The Existence and Uniqueness of the Velocity Correlation Derivative in Chandrasekhar's Theory of Turbulence

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1. Introduction. For an isotropically turbulent, incompressible, random fluid-velocity field CHANDRASEKHAR [1] has closed at the fourth moments the infinite sequence of moment equations analogous to the Karman-Howarth equation by assuming that certain correlations of four velocities are bilinear forms in the correlations of pairs of velocities, as if the four relevant velocities had a joint normal distribution. He obtains a fifth-order, non-linear partial differential equation in the velocity correlation $f(r, t) = \langle u_1(\mathbf{x} + \mathbf{r}, \tau + t) u_1(\mathbf{x}, \tau) \rangle_{av}$ (CHANDRASEKHAR [2]). The attempt to specify boundary conditions on f by applying to it Kolmogoroff's similarity principles leads to such difficulties that CHANDRASEKHAR concludes that these similarity principles apply not to f but only to $\chi(r, t) = -\partial f / \partial r$. In the limit of zero viscosity Kolmogoroff's prin-

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ciples imply that $\chi(r, t)$ has the form $r^{-\frac{1}{2}}\sigma(tr^{-\frac{1}{2}})$, and Chandrasekhar's partial differential equation for χ then becomes a third-order ordinary equation for $\sigma(x)$:

$$(1) \quad \frac{1}{6}\sigma = \left(\frac{\sigma''}{44\sigma + 10x\sigma' - 4x^2\sigma''} \right) - x \frac{d}{dx} \left(\frac{\sigma''}{44\sigma + 10x\sigma' - 4x^2\sigma''} \right).$$

In order to determine σ from equation (1) CHANDRASEKHAR [2] introduces and justifies the following boundary conditions:

$$(2) \quad \sigma(x) = \sigma(-x),$$

$$(3) \quad \lim_{x \rightarrow \infty} \sigma(x) = 0,$$

$$(4) \quad \sigma(0) > 0.$$

He further assumes that if the physical solution, σ , of equation (1) is given in an open interval (a, b) then the physical solution in any interval (c, d) containing (a, b) is that one among all mathematical continuations of σ via equation (1) which has the largest number of continuous derivatives in (c, d) . We shall call this the assumption of maximal smoothness. It means, in particular, that σ will be analytic except at the fixed and movable singular points of equation (1) (INCE [3]).

With the usual techniques of numerical integration CHANDRASEKHAR [2] tabulates a one-parameter family of homologous solutions of equation (1) satisfying the above smoothness and boundary conditions. In the present paper we will justify the application of his numerical methods near the singular points of equation (1), thus proving an existence theorem. We will also show that his solutions are the only continuously differentiable ones. If, as CHANDRASEKHAR and the author expect, Taylor's energy dissipation formula (TAYLOR [4]) selects one member of Chandrasekhar's homologous family, we shall have an existence and uniqueness theorem for the function σ . Such a theorem must be among the consequences of Chandrasekhar's theory if that theory describes real turbulence.

2. The general plan of attack. Following CHANDRASEKHAR [2] we define

$$(5) \quad \phi = \frac{9\sigma''}{44\sigma + 10x\sigma' - 4x^2\sigma''}.$$

Then equation (1) becomes

$$(6) \quad \frac{3}{2}\sigma = \phi - x\phi'.$$

Eliminating σ from equation (5) we get

$$(7) \quad x(9 + 4x^2\phi)\phi''' = -(9 - 6x^2\phi)\phi'' + 44x\phi\phi' - 44\phi^2.$$

If at some point x_0 we have $44\sigma + 10x\sigma' - 4x^2\sigma'' = 0$ while σ is analytic, then from equation (5) ϕ either is analytic or has a pole at x_0 . Equation (6) excludes that ϕ have a pole at x_0 , so ϕ is analytic whenever σ is so. The converse being obvious from equation (6), we conclude that ϕ and σ fail to be analytic

at the same places, the isolated fixed and movable singularities of equation (7) or equation (1).

Our problem now is this: How many solutions has equation (7) which are analytic except at the fixed and the possible movable singular points of equation (7) and which provide, via equation (6), continuously differentiable functions σ satisfying equations (1), (2), (3), (4), and the principle of maximal smoothness? In our attack on this problem we shall investigate the general solution of equation (7) near each of the fixed singular points $x = 0$, $9 + 4x^2\phi = 0$, $x = \infty$, and we shall impose the smoothness and boundary conditions for σ on these general solutions. The acceptable solutions will be carried from one singular point to the next by Chandrasekhar's numerical integration of equation (7) since there is no question of the validity of this numerical integration except near the singular points. Chandrasekhar's curves show, incidentally, that we shall not encounter any movable singularities in the functions ϕ which are of interest to us here.

Figure 1 is a reproduction of Chandrasekhar's $\phi(x)$, which must be referred to occasionally to make the following discussion clear.

FIGURE 1. Illustrating the solution for $\phi(x)$: the curve labelled 3 is the locus $9 + 4x^2\phi = 0$ across which a solution for ϕ cannot be continued simply. The part of the solution labelled 1 is obtained by integrating from $x = 0$ outwards, while the part of the solution labelled 2 is obtained by integrating the solution from infinity inwards along a suitably determined linear asymptote.

3. Preliminary mathematical remarks. Besides the existence and uniqueness theorem in the appendix of the present paper, hereafter called T, we need three simple lemmas.

Lemma 1. *Let $\mathfrak{D}_\alpha$ denote the class of continuously differentiable non-vanishing real functions f on the half-open interval $(0, \delta]$ which satisfy*

$$\lim_{x \rightarrow 0+} \frac{xf'(x)}{f(x)} = \alpha.$$

Let $\mathfrak{D}_\alpha^{-1}$ be the class of reciprocals of members of $\mathfrak{D}_\alpha$. Let $\mathfrak{S}_\alpha$ be the class of functions f on $(0, \delta]$ which can be written in the form

$$f(x) = C \exp \int_0^x \frac{\alpha + s(\xi)}{\xi} d\xi$$

where C is a non-zero constant while $s(x)$ is continuous on the closed interval $[0, \delta]$ and $s(0) = 0$. Then

- (i) $\mathfrak{D}_\alpha = \mathfrak{S}_\alpha$.
- (ii) $\mathfrak{D}_\alpha^{-1} = \mathfrak{D}_{-\alpha}$.
- (iii) *If $\alpha > 1$, any member f of $\mathfrak{D}_\alpha$ is continuously differentiable on the closed interval $[0, \delta]$, and $f(0) = f'(0) = 0$.*
- (iv) *If $\alpha > -1$, for any member f of $\mathfrak{D}_\alpha$, $\int_0^\delta |f(\xi)| d\xi$ converges and $\lim_{x \rightarrow 0+} xf(x) = 0$.*

The proof of lemma 1 is simply an examination of the definitions of $\mathfrak{D}_\alpha$ and $\mathfrak{S}_\alpha$.

