Extended Asymptotic Eigenvalue Distributions for Bounded Domains in n -Space

F. H. BROWNELL

Communicated by H. LEWY

1. Introduction

Let D be a bounded, open, connected set with boundary B in R_n , n -dimensional Euclidean space. Letting $\nabla^2 u$ denote the Laplacian and ∇u the gradient of u , consider the problem with eigenvalue λ

$$(1.1) \quad \nabla^2 u + \lambda u = 0 \quad \text{on } D,$$

where $u(\mathbf{x})$ is to be real-valued with continuous second partials over $\mathbf{x} \in D$. For the Dirichlet condition u is to possess a continuous extension over the closure $\bar{D}$ such that

$$(1.2) \quad u = 0 \quad \text{on } B.$$

For the Neumann condition ∇u is to possess a continuous extension over $\bar{D}$ such that its normal component

$$(1.3) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } B,$$

at least if B is sufficiently smooth; if not, (1.3) will be appropriately modified (see (2.3)).

Let $\nu \lambda_i \leq \nu \lambda_{i+1}$ be the eigenvalues of the first of these problems if $\nu = 0$ and of the second if $\nu = 1$, the $\nu \lambda_i$ being repeated according to their multiplicities. We want to study the asymptotic behavior of $N_\nu(\lambda) = \sum_{\nu \lambda_i \leq \lambda} 1$, the number of eigenvalues $\leq \lambda$, as $\lambda \rightarrow +\infty$. The original result of WEYL was later obtained by CARLEMAN for dimension $n = 2$ or 3 by applying Tauberian theorems to an asymptotic evaluation of $\sum_{i=1}^{\infty} \nu \lambda_i^{-1} (\nu \lambda_i + \omega^2)^{-1}$ as $\omega \rightarrow +\infty$. This evaluation was found by utilizing the Green function of the "meson" equation operator $-\nabla^2 + \omega^2$, ([16], see bibliography). In two recent papers [13] and [14] PLEIJEL has found, in addition to CARLEMAN's leading term, the asymptotic series for $\sum_{i=1}^{\infty} \nu \lambda_i^{-1} (\nu \lambda_i + \omega^2)^{-1}$ in powers of ω^{-1} to any desired order for $n = 2$ or 3 . This work assumes the boundary B to be infinitely differentiable everywhere,

and utilizes the known Neumann-Poincaré construction of the Green function as the solution of an integral equation on the boundary. In a short previous paper [3], we have shown how PLEIJEL's asymptotic series for $\sum_{i=1}^{\infty} \lambda_i^{-1} (\lambda_i + \omega^2)^{-1}$ yields an asymptotic series for $N_r(\lambda)$ in the sense of a logarithmic Gaussian averaged error estimate, denoted by $\tilde{O}$.

In the present work [4], we determine what happens to PLEIJEL's series under a variety of weaker smoothness conditions on the boundary B ; in particular, one condition allows B to have a finite number of corners. We also obtain asymptotic series for $\sum_{i=1}^{\infty} \exp(-\lambda_i t)$, and the resulting $\tilde{O}$ -asymptotic series for $N_r(\lambda)$, from an analogous construction of the heat Green function, which was used previously by MINAKSHISUNDARAM [12] to obtain only the leading term. Also the theory of $\tilde{O}$ -estimates is extended from that given in the previous papers quoted above in order to take care of exponents of negative order.

The present paper is divided into several sections. The first two following the introduction are devoted to notation, some preliminary standard material, and the geometric smoothness estimates on the boundary B . The next two sections present well known constructions of the meson and heat Green functions and give their relation with the eigenvalues λ_i . We wish to apologize in advance for the lengthy detail* with which we have presented the well known material in these two sections, but we found this procedure necessary in order to tie down our resulting asymptotic expansions to the variety of smoothness conditions assumed for the boundary. The next two sections actually compute these asymptotic series, the following section gives the connection with a lattice point problem and the associated Epstein zeta function, and the last section summarizes and discusses our results.

We are indebted to Professors A. PLEIJEL, M. RIESZ, M. SCHIFFER, and M. ARSOVE for several fruitful discussions concerning this material.

2. Preliminaries and Notation

Throughout this paper μ_n will stand for ordinary Lebesgue measure on n -dimensional Euclidean space R_n , and $\ominus$ will be used for set differences. We denote $(x_1, x_2, \dots, x_n) \in R_n$ by $\mathbf{x}$. The inner product of $\mathbf{x}$ and $\mathbf{y} \in R_n$ we denote by $(\mathbf{x} \cdot \mathbf{y}) = \sum_{i=1}^n x_i y_i$ and let $|\mathbf{x}| = \sqrt{(\mathbf{x} \cdot \mathbf{x})}$. $(u, v) = \int_D u(\mathbf{x}) v(\mathbf{x}) d\mu_n(\mathbf{x})$ and $\|u\| = \sqrt{(u, u)}$ denote the inner product and norm respectively of u and $v \in L_2(D)$, D a Borel subset of R_n . For R_n -vector-valued functions $\mathbf{f}(\mathbf{x})$ of $\mathbf{x} \in R_n$, $(\nabla \cdot \mathbf{f}(\mathbf{x})) = \sum_{i=1}^n \partial f_i(\mathbf{x}) / \partial x_i$ denotes the divergence; for such functions of a real variable s , we define $d\mathbf{f}(s)/ds = (df_1(s)/ds, df_2(s)/ds, \dots, df_n(s)/ds)$; also integrals of R_n -vector-valued functions are likewise defined componentwise.

* This detail has been largely eliminated in the present abbreviated version, in which proofs for the first six sections have only been sketched. The complete manuscript, denoted by (M) hereafter, is available free of charge upon writing to the Mathematics Department, University of Washington, Seattle. Since we have maintained the numeration in (M), there will be large gaps here in the numbers of the formulæ.

We will need an "area" measure σ_n on sufficiently smooth, $(n-1)$ -dimensional subsets S of R_n . Let S be a finite, disjoint union of bounded sets S_q such that for each q the closure $\bar{S}_q$ in R_n of S_q is homeomorphic, according to projection along a coordinate axis k_q of R_n , to a compact set C_q in R_{n-1} , that the resulting function $x_{k_q} = g_q(x_1, x_2, \dots, x_{k_q-1}, x_{k_q+1}, \dots, x_n)$ defining the inverse map over C_q has bounded continuous first partials over the interior of C_q in R_{n-1} , that $\mu_{n-1}(C_q \ominus \text{int } C_q) = 0$, and that the projection of S_q contains $\text{int } C_q$. Letting χ_A be the characteristic function of A , we define over the Borel subsets A of such a set S

$$(2.1) \quad \sigma_n(A) = \sum_q \int_{C_q} \chi_A(\mathbf{x}) \left\{ 1 + \sum_{k \neq k_q} \left(\frac{\partial}{\partial \xi_k} g_q(\xi) \right)^2 \right\}^{\frac{1}{2}} d\mu_{n-1}(\xi)$$

with $x_k = \xi_k$ for $k \neq k_q$ and $x_{k_q} = g_q(\xi)$ defining $\mathbf{x} \in \bar{S}_q$ for $\xi \in C_q \subseteq R_{n-1}$. As one readily verifies, σ_n is a Borel measure on such S , is independent of the particular k_q -axes chosen, and is invariant under translation and proper or improper rotation of the coordinate axes. It is easy to compute

$$\sigma_n(\{\mathbf{x} \in R_n \mid |\mathbf{x}| = 1\}) = 2\pi^{\frac{1}{2}n} \{\Gamma(\frac{1}{2}n)\}^{-1},$$

which we will need later.

For the Gauss formula, we note that if D is a bounded, open, connected subset of R_n whose boundary B satisfies the above conditions, if $\mathbf{f}(\mathbf{x}) \in R_n$ is defined and continuous over $\mathbf{x} \in \bar{D}$, and if each partial $\partial f_i(\mathbf{x})/\partial x_i$ exists, is continuous over D , and is in $L_1(D)$, then

$$(2.2) \quad \int_D (\nabla \cdot \mathbf{f}(\mathbf{x})) d\mu_n(\mathbf{x}) = \int_B (\mathbf{f}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})) d\sigma_n(\mathbf{x})$$

where $\mathbf{n}(\mathbf{x})$ is the unit normal to B at $\mathbf{x}$ outward from D . $\mathbf{n}(\mathbf{x})$ must exist almost everywhere (σ_n) by the conditions on B .

We now formulate a modified Neumann condition for the given bounded, open, connected set D with boundary B in R_n . For our purposes we only need to do this for rather smooth functions on D , with at most a single singular point in D .

Definition (2.3)

A real-valued function $u(\mathbf{x})$ defined over $\mathbf{x} \in \bar{D} \ominus \{\mathbf{o}\mathbf{x}\}$ and possessing continuous second partials over $D \ominus \{\mathbf{o}\mathbf{x}\}$ for some $\mathbf{o}\mathbf{x} \in D$ is said to satisfy the modified Neumann condition on B if $u(\mathbf{x})$ is continuous over $\bar{D} \ominus \{\mathbf{o}\mathbf{x}\}$, if for some $r > 0$ having $\{\mathbf{x} \in R_n \mid |\mathbf{x} - \mathbf{o}\mathbf{x}| < 2r\} \subseteq D$ both $|\nabla u(\mathbf{x})| \in L_2$ and $\nabla^2 u(\mathbf{x}) \in L_1$ over $D \ominus \{\mathbf{x} \mid |\mathbf{x} - \mathbf{o}\mathbf{x}| \leq r\}$, and if every real-valued, continuous $v(\mathbf{x})$ over $\bar{D}$ which possesses continuous first partials over D with $|\nabla v(\mathbf{x})| \in L_2(D)$ also satisfies

$$(2.4) \quad - \int_{\{\mathbf{x} \mid |\mathbf{x} - \mathbf{o}\mathbf{x}| = r\}} v(\mathbf{x}) \left(\frac{\mathbf{x} - \mathbf{o}\mathbf{x}}{|\mathbf{x} - \mathbf{o}\mathbf{x}|} \cdot \nabla u(\mathbf{x}) \right) d\sigma_n(\mathbf{x}) \\ = \int_{D \cap \{\mathbf{x} \mid |\mathbf{x} - \mathbf{o}\mathbf{x}| > r\}} [v(\mathbf{x}) \nabla^2 u(\mathbf{x}) + (\nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x}))] d\mu_n(\mathbf{x}).$$

It is clear that if $u(\mathbf{x})$ is actually defined at any such ${}_0\mathbf{x}$ with continuous second partials there, then (2.4) is equivalent to

$$(2.5) \quad 0 = \int_D [v(\mathbf{x}) \nabla^2 u(\mathbf{x}) + (\nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x}))] d\mu_n(\mathbf{x})$$

by using (2.2). Also, if $\nabla u(\mathbf{x})$ possesses a continuous extension from $D \ominus \{{}_0\mathbf{x}\}$ to $\bar{D} \ominus \{{}_0\mathbf{x}\}$, if B satisfies the conditions for the definition of σ_n , and if $(\nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})) = 0$ almost everywhere (σ_n) on B , then (2.4) follows by using (2.2). Thus, for u with $\nabla^2 u \in L_1(D \ominus \{\mathbf{x} \mid |\mathbf{x} - {}_0\mathbf{x}| < r\})$, definition (2.3) is weaker than the classical Neumann condition (1.3).

We conclude this preliminary section with two lemmas which will give us the well known uniqueness of the Green function for either the Dirichlet or modified Neumann boundary condition for either the meson or heat equation differential operators. D is assumed bounded, connected, and open in R_n here as usual.

Lemma (2.6)

If $u(\mathbf{x})$ is real-valued and continuous over $\mathbf{x} \in \bar{D}$, if $u(\mathbf{x})$ has continuous second partials and satisfies

$$-\nabla^2 u(\mathbf{x}) + \omega^2 u(\mathbf{x}) = 0$$

over $\mathbf{x} \in D$ for some real constant $\omega > 0$, and if $u(\mathbf{x})$ satisfies the boundary condition (1.2) or (2.3) on B , then $u(\mathbf{x}) = 0$ over $\mathbf{x} \in \bar{D}$.

Lemma (2.7)

If $u(x, t)$ is real-valued and continuous over $(\mathbf{x}, t) \in \bar{D} \times (0, T]$ for some real $T > 0$, if $u(\mathbf{x}, t)$ has continuous second partials in $\mathbf{x}$ and a continuous first partial in t and satisfies

$$-\nabla^2 u(\mathbf{x}, t) + \frac{\partial}{\partial t} u(\mathbf{x}, t) = 0$$

over $(\mathbf{x}, t) \in D \times (0, T)$, if

$$\lim_{t \rightarrow 0^+} \int_D [u(\mathbf{x}, t)]^2 d\mu_n(\mathbf{x}) = 0,$$

if $u(\mathbf{x}, t)$ satisfies in $\mathbf{x}$ conditions (1.2) or (2.3) on B for every $t \in (0, T]$, and if in case of the (1.2) condition $u(\mathbf{x}, t)$ possesses a continuous, real-valued extension over $\bar{D} \times [0, T]$ and in case of the (2.3) condition $\partial u(\mathbf{x}, t)/\partial t$ is in $L_1(D \times (t', T))$ for every $t' \in (0, T)$, then $u(\mathbf{x}, t) = 0$ over $\bar{D} \times (0, T]$.

The proof of lemma (2.6) for the Dirichlet condition (1.2) is clear from the maximum principle. In the case of the modified Neumann condition (2.3), this implies (2.5) with u an admissible v . Thus

$$(2.8) \quad 0 = \omega^2 \int_D [u(\mathbf{x})]^2 d\mu_n(\mathbf{x}) + \int_D |\nabla u(\mathbf{x})|^2 d\mu_n(\mathbf{x}),$$

which for $\omega > 0$ implies $u \equiv 0$ as desired.

The proof of lemma (2.7) follows WEYL (see [12], appendix) for the Dirichlet condition (1.2). See (M) for details.

For the modified Neumann condition with (2.7), from (2.5) we now have in place of (2.8) for $0 < t \leq T$

$$(2.9) \quad 0 = \int_D u(\mathbf{x}, t) \left[\frac{\partial}{\partial t} u(\mathbf{x}, t) \right] d\mu_n(\mathbf{x}) + \int_D |\nabla u(\mathbf{x}, t)|^2 d\mu_n(\mathbf{x}).$$

Thus (see (M) for details) we have

$$\int_D ([u(\mathbf{x}, t_2)]^2 - [u(\mathbf{x}, t_1)]^2) d\mu_n(\mathbf{x}) = -2 \int_{t_1}^{t_2} \int_D |\nabla u(\mathbf{x}, t)|^2 d\mu_n(\mathbf{x}) dt \leq 0.$$

Thus

$$\int_D [u(\mathbf{x}, t)]^2 d\mu_n(\mathbf{x}) \leq \lim_{t_1 \rightarrow 0^+} \int_D [u(\mathbf{x}, t_1)]^2 d\mu_n(\mathbf{x}) = 0$$

makes $u(\mathbf{x}, t) \equiv 0$ on $\bar{D} \times (0, T]$, and the proof of (2.7) is completed.

3. Geometry of the Boundary B

As is assumed throughout, D is a bounded, open, connected set with boundary $B = \bar{D} \ominus D$ in Euclidean n -space R_n , with integer $n \geq 2$. In this section we formulate three different additional conditions concerning the smoothness of B , each of which will suffice for the Neumann-Poincaré construction of our various Green functions. The remainder of this section is devoted to recording asymptotic expansions of various geometric quantities under these conditions, which will be used in determining asymptotics of the constructed Green functions. The three smoothness conditions we have chosen by no means exhaust all possible ones of interest, but the results we will obtain with them are indicative of the general situation. We will see that the third condition implies the first, and also the second, if $n = 2$. As stated, there are no other implications among the three conditions.

Condition (3.1)

B is a finite union of sets V_a relatively open in B such that each V_a is homeomorphic, according to projection along a coordinate axis k_a , to an open solid sphere W_a in R_{n-1} , and that the resulting function $x_{k_a} = g_a(x_1, x_2, \dots, x_{k_a-1}, x_{k_a+1}, \dots, x_n)$ defining the inverse map over W_a has bounded, continuous first partials there. Moreover, the unit normal $\mathbf{n}(\mathbf{x})$ to B outward from D (whose unique existence at every $\mathbf{x} \in B$ is guaranteed by the preceding sentence) satisfies the Hölder condition of order η

$$(3.2) \quad |\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{x}')| \leq M |\mathbf{x} - \mathbf{x}'|^\eta$$

over $\mathbf{x}$ and $\mathbf{x}' \in B$ for some η and M satisfying $0 < \eta \leq 1$ and $0 < M < +\infty$. Also, for some $\delta > 0$ defining the set $D_\delta = \{\mathbf{x} \in D \mid (\inf_{\mathbf{z} \in B} |\mathbf{x} - \mathbf{z}|) < \delta\}$, there exists some continuous $\rho(\mathbf{x}) \geq 0$ defined over $\mathbf{x} \in D_\delta$ such that

$$\mu_n(A) = \int_B \int_0^\delta \chi_A(-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) \rho(-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) dh d\sigma_n(\mathbf{y})$$

for every Borel set $A \subseteq D_\delta$, χ_A being the characteristic function of A . Moreover, this $\rho(x)$ satisfies

$$(3.3) \quad |\rho(-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) - 1| \leq M'h^\eta$$

over $\mathbf{y} \in B$ and $0 < h < \delta$ for some $M' < +\infty$.

We remark for the third sentence of (3.1) that the existence of δ , of σ_n on B , and of the unique continuous density $\rho(\mathbf{x})$ is implied by the first two sentences of (3.1). However, (3.2) does not imply (3.3), even for some $\eta' < \eta$. For if we consider $n = 2$ and D bounded below by the curve $y_2 = |y_1|^{1+\eta}$ for $0 < \eta < 1$, then (3.2) is satisfied; but a short computation shows that we have $\rho(\mathbf{x}) \rightarrow \frac{1}{2}(1 - \eta) < 1$ as $\xi \rightarrow 0^+$ for $\mathbf{x} = (0, \xi)$, and (3.3) must fail.

Condition (3.4)

$n = 2$. B is a finite, disjoint union of $p + 1$ Jordan curves, each one of which is a finite union of closed Jordan arcs A_i with endpoints ${}_i\mathbf{x}$ such that $\{{}_i\mathbf{x}\} = A_i \cap A_{i'}$, with unique $j' \neq j''$ existing for each endpoint ${}_i\mathbf{x}$ and that the A_i are disjoint except for endpoints. Each A_i is rectifiable, and $\mathbf{x} \in A_i$ is everywhere a twice differentiable function $\mathbf{f}(s)$ of the arc length s , s being taken to increase along the unit tangent vector $\mathbf{t}(\mathbf{x})$ directed $\frac{1}{2}\pi$ radians counter-clockwise from the outward from D unit normal $\mathbf{n}(\mathbf{x})$. The resulting curvature $c(s) = (\mathbf{t}(\mathbf{x}) \cdot d\mathbf{n}(\mathbf{x})/ds)$ with $\mathbf{x} = \mathbf{f}(s)$ satisfies the Hölder condition of order η

$$(3.5) \quad |c(s) - c(s')| \leq M |s - s'|^\eta$$

over $\mathbf{f}(s)$ and $\mathbf{f}(s') \in A_i$ for every A_i for some η and M satisfying $0 < \eta \leq 1$ and $0 < M < +\infty$. At the finite number of common endpoints ${}_i\mathbf{x}$ of the arcs A_i , corners are permitted; more precisely, the angle of $\mathbf{t}(\mathbf{x})$ with $\mathbf{x} = \mathbf{f}(s)$ has a jump increase α_i , $-\pi < \alpha_i < \pi$, at ${}_i\mathbf{x}$ as s increases.

We say that such an ${}_i\mathbf{x}$ is a corner of B if and only if $\alpha_i \neq 0$.

Condition (3.6)

$n \geq 2$. B satisfies the first sentence of condition (3.1) and the resulting function g_q has continuous second partials $g''_{q;k,k'}$ over W_q satisfying the Hölder condition of order η

$$(3.7) \quad |g''_{q;k,k'}(\tilde{\mathbf{x}}) - g''_{q;k,k'}(\tilde{\mathbf{x}}')| \leq M |\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'|^\eta$$

over $\tilde{\mathbf{x}} = (x_1, x_2, \dots, x_{k_q-1}, x_{k_q+1}, \dots, x_n) \in W_q$ and $\tilde{\mathbf{x}}' \in W_q$ for every q , for all $k, k' \neq k_q$, and for some η and M satisfying $0 < \eta \leq 1$ and $0 < M < +\infty$.

It is clear that each of these three conditions implies the existence of the Borel measure σ_n on B with $\sigma_n(B) < +\infty$. Consider $\mathbf{x} = (-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) \in \bar{D}$, $h \geq 0$, $\mathbf{y} \in B$. Then for $\mathbf{z} \in B$ we want to expand $|\mathbf{z} - \mathbf{x}|^2$ and $(\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{x}))$. In these expansions which follow it is to be understood that M is some real constant,

$0 < M < +\infty$, depending only upon the domain D , but in different statements M may have different values which we have not bothered to distinguish with subscripts. Also θ , a real number, and $\theta \in R_n$ denote the values of various functions, similarly undistinguished by subscripts, of unspecified variables which everywhere satisfy the inequalities $|\theta| \leq 1$ and $|\theta| \leq 1$.

