distinct choices of the functionals C_0 and C_1 yield distinct contractions $L_1 \subset K$. The proof is so similar to that of Corollary 1 of Theorem 11 that it will not be presented.

The proof that the contraction L_1 determined by (λ_0, C_0, C_1) is equally well determined, for arbitrary $\lambda_1 \in \rho(L_1)$, by a second pair of functionals C'_0 and C'_1 is an obvious extension of Corollary 2 of Theorem 11. It will not be presented here. The corresponding corollary to Theorem 12 is merely stated below.

Corollary 1. *If $L_1 \subset K$ is the contraction determined by λ_0, C_0 , and C_1 , then L_1 is also determined for arbitrary $\lambda_1 \in \rho(L_1)$ by C'_0, C'_1 , and λ_1 , where $C'_0(f) = \int_a^b f h_0 dx$ and $C'_1(f) = \int_a^b f h_1 dx$ and*

$$h_0 = \frac{U(\lambda_0)R_0(\lambda_1)}{\Delta(\lambda_1)} \{g_1(\lambda_0)f_0 - C_0[U(\lambda_0)\psi_1(x, \lambda_1)]f_1\}$$

and

$$h_1 = \frac{U(\lambda_0)R_0(\lambda_1)}{\Delta(\lambda_1)} \{C_1[U(\lambda_0)\psi_0(x, \lambda_1)]f_0 - g_0(\lambda_1)f_1\}.$$

9. Classification of Initial Condition Domains. Boundary Conditions. In the preceding two sections we saw, except for minor restrictions, that essentially all secondary condition domains $D_r \subset D^*$ could be generated from a well

chosen base domain D_0 . This base domain D_0 corresponded to the situation when $f_0 \equiv 0$ and was independent of the choice of λ_0 , $I(\lambda_0) > 0$. The generated domains D_τ furnished suitable linear secondary conditions for the differential equation $-y'' + (\lambda - q)y = f$ provided $f \in X$, i.e., as long as $\lambda \in \rho(L_\tau)$; then for each $f \in X$ there existed a unique solution $y = R_\tau(\lambda)f \in D_\tau$ such that $\|y\| \leq \|R_\tau(\lambda)\| \|f\|$. In other words, if we are given a property P , then to each $f \in X$ we are assured a unique solution with property P provided the property is that of being in D_τ . The secondary conditions are called linear since D_τ is a linear set.

The base domains D_0 yield a familiar secondary condition, a boundary condition. Membership in D_0 depends only upon one or both end points according to the case. This is by no means typical of the general situation. For example, in case (I, II) or (I, III) a function f is in D_τ if and only if $A[f, \lambda_0] + \int_a^b [U(\lambda_0)f]f_0 dx = 0$. Examination of this last condition shows that membership of f in D_τ ordinarily depends *both* upon the behavior of f and its derivative at $x = b$ and upon the behavior of $U(\lambda_0)f$ (hence upon f) throughout the interval (a, b) . On the other hand, there are situations, other than the one where $f_0 \equiv 0$, that depend only upon the end points of (a, b) . We will refer to such secondary conditions as boundary conditions. In order to give a precise definition of the boundary domains let $D_1(\lambda_0, f_0)$ stand for the secondary condition domain determined by the pair λ_0, f_0 . Let $H = \{f|f \in D^*, f = 0 \text{ in a neighborhood of } b\}$. For any element of a boundary condition domain D_τ , if one alters this element interior to the interval (a, b) in such a fashion that the altered element is in D^* then this new element must remain in D_τ , since membership in such a D_τ depends only upon the behavior of member elements at the end points of (a, b) . With this last observation in mind the plausible definition is that D_τ is a boundary condition domain if and only if $H \subset D_\tau$. This definition is employed henceforth. One notices in passing that $H \subset D_0$.

Theorem 13. *In Cases (I, II) and (I, III) $D_1(\lambda_0, f_0)$ is a boundary condition domain if and only if $f_0 = \alpha\psi_0(x, \lambda_0)$ for some complex number α .*

Let $\psi_0 = \psi_0(x, \lambda_0)$ and let $M_\epsilon = \{f|f \in X, (f, \bar{\psi}_0) = 0, f = 0 \text{ in an } \epsilon\text{-neighborhood of } b\}$. Designate the union of all such sets M_ϵ by M . First we will show that $\bar{M} \subset \overline{U(\lambda_0)H}$. Let $g \in \bar{M}$. There exists a sequence $\{g_n\} \subset M$, $\|g - g_n\| \rightarrow 0$. It follows that $R_0(\lambda_0)g_n \rightarrow R_0(\lambda_0)g$ and by the representation (25) that $R_0(\lambda_0)g_n \in H$. We conclude that $g_n \in U(\lambda_0)H$, $g \in \overline{U(\lambda_0)H}$, and finally that $\bar{M} \subset \overline{U(\lambda_0)H}$.