Lemma 2. *If $f(x) \in \mathfrak{D}_\alpha$ with $\alpha > 1$ and if $g(x)$ is continuous on the closed interval $[0, \delta]$, while*

$$G(x) = f(x) \int_0^x \frac{g(\xi)}{f(\xi)} d\xi,$$

then $G(x)$ is continuously differentiable on the closed interval $[0, \delta]$, $G(0) = 0$, and $G'(0) = g(0)/(1 - \alpha)$.

The proof of lemma 2 is simply an application to the function $G(x)/x$ of part (iii) of lemma 1 and a strong version of L'Hospital's rule.*

* The version of this rule used here is that

$$\lim_{x \rightarrow 0+} \frac{g(x)}{f(x)} = \lim_{x \rightarrow 0+} \frac{g'(x)}{f'(x)}$$

whenever there is a δ such that $f'(x)$ never vanishes on $(0, \delta]$ and either

$$\lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} g(x) = 0$$

or

$$\lim_{x \rightarrow 0+} f(x) = \pm \infty.$$

See PERRON [6] and WIDDER [7] for proofs of this form of the rule.

The same technique, via part (iv) of lemma 1, gives

Lemma 3. *If $f(x) \in \mathfrak{D}_\alpha$ with $\alpha > -1$, and $g(x)$ is continuous on the closed interval $[0, \delta]$, while*

$$G(x) = \frac{1}{f(x)} \int_0^x f(\xi)g(\xi) d\xi,$$

then $G(x)$ is continuously differentiable on $[0, \delta]$, $G(0) = 0$, and $G'(0) = g(0)/(1 + \alpha)$.

Now we are prepared to deal with the singularities of equation (7).

4. The singularity at $x = 0$. Since we demand that σ be continuous everywhere, equation (6) can be integrated to give

$$(8) \quad \phi(x) = C^*x + \frac{3}{2}x \int_x^{+\infty} \frac{\sigma(\xi)}{\xi^2} d\xi$$

where the ambiguous sign must agree with the sign of x . From this equation and lemma 2, ϕ is continuous at $x = 0$ and $\phi(0) = \frac{3}{2}\sigma(0) > 0$.

Therefore, our aim in this section is to characterize those solutions of equation (7) which are analytic near but not necessarily at $x = 0$, and which are continuous and positive at $x = 0$. Because $a^2\phi(ax)$ is a solution of equation (7) whenever $\phi(x)$ is, we need consider only the case $\phi(0) = 1$. (In the future we shall say that the solution $a^2\phi(ax)$ was obtained from $\phi(x)$ by a homology transformation.)

Now we define y_1, y_2, y_3 as follows:

$$(9) \quad \phi = 1 + y_1 - xy_2 + xy_3; \quad \phi' = y_3; \quad x\phi'' = y_2.$$

Then equation (7) and the definitions become

$$(10) \quad \begin{aligned} \frac{dy_1}{dx} &= x^2 \left(\frac{10\phi}{9 + 4x^2\phi} \right) y_2 + x^2 \left(\frac{44\phi}{9 + 4x^2\phi} \right) y_3 - x \left(\frac{44\phi^2}{9 + 4x^2\phi} \right) \\ \frac{dy_2}{dx} &= x \left(\frac{10\phi}{9 + 4x^2\phi} \right) y_2 + x \left(\frac{44\phi}{9 + 4x^2\phi} \right) y_3 - \left(\frac{44\phi^2}{9 + 4x^2\phi} \right) \\ \frac{dy_3}{dx} &= x^{-1}y_2. \end{aligned}$$

Suppose now that $\hat{\mathbf{y}}(x) = (\hat{y}_1(x), \hat{y}_2(x), \hat{y}_3(x))$ is a solution of equations (10) which makes $\hat{\phi}(0) = 0$, and makes $\hat{\phi}$ continuous at $x = 0$ and analytic near $x = 0$. The $\hat{\phi}(x)$ given by this solution can be regarded as a fixed continuous function of x , in which case equations (10) can be regarded as a system of three inhomogeneous linear equations in y_1, y_2, y_3 . By the theorem T this system has precisely one solution, say $\mathbf{y}_0(x)$, which is continuously differentiable in a

closed neighborhood of $x = 0$ and vanishes at $x = 0$. From equations (10) it is clear that $\mathbf{y}'_0(0) = \mathbf{0}$. Define $\mathbf{h}(x) = \hat{\mathbf{y}}(x) - \mathbf{y}_0(x)$. Then $\mathbf{h}$ is a solution of

$$(11) \quad \begin{aligned} \frac{dh_1}{dx} &= x^2 \left(\frac{10\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) h_2 + x^2 \left(\frac{44\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) h_3 \\ \frac{dh_2}{dx} &= x \left(\frac{10\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) h_2 + x \left(\frac{44\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) h_3 \\ \frac{dh_3}{dx} &= \frac{h_2}{x}. \end{aligned}$$

Choose arbitrary constants C_1, C_2, C_3 and define $\theta_m(x)$ by

$$(12) \quad h_1(x) = C_1 + \theta_1(x), \quad h_2(x) = C_2 + \theta_2(x), \quad h_3(x) = C_3 + C_2 \ln |x| + \theta_3(x).$$

Then $\theta(x) = (\theta_1(x), \theta_2(x), \theta_3(x))$ satisfies

$$(13) \quad \begin{aligned} \frac{d\theta_1}{dx} &= x^2 \left(\frac{10\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) (C_2 + \theta_2) + x^2 \left(\frac{44\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) (C_3 + C_2 \ln |x| + \theta_3) \\ \frac{d\theta_2}{dx} &= x \left(\frac{10\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) (C_2 + \theta_2) + x \left(\frac{44\hat{\phi}}{9 + 4x^2\hat{\phi}} \right) (C_3 + C_2 \ln |x| + \theta_3) \\ \frac{d\theta_3}{dx} &= \frac{\theta_2}{x}. \end{aligned}$$