Lemma (3.8)

Let condition (3.1) be satisfied, let $\mathbf{z}$ and $\mathbf{y} \in B$ with $\mathbf{z}^* - \mathbf{y} = (\mathbf{z} - \mathbf{y}) - (\mathbf{n}(\mathbf{y}) \cdot (\mathbf{z} - \mathbf{y}))\mathbf{n}(\mathbf{y})$ defining the projection of $\mathbf{z} - \mathbf{y}$ onto the tangent hyperplane to B at $\mathbf{y}$, and let $\mathbf{x} = (-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) \in \bar{D}$, $h \geq 0$. Then

$$(3.9) \quad (\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{x})) = h + (h + |\mathbf{z}^* - \mathbf{y}|)^{1+\eta} M\theta,$$

$$(3.10) \quad |\mathbf{z} - \mathbf{x}|^2 = h^2 + |\mathbf{z}^* - \mathbf{y}|^2 + (h + |\mathbf{z}^* - \mathbf{y}|)^{2+\eta} M\theta'.$$

Lemma (3.11)

Let condition (3.4) be satisfied, let both $\mathbf{z}$ and $\mathbf{y} \in A_i$ for some A_i with $\mathbf{z} = \mathbf{f}(s)$ and $\mathbf{y} = \mathbf{f}(s_0)$, and let $\mathbf{x} = (-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) \in \bar{D}$, $h \geq 0$. Then

$$(3.12) \quad (\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{x})) = h + \frac{1}{2}(s - s_0)^2 c(s_0) + (h + |s - s_0|)^{2+\eta} M\theta,$$

$$(3.13) \quad |\mathbf{z} - \mathbf{x}|^2 = [h^2 + (s - s_0)^2] - h(s - s_0)^2 c(s_0) + (h + |s - s|)^{3+\eta} M\theta',$$

and with $\mathbf{x}' = -h\mathbf{n}(\mathbf{z}) + \mathbf{z}$ the Jacobian $J((x'_1, x'_2)/(h, s))$ exists satisfying

$$(3.14) \quad \left| \left[J \left(\frac{x'_1, x'_2}{h, s} \right) \right]_{s=s_0} \right| = |1 - hc(s_0)|.$$

In order to get similar expansions under condition (3.6), $n \geq 2$, consider $\mathbf{y} \in B$ and define real s_i by the projection of $\mathbf{z} \in B$ onto the tangent hyperplane to B at $\mathbf{y}$ given by the equation

$$\sum_{i=1}^{n-1} s_i \mathbf{t}_i = (\mathbf{z} - \mathbf{y}) - (\mathbf{n}(\mathbf{y}) \cdot (\mathbf{z} - \mathbf{y}))\mathbf{n}(\mathbf{y}),$$

where $\{\mathbf{t}_i\}$ is an arbitrary set of $n - 1$ orthonormal tangents to B at $\mathbf{y}$, $(\mathbf{t}_i \cdot \mathbf{t}_j) = 0$ if $i \neq j$ and $= 1$ if $i = j$, $(\mathbf{t}_i \cdot \mathbf{n}(\mathbf{y})) = 0$, and $\mathbf{n}(\mathbf{y})$ is the unit normal to B at $\mathbf{y}$ outward from D as usual. From condition (3.6) and the compactness of B , it is clear that there exists some $\delta_0 > 0$ such that over $\{\mathbf{z} \in B \mid |\mathbf{z} - \mathbf{y}| < \delta_0\}$, for every $\mathbf{y} \in B$, and for such set $\{\mathbf{t}_i\}$ this projection is one-to-one, bicontinuous, and has the inverse map $\mathbf{z} = \mathbf{f}(s_1, \dots, s_{n-1}; \mathbf{y}, \mathbf{t}_1, \dots, \mathbf{t}_{n-1})$ defined over at least

$$\left\{ (s_1, \dots, s_{n-1}) \in R_{n-1} \mid \left[\sum_{i=1}^{n-1} s_i^2 \right]^{\frac{1}{2}} < \frac{1}{2}\delta_0 \right\}$$

with continuous second partial derivatives in s_i there. Thus for $1 \leq i, j \leq n - 1$ we can define, the right hand side derivatives existing,

$$(3.15) \quad c_{ij}(\mathbf{y}; \mathbf{t}_1, \dots, \mathbf{t}_{n-1}) = \left(\mathbf{t}_i \cdot \left[\frac{\partial}{\partial s_j} \mathbf{n}(\mathbf{f}(s_1, \dots, s_{n-1}; \mathbf{y}, \mathbf{t}_1, \dots, \mathbf{t}_{n-1})) \right]_{s_k=0} \right)$$

as the elements of the $(n-1) \times (n-1)$ curvature matrix $C(\mathbf{y})$ of B at $\mathbf{y} \in B$ with respect to the tangent basis $\{\mathbf{t}_i\}$. The usual curvature tensor reduces to this at $\mathbf{y}$ for the s_i -coordinate system. Letting $I_{ij} = \delta_{ij}$ define the identity matrix I , we define the curvature invariants $q_k(\mathbf{y})$ in terms of the curvature matrix $C(\mathbf{y})$ by

$$(3.16) \quad \det(I - \beta C(\mathbf{y})) = 1 + \sum_{k=1}^{n-1} (-\beta)^k q_k(\mathbf{y}),$$

the $q_k(\mathbf{y})$ being independent of the choice of the tangent basis $\{\mathbf{t}_i\}$. Notice that (3.16) makes $q_1(\mathbf{y}) = \sum_{i=1}^{n-1} C_{ii}(\mathbf{y}; \mathbf{t}_1, \dots, \mathbf{t}_{n-1})$, and $(n-1)^{-1}$ times this quantity is often called the mean curvature. With these definitions and with the same convention about M and θ as in (3.8) and (3.11), we have the following expansion. Since (3.20) following implies (3.3), we see that condition (3.6) on B implies condition (3.1).

Lemma (3.17)

Let condition (3.6) be satisfied, let

$$\sum_{i=1}^{n-1} s_i \mathbf{t}_i = (\mathbf{z} - \mathbf{y}) - (\mathbf{n}(\mathbf{y}) \cdot (\mathbf{z} - \mathbf{y}))\mathbf{n}(\mathbf{y})$$

as above for $\mathbf{z}$ and $\mathbf{y} \in B$, and let $\mathbf{x} = (-h\mathbf{n}(\mathbf{y}) + \mathbf{y}) \in \bar{D}$ with $h \geq 0$. Then

$$(3.18) \quad \begin{aligned} (\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{x})) &= h + \frac{1}{2} \sum_{i,j=1}^{n-1} s_i s_j c_{ij}(\mathbf{y}; \mathbf{t}_1, \dots, \mathbf{t}_{n-1}) \\ &\quad + \left(h + \sum_{i=1}^{n-1} |s_i| \right)^{2+\eta} M\theta, \end{aligned}$$

$$(3.19) \quad \begin{aligned} |\mathbf{z} - \mathbf{x}|^2 &= \left(h^2 + \sum_{i=1}^{n-1} s_i^2 \right) - h \sum_{i,j=1}^{n-1} s_i s_j c_{ij}(\mathbf{y}; \mathbf{t}_1, \dots, \mathbf{t}_{n-1}) \\ &\quad + \left(h + \sum_{i=1}^{n-1} |s_i| \right)^{3+\eta} M\theta', \end{aligned}$$

and with $\mathbf{x}' = (-h\mathbf{n}(\mathbf{z}) + \mathbf{z})$ the Jacobian

$$J\left(\frac{x'_1, \dots, x'_n}{h, s_1, \dots, s_{n-1}}\right)$$

exists satisfying

$$(3.20) \quad \begin{aligned} \left| \left[J\left(\frac{x'_1, \dots, x'_n}{h, s_1, \dots, s_{n-1}}\right) \right]_{s_i=0, \atop \mathbf{z}=\mathbf{y}} \right| &= |\det(I - hC(\mathbf{y}))| \\ &= \left| 1 + \sum_{k=1}^{n-1} (-h)^k q_k(\mathbf{y}) \right|. \end{aligned}$$

The proofs of lemmas (3.8), (3.11), and (3.17) are rather obvious; see (M) for details.

4. Green Function Integrals and Eigenvalue Series

We define the various Green functions for D in the usual way as follows. For the heat equation on R_n the fundamental singularity is

$$Q_n(r; t) = (2\sqrt{\pi t})^{-n} e^{-r^2/4t}$$

over $r \geq 0, t > 0$. For the meson equation the fundamental singularity ${}_nK_\omega(r)$ is most conveniently defined (see for example [5], p. 555) for $\omega > 0$ and $r > 0$ by the requirements that ${}_nK_\omega(|\mathbf{x}|) \in L_1(R_n)$, that ${}_nK_\omega(r)$ be continuous in $r > 0$, and that

$$\int_{R_n} {}_nK_\omega(|\mathbf{x}|) e^{i(\mathbf{x} \cdot \mathbf{y})} d\mu_n(\mathbf{x}) = \frac{1}{\omega^2 + |\mathbf{y}|^2}$$

over $\mathbf{y} \in R_n$. Since the meson Green function is convenient to use only when $n = 2$ or $n = 3$, we note the explicit formulæ

$$\begin{aligned} {}_3K_\omega(r) &= \frac{1}{4\pi r} e^{-\omega r}, \\ {}_2K_\omega(r) &= \int_{-\infty}^{\infty} {}_3K_\omega(\sqrt{r^2 + v^2}) dv. \end{aligned}$$

Definition (4.1)

The heat equation Green function $H_\nu(\mathbf{x}, \mathbf{y}; t) = Q_n(|\mathbf{x} - \mathbf{y}|; t) - h_\nu(\mathbf{x}, \mathbf{y}; t)$ over $0 < t \leq T, \mathbf{x} \in D, \mathbf{y} \in \bar{D}$ is defined for $\nu = 0$ or 1 and some $T > 0$ by the requirements: that $h_\nu(\mathbf{x}, \mathbf{y}; t)$ be real-valued and continuous over $(\mathbf{x}, \mathbf{y}; t) \in D \times \bar{D} \times (0, T]$; that, for every fixed $\mathbf{x} \in D$, both

$$\lim_{t \rightarrow 0^+} \int_D [h_1(\mathbf{x}, \mathbf{y}; t)]^2 d\mu_n(\mathbf{y}) = 0$$

and $h_0(\mathbf{x}, \mathbf{y}; t)$ possess a continuous extension in $(\mathbf{y}; t)$ from $\bar{D} \times (0, T]$ to $\bar{D} \times [0, T]$ which vanishes at $t = 0$; that, for every fixed $\mathbf{x} \in D$, $h_\nu(\mathbf{x}, \mathbf{y}; t)$ have continuous second partials in the components of $\mathbf{y}$, a continuous first partial in t , and satisfy

$$-\nabla_{\mathbf{y}}^2 h_\nu(\mathbf{x}, \mathbf{y}; t) + \frac{\partial}{\partial t} h_\nu(\mathbf{x}, \mathbf{y}; t) = 0$$

over $(\mathbf{y}; t) \in D \times (0, T]$; that $\partial h_1(\mathbf{x}, \mathbf{y}; t)/\partial t \in L_1(D \times (t_1, T))$ in $(\mathbf{y}; t)$ for every fixed $\mathbf{x} \in D$ and $t_1 \in (0, T)$; and that, for every fixed $\mathbf{x} \in D$ and $t \in (0, T]$, $H_\nu(\mathbf{x}, \mathbf{y}; t)$ satisfy on $\mathbf{y} \in B$ the Dirichlet condition (1.2) or the modified Neumann condition (2.3) according as $\nu = 0$ or 1 .

Definition (4.2)

The meson Green function $G_\nu(\mathbf{x}, \mathbf{y}; \omega) = {}_nK_\omega(|\mathbf{x} - \mathbf{y}|) - g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ over $\mathbf{x} \in D, \mathbf{y} \in \bar{D}$, and $\mathbf{y} \neq \mathbf{x}$, is defined for $\omega > 0$ and $\nu = 0$ or 1 by the requirements: that $g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ be real-valued and continuous over $(\mathbf{x}, \mathbf{y}) \in D \times \bar{D}$; that, for every fixed

$\mathbf{x} \in D$, $g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ have continuous second partials in $\mathbf{y}$ and satisfy

$$-\nabla_{\mathbf{y}}^2 g_\nu + \omega^2 g_\nu = 0$$

over $\mathbf{y} \in D$; and that, for every fixed $\mathbf{x} \in D$, $G_\nu(\mathbf{x}, \mathbf{y}; \omega)$ satisfy on $\mathbf{y} \in B$ the Dirichlet condition (1.2) or the modified Neumann condition (2.3) according as $\nu = 0$ or 1.

From lemmas (2.6) and (2.7) it is clear that g_ν and h_ν , and hence G_ν and H_ν , must be unique if they exist. Although this existence is well known under much weaker conditions, we give in the next section only the appropriately modified Neumann-Poincaré construction of g_ν and h_ν under the smoothness conditions (3.1), (3.4), or (3.6) on B . This construction will verify the additional assumptions about G_ν and H_ν made in the following lemmas. These lemmas afford the well known relations between G_ν and H_ν and the eigenfunctions and eigenvalues to be studied. We remark that no use is made here of the ordinary Green and Neumann functions (G_ν with $\omega = 0$). This is because, although $G_0(\mathbf{x}, \mathbf{y}; 0)$ has the expected relation with the eigenfunctions for the Dirichlet condition on B , $G_1(\mathbf{x}, \mathbf{y}; 0)$ does not satisfy the Neumann condition (2.3) on B ([11], p. 246-247) and so does not behave similarly with the Neumann condition eigenfunctions.

Lemma (4.3)

Let the meson Green function $G_\nu(\mathbf{x}, \mathbf{y}; \omega)$ for D exist for all $\omega \geq \omega_0$, for some $\omega_0 > 0$, and for both $\nu = 0$ and 1. In addition to the requirements of definition (4.2), for $\omega \geq \omega_0$ and $\nu = 0$ or 1 let $\mu_n(B) = 0$; let $G_\nu(\mathbf{x}, \mathbf{y}; \omega) = G_\nu(\mathbf{y}, \mathbf{x}; \omega)$ over $\mathbf{x}, \mathbf{y} \in D$, $\mathbf{x} \neq \mathbf{y}$; let $\int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ be bounded and continuous over $\mathbf{x} \in D$; let $\int_D g_1(\mathbf{x}, \mathbf{y}; \omega) u(\mathbf{y}) d\mu_n(\mathbf{y})$ possess a continuous extension in $\mathbf{x}$ from D to $\bar{D}$ for every essentially bounded, measurable $u(\mathbf{y})$ over $\mathbf{y} \in D$; let

$$\int_D \int_D \int_D |\nabla_{\mathbf{x}} g_1(\mathbf{x}, \mathbf{y}; \omega)| |\nabla_{\mathbf{x}} g_1(\mathbf{x}, \mathbf{y}'; \omega)| d\mu_n(\mathbf{x}) d\mu_n(\mathbf{y}) d\mu_n(\mathbf{y}')$$

be finite; and let there be for every $\mathbf{x}' \in D$ some open $V \subseteq D$ containing $\mathbf{x}'$ such that the partial derivatives of order ≤ 2 of $g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ with respect to the $\mathbf{x}$ -coordinates are bounded over $(\mathbf{x}, \mathbf{y}) \in V \times D$. Then there is a complete orthonormal system of real-valued functions ${}_nu_j$ in $L_2(D)$ with associated real constants ${}_nu_{\lambda_j} \geq 0$, for integer $j \geq 1$ if $\nu = 0$ and integer $j \geq 0$ if $\nu = 1$, such that ${}_nu_{\lambda_j} \leq {}_nu_{\lambda_{j+1}}$, that ${}_nu_j(\mathbf{x})$ has continuous second partials and satisfies

$$-\nabla^2 {}_nu_j(\mathbf{x}) = {}_nu_{\lambda_j} {}_nu_j(\mathbf{x})$$

over $\mathbf{x} \in D$, and that ${}_nu_j(\mathbf{x})$ is continuous over $\mathbf{x} \in \bar{D}$ and satisfies the Dirichlet condition (1.2) or modified Neumann condition (2.3) on B according as $\nu = 0$ or 1. Such ${}_nu_{\lambda_j}$ and the linear subspaces of $L_2(D)$ spanned by ${}_nu_j$ with ${}_nu_{\lambda_j} = {}_nu_{\lambda_j}$ are unique. Moreover for $\omega \geq \omega_0$, ${}_nu_{\lambda_1} > 0$, ${}_nu_{\lambda_0} = 0$, ${}_nu_{\lambda_0}(\mathbf{x}) = [\mu_n(D)]^{-\frac{1}{2}}$, $\lim_{j \rightarrow \infty} {}_nu_{\lambda_j} = +\infty$,

$$(4.4) \quad \frac{{}_nu_j(\mathbf{x})}{{}_nu_{\lambda_j} + \omega^2} = \int_D G_\nu(\mathbf{x}, \mathbf{y}; \omega) {}_nu_j(\mathbf{y}) d\mu_n(\mathbf{y})$$

over $\mathbf{x} \in D$,

$$(4.5) \quad \frac{\delta_{1,\nu}}{\omega^4} + \sum_{i=1}^{\infty} \frac{1}{(\nu\lambda_i + \omega^2)^2} = \int_D \int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y}) d\mu_n(\mathbf{x}) < +\infty,$$

$$\sum_{i=1}^{\infty} \frac{1}{\nu\lambda_i^2} < +\infty,$$

$$(4.6) \quad \begin{aligned} G_\nu(\mathbf{x}, \mathbf{y}; \omega) - G_\nu(\mathbf{x}, \mathbf{y}; \omega_0) \\ = \frac{1}{\mu_n(D)} \left(\frac{1}{\omega^2} - \frac{1}{\omega_0^2} \right) \delta_{1,\nu} - \omega^2 \sum_{i=1}^{\infty} \frac{\nu u_i(\mathbf{x}) \nu u_i(\mathbf{y})}{\nu\lambda_i(\nu\lambda_i + \omega^2)} \\ + \omega_0^2 \sum_{i=1}^{\infty} \frac{\nu u_i(\mathbf{x}) \nu u_i(\mathbf{y})}{\nu\lambda_i(\nu\lambda_i + \omega_0^2)} \end{aligned}$$

over $\mathbf{x} \neq \mathbf{y} \in D$,

$$(4.7) \quad \begin{aligned} \lim_{\mathbf{y} \rightarrow \mathbf{x}} [G_\nu(\mathbf{x}, \mathbf{y}; \omega) - G_\nu(\mathbf{x}, \mathbf{y}; \omega_0)] \\ = \frac{1}{\mu_n(D)} \left(\frac{1}{\omega^2} - \frac{1}{\omega_0^2} \right) \delta_{1,\nu} - \omega^2 \sum_{i=1}^{\infty} \frac{(\nu u_i(\mathbf{x}))^2}{\nu\lambda_i(\nu\lambda_i + \omega^2)} \\ + \omega_0^2 \sum_{i=1}^{\infty} \frac{(\nu u_i(\mathbf{x}))^2}{\nu\lambda_i(\nu\lambda_i + \omega_0^2)} \end{aligned}$$

over $\mathbf{x} \in D$, and the series on the right hand side of (4.6) and (4.7) are absolutely convergent uniformly over $\mathbf{x}$ and $\mathbf{y} \in C$ for every compact subset C of D , and

$$(4.8) \quad \begin{aligned} \int_D [\lim_{\mathbf{y} \rightarrow \mathbf{x}} (G_\nu(\mathbf{x}, \mathbf{y}; \omega) - G_\nu(\mathbf{x}, \mathbf{y}; \omega_0))] d\mu_n(\mathbf{x}) \\ = \left(\frac{1}{\omega^2} - \frac{1}{\omega_0^2} \right) \delta_{1,\nu} - \omega^2 \sum_{i=1}^{\infty} \frac{1}{\nu\lambda_i(\nu\lambda_i + \omega^2)} + \omega_0^2 \sum_{i=1}^{\infty} \frac{1}{\nu\lambda_i(\nu\lambda_i + \omega_0^2)}. \end{aligned}$$

We remark first that $\int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ can be bounded and continuous over $\mathbf{x} \in D$ only if ${}_nK_\omega(|\mathbf{y}|)$ is in L_2 over $|\mathbf{y}| \leq 1$, and in the next section we will see that this is also sufficient. But $\int_0^1 [{}_nK_\omega(r)]^2 r^{n-1} dr$ is finite only for $n = 2$ or 3 ([5], p. 555). Thus lemma (4.3) does not apply for $n > 3$, and we use the following substitute. Here we will not try to obtain relations corresponding to (4.5) through (4.8), since it is more convenient to use the corresponding relations for the heat Green function. In this following lemma the k th iterated kernel $G_\nu^{(k)}(\mathbf{x}, \mathbf{y}; \omega)$ is defined by the integral

$$\begin{aligned} G_\nu^{(k)}(\mathbf{x}, \mathbf{y}; \omega) = \int_D \int_D \cdots \int_D G_\nu(\mathbf{x}, {}_1\xi; \omega) G_\nu({}_1\xi, {}_2\xi; \omega) \\ \cdots G_\nu({}_k\xi, \mathbf{y}; \omega) d\mu_n({}_k\xi) \cdots d\mu_n({}_1\xi). \end{aligned}$$

Lemma (4.9)

Let all the hypotheses of lemma (4.3) be satisfied except for the condition that ${}_nF_\omega(\mathbf{x}) = \int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ be bounded and continuous over D . In place of

this condition for $\omega \geq \omega_0$ let $\sup_{\mathbf{x} \in D} \int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)| d\mu_n(\mathbf{y})$ be finite and let $\sup_{\mathbf{x} \in D} \int_D |G_\nu^{(k)}(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ be finite for some integer $k \geq 1$. Then all the conclusions of lemma (4.3) remain valid except for the relations (4.5) through (4.8).

To prove lemma (4.3), we essentially follow CARLEMAN's argument, see (M) for details. First, since G_ν is real-valued, symmetric, and in $L_2(D \times D)$, it defines a bounded, Hermitian integral operator $\mathbf{G}_{\nu, \omega}$ which is compact on the Hilbert space $L_2(D)$. Thus this operator has a countable point spectrum $\{\nu\beta_i(\omega)\}$, $\lim_{i \rightarrow \infty} \nu\beta_i(\omega) = 0$, with eigenfunctions νu_i , $\nu\beta_i(\omega)\nu u_i = \mathbf{G}_{\nu, \omega}(\nu u_i)$, and these νu_i form a complete orthonormal system in $L_2(D)$.

Next, every $\nu\beta_i(\omega) \neq 0$. For otherwise there would exist $u \in L_2(D)$ with $\|u\| = 1$ and $\mathbf{G}_{\nu, \omega}(u) = 0$. Hence, using $\mu_n(B) = 0$, there would also exist $\tilde{u}(\mathbf{x})$ having continuous second partials over D , vanishing in a neighborhood of B , and satisfying $\|u - \tilde{u}\| < \frac{3}{4}$. Thus $\tilde{u} = \mathbf{G}_{\nu, \omega}(w)$ for $w(\mathbf{x}) = \omega^2 \tilde{u}(\mathbf{x}) - \nabla^2 \tilde{u}(\mathbf{x})$ would yield $0 = (\mathbf{G}_{\nu, \omega}(u), w) = (u, \mathbf{G}_{\nu, \omega}(w)) = (u, \tilde{u})$ and hence the contradiction $1 = \|u\|^2 = (u, \tilde{u}) + (u, u - \tilde{u}) \leq \|u - \tilde{u}\| < \frac{3}{4}$.

Next, $\nu\beta_i(\omega)\nu u_i(\mathbf{x}) = \int_D G_\nu(\mathbf{x}, \mathbf{y}; \omega)\nu u_i(\mathbf{y}) d\mu_n(\mathbf{y})$ over $\mathbf{x} \in D$, $\nu\beta_i(\omega) \neq 0$, and the given boundedness of $\nu F_\omega(\mathbf{x})$ over $\mathbf{x} \in D$ show by a standard argument ([5], theorem (5.5), p. 573 and [7], vol. II, p. 228) that $\nu u_i(\mathbf{x})$ is bounded, has continuous second partials, and satisfies

$$(4.10) \quad -\nabla^2 \nu u_i(\mathbf{x}) = \left\{ -\omega^2 + \frac{1}{\nu\beta_i(\omega)} \right\} \nu u_i(\mathbf{x}) = \nu\lambda_i \nu u_i(\mathbf{x})$$

over $\mathbf{x} \in D$.

We also verify that $\nu u_i(\mathbf{x})$ satisfies the Dirichlet condition (1.2) or the modified Neumann condition (2.3) on B according as $\nu = 0$ or $\nu = 1$.

Now consider real-valued $u \in L_2(D)$ with $\|u\| = 1$ such that $u(\mathbf{x})$ has continuous second partials and satisfies $-\nabla^2 u(\mathbf{x}) = \lambda u(\mathbf{x})$ over $\mathbf{x} \in D$ and that $u(\mathbf{x})$ satisfies (1.2) or (2.3) on B according as $\nu = 0$ or 1. We have just shown that νu_i and $\nu\lambda_i$ satisfy such conditions. If $\nu = 0$, then condition (1.2) on B and the maximum principle require $\lambda > 0$. If $\nu = 1$, then $\lambda = \int_D |\nabla u(\mathbf{x})|^2 d\mu_n(\mathbf{x}) \geq 0$, and $= 0$ only for $u(\mathbf{x})$ constant on D , follows from equation (2.5) under condition (2.3) on B . Thus the $\nu\lambda_i$ can be ordered as stated. Finally to prove the stated uniqueness of the $\nu\lambda_i$ and νu_i , and thus their independence of $\omega \geq \omega_0$, we need only show that the assumed λ and u satisfy

$$\frac{u(\mathbf{x})}{\lambda + \omega^2} = \int_D G_\nu(\mathbf{x}, \mathbf{y}; \omega) u(\mathbf{y}) d\mu_n(\mathbf{y})$$

over all $\mathbf{x} \in D$ and $\omega \geq \omega_0$, and thus occur among the $\nu\lambda_i$ and νu_i . Here, as shown above, $w(\mathbf{x}) = \int_D G_\nu(\mathbf{x}, \mathbf{y}; \omega) u(\mathbf{y}) d\mu_n(\mathbf{y})$ has continuous second partials with $-\nabla^2 w(\mathbf{x}) = u(\mathbf{x}) - \omega^2 w(\mathbf{x})$ over $\mathbf{x} \in D$ and satisfies condition (1.2) or (2.3) on B according as $\nu = 0$ or 1. Thus $(-\nabla^2 + \omega^2)[w - (\lambda + \omega^2)^{-1}u] = u - u = 0$ over D , and hence lemma (2.6) gives $w - (\lambda + \omega^2)^{-1}u = 0$ over D as desired.