Next we will show that $X \ominus \bar{M}$, the orthogonal complement of M , is one-dimensional and contains the element ψ_0 . The definition of M implies that $\bar{\psi}_0 \in X \ominus \bar{M}$. It remains to be proved that every element of $X \ominus \bar{M}$ is of the form $\alpha\psi_0$ where α is complex. Let $g \in X \ominus \bar{M}$. Let $g_\epsilon = 0$ in an ϵ -neighborhood of b and let $g_\epsilon = g$ elsewhere in (a, b) . Let $\psi_{0,\epsilon} = 0$ in an ϵ -neighborhood of b and let $\psi_{0,\epsilon} = \psi_0$ elsewhere in (a, b) . Let $p_\epsilon = g_\epsilon - [(g_\epsilon, \psi_{0,\epsilon})/(\psi_{0,\epsilon}, \bar{\psi}_{0,\epsilon})]\bar{\psi}_{0,\epsilon}$. On the one hand, p_ϵ is the sum of two elements of $X \ominus M_\epsilon$ and is, therefore, in $X \ominus M_\epsilon$. On the other hand, it is easy to see, by direct calculation, that $p_\epsilon \in M_\epsilon$. One concludes that $p_\epsilon = 0$ for arbitrary ϵ and it follows that $g = \alpha\psi_0$.

Now suppose for some pair λ_0, f_0 that $D_1(\lambda_0, f_0)$ is a boundary condition domain, i.e., $H \subset D_1(\lambda_0, f_0)$. Since $H \subset D_1(\lambda_0, f_0)$ we have $A[f, \lambda_0] + C[U(\lambda_0)f] = 0$ for all $f \in H$ (Section 7, Theorem 11, Lemma 1). Since $H \subset D_0$ we have that $A[f, \lambda_0] = 0$; therefore $\int_a^b U(\lambda_0)ff_0 dx = 0$ for all $f \in H$. One concludes that $\bar{f}_0 \in X \ominus U(\lambda_0)\bar{H}$. From the first part of the proof $\bar{M} \subset \overline{U(\lambda_0)H}$, hence $X \ominus \overline{U(\lambda_0)H} \subset X \ominus \bar{M}$. Therefore $\bar{f}_0 \in X \ominus U(\lambda_0)H \subset X \ominus \overline{U(\lambda_0)H} \subset X \ominus \bar{M}$ and by the second part of the proof above $f_0 = \alpha\psi_0$ for some complex number α .

The converse has a shorter proof. Let λ_0 and $f_0 = \alpha\psi_0(x, \lambda_0)$ determine the secondary domain $D_1(\lambda_0, f_0)$. As we have just seen $A[f, \lambda_0] = 0$ and $\int_a^b U(\lambda_0)f\psi_0(x, \lambda_0) dx = 0$ for all $f \in H$, hence by the definition of $D_1(\lambda_0, f_0)$ (Theorem 11, Lemma 1) we have that $H \subset D_1(\lambda_0, f_0)$. *Q.e.d.*

By modifying the definition of H to include functions of D^* that are equal to zero in neighborhoods of both a and b and by modifying the definition of M_* appearing in the first part of the previous proof to read $M_* = \{f | f \in X, (f, \alpha\psi_0 + \beta\psi_1) = 0, f = 0 \text{ in a neighborhood of } a \text{ and of } b\}$ one obtains a theorem similar to Theorem 13 but pertaining to the remaining cases. This theorem appears below.

Theorem 14. *In Cases (II, II), (II, III), and (III, III) $D_1(\lambda_0, f_0, f_1)$ is a boundary condition domain if and only if there exist complex numbers α, β, γ , and δ such that $f_0 = \alpha\psi_0(x, \lambda_0) + \beta\psi_1(x, \lambda_0)$ and $f_1 = \gamma\psi_0(x, \lambda_0) + \delta\psi_1(x, \lambda_0)$.*

In the previous sections it has been shown that for a fixed λ_0 the secondary condition domains are in one to one correspondence with the elements of the space X or $\{X, X\}$ according as $q(x)$ belongs to Cases (I, II) and (I, III) or to Cases (II, II), (II, III), and (III, III). The last two theorems show that the boundary condition domains correspond to either a one-dimensional subspace of X or a two-dimensional subspace of $\{X, X\}$ depending upon the general category into which $q(x)$ falls. This observation leads one to the conclusion that boundary conditions are rather exceptional examples of suitable secondary conditions for the differential equation $-y'' + [q(x) - \lambda]y = 0$.

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