By the theorem T equations (13) have a unique solution θ which is continuously differentiable and vanishes at $x = 0$; clearly $\theta'(0) = \mathbf{0}$. When $C_k = \delta_{kr}$ with $r, k = 1, 2, 3$, denote this solution by $\theta^{(r)}$. From equation (12)

$$(14) \quad \begin{bmatrix} h_1^{(1)} \\ h_2^{(1)} \\ h_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \theta_1^{(1)} \\ \theta_2^{(1)} \\ \theta_3^{(1)} \end{bmatrix}; \quad \begin{bmatrix} h_1^{(2)} \\ h_2^{(2)} \\ h_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \ln |x| \end{bmatrix} + \begin{bmatrix} \theta_1^{(2)} \\ \theta_2^{(2)} \\ \theta_3^{(2)} \end{bmatrix}; \quad \begin{bmatrix} h_1^{(3)} \\ h_2^{(3)} \\ h_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \theta_1^{(3)} \\ \theta_2^{(3)} \\ \theta_3^{(3)} \end{bmatrix}.$$

These are three linearly independent solutions of equations (11). Hence every solution of equations (11) is a linear combination of these three solutions. Consequently, there exist constants D_1^+, D_2^+, D_3^+ and D_1^-, D_2^-, D_3^- such that

$$(15) \quad \hat{\mathbf{y}}(x) = \sum_{r=1}^3 D_r^* \mathbf{h}^{(r)}(x) + \mathbf{y}_0(x)$$

where the ambiguous sign must agree with the sign of x . We cannot assert that $D_r^+ = D_r^-$ because we do not yet know whether $\mathbf{y}(x)$ is continuous at $x = 0$. If we define

$$\mathbf{n}(x) = \mathbf{y}_0(x) + \sum_{r=1}^3 D_r^* \theta^{(r)}(x)$$

then $\mathbf{n}$ is continuously differentiable in a closed neighborhood of $x = 0$, $\mathbf{n}(0) = \mathbf{n}'(0) = \mathbf{0}$, and

$$(16) \quad \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} D_1^* \\ D_2^* \\ D_2^* \ln |x| + D_3^* \end{bmatrix} + \begin{bmatrix} \eta_1(x) \\ \eta_2(x) \\ \eta_3(x) \end{bmatrix}.$$

Then from equations (9)

$$(17) \quad \phi = 1 + D_1^* - xD_2^* + D_2^*x \ln |x| + D_3^*x + \eta_1(x) - x\eta_2(x) + x\eta_3(x).$$

Now we cease to treat ϕ in equations (10) as the fixed function $\hat{\phi}$. Substituting equations (16) and (9) into (10), we obtain a system of equations for η_1 , η_2 , and η_3 . An application of the theorem T to this system of equations shows that it has a unique solution $\mathbf{n}(x)$ continuously differentiable in a closed neighborhood of $x = 0$ and vanishing at $x = 0$.

We conclude that the solutions of equation (7) which are continuous at $x = 0$ and thrice continuously differentiable in a deleted neighborhood of $x = 0$ are given by the two three-parameter families of equation (17). Now if $\phi(0) = 1$, $D_1^+ = D_1^- = 0$. Further, from equations (6) and (17)

$$(18) \quad \sigma(x) = 1 - xD_2^* + \eta_1(x) - x\eta_2(x).$$

Since σ is even and $\sigma'(0)$ exists, we must have $\sigma'(0) = 0$, so $D_2^+ = D_2^- = 0$. By equation (5) ϕ is also even, so $D_3^+ + D_3^- = 0$. Thus the solutions ϕ of equation (7) which yield a continuously differentiable even σ via equation (6), which are continuous at $x = 0$ and thrice continuously differentiable in a deleted neighborhood of $x = 0$, all must have the form

$$(19) \quad \phi(x) = 1 + D|x| + \zeta(x, D)$$

where $\zeta(x, D)$ is a continuously differentiable function of x in a closed neighborhood of $x = 0$ which vanishes with its first derivative at $x = 0$, and is uniquely determined by the value of D .

We now show that for every D there is an analytic solution ϕ of equation (7) satisfying $\phi(0) = 1$ and $\phi'(0) = D$. To this end let $\psi = \frac{2}{3}\phi$. Then equation (7) becomes

$$(20) \quad x\psi''' + \psi'' + \psi(2x^3\psi''' - 3x^2\psi'' - 22x\psi' + 22\psi) = 0.$$

If $\psi(x)$ is a solution of equation (20), so is $a^2\psi(ax)$; thus we may assume $\psi(0) = a_0 = 1$, $\psi'(0) = a_1 = 3D/\sqrt{2}$. We seek constants a_n such that

$$(21) \quad \psi(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Equation (20) leads to this recursion relation for a_n :

$$(22) \quad a_{n+2} = - \sum_{r=0}^n a_r a_{n-r} \frac{f(r)}{g(n)}, \quad n = 0, 1, \dots,$$

where $f(r) = (r-1)(r+2)(2r-11)$ and $g(n) = (n+2)(n+1)^2$. From equation (22), a_0 and a_1 can be chosen arbitrarily, after which all a_n are determined. If we can prove that equation (21) has a non-zero radius of convergence, we have established the existence of the desired analytic solution of equation (20). In this connection note that for all $n \geq 1$ and all r between 0 and n

$$(23) \quad |f(r)| + |f(n-r)| \leq 2g(n).$$

Thus

$$(24) \quad |a_{n+2}| \leq \sum_{r=0}^n |a_r| |a_{n-r}| \quad \text{if } n \geq 1.$$

Now suppose b_n are positive numbers satisfying $b_0 = |a_0|, \dots, b_m = |a_m|$, and

$$(25) \quad b_{n+2} = \sum_{r=0}^n b_{n-r} b_r \quad \text{if } n \geq m-1.$$

Then for all n , $|a_n| \leq b_n$, and if

$$(26) \quad \omega(x) = \sum_{n=0}^{\infty} b_n x^n,$$

then the series (21) for $\psi(x)$ certainly converges inside the circle of convergence of equation (26). We choose $b_0 = 1$, $b_1 = |a_1| = b$ and let $b_n = |a_n|$; for more initial b 's would give a better estimate of the radius of convergence of ψ , but would encumber the construction of ω . As it is, equations (25) with $m = 1$ imply

$$(27) \quad \omega(x) = \frac{1 - \sqrt{1 - 4x^2 + 4bx^3}}{2x^2},$$

which is analytic in any circle centered on $x = 0$ and containing no zeros of the radical.

