A Random Model of the Sea Surface Generated by a Hurricane

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1. Introduction. The object of this paper is to construct a random model of the sea surface (fully developed) generated by a hurricane. The storm area is assumed to be fixed. Let the ocean surface at $Re^{i\psi}$ at time t be given by $\eta(Re^{i\psi}, t)$ (a complex-valued coordinate system is used for convenience). The zero point is assumed to be the center of the storm area. It seems reasonable to require that $\eta(Re^{i\psi}, t)$ be a random process stationary in ψ and t . The character of a process stationary with respect to the angular coordinate ψ is discussed in Section 2. Of course, we want the surface to satisfy the linearized equations of motion. A representation of a process $\eta(Re^{i\psi}, t)$ stationary in ψ, t and satisfying the equations of motion is developed in Section 3. At a large distance from the storm center the surface should appear to be a superposition of waves diverging radially from the storm center. There must then be some singularity of the solution, the energy being fed into the sea surface at this singularity. The representation derived in Section 3 does have a singularity but does not appear to be too realistic. Two other models that appear to be more realistic are obtained in Sections 4 and 5. The character of the representations are investigated at a large distance from the storm center. One then obtains a representation that to the first order looks like PIERSON's representation of a storm-generated ocean surface [3]. The analysis indicates that one ought to be able to obtain information about the surface by carrying out a random spectral analysis of the surface at some distance from the storm center.

2. A process stationary with respect to angular displacement. First consider a process $\eta(re^{i\theta})$, $0 \leq \theta < 2\pi$ with r fixed, weakly stationary on the circle $0 \leq \theta < 2\pi$; that is, we are assuming that

$$(1) \quad \begin{aligned} E\eta(re^{i\theta}) &\equiv 0 \\ E\eta(re^{i\theta})\overline{\eta(re^{i\phi})} &= \rho(\theta - \phi), \quad \theta - \phi \bmod (2\pi). \end{aligned}$$

It is clear that $\rho(\theta)$ is finite since

$$\rho(\theta) \leq \rho(0) < \infty.$$

Let

$$(2) \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \rho(\theta) d\theta \geq 0.$$

It is clear that $a_n \geq 0$ since $\rho(\theta)$ is a positive semi-definite function. Note that since $\rho \in L^2$

$$(3) \quad \rho(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

A theorem of KARHUNEN [1] then implies that

$$(4) \quad \eta(re^{i\theta}) = \sum_{n=-\infty}^{\infty} Z_n e^{in\theta} \\ EZ_n \overline{Z_m} = \delta_{nm} a_n.$$

If $\eta(re^{i\theta})$ is real-valued $\rho(\theta) = \rho(-\theta)$ so that $a_n = a_{-n}$ and $Z_n = \overline{Z_{-n}}$.

We are really interested in a process $\eta(re^{i\theta}, t)$ stationary with respect to angular displacement and time t . As before we can write

$$\eta(re^{i\theta}, t) = \sum_{n=-\infty}^{\infty} Z_n(r, t) e^{in\theta}$$

where

$$Z_n(r, t) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \eta(re^{i\theta}, t) d\theta.$$

Let

$$E\eta(re^{i\theta}, t) \overline{\eta(r'e^{i\theta'}, t')} = \rho(r, r'; t, t'; \theta - \theta')$$

We can readily verify that the processes $Z_n(r, t)$ are orthogonal for

$$\begin{aligned} EZ_n(r, t) \overline{Z_m(r', t')} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-in\theta} e^{im\theta'} \rho(r, r'; t, t'; \theta - \theta') d\theta d\theta' \\ &= \frac{1}{4\pi^2 T^2} \int_0^{2\pi T} \int_0^{2\pi T} e^{-in\theta} e^{im\theta'} \rho(r, r'; t, t'; \theta - \theta') d\theta d\theta' \\ &= \frac{1}{2\pi T} \int_0^{2\pi T} e^{i(m-n)\theta'} \frac{1}{2\pi T} \int_{-\theta'}^{2\pi T - \theta'} e^{-inu} \rho(r, r'; t, t'; u) du d\theta' \rightarrow 0 \end{aligned}$$

as $T \rightarrow \infty$ if $n \neq m$. Thus

$$EZ_n(r, t) \overline{Z_m(r', t')} \equiv 0$$

if $n \neq m$. Note that the processes $Z_n(r, t)$ are jointly stationary in time.