Only (4.5) through (4.8) remain to be shown. (4.5) follows clearly from (4.4)

and the completeness of $\{,u_i(\mathbf{x}), ,u_i(\mathbf{y})\}$ in $L_2(D \times D)$, and $\sum_{i=1}^{\infty} (, \lambda_i)^{-2} < +\infty$ follows from $\sum_{i=1}^{\infty} (, \lambda_i + \omega^2)^{-1} < +\infty$ and $, \lambda_i > 0$ for $j \geq 1$. Now notice that

$$\frac{1}{, \lambda_i + \omega^2} - \frac{1}{, \lambda_i + \omega_0^2} = -\frac{\omega^2}{, \lambda_i(, \lambda_i + \omega^2)} + \frac{\omega_0^2}{, \lambda_i(, \lambda_i + \omega_0^2)}.$$

Thus, since $\int_D |G_{,}(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y}) = ,F_{\omega}(\mathbf{x}) < +\infty$, for fixed $\mathbf{x} \in D$ we have (4.6) valid with convergence in $L_2(D)$ -norm over $\mathbf{y}$ by using (4.4) and the completeness of $\{,u_i\}$. Also the relation $, \lambda_i^{-1} \leq (1 + , \lambda_i^{-1}\omega^2)(, \lambda_i + \omega^2)^{-1}$, the Schwarz inequality, and the Bessel equality show that

$$(4.12) \quad \sum_{i=N+1}^{\infty} \left| \frac{,u_i(\mathbf{x}), u_i(\mathbf{y})}{, \lambda_i(, \lambda_i + \omega^2)} \right| \leq \left(1 + \frac{\omega^2}{, \lambda_1}\right) \left[,F_{\omega}(\mathbf{x}) - \frac{\delta_{1, ,}}{\omega^4 \mu_n(D)} \right. \\ \left. - \sum_{i=1}^N \left(\frac{,u_i(\mathbf{x})}{, \lambda_i + \omega^2} \right)^2 \right]^{\frac{1}{2}} \cdot \left[,F_{\omega}(\mathbf{y}) - \frac{\delta_{1, ,}}{\omega^4 \mu_n(D)} - \sum_{i=1}^N \left(\frac{,u_i(\mathbf{y})}{, \lambda_i + \omega^2} \right)^2 \right]^{\frac{1}{2}}$$

for every $\mathbf{x}$ and $\mathbf{y} \in D$. Also each of the square root factors on the right side of (4.12) is continuous over D , since $,F_{\omega}$ is given to be so, and has the limit 0 as $N \rightarrow +\infty$ everywhere over D , again by (4.4) and the completeness of $\{,u_i\}$. Since the limit function of an everywhere monotonely convergent sequence of continuous functions over a compact space is continuous if and only if the convergence is uniform ([9], p. 121, theorem 26), we have established (4.6) and (4.7) with uniform, absolute convergence over compact subsets of D as stated. (4.8) follows from (4.7) by using $\|,u_i\|^2 = 1$ and the monotone convergence theorem.

We now pass to the proof of lemma (4.9), having completed that of lemma (4.3). Here, as is well known ([6], p. 112, $A(x) = 1$), the condition

$$\sup_{\mathbf{x} \in D} \int_D |G_{,}(\mathbf{x}, \mathbf{y}; \omega)| d\mu_n(\mathbf{y}) < +\infty$$

guarantees that the real, symmetric kernel $G_{,}$ defines a bounded Hermitian operator on the Hilbert space $L_2(D)$. Also, since $G_{,}^{(k)} \in L_2(D \times D)$, the k th iterate of this operator is compact. Thus this operator has a countable point spectrum $\{, \beta_i(\omega)\}$ with $\lim_{i \rightarrow \infty} , \beta_i(\omega) = 0$, with a complete orthonormal system of eigenfunctions $,u_i$ in $L_2(D)$, and with

$$, \beta_i(\omega), u_i(\mathbf{x}) = \int_D G_{,}(\mathbf{x}, \mathbf{y}; \omega), u_i(\mathbf{y}) d\mu_n(\mathbf{y})$$

for almost all $\mathbf{x} \in D$. Actually, this last equation can be taken to hold for all $\mathbf{x} \in D$, since the boundedness of $\int_D |G_{,}^{(k)}(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ over $\mathbf{x} \in D$ requires $,u_i$ to be essentially bounded there. The remaining part of the proof of lemma (4.9) follows exactly like that of lemma (4.3).

Now suppose the hypotheses for lemma (4.9) are satisfied. Since we have shown zero not to be in the point spectrum of the bounded, Hermitian operator $G_{, , \omega}$ on $L_2(D)$ defined by $G_{,}(\mathbf{x}, \mathbf{y}; \omega)$, the operator $T_{, , \omega} = \omega^2 \mathbf{I} - G_{, , \omega}^{-1}$ exists

and is self-adjoint with domain $\mathfrak{D}_\nu = \mathbf{G}_{\nu, \omega}(L_2(D))$. Using (4.4), it is not hard to see that

$$\mathfrak{D}_\nu = \left\{ u \in L_2(D) \mid \sum_{j=1}^{\infty} \nu \lambda_j^2 |\langle u, \nu u_j \rangle|^2 < +\infty \right\}$$

and is independent of $\omega \geq \omega_0$. In order to relate the heat Green function to $\nu \lambda_j$ and $\nu u_j(\mathbf{x})$, we will use the following modified form of the uniqueness lemma (2.7). Here let $\mathfrak{C}_1$ denote the set of those functions in $L_2(D)$ which are continuous and possess continuous first partial derivatives over D .

Lemma (4.13)

Let the hypotheses of lemma (4.9) be satisfied for D . Then for every $\omega \geq \omega_0$ the proper subset $\mathbf{G}_{\nu, \omega}(\mathfrak{C}_1)$ of $\mathfrak{D}_\nu$ contains every u which is bounded over D , which possesses continuous third partials at every $\mathbf{x} \in D$, for which $\nabla^2 u$ is bounded over D , and which satisfies condition (1.2) or (2.3) on B according as $\nu = 0$ or 1. Also $[\mathbf{T}_{\nu, \omega}(u)](\mathbf{x}) = \nabla^2 u(\mathbf{x})$ almost everywhere (μ_n) over $\mathbf{x} \in D$ for every $\omega \geq \omega_0$ and $u \in \mathbf{G}_{\nu, \omega}(\mathfrak{C}_1)$. If $u_t \in \mathfrak{D}_\nu$ and $\lim_{\tau \rightarrow 0} \|\tau^{-1}(u_{t+\tau} - u_t) - \mathbf{T}_{\nu, \omega}(u_t)\| = 0$ over $0 < t \leq T$, and if $\lim_{t \rightarrow 0^+} \|u_t\| = 0$, then $u_t = 0$ in $L_2(D)$ throughout $0 < t \leq T$.

First, $\mathfrak{C}_1 \subset L_2(D)$ properly and the fact that $\mathbf{G}_{\nu, \omega}$ is one-to-one make $\mathbf{G}_{\nu, \omega}(\mathfrak{C}_1) \subset \mathfrak{D}_\nu$ properly. Second, if $u(\mathbf{x})$ is bounded and possesses continuous third partials over $\mathbf{x} \in D$, and if $\nabla^2 u$ is bounded over D , then $v = \mathbf{G}_{\nu, \omega}(\omega^2 u - \nabla^2 u) \in L_2(D)$ exists and, as we have seen in the proof of lemma (4.3), $v(\mathbf{x})$ can be taken to have continuous second partials in $\mathbf{x} \in D$ with $\nabla^2 v(\mathbf{x}) = \omega^2 v(\mathbf{x}) - [\omega^2 u(\mathbf{x}) - \nabla^2 u(\mathbf{x})]$. Hence $(-\nabla^2 + \omega^2)(u(\mathbf{x}) - v(\mathbf{x})) = 0$ over $\mathbf{x} \in D$, and, as shown in the proof just quoted, the boundedness of $\omega^2 u(\mathbf{x}) - \nabla^2 u(\mathbf{x})$ guarantees that $v(\mathbf{x})$ satisfies conditions (1.2) or (2.3) on B according as $\nu = 0$ or 1. Thus, if $u(\mathbf{x})$ also satisfies such conditions, lemma (2.6) yields $u(\mathbf{x}) = v(\mathbf{x}) = \int_D G_\nu(\mathbf{x}, \mathbf{y}; \omega) [\omega^2 u(\mathbf{y}) - \nabla^2 u(\mathbf{y})] d\mu_n(\mathbf{y})$, and $u \in \mathbf{G}_{\nu, \omega}(\mathfrak{C}_1)$ as desired for the second statement.

Third, assume $u \in \mathbf{G}_{\nu, \omega}(\mathfrak{C}_1)$; thus $u = \mathbf{G}_{\nu, \omega}(\tilde{u})$ for some $\tilde{u} \in \mathfrak{C}_1$. Hence, by the argument quoted above, $u(\mathbf{x})$ can be taken to have continuous second partials and $\nabla^2 u(\mathbf{x}) = \omega^2 u(\mathbf{x}) - \tilde{u}(\mathbf{x})$ at every $\mathbf{x} \in D$. But $u = \mathbf{G}_{\nu, \omega}(\tilde{u})$ yields $\mathbf{T}_{\nu, \omega}(u) = \omega^2 u - \tilde{u}$ by definition, thus $[\mathbf{T}_{\nu, \omega}(u)](\mathbf{x}) = \omega^2 u(\mathbf{x}) - \tilde{u}(\mathbf{x}) = \nabla^2 u(\mathbf{x})$ almost everywhere (μ_n) over $\mathbf{x} \in D$ as desired.

Finally, assume $u_t \in \mathfrak{D}_\nu$ and $\lim_{\delta \rightarrow 0} \|\delta^{-1}(u_{t+\delta} - u_t) - \mathbf{T}_{\nu, \omega}(u_t)\| = 0$ over $t \in (0, T]$, and that $\lim_{t \rightarrow 0^+} \|u_t\| = 0$. The first condition clearly shows that $d(u_t, u_t)/dt = \langle \mathbf{T}_{\nu, \omega}(u_t), u_t \rangle + \langle u_t, \mathbf{T}_{\nu, \omega}(u_t) \rangle = 2\langle \mathbf{T}_{\nu, \omega}(u_t), u_t \rangle$ exists for $0 < t \leq T$. But (4.4) and the Parseval relation yield

$$\begin{aligned} \langle \mathbf{T}_{\nu, \omega}(u_t), u_t \rangle &= \sum_{j=1}^{\infty} \langle \mathbf{T}_{\nu, \omega}(u_t), \nu u_j \rangle \overline{\langle u_t, \nu u_j \rangle} = \sum_{j=1}^{\infty} \langle u_t, \mathbf{T}_{\nu, \omega}(\nu u_j) \rangle \overline{\langle u_t, \nu u_j \rangle} \\ &= \sum_{j=1}^{\infty} (-\nu \lambda_j) |\langle u_t, \nu u_j \rangle|^2 \leq 0, \end{aligned}$$

so $d \|u_t\|^2/dt \leq 0$ over $0 < t \leq T$. Thus $\|u_t\|^2 \leq \lim_{t' \rightarrow 0^+} \|u_{t'}\|^2 = 0$ makes $\|u_t\| = 0$ and $u_t = 0$ in $L_2(D)$ over $0 < t \leq T$ as desired.

Lemma (4.14)

Let the hypotheses of lemma (4.9) be satisfied for D and let the heat Green function $H_\nu(\mathbf{x}, \mathbf{y}; t)$ exist over $D \times \bar{D} \times (0, T]$ for some $T > 0$ and for $\nu = 0$ or 1 . In addition to the requirements of definition (4.1), let $H_\nu(\mathbf{x}, \mathbf{y}; t)$ be bounded and continuous in $(\mathbf{x}, \mathbf{y}) \in D \times D$ for every $t \in (0, T]$; and for every $u(\mathbf{x})$ continuous over D and vanishing over some neighborhood of B let $w(\mathbf{y}; t) = \int_D u(\mathbf{x}) H_\nu(\mathbf{x}, \mathbf{y}; t) d\mu_n(\mathbf{x})$ have continuous third partials in $\mathbf{y}$ and a continuous first partial in t over $D \times (0, T]$, let $w(\mathbf{y}; t)$ satisfy conditions (1.2) or (2.3) in $\mathbf{y}$ on B according as $\nu = 0$ or 1 for every $t \in (0, T]$, let $\nabla_{\mathbf{y}}^2 w(\mathbf{y}; t) = \partial w(\mathbf{y}; t)/\partial t$ and be bounded over $\mathbf{y} \in D$ and $t \in [t', T]$ for every $t' \in (0, T]$, and let

$$\lim_{t \rightarrow 0^+} \int_D \left| \int_D u(\mathbf{x}) h_\nu(\mathbf{x}, \mathbf{y}; t) d\mu_n(\mathbf{x}) \right|^2 d\mu_n(\mathbf{y}) = 0.$$

Then ${}_\nu \lambda_i$ and ${}_\nu u_i(\mathbf{x})$ given by lemma (4.9) satisfy

$$(4.15) \quad e^{-{}_\nu \lambda_i t} {}_\nu u_i(\mathbf{x}) = \int_D H_\nu(\mathbf{x}, \mathbf{y}; t) {}_\nu u_i(\mathbf{y}) d\mu_n(\mathbf{y}), \quad \mathbf{x} \in D, \quad t \in (0, T];$$

$$(4.16) \quad H_\nu(\mathbf{x}, \mathbf{y}; t) = H_\nu(\mathbf{y}, \mathbf{x}; t) = \frac{\delta_{1,\nu}}{\mu_n(D)} + \sum_{i=1}^{\infty} e^{-{}_\nu \lambda_i t} {}_\nu u_i(\mathbf{x}) {}_\nu u_i(\mathbf{y})$$

over $(\mathbf{x}, \mathbf{y}; t) \in D \times D \times (0, T]$ with absolute convergence uniformly over $\mathbf{x}$ and $\mathbf{y} \in C$ for every compact subset C of D and every $t \in (0, T]$; and for every $t \in (0, T]$

$$(4.17) \quad \int_D H_\nu(\mathbf{x}, \mathbf{x}; t) d\mu_n(\mathbf{x}) = \delta_{1,\nu} + \sum_{i=1}^{\infty} e^{-{}_\nu \lambda_i t}.$$

To prove lemma (4.14), consider any given $u(\mathbf{x})$ continuous over D and vanishing outside some compact subset of D , and thus over some neighborhood of B . Our proof depends upon establishing for $w(\mathbf{y}; t) = \int_D u(\mathbf{x}) H_\nu(\mathbf{x}, \mathbf{y}; t) d\mu_n(\mathbf{x})$ that

$$(4.18) \quad w(\mathbf{y}; t) = \frac{\delta_{1,\nu}}{\mu_n(D)} \int_D u(\mathbf{x}) d\mu_n(\mathbf{x}) + \sum_{i=1}^{\infty} e^{-{}_\nu \lambda_i t} (u, {}_\nu u_i) {}_\nu u_i(\mathbf{y})$$

in $L_2(D)$ over $\mathbf{y}$ for every $t \in (0, T]$. Since $e^{-{}_\nu \lambda_i t} \leq 1$ and $u \in L_2(D)$, the Bessel inequality shows that the right side of (4.18) is an $L_2(D)$ -norm convergent series. We denote its sum by $w^*(\mathbf{y}; t) \in L_2(D)$ over $\mathbf{y}$ for $t \geq 0$, set $\tilde{w}(\mathbf{y}; t) = w(\mathbf{y}; t) - w^*(\mathbf{y}; t)$, and use lemma (4.13) to show $\tilde{w}(\cdot; t) = 0$ and obtain (4.18).

First, $\lambda e^{-\lambda t} < t^{-1}$ over $\lambda \geq 0$ yields $\sum_i {}_\nu \lambda_i^2 |(w^*, {}_\nu u_i)|^2 < t^{-2} \|u\|^2$ and $w^*(\cdot; t) \in \mathfrak{D}_\nu$ over $t > 0$. Since our hypotheses here and lemma (4.13) yield $w(\cdot; t) \in \mathfrak{D}_\nu$ over $0 < t \leq T$, so also is $\tilde{w}(\cdot; t)$. Second,

$$\mathbf{T}_{\nu, \omega}(w^*(\cdot; t)) = \sum_{i=1}^{\infty} (-{}_\nu \lambda_i e^{-{}_\nu \lambda_i t})(u, {}_\nu u_i) {}_\nu u_i = \lim_{\tau \rightarrow 0} \tau^{-1} [w^*(\cdot; t + \tau) - w^*(\cdot; t)]$$

in the sense of $L_2(D)$ -norm limits. Again our hypotheses and lemma (4.13) show that $[\mathbf{T}_{\nu, \omega}(w(\cdot; t))](\mathbf{y}) = \nabla^2 w(\mathbf{y}; t) = \partial w(\mathbf{y}; t)/\partial t$, which in turn equals the $L_2(D)$ -norm limit of $\tau^{-1}[w(\cdot; t + \tau) - w(\cdot; t)]$ as $\tau \rightarrow 0$. Hence

$$\lim_{\tau \rightarrow 0} \|\tau^{-1}[\mathcal{W}(\cdot; t + \tau) - \mathcal{W}(\cdot; t)] - \mathbf{T}_{\nu, \omega}(\mathcal{W}(\cdot; t))\| = 0$$

over $0 < t \leq T$. Finally, $\|w^*(\cdot; t) - u\| \rightarrow 0$ and $\|w(\cdot; t) - u\| \rightarrow 0$ as $t \rightarrow 0^+$ show that $\mathcal{W}(\cdot; t)$ satisfies all the conditions of the last statement of lemma (4.13), and hence vanishes over $0 < t \leq T$ establishing (4.18) as desired (see (M) for more details).

From (4.18) and the Fubini theorem, by taking the inner product with νu_i , follows for u as above

$$(4.19) \quad 0 = \int_D u(\mathbf{x}) \left\{ e^{-\nu \lambda_i t} \nu u_i(\mathbf{x}) - \int_D H_\nu(\mathbf{x}, \mathbf{y}; t) \nu u_i(\mathbf{y}) d\mu_n(\mathbf{y}) \right\} d\mu_n(\mathbf{x}),$$

which proves (4.15). This in turn yields (4.16) and (4.17), the convergence estimate (4.12) being replaced analogously by

$$(4.20) \quad \begin{aligned} & \sum_{i=N+1}^{\infty} |e^{-\nu \lambda_i t} \nu u_i(\mathbf{x}) \nu u_i(\mathbf{y})| \\ & \leq \left[\int_D |H_\nu(\mathbf{x}, \boldsymbol{\xi}; \tfrac{1}{2}t)|^2 d\mu_n(\boldsymbol{\xi}) - \frac{\delta_{1, \nu}}{\mu_n(D)} - \sum_{i=1}^N e^{-\nu \lambda_i t} (\nu u_i(\mathbf{x}))^2 \right]^{\frac{1}{2}} \\ & \quad \cdot \left[\int_D |H_\nu(\mathbf{y}, \boldsymbol{\xi}; \tfrac{1}{2}t)|^2 d\mu_n(\boldsymbol{\xi}) - \frac{\delta_{1, \nu}}{\mu_n(D)} - \sum_{i=1}^N e^{-\nu \lambda_i t} (\nu u_i(\mathbf{y}))^2 \right]^{\frac{1}{2}} \end{aligned}$$

over $D \times D \times (0, T]$.

5. Construction of the Green Functions

In this section we shall give the details of the explicit Neumann-Poincaré construction of the Green functions G_ν and H_ν under our various smoothness conditions on the boundary B of D , and then observe that this construction enables us to verify the hypotheses of lemma (4.14) and (4.3) or (4.9). Thus, in the following section, we can use the relations (4.8) and (4.17) with this explicit construction to get asymptotic information about $\nu \lambda_i$.

It will be convenient in the following theorem to let $r_0(\mathbf{y})$ be the diameter of B if B has no corners, and otherwise to let $r_0(\mathbf{y})$ be the minimum of $|\mathbf{y} - \mathbf{z}|$ over the finite, non-empty set of $\mathbf{z}$ which are corners of B .

Theorem (5.1)

Let the boundary B of D satisfy either condition (3.1), (3.4), or (3.6). Then g_ν and G_ν exist satisfying definition (4.2) over $\omega \geq \omega_0$, $\nu = 0$ or 1 , for some ω_0 , $0 < \omega_0 < +\infty$. Moreover, for $(\mathbf{x}, \mathbf{y}; \omega) \in D \times D \times [\omega_0, +\infty)$,

$$(5.2) \quad g_0(\mathbf{x}, \mathbf{y}; \omega) = - \int_B (\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{y} - \mathbf{z}|)) v_0(\mathbf{z}; \mathbf{x}, \omega) d\sigma_n(\mathbf{z}),$$

$$g_1(\mathbf{x}, \mathbf{y}; \omega) = - \int_B {}_n K_\omega(|\mathbf{y} - \mathbf{z}|) v_1(\mathbf{z}; \mathbf{x}, \omega) d\sigma_n(\mathbf{z}).$$

Here, for every fixed $\mathbf{x} \in D$ and $\omega \geq \omega_0$, $v_0(\mathbf{z}; \mathbf{x}, \omega)$ is continuous in $\mathbf{z}$ over B and $v_1(\mathbf{z}; \mathbf{x}, \omega)$ is continuous over $C_\delta = \{\mathbf{z} \in B | r_0(\mathbf{z}) \geq \delta\}$ and satisfies there a Hölder condition of order η' , $0 < \eta' < 1$, for every $\delta > 0$. For every $\mathbf{x} \in D$, $\omega \geq \omega_0$, and ${}_0\mathbf{z} = \mathbf{z} \in B$ which is not a corner of B ,

$$(5.3) \quad \begin{aligned} v_0(\mathbf{z}; \mathbf{x}, \omega) = & 2 {}_n K_\omega(|\mathbf{z} - \mathbf{x}|) \\ & + \sum_{m=1}^{\infty} \int_{(\times B)^m} \left[\prod_{k=1}^m 2(\mathbf{n}({}_k\mathbf{z}) \cdot \nabla_{{}_k\mathbf{z}} {}_n K_\omega(|{}_{k-1}\mathbf{z} - {}_k\mathbf{z}|)) \right] \\ & \cdot 2 {}_n K_\omega(|{}_m\mathbf{z} - \mathbf{x}|) d(\times \sigma_n)^m({}_1\mathbf{z}, \dots, {}_m\mathbf{z}), \end{aligned}$$

$$(5.4) \quad \begin{aligned} v_1(\mathbf{z}; \mathbf{x}, \omega) = & -2(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{z} - \mathbf{x}|)) \\ & + \sum_{m=1}^{\infty} (-1)^{m+1} \int_{(\times B)^m} \left[\prod_{k=1}^{m+1} 2(\mathbf{n}({}_{k-1}\mathbf{z}) \cdot \nabla_{{}_{k-1}\mathbf{z}} {}_n K_\omega(|{}_{k-1}\mathbf{z} - {}_k\mathbf{z}|)) \right]_{m+1} {}_m\mathbf{z} = \mathbf{x} \\ & \cdot d(\times \sigma_n)^m({}_1\mathbf{z}, \dots, {}_m\mathbf{z}); \end{aligned}$$

the right side of (5.3) converges absolutely and uniformly over all $\mathbf{z} \in B$; and the right side of (5.4) converges in $L_1(B, \sigma_n)$ -norm over $\mathbf{z}$, and absolutely and uniformly over $\mathbf{z} \in C_\delta$ for every $\delta > 0$. Also if $n = 2$ and $\omega \geq \omega_0$, then $g_\nu(\mathbf{x}, \mathbf{x}; \omega) \in L_1(D)$ and

$$(5.5) \quad \begin{aligned} - \int_D g_0(\mathbf{x}, \mathbf{x}; \omega) d\mu_2(\mathbf{x}) &= (-1) \sum_{m=0}^{\infty} (-1)^m L_m(\omega), \\ - \int_D g_1(\mathbf{x}, \mathbf{x}; \omega) d\mu_2(\mathbf{x}) &= \sum_{m=0}^{\infty} L_m(\omega), \end{aligned}$$

both absolutely convergent, with

$$(5.6) \quad \begin{aligned} L_m(\omega) = & \int_D \int_{(\times B)^{m+1}} \left\{ (-2)(\mathbf{n}({}_0\mathbf{z}) \cdot \nabla_{{}_0\mathbf{z}} {}_2 K_\omega(|\mathbf{x} - {}_0\mathbf{z}|)) {}_2 K_\omega(|{}_m\mathbf{z} - \mathbf{x}|) \right. \\ & \cdot \left. \left[\prod_{k=1}^m (-2)(\mathbf{n}({}_k\mathbf{z}) \cdot \nabla_{{}_k\mathbf{z}} {}_2 K_\omega(|{}_{k-1}\mathbf{z} - {}_k\mathbf{z}|)) \right] \right\} d(\times \sigma_2)^{m+1}({}_0\mathbf{z}, \dots, {}_m\mathbf{z}) d\mu_2(\mathbf{x}). \end{aligned}$$

Before proving the various statements of this theorem, we remark that if $n = 3$ we can only say that $[g_\nu(\mathbf{x}, \mathbf{x}; \omega) - g_\nu(\mathbf{x}, \mathbf{x}; \omega_0)] \in L_1(D)$ instead of $g_\nu(\mathbf{x}, \mathbf{x}; \omega)$ itself. The integral of this difference can easily be evaluated like (5.5), but we will not bother to write out the details.