Since $\zeta(x, D)$ in equation (19) was unique, the recursion relations (22) can be used to obtain a power series for the function ϕ of equation (19):

$$(28) \quad \phi(x) = 1 + D|x| - \frac{2}{9}x^2 - \frac{11}{27}D|x|^3 + \dots$$

Therefore

$$(29) \quad \frac{3}{2}\sigma(x) = 1 + \frac{2}{9}x^2 + \frac{2}{27}D|x|^3 + \dots$$

Since every solution (29) of equation (1) can be continued analytically across the point $x = 0$, the principle of maximal smoothness conflicts with the evenness of $\sigma(x)$ unless $D = 0$.

We conclude that if we demand that σ be continuously differentiable, even, positive at $x = 0$, and that it satisfy the principle of maximal smoothness and equation (1), σ is one of a family of functions determined via equation (6) from the homologues $a^2\phi(ax)$ of the unique analytic function $\phi(x)$ satisfying equation (7), $\phi(0) = 1$, and $\phi'(0) = 0$, whose power series is given by equations (22). This justifies Chandrasekhar's use of that power series to start his solution near $x = 0$.

5. Singular points on $9 + 4x^2\phi = 0$. At any point $x_0 > 0$ where ϕ crosses the singular hyperbola $9 + 4x^2\phi = 0$, ϕ must be continuous. If there were a discontinuity in ϕ , it would have to be a saltus on account of equation (8) and the continuity of σ . But if $\lim_{x \rightarrow x_0-} \phi(x)$ is on the singular hyperbola, a saltus in ϕ at x_0 means $\lim_{x \rightarrow x_0+} \phi(x)$ is not on the singular hyperbola. Then the values of ϕ in $x > x_0$ can be extended analytically into an interval $[x_0 - \epsilon, x_0]$. Because of equation (6) this provides an analytic extension of σ from $x > x_0$ into $[x_0 - \epsilon, x_0]$ and violates the principle of maximal smoothness, unless this σ is the same as the σ already known in that interval. But the same analytic σ cannot give two different functions ϕ because of equation (5). Hence follows the continuity of ϕ at x_0 . But if σ and ϕ are continuous at x_0 , equation (6) shows that ϕ' is continuous there.

To examine the nature of ϕ at points where $9 + 4x^2\phi = 0$ it is expedient to use Chandrasekhar's variables

$$(30) \quad u = x^2\phi, \quad v = x^3\phi', \quad w = x^4\phi''$$

in terms of which the singular hyperbola becomes $u = -\frac{9}{4}$. Since ϕ and ϕ' are continuous functions of x , so are u and v ; and

$$\frac{du}{dx} = \frac{2u + v}{x},$$

so if $2u + v \neq 0$, the implicit function theorem makes x a continuous function of u . Thus at every crossing of the singular line where $2u + v \neq 0$, v is a continuous function of u . The equations for u, v, w equivalent to equation (7) are (CHANDRASEKHAR [2])

$$(31) \quad \frac{dv}{du} = \frac{3v + w}{2u + v}, \quad \frac{dw}{du} = \frac{27w + 22uv + 44w - 44u^2}{(9 + 4u)(v + 2u)}.$$

Let u_0, v_0, w_0 be the values of u, v, w at x_0 , the point where ϕ crosses the singular hyperbola, and define new variables κ, λ, μ by

$$(32) \quad u = -\frac{9}{4} + A\kappa, \quad v = v_0 + A\lambda, \quad \kappa w = \mu,$$

where $A = -(2u_0 + v_0)$. Chandrasekhar's numerical integration of equation (7) in the range $0 \leq x < x_0$ shows that for his function A is > 0 .

In terms of $\lambda(\kappa)$ and $\mu(\kappa)$ equations (31) become

$$(33) \quad \begin{aligned} \frac{d\lambda}{d\kappa} &= -\frac{\mu}{\kappa A} \left(\frac{1}{1-2\kappa-\lambda} \right) - \frac{3A\lambda + 3v_0}{A(1-2\kappa-\lambda)}, \\ \frac{d\mu}{d\kappa} &= \frac{\mu}{\kappa} \left(1 + \frac{45}{8A} + \frac{90\kappa + 45\lambda + 44A\kappa}{8A(1-2\kappa-\lambda)} \right) \\ &\quad + \frac{11}{4A} \left(\frac{(9-4A\kappa)(v_0 + A\lambda) + 4(\frac{9}{4} - A\kappa)^2}{1-2\kappa-\lambda} \right). \end{aligned}$$

Furthermore, $\lambda(\kappa)$ is a continuous function of κ in some interval $[-\delta, \delta]$, and $\lambda(0) = 0$. The second of equations (33) can be written

$$(34) \quad \frac{d\mu}{d\kappa} = \mu \left(\frac{\alpha + s(\kappa)}{\kappa} \right) + \nu(\kappa)$$

where $s(\kappa)$ and $\nu(\kappa)$ are continuous in $[-\delta, \delta]$ and $s(0) = 0$. Now let

$$(35) \quad \theta(\kappa) = \exp \int_{\delta}^{\kappa} \frac{\alpha + s(\kappa)}{\kappa} d\kappa.$$

Then $\theta \in \mathfrak{D}_{\alpha}$ with $\alpha = 1 + 45/8A$. When $u_0 + 2v_0 < 0$, $A > 0$, so $\alpha > 1$. Furthermore, since

$$(36) \quad \begin{aligned} \frac{d\mu}{d\kappa} &= \mu \frac{1}{\theta} \frac{d\theta}{d\kappa} + \nu, \\ \mu &= C^* \theta(\kappa) + \theta(\kappa) \int_{-\delta}^{\kappa} \frac{\nu(\kappa)}{\theta(\kappa)} d\kappa \end{aligned}$$

where the ambiguous signs must agree with the sign of κ . Since $\theta'(0) = 0$, equation (35) and lemma 2 imply that $\mu(\kappa)$ is continuously differentiable and $\mu(0) = 0$, $\mu'(0) = \nu(0)/(1-\alpha) = -\nu(0)(8A/45)$. But then from equation (33) w and hence ϕ'' are continuous across the singular hyperbola, and

$$(37) \quad w_0 = \frac{2}{5}(u_0 - v_0).$$

Note that by the theorem T the equations (33) have a one-parameter family of continuously differentiable solutions $\lambda(\kappa)$, $\mu(\kappa)$ vanishing at $\kappa = 0$. Therefore, ϕ has a one-parameter family of continuations across the singular hyperbola all of which make ϕ , ϕ' , ϕ'' continuous at the crossing point.

CHANDRASEKHAR chooses from among this one parameter family a solution which satisfies equation (3). Such a choice is dubious, and certainly violates the principal of maximal smoothness, in case Chandrasekhar's function ϕ for $x < x_0$ can be continued analytically across the singular hyperbola. We conclude this section by proving that such a continuation is impossible.