3. The equations of motion. We now look for a process $\eta(re^{i\theta}, t)$ stationary in θ and t that satisfies the equations of motion. Assume that the ocean depth

is h units. The linearized problem is characterized by a potential $\Phi(x, y, z, t) = \Phi(re^{i\theta}, z, t)$ satisfying the potential equation (see LAMB [2])

$$(5) \quad \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

and the boundary conditions

$$(6) \quad \begin{aligned} \Phi_z &= 0 & \text{at } z &= -h \\ \eta &= \frac{1}{g} \Phi_t & \text{at } z &= 0 \\ \eta_t &= -\Phi_z & \text{at } z &= 0. \end{aligned}$$

The function $\eta = \eta(re^{i\theta}, t)$ is the ocean surface at time t and g is the gravitational constant. The pressure p can readily be obtained from the potential function by using the equation

$$(7) \quad \frac{p}{\rho} = \Phi_t - gz$$

where ρ is the water density. The approach we use in solving the problem is that of separation of variables. Since we are looking for a process η stationary in θ and t it seems reasonable to require that Φ be stationary in θ, t . The potential function Φ (which is a random process) can be written

$$(8) \quad \Phi(x, y, z, t) = \Phi(re^{i\theta}, z, t) = \sum_{n=-\infty}^{\infty} e^{in\theta} Z_n(r, z, t)$$

where the processes $Z_n(r, z, t)$, $n = \dots, -1, 0, 1, \dots$, are orthogonal to each other and stationary in t . Rewrite the potential equation in terms of the variables r, θ, z

$$(9) \quad \Delta\Phi = \frac{1}{r} \left\{ \frac{\partial}{\partial r} (r\Phi_r) + \frac{1}{r} \Phi_{\theta\theta} \right\} + \Phi_{zz} = 0.$$

On integrating the expression $e^{-in\theta} \Delta\Phi$ from 0 to 2π with respect to θ one sees that $Z_n(r, z, t)$ must satisfy the equation

$$(10) \quad \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} Z_n(r, z, t) \right) - \frac{n^2}{r^2} Z_n(r, z, t) + \frac{\partial^2}{\partial z^2} Z_n(r, z, t) = 0.$$

Let us first look for solutions of the above equation that are of the product form $A(r)B(z)C(t)$ with period T in t . Then clearly

$$\frac{1}{A} \frac{1}{r} \left[\frac{d}{dr} \left(r \frac{dA}{dr} \right) - \frac{n^2}{r} A \right] = -\lambda = -\frac{1}{B} \frac{d^2 B}{dz^2}.$$

Since $C(t)$ has period T we can assume it is of the form

$$C(t) = e^{-2\pi i t/T}.$$

The product solutions are required to satisfy the boundary conditions. The random solutions desired will be obtained in terms of these deterministic product solutions. The first boundary condition implies that

$$\frac{dB(z)}{dz} = 0 \quad \text{at } z = -h.$$

The last two boundary conditions imply that

$$\begin{aligned} \eta_{t|z=0} &= -\frac{4\pi^2}{T^2 g} A(r)B(0)e^{-2\pi i t/T} \\ &= -\Phi_{z|z=0} = -A(r)B'(0)e^{-2\pi i t/T} \end{aligned}$$

or

$$\frac{4\pi^2}{T^2 g} B(0) = B'(0).$$

The equation

$$\frac{d^2 B}{dz^2} - \lambda B = 0$$

coupled with the boundary condition $B'(-h) = 0$ indicates that

$$B(z) = K \cosh \{ \sqrt{\lambda} (z + h) \}.$$

The last two boundary conditions imply that

$$\frac{4\pi^2}{T^2 g} \cosh \{ \sqrt{\lambda} h \} = \sqrt{\lambda} \sinh \{ \sqrt{\lambda} h \}$$

or that

$$(11) \quad \sqrt{\lambda} \tanh \{ \sqrt{\lambda} h \} = \frac{4\pi^2}{T^2 g}.$$

Note that if $\sqrt{\lambda} h$ is small, this equation can be rewritten in the approximate form

$$(12) \quad \lambda h = \frac{4\pi^2}{T^2 g},$$

while if h is large it becomes

$$(13) \quad \sqrt{\lambda} = \frac{4\pi^2}{T^2 g}.$$

The case of large depth is of especial interest to us.