To prove theorem (5.1) we first establish a few lemmas.

Lemma (5.7)

Let B satisfy conditions (3.1), (3.4), or (3.6), and define over $\omega > 0$

$$(5.8) \quad R_\omega = \sup_{\mathbf{y} \in B} \int_B |2(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{y} - \mathbf{z}|))| d\sigma_n(\mathbf{z}),$$

$$\tilde{R}_\omega = \sup_{\mathbf{y} \in \tilde{B}} \int_B |2(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{y} - \mathbf{z}|))| d\sigma_n(\mathbf{z}).$$

Then $\lim_{\omega \rightarrow +\infty} R_\omega = 0$ under condition (3.1) or (3.6), $\lim_{\omega \rightarrow +\infty} R_\omega = \pi^{-1}(\max_i |\alpha_i|) < 1$ under condition (3.4), and in either case both R_ω and $\tilde{R}_\omega$ are bounded and decreasing over $\omega > 0$.

These results depend upon information about the derivative ${}_n K'_\omega(r)$ of the meson fundamental singularity. With $S_1 = \{\mathbf{x} \in R_n \mid |\mathbf{x}| = 1\}$, we know ([5], p. 555, (2.1) through (2.10)) that ${}_n K'_\omega(r) = -(2\pi r)_{n+2} K_\omega(r) = -(\sigma_n(S_1) r^{n-1})^{-1} p_n(\omega r) < 0$, and from ${}_n K''_\omega(r) + (n-1)r^{-1} {}_n K'_\omega(r) = \omega^2 {}_n K_\omega(r)$ follows $p'_n(v) = -(\sigma_n(S_1) v^{n-1}) {}_n K_1(v) < 0$. Thus

$$(5.9) \quad {}_n K'_\omega(r) = -\frac{p_n(\omega r)}{\sigma_n(S_1) r^{n-1}},$$

$p_n(v)$ is decreasing over $v > 0$, $0 < p_n(v) < 1$, and $p_n(v) = O(e^{-\frac{1}{2}v})$ there, and $\lim_{v \rightarrow 0+} p_n(v) = 1$. Now from (5.9) we see that R_ω and $\tilde{R}_\omega$ are both decreasing over $\omega > 0$. Also we see for every $\delta > 0$ that

$$\sup_{\mathbf{y} \in B} \int_{B \cap \{\mathbf{z} \mid |\mathbf{z} - \mathbf{y}| \geq \delta\}} |2(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{y} - \mathbf{z}|))| d\sigma_n(\mathbf{z}) \leq \frac{2\sigma_n(B)p_n(\omega\delta)}{\sigma_n(S_1)\delta^{n-1}} \rightarrow 0$$

as $\omega \rightarrow +\infty$, and the same also holds for the supremum over $\mathbf{x} \in \tilde{B}$. Hence

$$(5.10) \quad (\limsup_{\omega \rightarrow +\infty} R_\omega) \leq \limsup_{\delta \rightarrow 0+} \left(\sup_{\mathbf{y} \in B} \int_{B \cap \{\mathbf{z} \mid |\mathbf{y} - \mathbf{z}| < \delta\}} \frac{2|(\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{y}))|}{\sigma_n(S_1)|\mathbf{x} - \mathbf{y}|^n} d\sigma_n(\mathbf{z}) \right),$$

and clearly R_ω will be bounded over $\omega > 0$ if the right side of (5.10) is finite. Obviously the same two statements about $\tilde{R}_\omega$ also follow if $\sup_{\mathbf{y} \in B}$ is replaced by $\sup_{\mathbf{y} \in \tilde{B}}$ in the right side of (5.10).

Now, by using the geometric estimates over B afforded by lemmas (3.8), (3.11), and (3.17), we find (see (M) for details) that the right side of (5.10) vanishes under condition (3.1) or (3.6) on B , and is $\leq \pi^{-1}(\max_i |\alpha_i|)$ under condition (3.4) on B . Similarly if $\sup_{\mathbf{y} \in B}$ here is replaced by $\sup_{\mathbf{y} \in \tilde{B}}$, the right side of (5.10) is increased by 1 at most. By estimating $\liminf_{\omega \rightarrow \infty} R_\omega$ similarly, the proof of lemma (5.7) is completed.

For the following additional lemmas, let $\gamma = 1$ under conditions (3.4) or (3.6) on B , and $\gamma = \eta$ under condition (3.1), so that always $0 < \gamma \leq 1$. Also let $C_\delta = \{\mathbf{z} \in B \mid r_0(\mathbf{z}) \geq \delta\}$ as in theorem (5.1) and let $B_0 = \{\mathbf{z} \in B \mid r_0(\mathbf{z}) > 0\}$.

Lemma (5.13)

Let B satisfy conditions (3.1), (3.4), or (3.6) and define $F_m(\mathbf{z}, \mathbf{y}; \omega)$ inductively over $B_0 \times \bar{D}$ for $\omega > 0$ and integer $m \geq 0$ by the equations

$$F_0(\mathbf{z}, \mathbf{y}; \omega) = |(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{z} - \mathbf{y}|))|$$

and

$$F_{m+1}(\mathbf{z}, \mathbf{y}; \omega) = \int_B F_m(\mathbf{z}, \xi; \omega) |(\mathbf{n}(\xi) \cdot \nabla_\xi {}_n K_\omega(|\xi - \mathbf{y}|))| d\sigma_n(\xi).$$

Then for every $\delta > 0$ and integer $m \geq 0$ there exists some finite real $M_{\delta, m}$ such that

$$(5.14) \quad F_m(\mathbf{z}, \mathbf{y}; \omega) \leq M_{\delta, m}(\epsilon_m + |\mathbf{z} - \mathbf{y}|^{m\gamma + \gamma + 1 - n}), \quad \epsilon_0 = 0, \quad \epsilon_{m+1} = 1,$$

over $\mathbf{z} \in C_\delta$, $\mathbf{y} \in B$, $\mathbf{z} \neq \mathbf{y}$, $\omega > 0$, and F_m is bounded there for $m \geq -1 + \gamma^{-1}(n - 1)$.

Lemma (5.15)

Let B satisfy conditions (3.1), (3.4), or (3.6). Then for every $\delta > 0$ there exists some finite real M_δ such that

$$(5.16) \quad \int_B |(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{z} - \xi|)) - (\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} {}_n K_\omega(|\mathbf{y} - \xi|))| d\sigma_n(\xi) \leq M_\delta |\mathbf{z} - \mathbf{y}|^{\frac{1}{2}\gamma}$$

and, with $\{\xi \in B \mid |\mathbf{z} - \xi| \geq \delta \text{ and } |\mathbf{y} - \xi| \geq \delta\} = Q_\delta(\mathbf{z}, \mathbf{y})$,

$$(5.17) \quad \left[\sup_{\xi \in Q_\delta(\mathbf{z}, \mathbf{y})} |(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_n K_\omega(|\mathbf{z} - \xi|)) - (\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} {}_n K_\omega(|\mathbf{y} - \xi|))| \right] \leq M_\delta |\mathbf{z} - \mathbf{y}|^\gamma$$

over $\mathbf{z}$ and $\mathbf{y} \in C_\delta$ and $\omega > 0$.

Lemma (5.18)

Let B satisfy condition (3.4), $n = 2$, and define $\ln^+ v = (\max 0, \ln v)$. Then for some finite real M

$$(5.19) \quad \int_B \ln^+ \left(\frac{1}{|\mathbf{z} - \xi|} \right) |2(\mathbf{n}(\xi) \cdot \nabla_\xi {}_2 K_\omega(|\xi - \mathbf{y}|))| d\sigma_2(\xi) \leq M + R_\omega \ln^+ \left(\frac{1}{r_0(\mathbf{y})} \right)$$

over all $\mathbf{z} \in \bar{D}$, $\mathbf{y} \in B$, and $\omega > 0$; the same statement is also true if $\ln^+ (|\mathbf{z} - \xi|^{-1})$ is replaced by $\ln^+ ([r_0(\xi)]^{-1})$.

Since lemmas (5.13), (5.15), and (5.18) are used only for the $\nu = 1$ case of theorem (5.1), and since these estimates are obtained by straightforward but messy computation, we merely refer the reader to (M) for details.

Now returning to the proof of theorem (5.1), from lemma (5.7) we can choose $\omega_0 < +\infty$ so that $R_{\omega_0} < 1$, and hence $R_\omega \leq R_{\omega_0} < 1$ for $\omega \geq \omega_0$. Hence the m th term of the (5.3) series, with the integrands replaced by their absolute values, is $\leq 2(R_\omega)^m [\sup_{\xi \in B} {}_n K_\omega(|\xi - \mathbf{x}|)]$ over all $\mathbf{z} \in B$. Hence, for every $\mathbf{x} \in D$, the (5.3)

series converges absolutely and uniformly over $\mathbf{z} \in B$ and $\omega \geq \omega_0$. Also, from the analysis used in proving lemma (5.7), we see by induction on m that each of the terms of this (5.3) series is continuous over $\mathbf{z} \in B$ except for removable discontinuities at the corners of B . Thus $v_0(\mathbf{z}; \mathbf{x}, \omega)$ can be chosen both to be continuous over $\mathbf{z} \in B$ and to be the sum of the (5.3) series, except when $\mathbf{z}$ is a corner of B , for every $\mathbf{x} \in D$ and $\omega \geq \omega_0$.

Also $R_\omega \leq R_{\omega_0} < 1$ shows that the (5.4) series converges in $L_1(B, \sigma_n)$ -norm over $\mathbf{z} \in B$, and uniformly and absolutely over C_δ by using lemma (5.13) as well. Lemma (5.15) shows that the sum $v_1(\mathbf{z}; \mathbf{x}, \omega)$ satisfies the stated Hölder condition over C_δ , see (M) for details.

Now for the $v_\nu(\mathbf{z}; \mathbf{x}, \omega)$ which we have just constructed, $\nu = 0$ or 1 and $\omega \geq \omega_0$, we wish to show that $g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ given by (5.2) and the resulting $G_\nu(\mathbf{x}, \mathbf{y}; \omega) = {}_nK_\omega(|\mathbf{x} - \mathbf{y}|) - g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ satisfy all the conditions of definition (4.2) for the unique meson Green function. First, such $g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ defined by (5.2) over $\mathbf{x} \in D$, $\mathbf{y} \in D$, $\omega \geq \omega_0$ obviously has continuous second partials in $\mathbf{y}$ and satisfies $0 = -\nabla_{\mathbf{y}}^2 g_\nu(\mathbf{x}, \mathbf{y}; \omega) + \omega^2 g_\nu(\mathbf{x}, \mathbf{y}; \omega)$ there from the definition of ${}_nK_\omega(r)$. Also, for every fixed $\mathbf{x} \in D$ and $\omega \geq \omega_0$, it is easy to see from the continuity of $v_0(\mathbf{z}; \mathbf{x}, \omega)$ over $\mathbf{z} \in B$ and from (5.9) and the geometrical estimates of $|\mathbf{z} - \mathbf{y}|$ and $(\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{y}))$ that $g_0(\mathbf{x}, \mathbf{y}; \omega)$ defined by (5.2) over $\mathbf{y} \in D$ possesses a continuous extension, obviously unique, over $\mathbf{y} \in \bar{D}$, even including $\mathbf{y}$ at corners of B if such exist. Moreover, the Dirichlet condition (1.2), that the resulting extended $G_0(\mathbf{x}, \mathbf{y}; \omega) = 0$ over $\mathbf{y} \in B$, is equivalent by the same analysis to the integral equation

$$(5.23) \quad {}_nK_\omega(|\mathbf{x} - \mathbf{y}|) = \frac{1}{2}v_0(\mathbf{y}; \mathbf{x}, \omega) - \int_B (\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} {}_nK_\omega(|\mathbf{y} - \mathbf{z}|))v_0(\mathbf{z}; \mathbf{x}, \omega) d\sigma_n(\mathbf{z})$$

over those $\mathbf{y} \in B$ which are not corners of B , for fixed $\mathbf{x} \in D$ and $\omega \geq \omega_0$. But $R_\omega \leq R_{\omega_0} < 1$ and (5.3), with its right side uniformly convergent over B for fixed $\mathbf{x} \in D$ and $\omega \geq \omega_0$, show that (5.23) follows as desired. Hence the $g_0(\mathbf{x}, \mathbf{y}; \omega)$ and $G_0(\mathbf{x}, \mathbf{y}; \omega)$ constructed here satisfy all conditions of definition (4.2).

In order to show, for $\mathbf{x} \in D$ and $\omega \geq \omega_0$ fixed, that $g_1(\mathbf{x}, \mathbf{y}; \omega)$, defined by (5.2) over $\mathbf{y} \in D$, actually makes $G_1(\mathbf{x}, \mathbf{y}; \omega) = {}_nK_\omega(|\mathbf{x} - \mathbf{y}|) - g_1(\mathbf{x}, \mathbf{y}; \omega)$ satisfy the modified Neumann condition (2.3) on B , first notice from (5.4) that $v_1(\mathbf{y}; \mathbf{x}, \omega)$ satisfies in place of (5.23) the integral equation

$$(5.24) \quad \begin{aligned} (\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} {}_nK_\omega(|\mathbf{y} - \mathbf{x}|)) &= -\frac{1}{2}v_1(\mathbf{y}; \mathbf{x}, \omega) \\ &- \int_B (\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} {}_nK_\omega(|\mathbf{y} - \mathbf{z}|))v_1(\mathbf{z}; \mathbf{x}, \omega) d\sigma_n(\mathbf{z}) \end{aligned}$$

over those $\mathbf{y} \in B$ which are not corners of B , for every fixed $\mathbf{x} \in D$ and $\omega \geq \omega_0$. Also (for details see (M) and [11], pp. 162–165), from (5.2) and the Hölder condition on $v_1(\mathbf{z}; \mathbf{x}, \omega)$ over C_δ for every $\delta > 0$ it follows that $\nabla_{\mathbf{y}} g_1(\mathbf{x}, \mathbf{y}; \omega)$ possesses a continuous extension from D to all $\mathbf{y} \in \bar{D}$ which are not corners of B , and that for such $\mathbf{y} \in B$ under this extension $(\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} g_1(\mathbf{x}, \mathbf{y}; \omega))$ equals the

right side of (5.24). Thus the normal component of $\nabla_{\mathbf{y}} G_1(\mathbf{x}, \mathbf{y}; \omega)$ so extended vanishes over $\mathbf{y} \in B$ which are not corners of B . From this result, in case corners exist, using also lemma (5.18) and approximations to a general testing function $v(\mathbf{y})$ in (2.4) by testing functions vanishing near the corners of B , we conclude (see (M)) that $G_1(\mathbf{x}, \mathbf{y}; \omega)$ satisfies as a function of $\mathbf{y}$ the modified Neumann condition (2.3) on B for every $\mathbf{x} \in D$ and $\omega \geq \omega_0$. Hence $G_1(\mathbf{x}, \mathbf{y}; \omega)$ is the unique Neumann condition Green function of definition (4.2).

We have now proved all the statements of theorem (5.1) except for (5.5) when $n = 2$. But this is clear from (5.2), (5.3), and (5.4) by using the Fubini theorem if we can show $\sum_{m=0}^{\infty} \tilde{L}_m(\omega) < +\infty$ for $\tilde{L}_m(\omega)$ defined as the right side of (5.6) with the integrand replaced by its absolute value. Now under condition (3.1) on B , (3.10) shows that there exists $\delta > 0$ such that every $\mathbf{x} \in D$ for which $(\inf_{\mathbf{z} \in B} |\mathbf{x} - \mathbf{z}|) < \delta$, and thus $\mathbf{x} = -h\mathbf{n}(\mathbf{y}) + \mathbf{y}$ with $0 < h < \delta$ for some $\mathbf{y} \in B$, satisfies $(\inf_{\mathbf{z} \in B} |\mathbf{x} - \mathbf{z}|) \geq \frac{1}{2}h$ for all such $\mathbf{y}$ and h . Under condition (3.4) or (3.6), there exists $\delta > 0$ such that if $\mathbf{x} \in D$ and $(\inf_{\mathbf{z} \in B} |\mathbf{x} - \mathbf{z}|) < \delta$, then, except for a set of $\mathbf{x}$ of μ_n -measure zero arising only under condition (3.4), this infimum is uniquely attained at $\mathbf{z} = \mathbf{y} \in B$, $\mathbf{x} = -h\mathbf{n}(\mathbf{y}) + \mathbf{y}$, and $h = (\inf_{\mathbf{z} \in B} |\mathbf{x} - \mathbf{z}|)$. Also recall (see (5.9) and the preceding sentence) that $0 < {}_2K_{\omega}(r) = {}_2K_1(\omega r) \leq M[1 + \ln^+((\omega r)^{-1})] e^{-\omega r}$ and that ${}_2K_1(v)$ is monotone in $v > 0$. Thus, using (3.3), (3.14), and (3.20),

$$\begin{aligned} \int_D [\sup_{\mathbf{z} \in B} {}_2K_{\omega}(|\mathbf{x} - \mathbf{z}|)] d\mu_2(\mathbf{x}) &= \int_D {}_2K_1(\omega(\inf_{\mathbf{z} \in B} |\mathbf{x} - \mathbf{z}|)) d\mu_2(\mathbf{x}) \\ &\leq M \left[e^{-\omega\delta} + \int_0^{\delta} {}_2K_1(\tfrac{1}{2}\omega h) dh \right] \leq M'\omega^{-1} \end{aligned}$$

over $\omega \geq \omega_0$ by letting $h = \omega^{-1}s$ change the variable. Thus lemma (5.7) yields

$$(5.26) \quad \tilde{L}_m(\omega) \leq \tilde{R}_{\omega}(R_{\omega})^m \int_D [\sup_{\mathbf{z} \in B} {}_2K_{\omega}(|\mathbf{x} - \mathbf{z}|)] d\mu_2(\mathbf{x}) \leq \frac{M(R_{\omega})^m}{\omega}$$

over $\omega \geq \omega_0$ for some $M < +\infty$. Thus $0 < R_{\omega} \leq R_{\omega_0} < 1$ yields $\sum_{m=0}^{\infty} \tilde{L}_m(\omega) < +\infty$, (5.5) follows, and the proof of theorem (5.1) is complete.

Theorem (5.27)

Let the boundary B of D satisfy either condition (3.1), (3.4), or (3.6). Then the Green function $G_{\nu}(\mathbf{x}, \mathbf{y}; \omega)$ given by theorem (5.1) over $\omega \geq \omega_0$ and $\nu = 0$ or 1 satisfies all the conditions of lemma (4.9). If in addition $n = 2$, then also the conditions of lemma (4.3) are satisfied and with $L_m(\omega)$ defined by (5.6) over $\omega \geq \omega_0$,

$$\begin{aligned} (5.28) \quad &\omega^2 \sum_{i=1}^{\infty} \frac{1}{{}_0\lambda_i({}_0\lambda_i + \omega^2)} \\ &= \int_D (\lim_{\mathbf{y} \rightarrow \mathbf{x}} [G_0(\mathbf{x}, \mathbf{y}; \omega_0) - G_0(\mathbf{x}, \mathbf{y}; \omega)]) d\mu_n(\mathbf{x}) + \tilde{C}_0 \\ &= \frac{1}{2\pi} \mu_2(D) \ln \omega + \sum_{m=0}^{\infty} (-1)^m L_m(\omega) + C_0, \end{aligned}$$

$$\begin{aligned}
(5.29) \quad & -\frac{1}{\omega^2} + \omega^2 \sum_{i=1}^{\infty} \frac{1}{\lambda_i(\lambda_i + \omega^2)} \\
& = \int_D (\lim_{\mathbf{y} \rightarrow \mathbf{x}} [G_1(\mathbf{x}, \mathbf{y}; \omega_0) - G_1(\mathbf{x}, \mathbf{y}; \omega)]) d\mu_n(\mathbf{x}) + \tilde{C}_1 \\
& = \frac{1}{2\pi} \mu_2(D) \ln \omega - \sum_{m=0}^{\infty} L_m(\omega) + C_1.
\end{aligned}$$

The asymptotic evaluation of the right side of (5.28) and (5.29) as $\omega \rightarrow +\infty$ will give us the information about the distribution of λ_i for $n = 2$ which we seek in this paper, and a similar procedure would work for $n = 3$. (5.27) and (5.28) follow immediately from (4.8) and (5.5) after using (5.9) to obtain

$$\begin{aligned}
\lim_{r \rightarrow 0^+} [{}_2K_\omega(r) - {}_2K_{\omega_0}(r)] &= \lim_{r \rightarrow 0^+} \int_{\omega_0 r}^{\omega r} \frac{-p_2(v)}{2\pi v} dv = \lim_{r \rightarrow 0^+} \int_{\omega_0 r}^{\omega r} \frac{-1}{2\pi v} dv \\
&= -\frac{1}{2\pi} [\ln \omega - \ln \omega_0].
\end{aligned}$$

Thus only verification of the conditions of lemmas (4.9) and (4.3) as stated is needed to prove theorem (5.27).