At every point (x_0, ϕ_0) on the singular hyperbola $9 + 4x^2\phi = 0$ there is a one-parameter family of analytic solutions of equation (7), one for each* slope

* Except that if $5[4(1 - x_0^2\psi_0')^{-1}]$ is a positive integer, it is clear from equation (39) that in general there will be no analytic solution ψ with slope ψ_0' at $x = x_0$.

ϕ'_0 . To see this use equation (20). Let $x = x_0 + \xi$, and write

$$(38) \quad \psi = \sum_{n=0}^{\infty} a_n \xi^n.$$

Then from equation (20) the recursion relation for a_n is

$$(39) \quad \begin{aligned} & (n+1)(n+2)[4n(1-x_0^3 a_1) - 5]a_{n+2} \\ &= 4x_0^3 \sum_{s=0}^{n-2} a_{n-s} a_{s+3} (s+3)(s+2)(s+1) \\ &+ 6x_0^2 \sum_{s=0}^{n-1} a_{n-s} a_{s+2} (s+2)(s+1)(2s-1) \\ &+ 4x_0 \sum_{s=0}^n a_{n-s} a_{s+1} (s+1)(3s^2 - 3s - 11) \\ &+ 2 \sum_{s=0}^n a_{n-s} a_s (s-1)(s+2)(2s-11), \quad n = 0, 1, 2, \dots \end{aligned}$$

In this equation symbols like $\sum_{n=0}^{-1}$ are to be interpreted as 0. From equation (39) a_1 may be chosen arbitrarily; however, we must always have $2x_0^2 a_0 + 1 = 0$. The convergence of the series in equation (38) can be demonstrated by techniques like those used on equation (21). At the point where Chandrasekhar's solution crosses the singular hyperbola his numerical integration gives

$$x_0 = 1.131867, \quad \phi_0 = -1.756274, \quad \phi'_0 = -3.243235.$$

Using these figures we can compute the coefficients a_n in the series in equation (38). The results are

$$\begin{array}{lll} a_0 = -0.3902831 & a_3 = 1.205634 & a_6 = 0.1608713 \\ a_1 = -0.7207189 & a_4 = 0.7073155 & a_7 = 0.07857287 \\ a_2 = 0.7306445 & a_5 = 0.2008615 & a_8 = 0.1229635. \end{array}$$

The terms in the sum in equation (38) are separately dominated by the terms in the power series

$$(40) \quad f(\xi) = \sum_{n=0}^{\infty} b_n \xi^n$$

where $b_n = |a_n|$ if $n = 0, 1, \dots, 8$, and if $n \geq 7$,

$$(41) \quad b_{n+2} = x_0^3 \sum_{s=0}^{n-2} b_{n-s} b_{s+3} + 3x_0^2 \sum_{s=0}^{n-1} b_{n-s} b_{s+2} + 3x_0 \sum_{s=0}^n b_{n-s} b_{s+1} + \sum_{s=0}^n b_{n-s} b_s.$$

From equation (41) one can deduce that

$$(42) \quad f(\xi) = \sum_{n=0}^8 b_n \xi^n + \xi^9 h(\xi)$$

where

$$(43) \quad h(\xi) = \frac{2C(\xi)}{B(\xi) + \sqrt{B^2(\xi) - 4A(\xi)C(\xi)}}$$

and

$$\begin{aligned} 4A(\xi) &= \xi^8(5.80024 + 15.3735\xi + 13.5824\xi^2 + 4\xi^3), \\ B(\xi) &= 1 - 2.52054\xi - 7.23093\xi^2 - 8.66569\xi^3 - 7.66152\xi^4 - 4.51096\xi^5 \\ &\quad - 2.07481\xi^6 - 1.20035\xi^7 - 0.880420\xi^8 - 0.485171\xi^9 - 0.120253\xi^{10}, \\ C(\xi) &= 2.66350 + 1.87357\xi + 1.26196\xi^2 + 0.774281\xi^3 + 0.385265\xi^4 \\ &\quad + 0.164683\xi^5 + 0.0779870\xi^6 + 0.0405181\xi^7 + 0.0168958\xi^8 + 0.00361517\xi^9. \end{aligned}$$

Then $\xi^9 h(\xi)$ is larger than the difference between $\psi(\xi)$ and $\sum_{n=0}^8 a_n \xi^n$. By these methods one finds that the analytic solution of equation (7) which has the same values of ϕ_0 and ϕ'_0 at x_0 as has Chandrasekhar's ϕ takes at $x = 0.95$ the value

$$\phi_{\text{analytic}}(0.95) = -1.086994 \pm 0.000012.$$

Chandrasekhar's solution gives

$$\phi(0.95) = -1.098497.$$

Although we do not have an error estimate for Chandrasekhar's numerical integration, it seems extremely unlikely that his solution is in error by 0.01; therefore his solution is not the analytic solution at the crossing of the singular hyperbola.

6. The singular point at $x = \infty$. To study this point define $z = x^{-1}$ and

$$(44) \quad g(z) = z\phi\left(\frac{1}{z}\right).$$

Then equation (7) becomes

$$(45) \quad z^2(4g + 9z^3)g''' = -18z(z^3 + g)g'' + 44gg'.$$

From equation (6)

$$(46) \quad g'(z) = \frac{3}{2}\sigma\left(\frac{1}{z}\right).$$

Therefore our hypotheses on σ imply that $g(z)$ is continuously differentiable in some closed neighborhood of $z = 0$, and $g'(0) = 0$. If $g(z)$ is a solution of equation (45), so is $a^3 g(a^{-1}z)$, so we may assume without loss of generality that $g(0) = 0$ or $g(0) = 1$; in the present section we shall assume $g(0) = 1$.