The function $A(r)$ satisfies the equation

$$\frac{d^2 A}{dr^2} + \frac{1}{r} \frac{dA}{dr} + \left(\lambda - \frac{n^2}{r^2} \right) A = 0.$$

Our aim is to build up a random model of the sea surface generated by a hurricane. At an appreciable distance from the storm area the waves generated should diverge from the storm center. For this reason it is intuitively reasonable to write the solution $A(r)$ in terms of Hankel functions

$$A(r) = \alpha H_n^{(1)}(\sqrt{\lambda} r) + \beta H_n^{(2)}(\sqrt{\lambda} r)$$

where $H_n^{(1)}, H_n^{(2)}$ are Hankel functions of the first and second kinds.

A solution of the product form

$$[\alpha H_n^{(1)}(\sqrt{\lambda} r) + \beta H_n^{(2)}(\sqrt{\lambda} r)] \frac{\cosh \{ \sqrt{\lambda} (z + h) \}}{\cosh \{ \sqrt{\lambda} h \}} e^{-2\pi i t / T}$$

where

$$\sqrt{\lambda} \tanh \{ \sqrt{\lambda} h \} = \frac{4\pi^2}{T^2 g}$$

satisfies the potential equation and the boundary conditions. The product solution in the limit as $h \rightarrow \infty$ (infinite depth) becomes

$$\left[\alpha H_n^{(1)}\left(\frac{\mu^2}{g} r\right) + \beta H_n^{(2)}\left(\frac{\mu^2}{g} r\right) \right] e^{\mu^2 z / g} e^{-i \mu t}$$

where $\mu = 2\pi/T$. Here we have made use of the limiting relationship

$$\sqrt{\lambda} = \frac{4\pi^2}{T^2 g}.$$

We now build the desired random potential function out of these product solutions. The potential function $\Phi(re^{i\theta}, z, t)$ has the form

$$(14) \quad \begin{aligned} \Phi(re^{i\theta}, z, t) &= \sum_{n=-\infty}^{\infty} e^{in\theta} Z_n(r, z, t) \\ &= \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{-\infty}^{\infty} e^{-i\mu t} e^{\mu^2 z / g} \left[H_n^{(1)}\left(\frac{\mu^2}{g} r\right) dU_n^{(1)}(\mu) + H_n^{(2)}\left(\frac{\mu^2}{g} r\right) dU_n^{(2)}(\mu) \right]. \end{aligned}$$

The processes $Z_n(r, z, t)$ are orthogonal and stationary in t . The orthogonality of the Z_n processes implies that the processes

$$(U_n^{(1)}(\mu), U_n^{(2)}(\mu)) \quad n = \dots, -1, 0, 1, \dots$$

are orthogonal to each other. The stationary character of the Z_n processes implies that the U_n processes have orthogonal increments, that is

$$(15) \quad E dU_n^{(1)}(\mu) \overline{dU_n^{(1)}(\lambda)} = E dU_n^{(1)}(\mu) dU_n^{(2)}(\lambda) = E dU_n^{(2)}(\mu) \overline{dU_n^{(2)}(\lambda)} = 0.$$

We can now write down the representation of the sea surface making use of the equation

$$\eta = \frac{1}{g} \Phi_{t|z=0}.$$

The sea surface is given by

$$(16) \quad \eta(re^{i\theta}, t) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{-\infty}^{\infty} e^{-i\mu t} \left[H_n^{(1)}\left(\frac{\mu^2}{g}r\right) dV_n^{(1)}(\mu) + H_n^{(2)}\left(\frac{\mu^2}{g}r\right) dV_n^{(2)}(\mu) \right]$$

where

$$(17) \quad dV_n^{(j)}(\mu) = -i\mu dU_n^{(j)}(\mu), \quad j = 1, 2.$$

In a hurricane the wave patterns move counterclockwise. We must therefore have

$$(18) \quad dV_n^{(j)}(\mu) = \begin{cases} 0 & \text{if } n, -\mu > 0 \\ 0 & \text{if } n, -\mu < 0. \end{cases}$$