First, for the symmetry relation $G_\nu(\mathbf{x}, \mathbf{y}; \omega) = G_\nu(\mathbf{y}, \mathbf{x}; \omega)$, if $\nu = 1$ the Neumann condition (2.3) gives us this result from (2.4) and (2.2) by an obvious modification of the standard argument. If $\nu = 0$, define the obviously symmetric function P by

$$\begin{aligned}
(5.31) \quad P(\mathbf{x}, \mathbf{y}; \omega) &= \int_D [\omega^2 G_0(\mathbf{z}, \mathbf{x}; \omega) G_0(\mathbf{z}, \mathbf{y}; \omega) \\
&\quad + (\nabla_{\mathbf{z}} G_0(\mathbf{z}, \mathbf{x}; \omega) \cdot \nabla_{\mathbf{z}} G_0(\mathbf{z}, \mathbf{y}; \omega))] d\mu_n(\mathbf{z})
\end{aligned}$$

over $\mathbf{x}, \mathbf{y} \in D$, $\mathbf{x} \neq \mathbf{y}$, and $\omega \geq \omega_0$. Since the integrand here is $(\nabla_{\mathbf{z}} \cdot [G_0(\mathbf{z}, \mathbf{y}; \omega) \nabla_{\mathbf{z}} G_0(\mathbf{z}, \mathbf{x}; \omega)])$, appropriate use of (2.2) and the Dirichlet boundary condition (1.2) yields

$$\begin{aligned}
(5.33) \quad & \limsup_{\substack{\mathbf{y} \rightarrow \mathbf{y}', \\ \mathbf{y} \in D}} |P(\mathbf{x}, \mathbf{y}; \omega) - G_0(\mathbf{x}, \mathbf{y}; \omega)| \\
& \leq \limsup_{\rho \rightarrow 0^+} \left(\limsup_{\substack{\mathbf{y} \rightarrow \mathbf{y}', \\ \mathbf{y} \in D}} \int_{\substack{\mathbf{z} \in D \\ (\inf_{\xi \in B} |\mathbf{z} - \xi|) < \rho}} |(\nabla_{\mathbf{z}} G_0(\mathbf{z}, \mathbf{x}; \omega) \cdot \nabla_{\mathbf{z}} G_0(\mathbf{z}, \mathbf{y}; \omega))| d\mu_n(\mathbf{z}) \right)
\end{aligned}$$

for $\mathbf{x} \in D$, $\mathbf{y}' \in B$, and $\omega \geq \omega_0$. From (5.2) we can show directly, if B has no corners, that the right side of (5.33) vanishes and thus by lemma (2.6) $G_0(\mathbf{x}, \mathbf{y}; \omega) = P(\mathbf{x}, \mathbf{y}; \omega)$ over $\mathbf{x}, \mathbf{y} \in D$, $\mathbf{x} \neq \mathbf{y}$, and $\omega \geq \omega_0$. Thus G_0 is symmetric in this case, and by an approximation argument using the maximum principle this symmetry persists even in case B has corners (see (M)).

The hypotheses of lemmas (4.3) and (4.9) other than the symmetry relation can also be obtained from (5.2) (see (M)). In particular, to show that $\int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ is bounded and continuous over $\mathbf{x} \in D$ if $n = 2$ or 3, and that

$\int_D |G_\nu^{(k)}(\mathbf{x}, \mathbf{y}; \omega)|^2 d\mu_n(\mathbf{y})$ and $\int_D |G_\nu(\mathbf{x}, \mathbf{y}; \omega)| d\mu_n(\mathbf{y})$ are bounded there if $n > 3$, we use $0 < G_0(\mathbf{x}, \mathbf{y}; \omega) < {}_nK_\omega(|\mathbf{x} - \mathbf{y}|)$ and also estimate $g_1(\mathbf{x}, \mathbf{y}; \omega)$ if $\nu = 1$. Thus theorem (5.27) is proved.

We now give the construction of the heat Green function $H_\nu(\mathbf{x}, \mathbf{y}; t)$. For simplicity this has been done only under conditions (3.1) and (3.6), so that we have not considered corners at all. Recall from definition (4.1) that $H_\nu(\mathbf{x}, \mathbf{y}; t) = Q_n(|\mathbf{x} - \mathbf{y}|; t) - h_\nu(\mathbf{x}, \mathbf{y}; t)$ and $Q_n(r; t) = (2\sqrt{\pi t})^{-n} e^{-r^2/4t}$.

Theorem (5.42)

Let the boundary B of D satisfy either condition (3.1) or (3.6). Then, for some positive T , $h_\nu(x, y; t)$ and $H_\nu(x, y; t)$ exist satisfying definition (4.1) and

$$(5.43) \quad \begin{aligned} h_0(\mathbf{x}, \mathbf{y}; t) &= - \int_0^t \int_B (\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{y}|; t - \tau)) w_0(\mathbf{z}, \tau; \mathbf{x}) d\sigma_n(\mathbf{z}) d\tau, \\ h_1(\mathbf{x}, \mathbf{y}; t) &= - \int_0^t \int_B Q_n(|\mathbf{z} - \mathbf{y}|; t - \tau) w_1(\mathbf{z}, \tau; \mathbf{x}) d\sigma_n(\mathbf{z}) d\tau \end{aligned}$$

over $\mathbf{x}$ and $\mathbf{y} \in D$, $0 < t \leq T$, and $\nu = 0$ or 1 . For every fixed $\mathbf{x} \in D$ the functions $w_\nu(\mathbf{z}, \tau; \mathbf{x})$, $\nu = 0$ or 1 , are bounded and continuous over $(\mathbf{z}, \tau) \in B \times [0, T]$, vanish over $\mathbf{z} \in B$ for $\tau = 0$, and satisfy over $\mathbf{z} \in B$ and $0 < \tau \leq T$, with ${}_0\mathbf{z} = \mathbf{z}$ and $\tau_0 = \tau$ for notational convenience,

$$(5.44) \quad \begin{aligned} w_0(\mathbf{z}, \tau; \mathbf{x}) &= 2Q_n(|\mathbf{z} - \mathbf{x}|; \tau) \\ &+ \sum_{m=1}^{\infty} \int_0^{\tau_0} \int_B \int_0^{\tau_1} \int_B \cdots \int_0^{\tau_{m-1}} \int_B \left\{ \left[\prod_{k=1}^m 2(\mathbf{n}({}_k\mathbf{z}) \cdot \nabla_{{}_k\mathbf{z}} Q_n(|{}_k\mathbf{z} - {}_{k-1}\mathbf{z}|; \tau_{k-1} - \tau_k)) \right] \right. \\ &\quad \left. \cdot 2Q_n(|{}_m\mathbf{z} - \mathbf{x}|; \tau_m) \right\} d\sigma_n({}_m\mathbf{z}) d\tau_m \cdots d\sigma_n({}_1\mathbf{z}) d\tau_1, \end{aligned}$$

$$(5.45) \quad \begin{aligned} w_1(\mathbf{z}, \tau; \mathbf{x}) &= -2(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{x}|; \tau)) \\ &+ \sum_{m=1}^{\infty} (-1)^{m+1} \int_0^{\tau_0} \int_B \cdots \int_0^{\tau_{m-1}} \int_B \left[\prod_{k=1}^{m+1} 2 \right. \\ &\quad \left. (\mathbf{n}({}_{k-1}\mathbf{z}) \cdot \nabla_{{}_{k-1}\mathbf{z}} Q_n(|{}_{k-1}\mathbf{z} - {}_k\mathbf{z}|; \tau_{k-1} - \tau_k)) \right]_{m+1\mathbf{z}=\mathbf{x}, \tau_{m+1}=0} \\ &\quad d\sigma_n({}_m\mathbf{z}) d\tau_m \cdots d\sigma_n({}_1\mathbf{z}) d\tau_1, \end{aligned}$$

both being uniformly convergent over $\mathbf{z} \in B$ and $0 < \tau \leq T$. Furthermore

$$(5.46) \quad \begin{aligned} - \int_D h_0(\mathbf{x}, \mathbf{x}; t) d\mu_n(\mathbf{x}) &= \sum_{m=0}^{\infty} (-1)^{m+1} J_m(t) \quad \text{and} \\ - \int_D h_1(\mathbf{x}, \mathbf{x}; t) d\mu_n(\mathbf{x}) &= \sum_{m=0}^{\infty} J_m(t) \end{aligned}$$

exist, the right sides being absolutely convergent over $0 < t \leq T$ with

$$(5.47) \quad J_m(t) = \int_D \int_0^t \int_B \int_0^{\tau_0} \int_B \cdots \int_0^{\tau_{m-1}} \int_B \left[\prod_{k=0}^m (-2) \right. \\ \left. (\mathbf{n}(k\mathbf{z}) \cdot \nabla_{k\mathbf{z}} Q_n(|k\mathbf{z} - k_{-1}\mathbf{z}|; \tau_{k-1} - \tau_k)) \right]_{-1\mathbf{z}=\mathbf{x}, \tau_{-1}=t} \\ \cdot Q_n(|m\mathbf{z} - \mathbf{x}|; \tau_m) d\sigma_n(m\mathbf{z}) d\tau_m \cdots d\sigma_n(0\mathbf{z}) d\tau_0 d\mu_n(\mathbf{x}).$$

As the basic convergence estimate for this theorem we now have the following analogue of lemma (5.7).

Lemma (5.48)

Let B satisfy conditions (3.1) or (3.6) and define for $t > 0$

$$(5.49) \quad R(t) = \sup_{\mathbf{y} \in B} \int_0^t \int_B 2 |\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{y}|; t - \tau)| d\sigma_n(\mathbf{z}) d\tau, \\ \tilde{R}(t) = \sup_{\mathbf{y} \in \tilde{B}} \int_0^t \int_B 2 |\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{y}|; t - \tau)| d\sigma_n(\mathbf{z}) d\tau.$$

Then $\lim_{t \rightarrow 0^+} R(t) = 0$ and both $R(t)$ and $\tilde{R}(t)$ are monotone increasing and bounded over $t > 0$.

Since the behavior of the $Q_n(r; t)$ -derived kernel is somewhat less obvious than the analogous ${}_nK_\omega(r)$ behavior, we start by noting (see (M)) that

$$(5.50) \quad {}_nP_\beta(t, \delta, h) = \int_0^t \int_0^\delta Q_n(\sqrt{r^2 + h^2}; s) \frac{hr^\beta}{s} r^{n-2} dr ds, \\ {}_nP_\eta(t, \delta) = \int_0^t \int_0^\delta Q_n(r; s) \frac{r^{1+\eta}}{s} r^{n-2} dr ds,$$

defined over $t > 0, 1 \geq \delta > 0, h > 0, 1 \geq \beta \geq 0, 1 \geq \eta > 0$, satisfy there

$$(5.51) \quad {}_nP_\eta(t, \delta) \leq M_1 t^{1/\eta}, \quad {}_nP_\eta(t, \delta) \leq M_2 \delta^\eta, \\ M_1 = \frac{2^\eta}{\pi^{1/\eta}} \Gamma\left(\frac{n+\eta}{2}\right), \quad M_2 = \frac{1}{\pi^{1/\eta}} \Gamma\left(\frac{n}{2}\right),$$

$$(5.52) \quad \lim_{h \rightarrow 0^+} {}_nP_\beta(t, \delta, h) = \begin{cases} [\sigma_{n-1}(S'_1)]^{-1} & \text{if } \beta = 0, \\ 0 & \text{if } 0 < \beta \leq 1 \end{cases}$$

for $S'_1 = \{\xi \in R_{n-1} \mid |\xi| = 1\}$ and integer $n \geq 2$. Now using $\nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{y}|; s) = Q_n(|\mathbf{z} - \mathbf{y}|; s) [-(2s)^{-1}(\mathbf{z} - \mathbf{y})]$ and the geometric estimate (3.9) with $\eta' = \eta$, or (3.18) with $\eta' = 1$, on the $|\mathbf{z} - \mathbf{y}| < \delta$ part of the (5.49) integral, we see ((M) for more details) that

$$(5.53) \quad R(t) \leq M {}_nP_{\eta'}(t, \delta) + \frac{\sigma_n(B)(\text{diam } B)}{(2\sqrt{\pi})^n} \int_0^t \frac{ds}{e^{\delta^2/4s} s^{1/n+1}}$$

with $M < +\infty$ for $\delta > 0$ sufficiently small so that, for every $y \in B$, $(n(z) \cdot n(y)) \geq \frac{1}{2}$ and the tangent hyperplane projection is one-to-one over $\{z \in B \mid |z - y| < \delta\}$. Hence (5.51) shows that $\lim_{t \rightarrow 0^+} R(t) = 0$ under conditions (3.1) or (3.6) on B .

To get the analogue of (5.53) for $\tilde{R}(t)$ we again in (5.49) split B into two parts according as $|z - y| \geq \delta$ or $< \delta$ for each $y \in \bar{D}$. Thus

$$(5.54) \quad \begin{aligned} \tilde{R}(t) \leq M \left[\sup_{0 < h < \delta} (1 + h^{\eta'}) {}_n P_0(t, \delta, h) \right] \\ + MP_{\eta'}(t, \delta) + \frac{\sigma_n(B)(\text{diam } B)}{(2\sqrt{\pi})^n} \int_0^t \frac{ds}{e^{\delta^2/4s} s^{\frac{1}{2}n+1}} ds \end{aligned}$$

with $M < +\infty$ for $\delta > 0$ sufficiently small. The proof of (5.51) and (5.52) also gives the boundedness of the right sides of (5.53) and (5.54) over $t > 0$ (and actually $1 = \lim_{t \rightarrow 0^+} \tilde{R}(t)$ from (5.52)). Since the monotone behavior of $R(t)$ and $\tilde{R}(t)$ is clear after making the change of variable $\tau = t - s$ in (5.49), lemma (5.48) is completely proved.

In order to deal with the transpose of the derived kernel of $Q_n(r; t)$ we find it more convenient to use a corollary of lemma (5.48) instead of the analogue of lemma (5.13). Naturally we could have obtained the same thing with ${}_n K_\omega(r)$ if we had not wanted to deal with corners.

Corollary (5.55)

Under the conditions of lemma (5.48)

$$(5.56) \quad R^*(t) = \sup_{y \in B} \int_0^t \int_B 2 | (n(y) \cdot \nabla_y Q_n(|y - z|; t - \tau)) | d\sigma_n(z) d\tau$$

also satisfies $\lim_{t \rightarrow 0^+} R^*(t) = 0$, and $R^*(t)$ is bounded and monotone increasing over $t > 0$.

This follows since (5.53) is obtained for $R^*(t)$ in exactly the same way as for $R(t)$.

We also need the following analogue of lemma (5.15), for which as before we let $\gamma = 1$ under condition (3.6) and $\gamma = \eta$ under condition (3.1) on B , so that always $0 < \gamma \leq 1$.

Lemma (5.57)

Let B satisfy conditions (3.1) or (3.6). Then there exists some finite real M such that

$$(5.58) \quad \begin{aligned} \int_0^t \int_B | (n(z) \cdot \nabla_z Q_n(|z - \xi|; t - \tau)) \\ - (n(y) \cdot \nabla_y Q_n(|y - \xi|; t - \tau)) | d\sigma_n(\xi) d\tau \leq M |z - y|^{\frac{1}{2}\gamma} \end{aligned}$$

over z and $y \in B$ and $t > 0$.

To prove (5.58) we split up B exactly as in the proof of lemma (5.15) and proceed as there, see (M) for details.

We will find, unlike the boundedness of the right sides of (5.5) for large ω ,

that the right sides of (5.46) are not bounded over small $t > 0$. Hence it should not be surprising that we need the following special convergence estimate for (5.46), which has no analogue in theorem (5.1).

Lemma (5.59)

Let B satisfy conditions (3.1) or (3.6). Then for every $t > 0$ there exists a least non-negative real number $W(t)$ such that

$$(5.60) \quad \int_0^t \int_B 2 |(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{y}|; t - \tau))| Q_n\left(\frac{|\mathbf{z} - \mathbf{x}|}{2^m}; \tau\right) d\sigma_n(\mathbf{z}) d\tau \\ \leq W(t)(2^{n-1})^m Q_n\left(\frac{|\mathbf{y} - \mathbf{x}|}{2^{m+2}}; t\right)$$

over all $\mathbf{x} \in D$, $\mathbf{y} \in B$, and integers $m \geq 0$. Moreover, with $0 < \gamma \leq 1$ defined as in lemma (5.57), $\lim_{t \rightarrow 0^+} W(t) = 0$ and

$$(5.61) \quad W(t) \leq Mt^{\gamma}$$

over $t > 0$ for some $M < +\infty$. If $\tilde{W}(t)$ is also defined by (5.60) with the condition $\mathbf{y} \in B$ replaced by $\mathbf{y} \in \tilde{D}$, then $\tilde{W}(t)$ is bounded over $t > 0$.

In order to analyze (5.60) we first show that

$$(5.62) \quad \sup_{\mathbf{x} \in D} \int_0^t \int_B Q_n(|\mathbf{x} - \mathbf{z}|; \tau) d\sigma_n(\mathbf{z}) d\tau \leq Mt^{\gamma}$$

over $t > 0$ for some $M < +\infty$. We also note that (5.51) and (5.53) with $\eta' = \gamma$ yield

$$(5.63) \quad R(t) \leq Mt^{\gamma}$$

over $t > 0$ for some $M < +\infty$. Finally (see (M)) we obtain (5.61), and thus prove lemma (5.59), from (5.62) and (5.63), by dividing the range of integration on the left side of (5.60) into the four parts $\{\mathbf{z}, \tau | \frac{1}{2}t \leq \tau \leq t, |\mathbf{z} - \mathbf{y}| < \frac{1}{2}|\mathbf{x} - \mathbf{y}|\}$, $\{\mathbf{z}, \tau | \frac{1}{2}t \leq \tau \leq t, |\mathbf{z} - \mathbf{y}| \geq \frac{1}{2}|\mathbf{x} - \mathbf{y}|\}$, $\{\mathbf{z}, \tau | 0 < \tau < \frac{1}{2}t, |\mathbf{z} - \mathbf{x}| < \frac{1}{2}|\mathbf{x} - \mathbf{y}|\}$, and $\{\mathbf{z}, \tau | 0 < \tau < \frac{1}{2}t, |\mathbf{z} - \mathbf{x}| \geq \frac{1}{2}|\mathbf{x} - \mathbf{y}|\}$.

Now turning to the proof of theorem (5.42), lemma (5.48) and corollary (5.55) enable us to find some real $T > 0$ such that $0 < R(t) \leq R(T) < 1$ and $0 < R^*(t) \leq R^*(T) < 1$ over $0 < t \leq T$. Thus for every fixed $\mathbf{x} \in D$ the (5.44) and (5.45) series converge uniformly and absolutely over $\mathbf{z} \in B$ and $0 < \tau \leq T$; moreover from

$$\lim_{\tau \rightarrow 0^+} \left[\sup_{r \geq \rho, 0 < \tau' \leq \tau} Q_n(r; \tau') \right] = 0 = \lim_{\tau \rightarrow 0^+} \left[\sup_{r \geq \rho, 0 < \tau' \leq \tau} Q_n(r; \tau') \frac{r}{2\tau'} \right]$$

clearly these series sums $w_r(\mathbf{z}, \tau; \mathbf{x}) \rightarrow 0$ as $\tau \rightarrow 0^+$ uniformly over $\mathbf{z} \in B$ for such $\mathbf{x} \in D$. Also (see (M)) each term in the (5.44) and (5.45) series is continuous in $(\mathbf{z}, \tau)$ over $B \times (0, T]$; thus, for every $\mathbf{x} \in D$, $w_r(\mathbf{z}, \tau; \mathbf{x})$ can be taken to be continuous over $B \times [0, T]$ and to vanish at $\tau = 0$. The special estimate (5.58) shows that w_1 satisfies a Hölder condition of order $\frac{1}{2}\gamma$ in $\mathbf{z}$.

Next we need to verify that the $h_\nu(\mathbf{x}, \mathbf{y}; t)$ given by (5.43) satisfy definition (4.1) of the heat Green function. First, since

$$\nabla^2 Q_n(r; t) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) Q_n(r; t) = \left(-\frac{n}{2t} + \frac{r^2}{4t^2} \right) Q_n(r; t) = \frac{\partial}{\partial t} Q_n(r; t),$$

it is clear that the $h_\nu(\mathbf{x}, \mathbf{y}; t)$ given by (5.43) for fixed $\mathbf{x} \in D$ have continuous partials of all orders in $\mathbf{y}$ and t and satisfy $-\nabla_{\mathbf{y}}^2 h_\nu(\mathbf{x}, \mathbf{y}; t) + \partial h_\nu(\mathbf{x}, \mathbf{y}; t)/\partial t = 0$ over $(\mathbf{y}; t) \in D \times (0, T]$, since the continuous extension of the integrand in (5.43) vanishes at $\tau = t$. Also from (5.43), (5.62), and the boundedness of $\tilde{R}(t)$ in lemma (5.48), and from $w_\nu \rightarrow 0$ as $\tau \rightarrow 0^+$, clearly $h_\nu(\mathbf{x}, \mathbf{y}; t) \rightarrow 0$ as $t \rightarrow 0^+$ uniformly over $\mathbf{y} \in D$ for every fixed $\mathbf{x} \in D$.

To obtain the boundary conditions from (5.43), we see from (5.52) as in (5.54) (see (M)) that

$$(5.74) \quad \begin{aligned} & [\lim_{\substack{\mathbf{y} \rightarrow \mathbf{z}, \\ \mathbf{y} \in D}} h_0(\mathbf{x}, \mathbf{y}; t)] = \frac{1}{2} w_0(\mathbf{z}, t; \mathbf{x}) \\ & - \int_0^t \int_B (\mathbf{n}(\xi) \cdot \nabla_\xi Q_n(|\xi - \mathbf{z}|; t - \tau)) w_0(\xi, \tau; \mathbf{x}) d\sigma_n(\xi) d\tau \end{aligned}$$

exists for every $\mathbf{x} \in D$, $\mathbf{z} \in B$, and $t \in (0, T]$. Thus, for every $\mathbf{x} \in D$, $h_0(\mathbf{x}, \mathbf{y}; t)$ possesses a continuous extension in $(\mathbf{y}; t)$ from $D \times (0, T]$ to $\bar{D} \times [0, T]$ which vanishes at $t = 0$. Also (5.44) with uniform convergence shows that the right side of (5.74) equals $Q_n(|\mathbf{z} - \mathbf{x}|; t)$, and hence $H_0(\mathbf{x}, \mathbf{y}; t)$ satisfies in $\mathbf{y}$ the Dirichlet condition (1.2) on B for $\mathbf{x} \in D$ and $t \in (0, T]$.

In case $\nu = 1$, then we verify from (5.43) (see (M)) that $h_1(\mathbf{x}, \mathbf{y}; t)$ possesses a continuous extension in $(\mathbf{y}; t)$ from $D \times (0, T]$ to $\bar{D} \times [0, T]$ for every fixed $\mathbf{x} \in D$. Also, using the Hölder condition of order $\frac{1}{2}\gamma > 0$ on $w_1(\mathbf{z}, \tau; \mathbf{x})$ as a function of $\mathbf{z}$, we find that $\nabla_{\mathbf{y}} h_1(\mathbf{x}, \mathbf{y}; t)$ possesses a continuous extension in $\mathbf{y}$ from D to $\bar{D}$ for every $\mathbf{x} \in D$ and $t \in (0, T]$, and that for this extension

$$(5.76) \quad \begin{aligned} & (\mathbf{n}(\mathbf{z}) \cdot [\lim_{\substack{\mathbf{y} \rightarrow \mathbf{z}, \\ \mathbf{y} \in D}} \nabla_{\mathbf{y}} h_1(\mathbf{x}, \mathbf{y}; t)]) = -\frac{1}{2} w_1(\mathbf{z}, t; \mathbf{x}) \\ & - \int_0^t \int_B (\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\xi - \mathbf{z}|; t - \tau)) w_1(\xi, \tau; \mathbf{x}) d\sigma_n(\xi) d\tau \end{aligned}$$

over $\mathbf{z} \in B$, $\mathbf{x} \in D$, $t \in (0, T]$. (5.45) with uniform convergence shows that the right side of (5.76) equals $(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{x} - \mathbf{z}|; t))$, and hence the normal component of the continuous extension of $\nabla_{\mathbf{y}} H_1(\mathbf{x}, \mathbf{y}; t)$ in $\mathbf{y}$ from D to $\bar{D}$ vanishes over $\mathbf{y} \in B$. Thus, if $\nabla_{\mathbf{y}}^2 h_1(\mathbf{x}, \mathbf{y}; t) = \partial h_1(\mathbf{x}, \mathbf{y}; t)/\partial t \in L_1(D)$ in $\mathbf{y}$, $H_1(\mathbf{x}, \mathbf{y}; t)$ satisfies in $\mathbf{y}$ the modified Neumann condition (2.3) on B for $\mathbf{x} \in D$ and $t \in (0, T]$.