Define f_1, f_2, f_3 by

$$(47) \quad g = 1 + zf_1, \quad g' = f_2, \quad g'' = \frac{1}{z} f_3.$$

Then f_1 and f_2 are continuous at $z = 0$ and vanish there. Define y_1, y_2, y_3 by

$$(48) \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 0 & 2 & -1 \\ -27 & 5 & 2 \\ 0 & 11 & 2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Then equation (45) can be written

$$(49) \quad \frac{dy_i}{dz} = \frac{y_i}{z} (\alpha_i + z^3 s(z)) + z^2 F_i(z), \quad i = 1, 2, 3,$$

where $\alpha_1 = -\frac{11}{2}$, $\alpha_2 = -1$, and $\alpha_3 = 2$, while

$$\begin{aligned} s(z) &= \frac{45}{8 + 8zf_1 + 18z^3} \\ F_1(z) &= \frac{18f_2}{20 + 20zf_1 + 45z^3} \\ F_2(z) &= \frac{9(45f_1 - 23f_2)}{40 + 40zf_1 + 90z^3} \\ F_3(z) &= -\frac{297f_2}{40 + 40zf_1 + 90z^3}. \end{aligned}$$

Thus $s(z)$ and $F_i(z)$ are continuous at and near $z = 0$. Define $g_i(z)$ by

$$(50) \quad g_i(z) = \exp \int_z^z \frac{\alpha_i + \zeta^3 s(\zeta)}{\zeta} d\zeta.$$

Then equations (49) can be integrated to give

$$(51) \quad y_i(z) = C_i g_i(z) + g_i(z) \int_0^z \zeta^2 g_i^{-1}(\zeta) F_i(\zeta) d\zeta.$$

Since $g_i \in \mathfrak{D}_{\alpha_i}$, $z^2 g_i^{-1} \in \mathfrak{D}_{2-\alpha_i}$ and $2 - \alpha_i > -1$ for $i = 1, 2, 3$. Thus equations (51) say that

$$(52) \quad y_i(z) = C_i g_i(z) + z^2 \gamma_i(z)$$

where $\gamma_i(z)$ is continuously differentiable in a closed neighborhood of $z = 0$ and vanishes at $z = 0$. But from equation (50), $g_i(z) = z^{\alpha_i} h_i(z)$ where $h_i(0) \neq 0$. Since $y_1(z)$ and $y_2(z)$ must be continuous at $z = 0$, we must have $C_1 = C_2 = 0$. Hence

$$(53) \quad y_1(z) = z^2 \gamma_1(z), \quad y_2(z) = z^2 \gamma_2(z), \quad y_3(z) = z^2 \gamma_3(z) + Cz^2.$$

Then from equation (48),

$$(54) \quad \begin{aligned} f_1 &= \frac{z^2}{9} (-4\gamma_1(z) - 5\gamma_2(z) + 3\gamma_3(z) + C), \\ f_2 &= \frac{z^2}{9} (2\gamma_1(z) + \gamma_3(z) + C). \end{aligned}$$

If equations (53) and (54) are substituted into equations (49), a system of three equations for $\gamma_1, \gamma_2, \gamma_3$ is obtained which satisfies the hypotheses of the theorem T, and in the sense of that theorem has eigenvalues $-\frac{1}{2}, -3$, and 0. Therefore theorem 1 asserts that this system has a unique solution $(\gamma_1, \gamma_2, \gamma_3)$ continuously differentiable in a closed neighborhood of $z = 0$ and vanishing at $z = 0$. Consequently every solution (53) of equations (49) is uniquely specified by its value of C . From equation (47) every solution of equation (45) which is continuously differentiable at $z = 0$ and satisfies $g(0) = 1$ and $g'(0) = 0$ has the form

$$(55) \quad g(z) = 1 + \frac{1}{3}Cz^3 + \frac{1}{6}z^3m(z, C)$$

where $m(z, C)$ is continuously differentiable in a closed neighborhood of $z = 0$, vanishes at $z = 0$, and is uniquely determined by C . We shall prove that $m(z, C)$ is an analytic function of z by exhibiting for each C an analytic solution of equation (45) in the form (55).

To construct such analytic solutions define

$$\zeta = z^3 \quad \text{and} \quad h(\zeta) = \frac{2}{3}g(z).$$

Since $a^3g(a^{-1}z)$ is a solution of equation (45) if $g(z)$ is, we may take $h(0) = 1$. Then equation (45) becomes

$$(56) \quad (3\zeta^2h''' + 8\zeta h'' + 2h') + h(6\zeta h''' + 21h'') = 0.$$

If

$$(57) \quad h(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n,$$

then from equation (56) $a_2 = -a_1/21$, and for $n \geq 1$

$$(58) \quad a_{n+2} = -\frac{(n+1)^2(n+\frac{2}{3})a_{n+1} + \sum_{s=0}^{n-1} a_{n-s}a_{s+2}(s+2)(s+1)(2s+7)}{(n+1)(n+2)(2n+7)}.$$

Thus we can choose a_1 arbitrarily. From equation (58), if $n \geq 2$,

$$(59) \quad |a_{n+2}| \leq |a_{n+1}| + \sum_{s=0}^n |a_{n-s+1}| |a_{s+1}|.$$

If $b_0 = |a_0|, \dots, b_m = |a_m|$, and

$$(60) \quad b_{n+2} = b_{n+1} + \sum_{s=0}^n b_{n-s+1}b_{s+1}$$

for $n \geq m - 1$, then $0 \leq |a_n| \leq b_n$ for all n , so equation (57) converges inside the circle of convergence of

$$(60) \quad \rho(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n.$$

Now take $m = 1$, so $b_0 = 1$, $b_1 = |a_1|$, and from equation (60)

$$(61) \quad \rho(\zeta) = \frac{3 - \zeta - \sqrt{1 - (2 + 4b_1)\zeta + \zeta^2}}{2}.$$

Therefore the series in equations (60) and (57) converge inside any circle centered on $\zeta = 0$ and containing no zeros of the radical in equation (61).

7. An examination of equation (7) in the large. CHANDRASEKHAR solved equation (7) as follows: He started the power series at $x = 0$ given in section 4 of this paper and continued it by numerical integration until it intersected the singular hyperbola $9 + 4x^2\phi = 0$ at a point (x_0, ϕ_0) . To continue the solution across this hyperbola he started a power series solution at $z = 0$ ($x = \infty$) of the sort discussed in section 6 of this paper, and adjusted the two parameters in this power series until its continuation by numerical integration joined his other solution at the singular hyperbola with continuous ϕ and ϕ' . Chandrasekhar's calculations and the results of sections 4, 5, and 6 enable us to assert the existence of a twice continuously differentiable even solution of equation (7), analytic everywhere except at the singular hyperbola and at $x = \infty$, to which Chandrasekhar's calculations furnish an approximation.

The uniqueness of Chandrasekhar's solution is still in doubt, because of the one parameter family of possible extensions of ϕ across the singular hyperbola into the domain $x > x_0$. As Chandrasekhar's numerical integration and section 6 of this paper have shown, there is only one member ϕ of this family which produces, via equation 6, a maximally smooth, continuously differentiable, even $\sigma(x)$ vanishing at $x = \infty$, if one adds the conditions that ϕ never recrosses the singular hyperbola for any finite or infinite $x > x_0$, and that ϕ is analytic in (x_0, ∞) . The question is whether we still have a unique ϕ if we remove these last two conditions.