The condition that the process η be real-valued implies that the coefficient of $e^{in\theta}$ is the complex conjugate of the coefficient of $e^{-in\theta}$. This leads to the following condition on the processes $V_n^{(j)}(\mu)$, $j = 1, 2$,

$$dV_n^{(1)}(\mu) = (-1)^n \overline{dV_{-n}^{(2)}(-\mu)}.$$

When $n = 0$ the condition becomes

$$(19) \quad dV_0^{(1)}(\mu) = \overline{dV_0^{(2)}(-\mu)}.$$

At a large distance from the storm center the sea surface appears to be a superposition of waves moving radially outwards for

$$(20) \quad e^{-i\mu t} H_n^{(1)}\left(\frac{\mu^2}{g}r\right) \sim \left(\frac{2}{\pi r}\right)^{\frac{1}{2}} e^{i\mu^2 r/g} e^{-i\mu t}$$

and only nonnegative values for μ are of any importance. The same result is obtained on looking at the terms involving Hankel functions of the second kind. The model we have constructed here has an unpleasant feature in the superposition of singularities at the origin $r = 0$. However, more reasonable models will be constructed in the following section on the basis of the discussion given above.

4. First model of the sea surface. As a first attempt at a more realistic model consider a continuum of logarithmic sources on the circle of radius ρ about the center of the whole phenomenon. This suggests a representation of the form

$$(21) \quad \eta(Re^{i\psi}, t) = \int_{-\infty}^{\infty} \int_0^{2\pi} e^{-i\mu t} H_0\left(\frac{\mu^2}{g}\right) |Re^{i\psi} - \rho e^{i\phi}|$$

where $U(\mu, \phi)$ is a random weight function, and

$$(22) \quad H_0\left(\frac{\mu^2}{g}r\right) = \begin{cases} H_0^{(1)}\left(\frac{\mu^2}{g}r\right) & \text{if } \mu > 0 \\ H_0^{(2)}\left(\frac{\mu^2}{g}r\right) & \text{if } \mu < 0. \end{cases}$$

Let

$$(23) \quad U_i(\mu) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i i \phi} d_\phi U(\mu, \phi)$$

so that

$$(24) \quad U(\mu, \phi) = \sum_{i=-\infty}^{\infty} U_i(\mu) e^{i i \phi}.$$

Using GRAF's addition formula [4] we see that

$$(25) \quad \eta(R e^{i\psi}, t) = \begin{cases} \sum_{n=-\infty}^{\infty} e^{i n \psi} \int_{-\infty}^{\infty} e^{-i \mu t} H_n\left(\frac{\mu^2}{g}R\right) J_n\left(\frac{\mu^2}{g}\rho\right) dU_n(\mu) & \text{if } R \geq \rho \\ \sum_{n=-\infty}^{\infty} e^{i n \psi} \int_{-\infty}^{\infty} e^{-i \mu t} H_n\left(\frac{\mu^2}{g}\rho\right) J_n\left(\frac{\mu^2}{g}R\right) dU_n(\mu) & \text{if } R \leq \rho. \end{cases}$$

The stationary character of η in ψ and t implies that the processes $U_n(\mu)$ are orthogonal to each other and of orthogonal increments. The formal manipulations at the beginning of this section can all be justified under the assumption that the functions $F_n(\mu)$, $F_n(\mu) = E |U_n(\mu)|^2$, are of uniformly bounded variation. In fact the most interesting case is that in which

$$(26) \quad F_n(\mu) \equiv F(\mu).$$

We shall consider this case in some detail because this assumption seems to be appropriate in our context. The assumption $F_n(\mu) \equiv F(\mu)$ implies that the process $Z(\mu, \phi)$ is a process of orthogonal increments in μ, ϕ simultaneously and further that

$$(27) \quad E |d_{\mu, \phi} U(\mu, \phi)|^2 = \frac{1}{2\pi} dF(\mu) d\phi.$$

The model constructed here is obtained by considering a superposition of logarithmic sources with random amplitudes. The condition $F_n(\mu) \equiv F(\mu)$ implies that the sources are uncorrelated.