Finally, since unfortunately $\partial Q_n(|\mathbf{z} - \mathbf{y}|; t)/\partial t \notin L_1(D \times (0, T))$, we show that $\partial h_1(\mathbf{x}, \mathbf{y}; t)/\partial t \in L_1(D \times (0, T))$ in $(\mathbf{y}; t)$ for every $\mathbf{x} \in D$, and is an element of $L_1(D)$ in $\mathbf{y}$ for every $(\mathbf{x}; t) \in D \times (0, T]$, by showing (see (M)) that $\partial w_1(\mathbf{z}, \tau; \mathbf{x})/\partial t$ exists and is bounded and continuous over $B \times (0, T]$ as the sum of the (5.45)

term-by-term differentiated series for every $\mathbf{x} \in D$. Thus $H_\nu(\mathbf{x}, \mathbf{y}; t)$ and $h_\nu(\mathbf{x}, \mathbf{y}; t)$ satisfy all the conditions of definition (4.1).

To finish the proof of theorem (5.42) we only need to obtain (5.46). Assuming for the moment that $\sum_{m=0}^{\infty} \tilde{J}_m(t) < +\infty$, where $\tilde{J}_m(t)$ is defined as the right side of (5.47) with the integrand replaced by its absolute value, (5.46) follows from (5.43), (5.44), (5.45), and the Fubini theorem. Here we have used the change of variable $\tau_k = t - \tau'_{m-k}$ and ${}_k\mathbf{z} = {}_{m-k}\mathbf{z}'$ for $k = 0, 1, \dots, m$ in (5.47) and have switched the order of integration of the τ 's and $\mathbf{z}$'s to pass from the $\nu = 0$ to the $\nu = 1$ form of $J_m(t)$. Finally, in order to prove that $\sum_{m=0}^{\infty} \tilde{J}_m(t) < +\infty$, notice from (5.61) that there exists an $M_m < +\infty$ for each integer m satisfying $\gamma m \geq n$ such that

$$(5.80) \quad \left[\sup_{0 < \tau \leq t} W(\tau) \right]^m Q_n(r, t) \leq \left[\sup_{0 < \tau \leq t} W(\tau) \right]^m Q_n(0, t) \leq M_m$$

over $0 < t \leq T$ and $r \geq 0$. Defining m_0 as the least such integer, and using (5.49), (5.60), and (5.80) in (5.47), we have

$$(5.81) \quad \tilde{J}_m(t) \leq M \tilde{R}(t) (R(t))^{m-m_0}$$

over $m \geq m_0$ and $0 < t \leq T$ for some $M < +\infty$. For $0 \leq m < m_0$ from (5.60) alone

$$(5.82) \quad \begin{aligned} \tilde{J}_m(t) &\leq M \left[\sup_{0 < \tau \leq t} W(\tau) \right]^m \\ &\quad \cdot \int_D \int_0^t \int_B |(\mathbf{n}(\mathbf{z}) \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{x}|; t - \tau))| Q_n\left(\frac{|\mathbf{z} - \mathbf{x}|}{2^m}; \tau\right) \\ &\quad d\sigma_n(\mathbf{z}) d\tau d\mu_n(\mathbf{x}). \end{aligned}$$

But from (5.60), with $\tilde{W}(t)$ replacing $W(t)$, the integral on the right of (5.82) is $\leq M t^{-\frac{1}{2}n}$, (and actually $O(t^{-\frac{1}{2}(n-1)})$ will be seen in the next section). Hence (5.82) and $0 < R(t) \leq R(T) < 1$ in (5.81) show that $\sum_{m=0}^{\infty} \tilde{J}_m(t) < +\infty$ over $0 < t \leq T$, which completes the proof of theorem (5.42).

Theorem (5.83)

Let the boundary B of D satisfy either condition (3.1) or (3.6). Then the heat Green function $H_\nu(\mathbf{x}, \mathbf{y}; t)$ given by theorem (5.42) over $0 < t \leq T$ and $\nu = 0$ or 1 satisfies all the hypotheses of lemma (4.14). In addition, with $J_m(t)$ defined by (5.47) over $0 < t \leq T$,

$$(5.84) \quad \begin{aligned} \sum_{j=1}^{\infty} e^{-\lambda_j t} &= \int_D H_0(\mathbf{x}, \mathbf{x}; t) d\mu_n(\mathbf{x}) \\ &= \mu_n(D) Q_n(0; t) + \sum_{m=0}^{\infty} (-1)^{m+1} J_m(t), \end{aligned}$$

$$(5.85) \quad \begin{aligned} 1 + \sum_{j=1}^{\infty} e^{-\lambda_j t} &= \int_D H_1(\mathbf{x}, \mathbf{x}; t) d\mu_n(\mathbf{x}) \\ &= \mu_n(D) Q_n(0; t) + \sum_{m=0}^{\infty} J_m(t). \end{aligned}$$

First, theorem (5.27) shows that either condition (3.1) or (3.6) on B is sufficient to obtain the hypotheses of lemma (4.9). We verify (see (M)) the other hypotheses of lemma (4.14) by using (5.43) and the proof of theorem (5.42). Since (4.17) together with (5.46) gives (5.84) and (5.85), the proof of theorem (5.83) is complete.

6. Asymptotic Evaluation of Green Function Integrals

In this section we shall give asymptotic evaluations of the right sides of (5.28), (5.29) and (5.84), (5.85) under the various conditions (3.1), (3.4), and (3.6) on the boundary B . In the next section we shall see what information this gives about the asymptotic distribution of the eigenvalues. We start with the heat Green function.

Lemma (6.1)

Let the boundary B of D satisfy condition (3.1), with $0 < \eta \leq 1$, appearing in (3.2) and (3.3). Then over $0 < t \leq T$

$$(6.2) \quad J_0(t) = \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1}t^{\frac{1}{2}(n-1)}} + O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}),$$

$$(6.3) \quad \sum_{m=1}^{\infty} |J_m(t)| = O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}),$$

$$(6.4) \quad \sum_{i=1}^{\infty} e^{-\lambda_i t} = \frac{\mu_n(D)}{(2\sqrt{\pi})^n t^{\frac{1}{2}n}} - \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1}t^{\frac{1}{2}(n-1)}} + O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}),$$

$$(6.5) \quad 1 + \sum_{i=1}^{\infty} e^{-\lambda_i t} = \frac{\mu_n(D)}{(2\sqrt{\pi})^n t^{\frac{1}{2}n}} + \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1}t^{\frac{1}{2}(n-1)}} + O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}).$$

To prove (6.2) we have from (5.47) and $\nabla_y Q_n(|y|; s) = -(2s)^{-1}Q_n(|y|; s)y$

$$(6.6) \quad J_0(t) = \int_D \int_0^t \int_B \frac{(\mathbf{n}(z) \cdot (z - \mathbf{x}))}{t - \tau} Q_n(|z - \mathbf{x}|; t - \tau) Q_n(|z - \mathbf{x}|; \tau) d\sigma_n(z) d\tau d\mu_n(\mathbf{x}).$$

We divide the integration in (6.6) into two parts according as $|\mathbf{x} - \mathbf{z}| \geq \delta$ or as $|\mathbf{x} - \mathbf{z}| < \delta$, and notice that the integral over the first part is $O(e^{-\delta^2/8t})$.

For the second part of the (6.6) integral we have $\mathbf{x} = -h\mathbf{n}(\mathbf{y}) + \mathbf{y}$ with $0 < h < \delta$ for some $\mathbf{y} \in B$, we let $r = |\mathbf{z}^* - \mathbf{y}|$ be defined by projection onto the tangent hyperplane to B at $\mathbf{y}$ as in lemma (3.8), we use $(\mathbf{n}(z) \cdot \mathbf{n}(\mathbf{y})) = 1 + r^2 M \theta$ from (3.2) and (3.10) to change the σ_n -measure over $\mathbf{z}$ to μ_{n-1} -measure on the hyperplane, and we use (3.3), (3.9), and (3.10). Thus we obtain from (6.6) over $0 < t \leq T$

$$\begin{aligned}
(6.7) \quad J_0(t) &= O(e^{-\delta^2/8t}) + \frac{1}{(4\pi)^n} \int_B \int_0^\delta \int_0^t \int_{\{z \in B \mid r < 2\delta\}} \frac{h + (h+r)^{1+\eta} M \theta}{(t-\tau)^{\frac{1}{2}n+1} \tau^{\frac{1}{2}n}} \\
&\quad \cdot \exp\left(-t \frac{h^2 + r^2 + (h+r)^{2+\eta} M \theta'}{4(t-\tau)\tau}\right) d\sigma_n(z) d\tau dh d\sigma_n(y) \\
&= O(e^{-\delta^2/8t}) + \frac{\sigma_{n-1}(S_1)}{(4\pi)^n} \int_B \int_0^\delta \int_0^t \int_{\{z \in B \mid r < 2\delta\}} \frac{h + (h+r)^{1+\eta} M \theta}{(t-\tau)^{\frac{1}{2}n+1} \tau^{\frac{1}{2}n}} \\
&\quad \cdot \exp\left(-t \frac{h^2 + r^2 + (h+r)^{2+\eta} M \theta'}{4(t-\tau)\tau}\right) r^{n-2} dr d\tau dh d\sigma_n(y)
\end{aligned}$$

for $S_1 = \{\xi \in R_{n-1} \mid |\xi| = 1\}$ and sufficiently small $\delta > 0$, with the usual conventions about M and θ . Take the h integration inside the τ integration by using the Fubini theorem and let $h = [4\tau(t-\tau)t^{-1}]^{\frac{1}{2}}u^{\frac{1}{2}}$, $r = [4\tau(t-\tau)t^{-1}]^{\frac{1}{2}}v^{\frac{1}{2}}$, and $\tau = st$ in (6.7). Thus

$$\begin{aligned}
(6.8) \quad J_0(t) &= O(e^{-\delta^2/8t}) + \frac{\sigma_{n-1}(S_1)}{4^{\frac{1}{2}(n+1)} \pi^n t^{\frac{1}{2}(n-1)}} \int_B \int_0^1 \frac{s^{\frac{1}{2}}}{(1-s)^{\frac{1}{2}}} \\
&\quad \cdot \left\{ \int_0^{\delta^2/4ts(1-s)} \int_0^{\delta^2/ts(1-s)} \frac{u^{\frac{1}{2}} + (u^{\frac{1}{2}} + v^{\frac{1}{2}})^{1+\eta} [ts(1-s)]^{\frac{1}{2}\eta} M \theta}{u^{\frac{1}{2}}} v^{\frac{1}{2}(n-3)} \right. \\
&\quad \cdot \exp(-u - v + (u^{\frac{1}{2}} + v^{\frac{1}{2}})^{2+\eta} [ts(1-s)]^{\frac{1}{2}\eta} M \theta') dv du \Big\} ds d\sigma_n(y) \\
&= O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}) + \frac{\sigma_{n-1}(S_1)}{4^{\frac{1}{2}(n+1)} \pi^n t^{\frac{1}{2}(n-1)}} \int_B \int_0^1 \frac{s^{\frac{1}{2}}}{(1-s)^{\frac{1}{2}}} \\
&\quad \cdot \left\{ \int_0^{\delta^2/4ts(1-s)} \int_0^{\delta^2/ts(1-s)} v^{\frac{1}{2}(n-3)} \right. \\
&\quad \cdot \exp(-u - v + (u^{\frac{1}{2}} + v^{\frac{1}{2}})^{2+\eta} [ts(1-s)]^{\frac{1}{2}\eta} M \theta') dv du \Big\} ds d\sigma_n(y)
\end{aligned}$$

over $0 < t \leq T$, $\delta > 0$ being taken sufficiently small so that $(u^{\frac{1}{2}} + v^{\frac{1}{2}})^{2+\eta} [ts(1-s)]^{\frac{1}{2}\eta} M < \frac{1}{2}(u+v)$ over $0 < u(4ts(1-s)) < \delta^2$ and $0 < v(ts(1-s)) < \delta^2$. Thus we find (see (M) for details) that the error term in the exponential in (6.8) can be dropped and the upper limits of u and v integration replaced by $+\infty$ without altering the order of the error estimate $O(t^{-\frac{1}{2}(n-1)-\frac{1}{2}\eta})$. Hence (6.8) becomes

$$\begin{aligned}
(6.9) \quad J_0(t) &= O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}) + \frac{\sigma_n(B)\sigma_{n-1}(S_1)}{4^{\frac{1}{2}(n+1)} \pi^n t^{\frac{1}{2}(n-1)}} \\
&\quad \cdot \int_0^1 \sqrt{\frac{s}{1-s}} ds \cdot \int_0^\infty e^{-u} du \cdot \int_0^\infty e^{-v} v^{\frac{1}{2}(n-3)} dv \\
&= O(t^{-\frac{1}{2}(n-1)+\frac{1}{2}\eta}) + \frac{\sigma_n(B)}{t^{\frac{1}{2}(n-1)}} \left[\frac{\sigma_{n-1}(S_1) \Gamma\left(\frac{n-1}{2}\right)}{4^{\frac{1}{2}(n+1)} \pi^n} \int_0^1 \sqrt{\frac{s}{1-s}} ds \right].
\end{aligned}$$

Since $\sigma_{n-1}(\mathcal{S}_1) = 2\pi^{\frac{1}{2}(n-1)} \{\Gamma[\frac{1}{2}(n-1)]\}^{-1}$ and $\int_0^1 \sqrt{s/(1-s)} ds = \int_0^{\frac{1}{2}\pi} (1 - \cos 2\phi) d\phi = \frac{1}{2}\pi$ with $s = \sin^2 \phi$, (6.9) proves (6.2).

Clearly an analysis similar to that yielding (6.9) above shows that the integral on the right of (5.82) is $O(t^{-\frac{1}{2}(n-1)})$ over $0 < t \leq T$ for each integer $m = 1, 2, \dots, m_0 - 1$. Thus $|J_m(t)| \leq \tilde{J}_m(t)$, (5.81), and (5.61), with $\gamma = \eta$ put into (5.82), show that (6.3) follows. (6.2) and (6.3) put into (5.84) and (5.85) yield (6.4) and (6.5), and the proof of lemma (6.1) is complete.

Lemma (6.10)

Let the boundary B of D satisfy condition (3.6) with $0 < \eta \leq 1$ (appearing in (3.7) and with the curvature invariant $q_1(\mathbf{y})$ defined over B by (3.15) and (3.16)). Then over $0 < t \leq T$

$$(6.11) \quad J_0(t) = \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1}t^{\frac{1}{2}(n-1)}} + O(t^{-\frac{1}{2}(n-2)+\frac{1}{2}\eta}),$$

$$(6.12) \quad J_1(t) = \frac{\int_B q_1(\mathbf{y}) d\sigma_n(\mathbf{y})}{3(2\sqrt{\pi})^n t^{\frac{1}{2}(n-2)}} + O(t^{-\frac{1}{2}(n-2)+\frac{1}{2}\eta}),$$

$$(6.13) \quad \sum_{m=2}^{\infty} |J_m(t)| = O(t^{-\frac{1}{2}(n-3)}),$$

$$(6.14) \quad \sum_{i=1}^{\infty} e^{-\lambda_i t} = \frac{\mu_n(D)}{(2\sqrt{\pi})^n t^{\frac{1}{2}n}} - \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1}t^{\frac{1}{2}(n-1)}} + \frac{\int_B q_1(\mathbf{y}) d\sigma_n(\mathbf{y})}{3(2\sqrt{\pi})^n t^{\frac{1}{2}(n-2)}} + O(t^{-\frac{1}{2}(n-2)+\frac{1}{2}\eta}),$$

$$(6.15) \quad 1 + \sum_{i=1}^{\infty} e^{-\lambda_i t} = \frac{\mu_n(D)}{(2\sqrt{\pi})^n t^{\frac{1}{2}n}} + \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1}t^{\frac{1}{2}(n-1)}} + \frac{\int_B q_1(\mathbf{y}) d\sigma_n(\mathbf{y})}{3(2\sqrt{\pi})^n t^{\frac{1}{2}(n-2)}} + O(t^{-\frac{1}{2}(n-2)+\frac{1}{2}\eta}).$$

The proof of lemma (6.10) goes the same way as that of lemma (6.1). Naturally more work is necessary to preserve the next higher order terms beyond those in (3.8) appearing in the (3.17) geometric expansions and to compute the next lower order terms in t beyond the corresponding ones in (6.1). We refer to (M) for the details of these computations. We remark that the coefficient of the expected $t^{-\frac{1}{2}(n-2)}$ term in (6.11) turns out to be zero ((M) , (6.18)). Also, in evaluating the leading term in (6.12) we use (after proof)

$$(6.25) \quad \int_0^t \int_{R_{n-1}} (-2)(\mathbf{n} \cdot \nabla_{\mathbf{z}} Q_n(|\mathbf{z} - \mathbf{y}|; t - \tau)) Q_n(|\mathbf{z} - \mathbf{x}|; \tau) d\mu_{n-1}(\mathbf{z}) d\tau \\ = Q_n\left(\left[(x_n + y_n)^2 + \sum_{i=1}^{n-1} (x_i - y_i)^2\right]^{\frac{1}{2}}; t\right)$$

for ${}_0\mathbf{n} = (0, 0, \dots, 0, -1)$ and $\mathbf{x}$ and $\mathbf{y} \in R_n$ having $x_n > 0$ and $y_n > 0$. This is the analogue of the meson equation formula ([13], equation 27), using (5.9) notation,

$$(6.75) \quad \int_{-\infty}^{\infty} y_2 \frac{p_2(|\mathbf{y} - (\xi, 0)|) {}_2K_1(|\mathbf{x} - (\xi, 0)|) d\xi}{\pi |\mathbf{y} - (\xi, 0)|^2} = {}_2K_1(\sqrt{(x_2 + y_2)^2 + (x_1 - y_1)^2})$$

for $\mathbf{x}$ and $\mathbf{y} \in R_2$ having x_2 and $y_2 > 0$.

We now turn to the asymptotic evaluation of the right sides of the meson equation relations (5.28) and (5.29) in case $n = 2$ only, although, as we have remarked before, similar things can be done when $n = 3$.

Lemma (6.29)

Let the boundary B of D satisfy condition (3.1) with $0 < \eta \leq 1$ (appearing in (3.2) and (3.3)) and let $n = 2$. Then over $\omega \geq \omega_0$

$$(6.30) \quad L_0(\omega) = \frac{\sigma_2(B)}{8\omega} + O(\omega^{-1-\eta}),$$

$$(6.31) \quad \sum_{m=1}^{\infty} |L_m(\omega)| = O(\omega^{-1-\eta}),$$

$$(6.32) \quad \sum_{i=1}^{\infty} \frac{1}{{}_0\lambda_i({}_0\lambda_i + \omega^2)} = \frac{\mu_2(D) \ln \omega}{2\pi\omega^2} + \frac{C_0}{\omega^2} + \frac{\sigma_2(B)}{8\omega^3} + O(\omega^{-3-\eta}),$$

$$(6.33) \quad \sum_{i=1}^{\infty} \frac{1}{{}_1\lambda_i({}_1\lambda_i + \omega^2)} = \frac{\mu_2(D) \ln \omega}{2\pi\omega^2} + \frac{C_1}{\omega^2} - \frac{\sigma_2(B)}{8\omega^3} + O(\omega^{-3-\eta}).$$

To prove (6.30) we see from (5.6) and (5.9) with $\sigma_2(S_1) = 2\pi$ that

$$(6.34) \quad L_0(\omega) = \int_D \int_B \frac{p_2(\omega |\mathbf{z} - \mathbf{x}|)({}_0\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{x}))}{\pi |\mathbf{z} - \mathbf{x}|^2} {}_2K_{\omega}(|\mathbf{z} - \mathbf{x}|) d\sigma_2(\mathbf{z}) d\mu_2(\mathbf{x})$$

over $\omega \geq \omega_0$. Since ${}_2K_{\omega}(r) = {}_2K_1(\omega r)$ and since both $p_2(v)$ and ${}_2K_1(v)$ are $O(e^{-\frac{1}{2}v})$ over $v \geq 1$ (see (5.9) and [5], (2.1) through (2.10)), it is clear that the integral of the absolute value of the (6.34) integrand over the subregion where $|\mathbf{z} - \mathbf{x}| \geq \delta$ is $O(e^{-\omega\delta})$ for every $\delta > 0$. For the remaining part of the (6.34) integral we have $\mathbf{x} = -h\mathbf{n}(\mathbf{y}) + \mathbf{y}$ with $0 < h < \delta$ for some $\mathbf{y} \in B$; we let $r = |\mathbf{z}^* - \mathbf{y}|$ be defined by projection onto the tangent line to B at $\mathbf{y}$ as in lemma (3.8), and for sufficiently small $\delta > 0$ we can use (3.2), (3.3), (3.9), and (3.10) as in (6.7). Thus (6.34) becomes over $\omega \geq \omega_0$

$$L_0(\omega) - O(e^{-\omega\delta}) = 2 \int_B \int_0^{\delta} \int_0^{2\delta} \frac{[h + (r + h)^{1+\eta} M \theta]}{\pi(r^2 + h^2)} \cdot [\rho_2(\omega\rho) {}_2K_1(\omega\rho)]_{\rho=(1+(r+h)^{\eta} M \theta')} \sqrt{r^2 + h^2} dr dh d\sigma_2(\mathbf{y})$$

for $\delta > 0$ sufficiently small. Thus (see (M)) making the change of variable $r = \omega^{-1}s$ and $h = \omega^{-1}u$, our equation for $L_0(\omega)$ becomes over $\omega \geq \omega_0$

$$\begin{aligned}
L_0(\omega) &= O(\omega^{-1-\eta}) + \frac{2\sigma_2(B)}{\omega} \int_0^{\omega^\delta} \int_0^{2\omega^\delta} \frac{u[p_2(v) {}_2K_1(v)]_{v=\sqrt{s^2+u^2}} ds du}{\pi(s^2+u^2)} \\
&= O(\omega^{-1-\eta}) + 2\sigma_2(B) \left[\int_0^{\frac{1}{2}\pi} \int_0^\infty \frac{1}{\pi} (\sin \phi) p_2(r) {}_2K_1(r) dr d\phi \right] \frac{1}{\omega} \\
&= O(\omega^{-1-\eta}) + \sigma_2(B) \left[2 \int_0^\infty (-2r {}_2K_1'(r)) {}_2K_1(r) dr \right] \frac{1}{\omega} \\
(6.35) \quad &= O(\omega^{-1-\eta}) + \sigma_2(B) \left[2 \int_0^\infty [{}_2K_1(r)]^2 dr \right] \frac{1}{\omega} \\
&= O(\omega^{-1-\eta}) + \sigma_2(B) \left[\frac{1}{4} {}_1K_1(0) \right] \frac{1}{\omega} \\
&= O(\omega^{-1-\eta}) + \frac{\sigma_2(B)}{8\omega}.
\end{aligned}$$

In going from the first to the second line here, we have used $p_2(r) {}_2K_1(r) = O(e^{-r})$ over $r \geq 1$ to extend the range of integration to the whole first quadrant without altering the $O(\omega^{-1-\eta})$ error estimate. In the last two lines, $\int_0^\infty [{}_2K_1(|s|)]^2 ds = \frac{1}{4} {}_1K_1(0) = \frac{1}{8}$ follows from equations (2.11) and (2.4) of the paper quoted above [5]. Thus (6.30) has been proved.