An argument like the one in section 5 shows that if $\sigma'(x)$ is to be continuous in the domain $x_0 \leq x < \infty$, then $\phi(x)$, $\phi'(x)$, and $\phi''(x)$ must be continuous there. But as long as ϕ does not cross the singular hyperbola, equation (7) shows that $\phi'''(x)$ must also be continuous. Then the uniqueness theorem for ordinary differential equations at a regular point and the existence theorem for analytic ordinary differential equations imply that ϕ must be analytic. Thus we obtain the uniqueness of ϕ if we require the continuity of $\sigma'(x)$, the maximal smoothness and evenness of $\sigma(x)$, that $\lim_{x \rightarrow \infty} \sigma(x) = 0$, and that $\phi(x)$ never recrosses the singular hyperbola for any $x > x_0$. To prove uniqueness in the absence of this last restriction we will prove that any ϕ which recrosses the singular hyperbola makes $\sigma'(x)$ discontinuous.

Let $\zeta = x^{-3}$, $\zeta_0 = x_0^{-3}$, and

$$(62a) \quad p(\zeta) = -\frac{(9 + 4x^2\phi)}{18x^3},$$

$$(62b) \quad q(\zeta) = \zeta p'(\zeta),$$

$$(63c) \quad r(\zeta) = \zeta^2 p''(\zeta) + \zeta p'(\zeta).$$

Then in terms of p and ζ equation (7) becomes

$$(63) \quad 12\zeta p p''' = -(42p + 5\zeta)p'' + 4p' + 2.$$

The argument of section 4 defines $p(\zeta)$ uniquely (up to a homology transformation) in the domain $\zeta_0 \leq \zeta < \infty$, and Chandrasekhar's numerical integration gives for ζ_0 , $p(\zeta_0)$, $q(\zeta_0)$, and $r(\zeta_0)$ these values:

$$(64) \quad \zeta_0 = 0.6896, \quad p(\zeta_0) = 0, \quad q(\zeta_0) = -0.4701, \quad r(\zeta_0) = -0.5703.$$

The argument of section 5 shows that p , q , and r are continuous at ζ_0 , so if $\zeta < \zeta_0$ and ζ is near ζ_0 , it is apparent from figure 1 and equation (62) that $p(\zeta) > 0$, regardless of which member of the one-parameter family of continuations of $p(\zeta)$ into the domain $[0, \zeta_0]$ is chosen.

Now suppose instead of choosing Chandrasekhar's continuation of $p(\zeta)$ into $[0, \zeta_0]$ we choose one for which $p(\zeta) = 0$ somewhere in $[0, \zeta_0)$. Let ζ_1 be the largest point in $[0, \zeta_0)$ such that $p(\zeta_1) = 0$. Then if $\zeta \in (\zeta_1, \zeta_0)$, $p(\zeta) > 0$. Equation (63) and the definitions of q and r can be rewritten as

$$(65) \quad \frac{dp}{d\zeta} = \frac{12pq}{12p\zeta}, \quad \frac{dq}{d\zeta} = \frac{12pr}{12p\zeta}, \quad \frac{dr}{d\zeta} = \frac{\zeta(2\zeta + 9q - 5r) + 6p(3q - r)}{12p\zeta}.$$

Now if we define the function $t(\zeta)$ by

$$(66) \quad \frac{dt}{d\zeta} = \frac{1}{12p\zeta},$$

then, since $p(\zeta)$ is analytic and positive in the domain (ζ_1, ζ_0) , $t(\zeta)$ is analytic and increasing in that domain, so that its inverse function $\zeta(t)$ is analytic and increasing in the domain $t_1 = t(\zeta_1) < t < t_0 = t(\zeta_0)$. Hence $\zeta(t)$, $p(t)$, $q(t)$, and $r(t)$ are analytic in (t_1, t_0) and satisfy there

$$(67a) \quad \frac{d\zeta}{dt} = 12p\zeta$$

$$(67b) \quad \frac{dp}{dt} = 12pq$$

$$(67c) \quad \frac{dq}{dt} = 12pr$$

$$(67d) \quad \frac{dr}{dt} = \zeta(2\zeta + 9q - 5r) + 6p(3q - r).$$

Equation (67b) and Rolle's theorem establish that there is a t in (t_1, t_0) at which $q(t) = 0$. Let t'' be the largest such t . If $q'(t'') = 0$, then from equation (67c) $r(t'') = 0$, so from equation (67d) $r'(t'') > 0$. Then from equation (67c) $q''(t'') > 0$, contradicting the fact, obvious from equations (64) and the definition of t'' , that $q(t) < 0$ if $t'' < t < t_0$. Therefore $q'(t'') < 0$. If there is a t in $[t_1, t'')$ where $q(t) = 0$, let the largest such t be t' . Then if $t' < t < t''$, $q(t) > 0$. In (t', t'') there is a point t_0 at which dq/dt and hence $r(t)$ decrease through 0 as t increases through t_0 . At this point $r'(t_0) \leq 0$. But from equation (67d) $r'(t_0) > 0$. This contradiction shows that there is no point t in $[t_1, t'')$ where $q(t) = 0$. Thus if $t_1 \leq t < t''$, $q(t) > 0$.

Now if we define $g(z)$ as in equation (44), where $\zeta = z^3$, then the definitions (62) imply

$$(68a) \quad p(\zeta) = -\frac{1}{2}z^3 - \frac{2}{9}g(z),$$

$$(68b) \quad q(\zeta) = -\frac{1}{2}z^3 - \frac{2}{27}zg'(z),$$

$$(68c) \quad r(\zeta) = -\frac{1}{2}z^3 - \frac{2}{81}[zg'(z) + z^2g''(z)].$$

From equation (46) $q(0) = 0$. Therefore in the preceding paragraph we have shown that $\zeta_1 > 0$. Now at t'' , since $q'(t'') < 0$, $r(t'') < 0$ by equation (67c). If $t_1 < t < t''$, $p > 0$ and $q > 0$ so equation (67d) affirms that if $t_1 < t < t''$, $r(t) < 0$. Again using equation (67d) we conclude that if $t_1 < t \leq t''$, $r'(t) \geq 2\zeta_1^2 > 0$, so

$$(69) \quad r(t) \leq r(t'') - 2\zeta_1^2(t'' - t).$$

Now equations (46) and (68b) imply that $q(\zeta)$ is a continuous function of ζ at ζ_1 , and since $\zeta_1 > 0$, equation (62b) implies that $dp/d\zeta$ is a continuous function of ζ at ζ_1 . Therefore from equation (66) $t_1 = -\infty$, and from equation (69)

$$(70) \quad \lim_{\zeta \rightarrow \zeta_1+} r(\zeta) = -\infty.$$

But from equation (68c) this makes $g''(z)$ discontinuous at $z_1 = \zeta_1^{\frac{1}{3}}$, and from equation (46) $\sigma'(x)$ is discontinuous at $x_1 = \zeta_1^{-\frac{1}{3}}$.