Let us see what can be said about the random sea surface at an appreciable distance from the center of the hurricane area, that is, when R is large. Let the local coordinate system be centered at $R_0 e^{i\psi_0}$ which is assumed to be close to $R e^{i\psi}$. We use the representation of $\eta(R e^{i\psi}, t)$ in the form

$$\eta(R e^{i\psi}, t) = \int_{-\infty}^{\infty} \int_0^{2\pi} e^{-i \mu t} H_0\left(\frac{\mu^2}{g} |R e^{i\psi} - \rho e^{i\phi}| \right) d_{\mu, \phi} U(\mu, \phi).$$

DIAGRAM 1

We shall obtain an asymptotic representation of $\eta(Re^{i\psi}, t)$ at a large distance from the center of the storm. In Diagram 1 the center of the storm is the center of the left circle. The logarithmic sources are located on the circumference of the circle (which is of radius ρ). The center of the local coordinate system is the center of the right circle. Assume that ρ and r are comparable in magnitude and that they are small compared to R_0 . Now

$$H_0\left(\frac{\mu^2}{g} |Re^{i\psi} - \rho e^{i\phi}|\right) = H_0\left(\frac{\mu^2}{g} \omega\right) \cong \left(\frac{\mu^2}{g} \omega\right)^{-\frac{1}{2}} e^{i\mu^2 \omega/g} e^{-i\pi/4}$$

and

$$\begin{aligned} \omega &= (\omega_0^2 + r^2 + 2r\omega_0 \cos(\theta - \alpha_0))^{\frac{1}{2}} \\ (28) \quad &= \omega_0 \left(1 + \frac{r}{\omega_0} \cos(\theta - \alpha_0) + O\left(\frac{r}{\omega_0}\right)^2\right) \\ &\cong \omega_0 + r \cos(\theta - \alpha_0). \end{aligned}$$

As an approximation we can then write

$$(29) \quad H_0\left(\frac{\mu^2}{g} \omega\right) \cong \left(\frac{\mu^2}{g} R_0\right)^{-\frac{1}{2}} \exp\left\{i \frac{\mu^2}{g} [\omega_0 + r \cos(\theta - \alpha_0)]\right\} e^{-\frac{1}{2}i\pi}.$$

Then

$$(30) \quad \begin{aligned} \eta(Re^{i\psi}, t) &= \int_{-\infty}^{\infty} \int_0^{2\pi} e^{-i\mu t} \left(\frac{\mu^2}{g} R_0\right)^{-\frac{1}{2}} \\ &\quad \cdot \exp\left\{i \frac{\mu^2}{g} [\omega_0 + r \cos(\theta - \alpha_0)]\right\} e^{-\frac{1}{2}i\pi} d_{\mu, \phi} U(\mu, \phi). \end{aligned}$$

Note that this is the representation PIERSON has suggested for storm-generated ocean waves [3]. Of course we should like to look at the form of the spectrum at such a large distance from the storm center in terms of the local angular coordinate α_0 . Now

$$(31) \quad e^{i(\alpha_0 - \psi_0)} = \frac{R_0}{\omega_0} - \frac{\rho}{\omega_0} e^{i(\phi - \psi_0)}$$

so that

$$(32) \quad \begin{aligned} \sin(\alpha_0 - \psi_0) &= -\frac{\rho}{\omega_0} \sin(\phi - \psi_0) \\ \cos(\alpha_0 - \psi_0) &= \frac{R_0}{\omega_0} - \frac{\rho}{\omega_0} \cos(\phi - \psi_0). \end{aligned}$$

A few straightforward computations lead us to the relation

$$(33) \quad \frac{d\phi}{d\alpha_0} = \frac{R_0 \cos(\alpha_0 - \psi_0)}{\omega_0 - R_0 \cos(\alpha_0 - \psi_0)} + 1.$$

It is clear that the variation of $\alpha_0 - \psi_0$ is from $-\sin^{-1}(\rho/R_0)$ to $\sin^{-1}(\rho/R_0)$. Note that in the neighborhood of $\alpha_0 = \psi_0 + \sin^{-1}(\rho/R_0)$ (an analogous result is true in the neighborhood of $\alpha_0 = \psi_0 - \sin^{-1}(\rho/R_0)$)

$$(34) \quad \left| \frac{d\phi}{d\alpha_0} \right| \sim \left| \alpha_0 - \psi_0 - \sin^{-1} \frac{\rho}{R_0} \right|^{-\frac{1}{2}}.$$