To prove (6.31), notice from (5.9) that if the $p_n(\omega r)$ factor is not replaced by 1 the arguments used to obtain (5.10) and (5.11) actually yield for some $M < +\infty$

$$(6.36) \quad R_\omega \leq M_\delta \exp(-\tfrac{1}{2}\omega\delta) + M' \int_0^\delta r^{\eta'-1} p_n(\omega \tfrac{1}{2}r) dr \leq M\omega^{-\eta'}$$

by making the change of variable $r = \omega^{-1}v$. Thus $\eta' = \eta$ under condition (3.1) on B yields in (5.26) the estimate $\tilde{L}_m(\omega) \leq M(R_\omega)^{m-1}\omega^{-1-\eta}$ over $m \geq 1$ and $\omega \geq \omega_0$, with $0 < R_\omega \leq R_{\omega_0} < 1$. Hence $|L_m(\omega)| \leq \tilde{L}_m(\omega)$ yields (6.31), (6.32) and (6.33) follow from (6.30) and (6.31) put into (5.28) and (5.29), and the proof of lemma (6.29) is complete.

Condition (3.6) when $n = 2$ is equivalent to condition (3.4) on B with $0 < \eta \leq 1$ unchanged, with $q_1(z) = c(s)$ for $z = f(s) \in B$, with $c(s_k^-) = c(s_k^+)$, and with all $\alpha_k = 0$, no corners being permitted. Thus the following lemma stated for condition (3.4) also takes care of condition (3.6) when $n = 2$ as a special case.

Lemma (6.37)

Let $n = 2$ and let the boundary B of D satisfy condition (3.4) with $0 < \eta \leq 1$ appearing in (3.5), B being the union of $p + 1$ disjoint Jordan curves. Then $\int_B q_1(z) d\sigma_2(z) = \int_B c(s) ds = 2\pi(1 - p) - \sum_k \alpha_k$ with $\{\alpha_k\}$ the finite set of corner angles of B , and over $\omega \geq \omega_0$

$$(6.38) \quad L_0(\omega) = \frac{\sigma_2(B)}{8\omega} + \left[\sum_k F_0(\alpha_k) \right] \frac{1}{\omega^2} + O(\omega^{-2-\eta}),$$

$$(6.39) \quad L_1(\omega) = \frac{1-p}{6\omega^2} + \left[\sum_k F_1(\alpha_k) \right] \frac{1}{\omega^2} + O(\omega^{-2-\eta}),$$

$$(6.40) \quad \sum_{m=2}^{\infty} \left| L_m(\omega) - \left[\sum_k F_m(\alpha_k) \right] \frac{1}{\omega^2} \right| = O(\omega^{-3}),$$

$$(6.41) \quad E_\nu(\alpha) = \sum_{m=0}^{\infty} (-1)^{(m-1)(1-\nu)} F_m(\alpha), \quad \nu = 0 \text{ or } 1,$$

$$(6.42) \quad \sum_{i=1}^{\infty} \frac{1}{0\lambda_i(0\lambda_i + \omega^2)} = \frac{\mu_2(D) \ln \omega}{2\pi\omega^2} + \frac{C_0}{\omega^2} + \frac{\sigma_2(B)}{8\omega^3} \\ - \left[\frac{1}{6}(1-p) + \sum_k E_0(\alpha_k) \right] \frac{1}{\omega^4} + O(\omega^{-4-\eta}),$$

$$(6.43) \quad -\omega^{-4} + \sum_{i=1}^{\infty} \frac{1}{1\lambda_i(1\lambda_i + \omega^2)} = \frac{\mu_2(D) \ln \omega}{2\pi\omega^2} + \frac{C_1}{\omega^2} - \frac{\sigma_2(B)}{8\omega^3} \\ - \left[\frac{1}{6}(1-p) + \sum_k E_1(\alpha_k) \right] \frac{1}{\omega^4} + O(\omega^{-4-\eta}).$$

The corner terms $F_m(\alpha)$ over $|\alpha| < \pi$ are continuous in α and are given by the formulæ

$$(6.44) \quad F_0(\alpha) = -\frac{1}{(2\pi)^2} \left(1 - \frac{\alpha \cos \alpha}{\sin \alpha} \right), \quad \alpha \neq 0,$$

$$F_1(\alpha) = \omega^2 L_1(\omega; \alpha) - \frac{1}{6} \left(\frac{\alpha}{2\pi} \right),$$

$$(6.45) \quad F_m(\alpha) = \omega^2 L_m(\omega; \alpha), \quad m \geq 2.$$

Here $D_\alpha = \{\mathbf{x} \in R_2 \mid \mathbf{x} = r(\cos \phi, \sin \phi), r > 0, \alpha < \phi < \pi\}$, B_α is its boundary in R_2 , and $L_m(\omega; \alpha)$ is the right side of (5.6) with B and D replaced by B_α and D_α , $\omega^2 L_m(\omega; \alpha)$ being constant in $\omega > 0$ for $m \geq 1$. Also

$$(6.46) \quad F_m(0) = 0 = F'_m(0), \quad m \geq 0; \quad E_\nu(0) = 0 = E'_\nu(0), \quad \nu = 0 \text{ or } 1.$$

Corollary (6.47)

If the conditions of lemma (6.37) are satisfied, and if in addition the $p+1$ disjoint Jordan curves whose union is B are polygons, then the various error estimates $O(\omega^{-2-\eta})$, $O(\omega^{-3})$, and $O(\omega^{-4-\eta})$ in (6.38) through (6.43) can all be replaced by $O(e^{-\omega^\rho})$ for some $\rho > 0$ depending only on B .

In a later section we will see that $E_\nu(\alpha) \neq 0$ by finding indirectly that $E_0(\frac{1}{2}\pi) = \frac{1}{48} = E_1(\frac{1}{2}\pi)$. A direct computation of $E_\nu(\alpha)$ from (6.41), (6.44), and (6.45), using numerical integration by machine, can of course be done. However, preliminary estimates indicate that sufficient accuracy even to obtain a general idea of $E_\nu(\alpha)$ as a function of α may require considerable labor.

For the proof of lemma (6.37), as for lemma (6.10), we refer to (M) for details. (6.75), like (6.25) there, is used to evaluate $L_1(\omega)$. The proof depends upon splitting the (5.6) integrals up into several parts according as $\mathbf{x}$ is near a boundary arc A_k or a corner $k\mathbf{x}$.

7. Log Gaussian Averaged Error Estimates

This section is devoted to obtaining asymptotic expansions for $N_\nu(\lambda) = \sum_{\lambda_j \leq \lambda} 1$, $\nu = 0$ or 1 , from the asymptotic evaluations of Green function integrals obtained in the previous section. The error terms in these expansions will be found to satisfy certain log Gaussian averaged error estimates, the theory of which has been developed in an earlier paper [3]. We will need to develop this theory further to take care of the additional cases considered here, and in this development we will quote freely from this earlier paper.

These log Gaussian averaged error estimates, denoted by $\tilde{O}$, will now be defined. $|dF(y)|$ denotes $dV_F(y)$ for $V_F(y)$ the total variation of F over $[0, y]$. We define $\psi_{F,b}(s) = \int_b^\infty y^{-s} dF(y)$ over complex s satisfying $\Re[s] \geq r$ if $\int_b^\infty y^{-r} |dF(y)| < +\infty$; and $dF(y)$ and $dV_F(y)$ denote Lebesgue-Stieltjes integration.

Definition (7.1)

For fixed $b > 0$, consider a real-valued function $F(y)$ which is of bounded variation over every finite interval of $y \geq 0$, for which $F(b^-) = F(b) = F(b^+)$ and for which $\int_b^\infty y^{-r_0} |dF(y)| < +\infty$ for some $r_0 \geq 0$. Also let either $f_r(y) = y^r$ or $f_r(y) = y^r \ln y$. For such $F(y)$ and $f_r(y)$ with $r \geq 0$ we say that $F(y) = \tilde{O}(f_r(y))$ over $y \geq b$ if

$$(7.2) \quad \left| \int_b^\infty \exp\left(-\frac{1}{2}\rho^2 \left[\ln \frac{y}{b}\right]^2\right) dF(y) \right| \leq M_\rho f_r(y)$$

over $y \geq e$ for some $M_\rho < +\infty$ existing for every $\rho > 0$. For $r < 0$ we say that such $F(y) = \tilde{O}(f_r(y))$ over $y \geq b$ if in addition to (7.2) also $\psi_{F,b}(s)$ possesses an analytic continuation in s from $\Re[s] > r_0$ to $\Re[s] \geq 0$ such that

$$(7.3) \quad \psi_{F,b}(0) = -F(b).$$

Since

$$\psi_{F,b'}(0) = \int_{b'}^b dF(y) + \psi_{F,b}(0) = F(b) - F(b') + \psi_{F,b}(0),$$

it is clear that the additional condition (7.3) when $r < 0$ does not really depend upon the choice of b . This additional condition is needed to fix arbitrary additive constants in $F(y)$, which are undetermined by $dF(y)$ in (7.2).

We will use the following theorem to obtain such $\tilde{O}$ estimates.

Theorem (7.4)

Let $F(y)$ and b satisfy the first sentence of definition (7.1), let $\int_b^\infty y^{-r_0} |dF(y)| < +\infty$ for some $r_0 \geq 0$, let $\psi_{F,b}(s)$ possess an analytic continuation in s from $\Re[s] > r_0$ throughout $\Re[s] \geq r_1$ for some $r_1 \leq r_0$, let this continuation satisfy $|\psi_{F,b}(s)| \leq M'(r - r_1)^{-1} e^{h|s|}$ over $r_1 < r \leq r_0 + 1$ with $s = r + i\xi$ for some real constants M' and h , and let (7.3) be satisfied if $r_1 < 0$. Then $F(y) = \tilde{O}(y^{r_1} \ln y)$ over $y \geq b$, and in (7.2) we can take

$$M_\rho = 2M'e^{1+\frac{1}{2}\rho^2 h^2 + \frac{1}{2}(1+|r_1|)^2/\rho^2}.$$

We have, as in the previous paper ([3], equation (2.3)),

$$(7.5) \quad \int_{x=\ln b}^{\infty} e^{-\frac{1}{2}\rho^2(\omega-x)^2 - s(\omega-x)} e^{-sz} dF(e^x) \\ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\xi-is)^2/\rho^2} \psi_{F,b}(z + i\xi) e^{i\xi\omega} \rho^{-1} d\xi$$

over real ω and complex z having $\Re[z] \geq r_0$. This is found by taking Fourier transforms of the left side over ω and using the Fubini theorem. As in the previous paper also, the left side of (7.5) is an entire function of z for all real ω . The given estimate on $\psi_{F,b}(s)$ shows that the right side is analytic in those z satisfying $\Re[z] > r_1$, using the Cauchy integral formula and the Fubini theorem. Thus (7.5) actually holds over all real ω and complex z having $\Re[z] > r_1$, and the given estimate shows in (7.5) with real $z = r$ that over all real ω and $r_1 < r \leq r_0 + 1$

$$(7.6) \quad \left| \int_{x=\ln b}^{\infty} e^{-\frac{1}{2}\rho^2(\omega-x)^2} e^{-r\omega} dF(e^x) \right| \leq \frac{M'}{(r-r_1)\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(v^2-r^2)/\rho^2 + h|v|} \frac{1}{\rho} dv \\ \leq \frac{M'}{r-r_1} e^{\frac{1}{2}r^2/\rho^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2 + h\rho|w|} dw \\ \leq \frac{2M'}{r-r_1} e^{\frac{1}{2}r^2/\rho^2 + \frac{1}{2}\rho^2 h^2}.$$

Multiplying by $e^{r\omega}$ and letting $e^{\omega} = v$ and $e^x = y$, this shows that the left side of (7.2) is

$$\leq v^{r+1} 2M' \exp \left(\inf_{r_1 < r \leq r_0+1} \left[\frac{r^2}{2\rho^2} + \frac{1}{2}\rho^2 h^2 - \ln(r-r_1) + (r-r_1) \ln v \right] \right) \\ \leq v^{r+1} (\ln v) 2M' e^{\frac{1}{2}(1+r_1)^2/\rho^2 + \frac{1}{2}\rho^2 h^2 + 1}$$

over $v \geq e$ by taking $r = r_1 + (\ln v)^{-1}$, which satisfies $r_1 < r \leq r_1 + 1 \leq r_0 + 1$. Hence (7.2) with the stated value of M_p follows, and the proof of theorem (7.4) is complete.

The following list of properties greatly enhances the significance of $\tilde{O}$ -estimates as defined by (7.1). In particular, (IV) shows that if a function can be expanded both in an asymptotic series with an $\tilde{O}$ -estimate on the error and in an ordinary asymptotic series, then both series must agree term by term as far as they go together.

Theorem (7.7)

Let $F(y)$ and b satisfy the first sentence of definition (7.1).

- (I) If $F(y) = O(y^r)$ over $y \geq b$, then $F(y) = \tilde{O}(y^r)$ over $y \geq b$.
- (II) If $F(y) = \tilde{O}(y^r)$ over $y \geq b$, then $F(y) = \tilde{O}(y^r \ln y)$ over $y \geq b$.

(III) If $F(y) = \tilde{O}(y^r \ln y)$ over $y \geq b$, then $F(y) = \tilde{O}(y^{r'})$ over $y \geq b$ for every $r' > r$.

(IV) If $F(y) = c_1 y^r + o(y^r)$ as $y \rightarrow +\infty$, and if $F(y) = \tilde{O}(y^{r'} \ln y)$ over $y \geq b$ for some $r' < r$, then $c_1 = 0$.

Here (II) and (III) are obvious from (7.2). To prove (I), notice that $y = e^x$ gives $y^r \exp(-\frac{1}{2}\rho^2(\ln v/y)^2) = \exp(rx - \frac{1}{2}\rho^2(x - \ln v)^2) \rightarrow 0$ as $y \rightarrow +\infty$. Thus, using $|F(y)| \leq My^r$, we can integrate by parts and the left side of (7.2) is seen to be

$$\begin{aligned}
 & \leq |F(b)| \exp\left(-\frac{1}{2}\rho^2\left(\ln \frac{v}{b}\right)^2\right) + M \int_b^\infty \exp\left(-\frac{1}{2}\rho^2\left(\ln \frac{v}{y}\right)^2\right) \rho^2 \left|\ln \frac{v}{y}\right| y^{r-1} dy \\
 (7.8) \quad & \leq |F(b)| \exp\left(-\frac{1}{2}\rho^2\left(\ln \frac{v}{b}\right)^2\right) + v^r M \rho^2 \int_{-\infty}^\infty e^{-\frac{1}{2}\rho^2 \omega^2 + r\omega} |\omega| d\omega \\
 & \leq M_\rho v^r
 \end{aligned}$$

as desired for (7.2) by letting $y = v e^\omega$. To obtain the additional condition (7.3) in case $r < 0$, we also can integrate by parts in the definition of $\psi_{F,b}(s)$, getting

$$(7.9) \quad \psi_{F,b}(s) = -b^{-s} F(b) + \int_b^\infty F(y) s y^{-s-1} dy$$

over $\Re[s] \geq r_0$. $|F(y)| \leq My^r$ with $r < 0$ shows that the right side of (7.9) is analytic in s over $\Re[s] > r$ and has the value $-F(b)$ at $s = 0$. Hence condition (7.3) follows and (I) has been completely proved.

To prove (IV) (see [3], theorem 2), consider first the case where $r \neq 0$. Then integrating by parts in (7.2) with $v = e^\omega$ and $y = e^x$ yields over $\omega \geq 1$

$$(7.10) \quad \rho^2 \left| \int_{x=1 \ln b}^\infty F(e^x) e^{-\frac{1}{2}\rho^2(\omega-x)^2} (x-\omega) dx \right| \leq M_\rho \omega e^{r'\omega} + |F(b)| e^{-\frac{1}{2}\rho^2(\omega-1 \ln b)^2}.$$

Now $F(y) = [c_1 + g(y)]y^r$ with $r > r'$ and $\lim_{y \rightarrow \infty} g(y) = 0$. Hence multiplying (7.10) by $e^{-r'\omega}$ and letting $\omega \rightarrow \infty$, we obtain

$$\begin{aligned}
 0 &= \lim_{\omega \rightarrow \infty} \int_{x=1 \ln b}^\infty [c_1 + g(e^x)] e^{-\frac{1}{2}\rho^2(\omega-x)^2 + r(x-\omega)} (x-\omega) dx \\
 (7.11) \quad &= c_1 \int_{-\infty}^\infty t e^{-\frac{1}{2}\rho^2 t^2 + r t} dt + \lim_{\omega \rightarrow \infty} \int_{-\infty}^\infty g(e^{\omega+t}) t e^{-\frac{1}{2}\rho^2 t^2 + r t} dt \\
 &= c_1 \int_{-\infty}^\infty t e^{-\frac{1}{2}\rho^2 t^2 + r t} dt
 \end{aligned}$$

by defining $g(y) = 0$ for $y < b$ and using dominated convergence. Thus (7.11) and $\int_{-\infty}^\infty t e^{-\frac{1}{2}\rho^2 t^2 + r t} dt \neq 0$ for $r \neq 0$ yields $c_1 = 0$ as desired.

In case $0 = r > r'$, then $F(y) = \tilde{O}(y^{r'} \ln y)$ over $y \geq b$ requires condition (7.3) to hold. Also, by integration by parts and analytic continuation, $F(y) =$

$c_1 + g(y)$ with $\lim_{y \rightarrow \infty} g(y) = 0$ makes (7.9) hold over $\Re[s] > 0$. Thus (7.3) yields with real $s > 0$

$$(7.12) \quad 0 = \lim_{s \rightarrow 0^+} \int_b^\infty F(y) s y^{-s-1} dy = c_1 + \lim_{s \rightarrow 0^+} \int_b^\infty g(y) s y^{-s-1} dy = c_1,$$

since

$$\begin{aligned} \limsup_{s \rightarrow 0^+} \left| \int_b^\infty g(y) s y^{-s-1} dy \right| \\ \leq \limsup_{y' \rightarrow \infty} \left[\left(\limsup_{s \rightarrow 0^+} s \int_b^{y'} |g(y)| y^{-s-1} dy \right) + \left(\sup_{y \geq y'} |g(y)| \right) \left(\lim_{s \rightarrow 0^+} (y')^{-s} \right) \right] \\ = \lim_{y' \rightarrow \infty} [\sup_{y \geq y'} |g(y)|] = 0. \end{aligned}$$

Thus (7.12) yields $c_1 = 0$ in case $r = 0$ as well as $r \neq 0$, and the proof of (IV) is complete.

In order to apply these $\tilde{O}$ -estimates to the asymptotic formulæ found for the Green function integrals, we show in the next theorems how the hypotheses of theorem (7.4) can be obtained from information about $\int_b^\infty e^{-y^t} dF(y)$ over $t > 0$ and about $\int_b^\infty y^{-1}(y-z)^{-1} dF(y)$ over $z < 0$.

Theorem (7.13)

Let $0 < b < \lambda_j \leq \lambda_{j+1}$ and a_j be real for integer $j \geq 1$, let $N(\lambda) = \sum_{\lambda_i \leq \lambda} a_i$ over $\lambda \geq 0$ satisfy $\int_b^\infty \lambda^{-r_0} |dN(\lambda)| < +\infty$ for some $r_0 > 0$, and let

$$(7.14) \quad \int_b^\infty e^{-\lambda t} dN(\lambda) = \sum_{i=1}^k c_i t^{-r_i} + O(t^{-r_{k+1}})$$

over $0 < t \leq T$ for some real $T > 0$ and $r_{k+1} < r_k < \dots < r_1$. Then over $\lambda > b$

$$(7.15) \quad N(\lambda) = \sum_{i=1}^k \frac{c_i \lambda^{r_i}}{\Gamma(r_i + 1)} + \tilde{O}(\lambda^{r_{k+1}} \ln \lambda),$$

defining $[\Gamma(r_i + 1)]^{-1} = 0$ if r_i is a negative integer.

To prove this theorem, let

$$F(\lambda) = N(\lambda) - \sum_{i=1}^k \frac{c_i \lambda^{r_i}}{\Gamma(r_i + 1)}$$

over $\lambda \geq \frac{1}{2}b > 0$, let $F(\lambda) = 0$ for $0 \leq \lambda < \frac{1}{2}b$, and let $r'_0 = (\max r_0, 1 + r_1)$. Thus $\int_b^\infty \lambda^{-r'_0} |dF(\lambda)| < +\infty$ follows from $\int_b^\infty \lambda^{-r_0} |dN(\lambda)| < +\infty$, both $F(\lambda)$ and $N(\lambda)$ satisfy the other conditions of the first sentence of definition (7.1) for b , and $\psi_{F,b}(s)$ is analytic in $\Re[s] > r'_0$ and $\psi_{N,b}(s)$ in $\Re[s] > r_0$. Also clearly from the Fubini theorem follows

$$\int_0^\infty t^{s-1} \left[\int_b^\infty e^{-\lambda t} dF(\lambda) \right] dt = \int_b^\infty \int_0^\infty \lambda^{-s} e^{-\lambda t} t^{s-1} dt dF(\lambda) = \Gamma(s) \psi_{F,b}(s)$$

over $\Re[s] \geq r'_0$, and similarly

$$\int_0^\infty t^{s-1} \left[\int_b^\infty e^{-\lambda t} d(\lambda^r) \right] dt = \Gamma(s) \int_b^\infty \lambda^{-s} d(\lambda^r) = \Gamma(s) r(s-r)^{-1} b^{-s+r}$$

over $\Re[s] > r$. Thus from (7.14)

$$(7.16) \quad \begin{aligned} \psi_{F,b}(s) &= \frac{1}{\Gamma(s)} \sum_{i=1}^k \frac{c_i T^{s-r_i}}{s-r_i} + \frac{M'}{\Gamma(s)} \int_0^T \theta(t) t^{s-1-r_{k+1}} dt \\ &\quad + \frac{1}{\Gamma(s)} \int_r^\infty \int_b^\infty t^{s-1} e^{-\lambda t} dN(\lambda) dt - \sum_{i=1}^k \frac{c_i r_i b^{-s+r_i}}{\Gamma(r_i+1) \cdot (s-r_i)} \end{aligned}$$

follows over $\Re[s] \geq r'_0$, where $M'\theta(t)t^{-r_{k+1}}$ with $|\theta(t)| \leq 1$ is the error term in (7.14). Now $\Gamma(s)^{-1}$ is an entire function, $\int_0^T \theta(t)t^{s-1-r_{k+1}} dt$ is analytic in $\Re[s] > r_{k+1}$, $\int_r^\infty \int_b^\infty t^{s-1} e^{-\lambda t} dN(\lambda) dt$ is entire in s since $0 < e^{-\lambda t} \leq e^{-\frac{1}{2}bt} e^{-\frac{1}{2}\lambda T}$ here, the simple poles at $s = r_i$ arising in the first sum on the right side of (7.16) are cancelled by those in the last sum for r_i not a nonpositive integer, and for r_i a nonpositive integer the factors $\Gamma(s)^{-1}$ and $r_i[\Gamma(r_i+1)]^{-1} = [\Gamma(r_i)]^{-1}$ suppress these singularities. Thus the right side of (7.16) is analytic in $\Re[s] > r_{k+1}$ and gives the analytic continuation of $\psi_{F,b}(s)$ there.