We conclude that no ϕ in the one-parameter family obtained in section 5 along which $\sigma'(x)$ is continuous can return to the singular hyperbola $9 + 4x^2\phi = 0$, either at a finite $x > x_0$ or at $x = \infty$. This incidentally shows that there are no acceptable solutions ϕ to be obtained by taking $g(0) = 0$ in section 6.

8. Conclusion. To recapitulate, our mathematical result in this paper is that if $\sigma(x)$ is even, continuously differentiable, and maximally smooth for all finite real x , and vanishes at $x = \infty$, and is positive at $x = 0$, and satisfies equation (1), then σ is given by equation (6) in terms of one member $\phi(x)$ of a one-parameter family of even, twice continuously differentiable, homologous solutions of equation (7), all members of which cross the singular hyperbola $9 + 4x^2\phi = 0$ exactly once, are analytic for all finite x except at this crossing

point, and have a pole of order 1 at $x = \infty$. Chandrasekhar's methods of computing these functions ϕ converge.

The mathematical result of this paper is incomplete in that the author does not know whether equation (7) has any solutions which cross the singular hyperbola twice and along which $\sigma(x)$ is continuous.

The physical result of the present paper is the following: It can be proved from the physics of homogeneous isotropic turbulence that $\sigma(x) = \sigma(-x)$. (See CHANDRASEKHAR [2].) The assumptions $\sigma(0) > 0$ and $\lim_{x \rightarrow \infty} \sigma(x) = 0$ are physically reasonable because $\sigma(0)$ is essentially the means square of the vorticity and $\lim_{t \rightarrow \infty} r^{\frac{1}{2}} \chi(r, t) = 0$ says essentially that the velocities at two different points $\mathbf{r}_1$ and $\mathbf{r}_2$ at times t_1 and t_2 become statistically independent as $|t_2 - t_1| \rightarrow \infty$. If one imposes the further restriction that $\sigma'(x)$ be continuous, Chandrasekhar's theory of turbulence leads, in the case of zero viscosity, to a specification of the radial derivative of the second-order velocity correlation which is unique up to a homology transformation. The physical content of the requirement that $\sigma'(x)$ be continuous may be seen from the formulas

$$(71) \quad -\frac{\partial}{\partial r} \langle u_1 u'_1 \rangle_{Av} = r^{-\frac{1}{2}} \sigma(tr^{-\frac{2}{3}})$$

and

$$(72) \quad \langle \omega_1 \omega'_1 \rangle_{Av} = \frac{11}{3} r^{-4/3} \sigma(tr^{-\frac{2}{3}}) - \frac{2}{3} tr^{-2} \sigma'(tr^{-\frac{2}{3}}),$$

where u_1 is the x component of the velocity and ω_1 is the x -component of the vorticity, $\text{curl } \mathbf{u}$.

If one cannot demand that $\sigma'(x)$ be continuous at $x = 0$, the uniqueness theorem proved in this paper fails, and many more solutions ϕ of equation (7) become acceptable. At any regular point of equation (7) one can give up the demand that $\sigma'(x)$ be continuous without losing the uniqueness theorem, because the existence of a solution for which $\sigma'(x)$ is continuous and the maximal smoothness of $\sigma(x)$ will then imply the continuity of $\sigma'(x)$. For a similar reason one can give up the demand that $\sigma'(x)$ be continuous at x_0 without losing the uniqueness theorem. Because of the incompleteness of the mathematical results of section 7 the author does not know whether abandoning the demand for continuity of $\sigma'(x)$ at other places besides x_0 where ϕ crosses the singular hyperbola will destroy the uniqueness theorem.

9. Appendix. (Added in proof.) The existence and uniqueness theorem referred to in the present paper is stated below as theorem T. The assertion about existence contained in that theorem can be proved by a reference to theorem 1 of WAŻEWSKI [8]; the assertion about uniqueness follows from theorems 7 and 9 of PERRON [5]. The independent variable t used by PERRON and WAŻEWSKI is related to our x by $x = e^{-t}$. I am indebted to Professor AUREL WINTNER for these references to the literature.

Theorem T: Let $\mathbf{g}(x, \mathbf{y})$ be an n -dimensional vector-valued function of x

and the vector $\mathbf{y}$. Let $G_{ii}(x, \mathbf{y})$ be the matrix $\partial g_i(x, y)/\partial x_i$. Let $\mathfrak{C}_\lambda$ be the class of solutions $\mathbf{y}(x)$ of

$$(73) \quad x \frac{d\mathbf{y}}{dx} = \mathbf{g}(x, \mathbf{y})$$

which have the properties that $x^{-\lambda}\mathbf{y}(x)$ is continuous in some closed interval $[0, \delta]$ and $\mathbf{y}(0) = \mathbf{0}$. Suppose

- (i) when $0 \leq x, |y_1|, \dots, |y_n| \leq \epsilon$, $\mathbf{g}$ has continuous first derivatives with respect to $x, y_1, \dots, y_n$ and the derivatives with respect to $y_1, \dots, y_n$ of all first derivatives of $\mathbf{g}$ are bounded;
- (ii) $\mathbf{g}(0, \mathbf{0}) = \mathbf{0}$;
- (iii) $0 \leq \lambda \leq 1$;
- (iv) $G_{ii}(0, \mathbf{0})$ has p_λ characteristic roots $> \lambda$ and none $= \lambda$.

Then it follows that

- (i') $\mathfrak{C}_\lambda$ consists of a p_λ -parameter family of functions. If $p_\lambda = 0$, $\mathfrak{C}_\lambda$ contains precisely one member;
- (ii') if $\mathbf{y} \in \mathfrak{C}_\lambda$ and $\mathbf{y}$ is defined on $[0, \delta]$, then $\mathbf{z}(x) = x^{1-\lambda}\mathbf{y}(x)$ is continuously differentiable on $[0, \delta]$; $\mathbf{z}'(0)$ is the same for all members of $\mathfrak{C}_\lambda$ and vanishes unless $\lambda = 1$.

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