Thus the local spectrum at the point $R_0 e^{i\psi_0}$ (an appreciable distance from the storm area) is of the form

$$(35) \quad \begin{aligned} d_{\mu, \alpha_0} F(\mu, \alpha_0) &\cong \left(\frac{\mu^2}{g} R_0 \right)^{-1} \frac{1}{2\pi} \left\{ \left| \frac{R_0 \cos(\alpha_0 - \psi_0)}{\omega_0^{(1)} - R_0 \cos(\alpha_0 - \psi_0)} \right| \right. \\ &\quad \left. + \left| \frac{R_0 \cos(\alpha_0 - \psi_0)}{\omega_0^{(2)} - R_0 \cos(\alpha_0 - \psi_0)} \right| \right\} d\alpha_0 dF(\mu), \end{aligned}$$

$$|\alpha_0 - \psi_0| \leq \sin^{-1} \frac{\rho}{R_0},$$

where $\omega_0^{(1)}, \omega_0^{(2)} = \omega_0^{(1)}(\alpha_0), \omega_0^{(2)}(\alpha_0)$ are the two solutions of the equation $\omega_0 e^{i\alpha_0} = R_0 e^{i\psi_0} - \rho e^{i\phi}$. It is again worthwhile noting that the angular variation of the spectrum is from $\psi_0 - \sin^{-1}(\rho/R_0)$ to $\psi_0 + \sin^{-1}(\rho/R_0)$ with singularities at the boundary points of the region of variation (see (34)).

The sea surface $\eta(R e^{i\psi}, t)$ constructed in this section satisfies the equations of motion in the regions $R < \rho$ and $R > \rho$ but not on the circle $R = \rho$. This is to be expected since the model was obtained by integrating over singularities located on the circle. The surface $\eta(R e^{i\psi}, t)$ is continuous everywhere but the first partial derivatives with respect to the surface coordinates are discontinuous on the circle $R = \rho$.

The standard techniques of statistical spectral analysis can be used at a distance from the storm area to obtain information about the spectrum of the storm. As before, at an appreciable distance from the storm area the sea surface appears to be a superposition of waves diverging radially from the storm center.

5. Another model of the sea surface. Let us now construct a second model of the sea surface. The surface is obtained by integrating over a continuum of

logarithmic sources in the annulus between the two circles $\rho = \rho_0, \rho_1$ where $\rho_0 < \rho_1$. Let

$$(36) \quad \eta(Re^{i\psi}, t) = \int_{-\infty}^{\infty} e^{-i\mu t} \int_{\rho_0}^{\rho_1} \frac{1}{\rho^{\frac{1}{2}}} H_0\left(\frac{\mu^2}{g} |Re^{i\psi} - \rho e^{i\phi}|\right) d_{\mu, \phi, \rho} U(\mu, \phi, \rho)$$

where $U(\mu, \phi, \rho)$ is a function of random orthogonal increments,

$$(37) \quad E |d_{\mu, \phi, \rho} U(\mu, \phi, \rho)|^2 = \begin{cases} \frac{K}{2\pi} dF(\mu) d\phi d\rho & \text{if } \rho_0 < \rho < \rho_1 \\ 0 & \text{otherwise} \end{cases}$$

and K is a constant. As before GRAF's addition formula can be used to obtain the representation

$$(38) \quad \eta(Re^{i\psi}, t) = \begin{cases} \sum_{n=-\infty}^{\infty} e^{in\psi} \int_{-\infty}^{\infty} e^{-i\mu t} H_n\left(\frac{\mu^2}{g} R\right) \int_{\rho_0}^{\rho_1} \frac{1}{\rho^{\frac{1}{2}}} J_n\left(\frac{\mu^2}{g} \rho\right) d_{\mu, \rho} Z_n(\mu, \rho) & \text{if } R > \rho_1 \\ \sum_{n=-\infty}^{\infty} e^{in\psi} \int_{-\infty}^{\infty} e^{-i\mu t} \left\{ H_n\left(\frac{\mu^2}{g} R\right) \int_{\rho_0}^{\rho_1} \frac{1}{\rho^{\frac{1}{2}}} J_n\left(\frac{\mu^2}{g} \rho\right) \right. \\ \quad \left. + J_n\left(\frac{\mu^2}{g} R\right) \int_R^{\rho_1} \frac{1}{\rho^{\frac{1}{2}}} H_n\left(\frac{\mu^2}{g} \rho\right) d_{\mu, \rho} Z_n(\mu, \rho) \right\} & \text{if } \rho_0 < R < \rho_1 \\ \sum_{n=-\infty}^{\infty} e^{in\psi} \int_{-\infty}^{\infty} e^{-i\mu t} J_n\left(\frac{\mu^2}{g} R\right) \int_{\rho_0}^{\rho_1} \frac{1}{\rho^{\frac{1}{2}}} H_n\left(\frac{\mu^2}{g} \rho\right) d_{\mu, \rho} Z_n(\mu, \rho) & \text{if } R < \rho_0 \end{cases}$$

where

$$(39) \quad Z_n(\mu, \rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} d_{\phi} U(\mu, \phi, \rho)$$

is a process with random orthogonal increments and

$$(40) \quad E |d_{\mu, \rho} Z_n(\mu, \rho)|^2 = \begin{cases} K dF(\mu) d\rho & \text{if } \rho_0 < \rho < \rho_1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the representation just given that at an appreciable distance from the storm center the sea surface is a superposition of waves diverging radially from the storm center. Outside the annulus $\rho_0 < R < \rho_1$, $\eta(Re^{i\psi}, t)$ satisfies the equations of motion. In the annulus $\eta(Re^{i\psi}, t)$ and its first partial derivatives with respect to the surface coordinates are continuous. The second partial derivatives do not exist in the annulus.

We are again interested in the process at an appreciable distance from the storm area. Let the center of the local coordinate system again be $R_0 e^{i\psi_0}$ where $R_0 e^{i\psi_0}$ is close to $Re^{i\psi}$. ρ_0, ρ_1, r are assumed to be comparable in magnitude and

small compared to R, R_0 . Using the asymptotic formula for the Hankel function, we find that

$$(41) \quad \eta(Re^{i\psi}, t) \cong \int_{-\infty}^{\infty} \int_0^{2\pi} \int_{\rho_0}^{\rho_1} e^{-i\mu t} \frac{1}{\rho^{\frac{1}{2}}} \left(\frac{\mu^2}{g} R_0 \right)^{-\frac{1}{2}} \\ \cdot \exp \left\{ \frac{i\mu^2}{g} (\omega_0 + r \cos(\theta - \alpha_0)) \right\} e^{-\frac{1}{2}i\pi} d_{\mu, \phi, \rho} U(\mu, \phi, \rho),$$

a representation of the form that PIERSON obtained for storm-generated ocean waves. We again wish to obtain the spectrum obtained locally at $R_0 e^{i\psi_0}$ in terms of the coordinates μ, α_0 (see Diagram 2). Use is made of the fact that $U(\mu, \phi, \rho)$ is a process

DIAGRAM 2

with orthogonal increments. The vector from the storm center to the center of the local coordinate system is $R_0 e^{i\psi_0}$. Consider a line meeting this vector at the center of the local coordinate system and making an angle α_0 with it. We are interested in integrating $1/\rho$ over the line segments in the annulus between the circles. The integration is carried out easily and the spectrum of the storm at $R_0 e^{i\psi_0}$ is found to be

$$(42) \quad d_{\mu, \alpha_0} F(\mu, \alpha_0) \cong \begin{cases} \frac{1}{2\pi} \left(\frac{\mu^2}{g} R_0 \right)^{-1} \log \left[\left(\frac{\rho_1}{R_0 \sin \alpha_0} \right)^2 - 1 \right]^{\frac{1}{2}} \\ \quad + \frac{\rho_1}{R \sin \alpha_0} \Big| d\alpha_0 dF(\mu) \\ \quad \text{if } \sin^{-1} \frac{\rho_0}{R_0} < |\alpha_0 - \psi_0| < \sin^{-1} \frac{\rho_0}{R_0} \\ \frac{1}{2\pi} \left(\frac{\mu^2}{g} R_0 \right)^{-1} \log \left| \frac{(\rho_1^2 - R_0^2 \sin^2 \alpha_0)^{\frac{1}{2}} + \rho_1}{(\rho_0^2 - R_0^2 \sin^2 \alpha_0)^{\frac{1}{2}} + \rho_0} \right| d\alpha_0 dF(\mu) \\ \quad \text{if } |\alpha_1 - \psi_0| < \sin^{-1} \frac{\rho_0}{R_0} \\ 0 \quad \text{otherwise.} \end{cases}$$

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