Also, with $s = r + i\xi$, we see from analysis of the Stirling formula that for every $h > \frac{1}{2}\pi$ and real v there exists $M'' < +\infty$ such that $|\Gamma(s)^{-1}| \leq M'' e^{h|\xi|}$ over $r \geq v$. Since $\int_0^T t^{r-1-r_{k+1}} dt = (r-r_{k+1})^{-1} T^{r-r_{k+1}}$ over $r > r_{k+1}$, the above analytic continuation satisfies

$$(7.17) \quad |\psi_{F,b}(s)| \leq \frac{M_h e^{h|\xi|}}{r-r_{k+1}}$$

over $r_{k+1} < r \leq r'_0 + 1$ for every $h > \frac{1}{2}\pi$ and some $M_h < +\infty$ existing for such h .

In case $r_{k+1} < 0$, then, since the entire function $\Gamma(s)^{-1}$ takes the value 0 at $s = 0$, we see from (7.16) that, providing each $r_i \neq 0$,

$$(7.18) \quad \psi_{F,b}(0) = \sum_{i=1}^k \frac{c_i b^{r_i}}{\Gamma(r_i+1)} = N(b) - F(b) = -F(b),$$

since $b < \lambda_1$ makes $N(b) = 0$. If $r_q = 0$, then, since $[s\Gamma(s)]^{-1}$ is analytic in $s = 0$ and takes the value 1 there, (7.16) yields

$$(7.19) \quad \psi_{F,b}(0) = c_q + \sum_{\substack{1 \leq i \leq k \\ i \neq q}} \frac{c_i b^{r_i}}{\Gamma(r_i+1)} = \sum_{i=1}^k \frac{c_i b^{r_i}}{\Gamma(r_i+1)} = -F(b)$$

as in (7.18). Hence condition (7.3) is satisfied if $r_{k+1} < 0$. Thus (7.16) and (7.17) show that all the hypotheses of theorem (7.4) are satisfied; hence $F(\lambda) = \tilde{O}(\lambda^{r_{k+1}} \ln \lambda)$ over $\lambda \geq b$, (7.15) follows for $N(\lambda)$ from the definition of $F(\lambda)$, and the proof of theorem (7.13) is complete.

Theorem (7.20)

Let $0 < b < \lambda_i \leq \lambda_{i+1}$ and a_i be real for integer $j \geq 1$, let $N(\lambda) = \sum_{\lambda_i \leq \lambda} a_i$ over $\lambda \geq 0$, let $\sum_{i=1}^\infty \lambda_i^{-2} |a_i| < +\infty$ and $\sum_{i=1}^\infty \lambda_i^{-2} < +\infty$, and let

$$(7.21) \quad \int_b^\infty \frac{dN(\lambda)}{\lambda(\lambda + v)} = \sum_{i=1}^k \frac{c_i + e_i \ln v}{v^{2-r_i}} + O(v^{-(2-r_{k+1})})$$

over $v \geq v_0$ for some real $v_0 > 0$ and $r_{k+1} < r_k < \dots < r_1 < 2$ such that $e_i \sin(\pi r_i) = 0$, and $e_i = 0$ if $r_i = 0$. Then over $\lambda \geq b$

$$(7.22) \quad N(\lambda) = - \sum_{i=1}^k \left(c_i \frac{\sin(\pi r_i)}{\pi r_i} + \frac{e_i}{r_i} \cos(\pi r_i) \right) \lambda^{r_i} + \tilde{O}(\lambda^{r_{k+1}} \ln \lambda),$$

defining $\sin(\pi r)/\pi r = 1$ if $r = 0$ and $e_i r_i^{-1} \cos \pi r_i = 0$ if $e_i = 0$.

This theorem follows as in lemmas (3) and (4) and theorem (5) of the previous paper [3], after suitable modification to take care of the case $r_i \leq 0$ now permitted. More precisely, lemma (3) there needs no modification at all; thus we conclude ([3], equation (2.7)) that

$$(7.23) \quad \psi_{N,b}(s) = \sum_{i=1}^\infty a_i \lambda_i^{-s} = \frac{1}{2\pi i} \int_{C_\rho} \left[\int_b^\infty \frac{dN(\lambda)}{\lambda(\lambda - z)} \right] \frac{dz}{(z)_C^{s-1}}$$

over $\Re[s] > 2$. Replace equation (2.8) in the previous lemma (4) by (7.21) with $\omega = v$, and thus from (7.23) we obtain in place of (2.9) there and in the same way

$$(7.24) \quad \psi_{N,b}(s) = g_k(s) + \sum_{i=1}^k \left(-\frac{\sin(\pi s)}{\pi} \right) \left[\frac{c_i}{s - r_i} + \frac{e_i}{(s - r_i)^2} \right]$$

as the analytic continuation of $\psi_{N,b}(s)$ into $\Re[s] > r_{k+1}$ except for poles at r_i . In (7.24), with $s = r + i\xi$, $g_k(s)$ is analytic and $|g_k(s)| \leq M'(r - r_{k+1})^{-1} e^{\pi|\xi|}$ over $r_{k+1} < r \leq 3$, and also $g_k(0) = 0$ if $r_{k+1} < 0$. Since $e_i \sin(\pi r_i) = 0$, only simple poles really occur in (7.24) at r_i . Defining

$$F(\lambda) = N(\lambda) + \sum_{i=1}^k \left(c_i \frac{\sin(\pi r_i)}{\pi r_i} + \frac{e_i}{r_i} \cos(\pi r_i) \right) \lambda^{r_i}$$

yields

$$\psi_{F,b}(s) = \psi_{N,b}(s) + \sum_{i=1}^k \left(c_i \frac{\sin(\pi r_i)}{\pi} + e_i \cos(\pi r_i) \right) \frac{b^{r_i-s}}{s - r_i}$$

over $\Re[s] > r_1$ as in the previous theorem (5) proof. Hence (7.24) shows that $\psi_{F,b}(s)$ possesses an analytic continuation over $r = \Re[s] > r_{k+1}$ and satisfies $|\psi_{F,b}(s)| \leq M(r - r_{k+1})^{-1} e^{\pi|\xi|}$ over $r_{k+1} < r \leq 3$ for some $M < +\infty$.

Also if $r_{k+1} < 0$ and each $r_i \neq 0$, then $\psi_{N,b}(0) = 0 = N(b)$ and

$$\psi_{F,b}(0) = - \sum_{i=1}^k \left(c_i \frac{\sin(\pi r_i)}{\pi r_i} + \frac{e_i}{r_i} \cos(\pi r_i) \right) b^{r_i} = -F(b).$$

If $r_{k+1} < 0$ and $r_q = 0$, then $\psi_{N,b}(0) = -c_q$, $N(b) = 0$, and

$$\begin{aligned} \psi_{F,b}(0) &= -c_q - \sum_{\substack{1 \leq i \leq k, \\ i \neq q}} \left(c_i \frac{\sin(\pi r_i)}{\pi r_i} + \frac{e_i}{r_i} \cos(\pi r_i) \right) b^{r_i} \\ &= - \sum_{i=1}^k \left(c_i \frac{\sin(\pi r_i)}{\pi r_i} + \frac{e_i}{r_i} \cos(\pi r_i) \right) b^{r_i} = -F(b). \end{aligned}$$

Thus $F(\lambda)$ satisfies all the conditions of theorem (7.4), $F(\lambda) = \tilde{O}(\lambda^{r_{k+1}} \ln \lambda)$ over $\lambda \geq b$, (7.22) follows, and the proof of theorem (7.20) is complete.

We now apply theorems (7.13) and (7.20) to the lemmas of the preceding section 6 in order to obtain our following final results concerning $N_\nu(\lambda) = \sum_{\nu, \lambda_i \leq \lambda} 1$, $\nu = 0$ or 1 denoting the Dirichlet boundary condition (1.2) or the modified Neumann condition (2.3), respectively, for the eigenvalue problem (1.1).

Theorem (7.25)

Let the boundary B of D satisfy condition (3.1) with $n \geq 2$ and $0 < \eta \leq 1$ (appearing in (3.2) and (3.3)). Then the number of eigenvalues $N_\nu(\lambda)$ for $\nu = 0$ or 1 satisfies over $\lambda \geq \frac{1}{2}\lambda_1 > 0$

$$(7.26) \quad N_\nu(\lambda) = \frac{\mu_n(D)}{(2\sqrt{\pi})^n \Gamma\left(\frac{n}{2} + 1\right)} \lambda^{\frac{1}{2}n} + (-1)^{\nu+1} \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1} \Gamma\left(\frac{n+1}{2}\right)} \lambda^{\frac{1}{2}(n-1)} + \tilde{O}(\lambda^{\frac{1}{2}(n-1-\eta)} \ln \lambda).$$

Theorem (7.27)

Let the boundary B of D satisfy condition (3.6) with $n \geq 2$, $0 < \eta \leq 1$ (appearing in (3.7)), and $q_1(\mathbf{y})$ defined as in theorem (6.10). Then $N_\nu(\lambda)$ for $\nu = 0$ or 1 satisfies over $\lambda \geq \frac{1}{2}\lambda_1 > 0$

$$(7.28) \quad N_\nu(\lambda) = \frac{\mu_n(D)}{(2\sqrt{\pi})^n \Gamma\left(\frac{n}{2} + 1\right)} \lambda^{\frac{1}{2}n} + (-1)^{\nu+1} \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1} \Gamma\left(\frac{n+1}{2}\right)} \lambda^{\frac{1}{2}(n-1)} + \left[\int_B q_1(\mathbf{y}) d\sigma_n(\mathbf{y}) \right] \frac{1}{3(2\sqrt{\pi})^n \Gamma\left(\frac{n}{2}\right)} \lambda^{\frac{1}{2}(n-2)} + \tilde{O}(\lambda^{\frac{1}{2}(n-2-\eta)} \ln \lambda).$$

Theorem (7.29)

Let the boundary B of D satisfy condition (3.4) with $n = 2$, α_k and p as there, and $0 < \eta \leq 1$ (appearing in (3.5)). Then, with $E_\nu(\alpha)$ defined by (6.41), $N_\nu(\lambda)$ for $\nu = 0$ or 1 satisfies over $\lambda \geq \frac{1}{2}\lambda_1 > 0$

$$(7.30) \quad N_\nu(\lambda) = \frac{\mu_2(D)}{4\pi} \lambda + (-1)^{\nu+1} \frac{\sigma_2(B)}{4\pi} \lambda^{\frac{1}{2}} + \left[\frac{1-p}{6} + \sum_k E_\nu(\alpha_k) \right] + \tilde{O}(\lambda^{-\frac{1}{2}\eta} \ln \lambda).$$

Moreover, if B is polygonal as well, then $\tilde{O}(\lambda^{-\frac{1}{2}\eta} \ln \lambda)$ can be replaced by $\tilde{O}(\lambda^r)$ in (7.30) for every real $r < 0$.

Defining $N_r^*(\lambda) = \sum_{0 < \lambda_i \leq \lambda} 1$, for which $N_0^*(\lambda) = N_0(\lambda)$ and $N_1^*(\lambda) = N_1(\lambda) - 1$ over $\lambda > 0$, clearly

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_{\frac{1}{2}, \lambda_1}^{\infty} e^{-\lambda t} dN_r^*(\lambda)$$

and

$$\sum_{i=1}^{\infty} \lambda_i^{-1} (\lambda_i + \omega^2)^{-1} = \int_{\frac{1}{2}, \lambda_1}^{\infty} \lambda^{-1} (\lambda + \omega^2)^{-1} dN_r^*(\lambda).$$

Hence, providing we can show

$$\int_{\frac{1}{2}, \lambda_1}^{\infty} \lambda^{-2} |dN_r^*(\lambda)| = \sum_{i=1}^{\infty} \lambda_i^{-2}$$

to be finite for theorem (7.20) and

$$\int_{\frac{1}{2}, \lambda_1}^{\infty} \lambda^{-r_0} |dN_r^*(\lambda)| = \sum_{i=1}^{\infty} \lambda_i^{-r_0}$$

to be finite for some $r_0 > 0$ for theorem (7.13), all three theorems (7.25), (7.27), and (7.29) follow from the asymptotic expansions (6.4), (6.5), (6.14), (6.15), (6.32), (6.33), (6.42), and (6.43), and from corollary (6.47). Notice that when $n = 2$ the results obtained from (7.22) here agree with those from (7.15), using $q_1(\mathbf{y}) = c(\mathbf{y})$ and $(2\pi)^{-1} \int_B c(\mathbf{y}) d\sigma_2(\mathbf{y}) = 1 - p$ for condition (3.6) on B considered as a special case of (3.4) with all $\alpha_k = 0$. Finally $\sum_{i=1}^{\infty} \lambda_i^{-2} < +\infty$ follows from

$$\sum_{i=1}^{\infty} (\lambda_i + \omega^2)^{-2} \leq \int_D \int_D [G_r(\mathbf{x}, \mathbf{y}; \omega)]^2 d\mu_2(\mathbf{y}) d\mu_2(\mathbf{x}) < +\infty$$

over $\omega \geq \omega_0$ when $n = 2$ as shown by (4.5) and theorem (5.27). Likewise (4.4), lemma (4.9), and theorem (5.27) show that

$$\sum_{i=1}^{\infty} (\lambda_i + \omega^2)^{-2m} \leq \int_D \int_D [G_r^{(m)}(\mathbf{x}, \mathbf{y}; \omega)]^2 d\mu_n(\mathbf{y}) d\mu_n(\mathbf{x}) < +\infty$$

over $\omega \geq \omega_0$ for some integer $m \geq 1$ when $n \geq 2$, and thus $\sum_{i=1}^{\infty} \lambda_i^{-r_0} < +\infty$ for $r_0 = 2m$. Hence the proof of the three theorems stated immediately above is complete.

8. Lattice Point Problem Connections

In this section we consider the well known connection between the number of lattice points inside a solid n -sphere and $N_r(\lambda) = \sum_{\lambda_i \leq \lambda} 1$ for D an n -dimensional rectangular parallelepiped in the light of the extended asymptotic expansions for $N_r(\lambda)$ found in the previous section. In particular, we exploit this connection in order to compute $E_0(\frac{1}{2}\pi) = E_1(\frac{1}{2}\pi) = \frac{1}{4^8}$, as promised earlier after theorem (6.3).

Specifically, define $F_n(x)$ over real $x \geq 0$ and integer $n \geq 1$ to be the number of points $(m_1, m_2, \dots, m_n) \in R_n$ with integer coordinates m_i such that $\sum_{i=1}^n m_i^2 \leq x$. It is well known ([2], p. 825) that over $x > \rho > 0$

$$(8.1) \quad F_n(x) = \frac{\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2} + 1\right)} x^{\frac{1}{2}n} + B_n(x), \quad B_n(x) = O(x^{\frac{1}{2}(n-1)/(1+n^{-1})}).$$

We find it convenient also for such n and x to define $G_n(x)$ as the number of integer coordinate points $\mathbf{m} \in R_n$ satisfying $|\mathbf{m}|^2 \leq x$ such that each $m_i \neq 0$, ${}_0P_n(x)$ as that such that each $m_i > 0$, and ${}_1P_n(x)$ as that such that each $m_i \geq 0$. With $\binom{n}{p} = n!/p!(n-p)!$ the usual binomial coefficient, we have the relations over $x > 0$ and integer $n \geq 1$

$$(8.2) \quad \begin{aligned} {}_0P_n(x) &= 2^{-n} G_n(x), \\ {}_1P_n(x) &= \sum_{p=0}^{n-1} \binom{n}{p} {}_0P_{n-p}(x) + 1, \\ F_n(x) &= \sum_{p=0}^{n-1} \binom{n}{p} G_{n-p}(x) + 1, \\ G_n(x) &= \sum_{p=0}^{n-1} \binom{n}{p} (-1)^p {}_1P_{n-p}(x) + (-1)^n. \end{aligned}$$

The first three of these relations follow by obvious combinatorial considerations, and the fourth follows from the third by induction on n . It is easy to see from (8.1) and (8.2) that for $p = 0$ or 1 and integer $n \geq 1$

$$(8.3) \quad \begin{aligned} {}_1P_n(x) &= \frac{\pi^{\frac{1}{2}n}}{2^n \Gamma\left(\frac{n}{2} + 1\right)} x^{\frac{1}{2}n} + (-1)^{n+1} \frac{\pi^{\frac{1}{2}(n-1)} n}{2^n \Gamma\left(\frac{n-1}{2} + 1\right)} x^{\frac{1}{2}(n-1)} \\ &\quad + O(x^{\frac{1}{2}(n-1)/(1+n^{-1})}) \end{aligned}$$

over $x > \rho > 0$.

Now let $D = (0, b) \times (0, b) \times \dots \times (0, b) \subseteq R_n$, the n -dimensional cube of side length $b > 0$. It is well-known and easy to see that

$$\prod_{i=1}^n [2b^{-1} \sin(m_i \pi b^{-1} x_i)]$$

with each integer $m_i > 0$ forms a complete orthonormal system of eigenfunctions in $L_2(D)$ for (1.1) with Dirichlet boundary condition (1.2), the eigenvalues ${}_0\lambda_i$ being $|\mathbf{m}|^2 \pi^2 b^{-2}$. For the modified Neumann boundary condition (2.3), we can similarly take

$$\prod_{i=1}^n (2^{\epsilon_i} b^{-1} \cos(m_i \pi b^{-1} x_i)), \quad \epsilon_i = 0 \text{ if } m_i = 0, \quad \epsilon_i = 1 \text{ if } m_i > 0,$$

with integer $m_i \geq 0$ as the eigenfunctions and $|\mathbf{m}|^2 \pi^2 b^{-2}$ as the eigenvalues λ_i . Thus, with $x = b^2 \pi^{-2} \lambda$, clearly

$$(8.4) \quad N_\nu(\lambda) = {}_\nu P_n(b^2 \pi^{-2} \lambda)$$

over $\lambda > 0$ for $\nu = 0$ or 1 . Applying (8.3), we can summarize this analysis in the following trivial theorem.

Theorem (8.5)

Let $D = (0, b) \times (0, b) \times \cdots \times (0, b) \subseteq R_n$ for integer $n \geq 1$. Then for $\nu = 0$ or 1 the number of eigenvalues $N_\nu(\lambda) = \sum_{\lambda_i \leq \lambda} 1$ over $\lambda > 0$ satisfies (8.4) and

$$(8.6) \quad N_\nu(\lambda) = \frac{\mu_n(D)}{(2\sqrt{\pi})^n \Gamma\left(\frac{n}{2} + 1\right)} \lambda^{\frac{n}{2}} + (-1)^{\nu+1} \frac{\sigma_n(B)}{4(2\sqrt{\pi})^{n-1} \Gamma\left(\frac{n-1}{2}\right)} \lambda^{\frac{n-1}{2}} + O(\lambda^{\frac{1}{2}(n-1)/(1+n-1)}).$$

Here in (8.3) and (8.4) we have used $\mu_n(D) = b^n$ and $\sigma_n(B) = 2nb^{n-1}$ in order to obtain (8.6). Notice that (8.6) agrees with (7.26) and (7.28) as far as it goes, although B here does not satisfy condition (3.1) or (3.6). When $n = 2$, however, B is a square satisfying condition (3.4) with all four $\alpha_k = \frac{1}{2}\pi$; thus theorem (7.7(IV)) applied to (7.30) shows that the coefficients of λ and $\lambda^{\frac{1}{2}}$ in (8.6) could not be different. Also when $n = 2$ we use (8.4) to compute $E_\nu(\frac{1}{2}\pi)$ in (7.30) in the following theorem.

Theorem (8.7)

The remainder $B_n(x)$ in (8.1) satisfies

$$(8.8) \quad B_1(x) = \tilde{O}(x^r), \quad B_2(x) = \tilde{O}(x^r)$$

over $x \geq \frac{1}{4}$ for every real $r < 0$. Also $E_0(\frac{1}{2}\pi) = E_1(\frac{1}{2}\pi) = \frac{1}{4}8$, and, for $D = (0, b) \times (0, b) \subseteq R_2$, (7.30) becomes, with $\mu_2(D) = b^2$ and $\sigma_2(B) = 4b$,

$$(8.9) \quad N_\nu(\lambda) = \frac{\mu_2(D)}{4\pi} \lambda + (-1)^{\nu+1} \frac{\sigma_2(B)}{4\pi} \lambda^{\frac{1}{2}} + \frac{1}{4} + \tilde{O}(\lambda^r)$$

over $\lambda \geq \frac{1}{4}, \lambda_1$ for every real $r < 0$.

First, (7.30) and (8.4) yield for $\nu = 0$ or 1

$$(8.10) \quad {}_\nu P_2(x) = \frac{\pi}{4} x + (-1)^{\nu+1} \sqrt{x} + \left[\frac{1}{8} + 4E_\nu\left(\frac{\pi}{2}\right) \right] + \tilde{O}(x^r)$$

over $x \geq \frac{1}{4}$ for every $r < 0$. Applying (8.1) and (8.2) to (8.10) we obtain first

$$\begin{aligned} 2\sqrt{x} + B_1(x) &= F_1(x) = 2 {}_0P_1(x) + 1 = {}_1P_2(x) - {}_0P_2(x) \\ &= 2\sqrt{x} + 4[E_1(\frac{1}{2}\pi) - E_0(\frac{1}{2}\pi)] + \tilde{O}(x^r), \end{aligned}$$

and hence over $x \geq \frac{1}{4}$

$$(8.11) \quad B_1(x) = 4[E_1(\frac{1}{2}\pi) - E_0(\frac{1}{2}\pi)] + \tilde{O}(x^r)$$

for every $r < 0$. Second,

$$\begin{aligned} \pi x + B_2(x) &= F_2(x) = 2^2 {}_0P_2(x) + 2F_1(x) - 1 \\ &= 4[\frac{1}{4}\pi x - \sqrt{x} + (\frac{1}{6} + 4E_0(\frac{1}{2}\pi)) + \tilde{O}(x^r)] + 2(2\sqrt{x} + B_1(x)) - 1 \\ &= \pi x + (-\frac{1}{3} + 16E_0(\frac{1}{2}\pi)) + 2B_1(x) + \tilde{O}(x^r), \end{aligned}$$

and hence over $x \geq \frac{1}{4}$

$$(8.12) \quad B_2(x) = [16E_0(\frac{1}{2}\pi) - \frac{1}{3}] + 2B_1(x) + \tilde{O}(x^r)$$

for every $r < 0$.

In order to evaluate the constants on the right side of (8.11), and (8.12), notice that with $\mathbf{m} = (m_1, \dots, m_n) \in R_n$ the Epstein zeta function

$$(8.13) \quad Z_n(s) = \sum_{|\mathbf{m}| > 0} |\mathbf{m}|^{-s}$$

satisfies, letting $f_n(s) = \psi_{B_n, \frac{1}{2}}(s)$ and using (8.1),

$$(8.14) \quad Z_n(2s) = \int_{\frac{1}{2}}^{\infty} x^{-s} dF_n(x) = \frac{2\pi^{\frac{1}{2}n} 2^{2s-n}}{\Gamma(\frac{1}{2}n)(2s-n)} + f_n(s)$$

over $\Re[s] > \frac{1}{2}n$. But the well known formula ([8], (3) p. 626)

$$(8.15) \quad \begin{aligned} \pi^{-\frac{1}{2}n} \Gamma(\frac{1}{2}n) Z_n(s) &= 2 \left(\frac{1}{s-n} - \frac{1}{s} \right) \\ &\quad + \int_1^{\infty} (z^{\frac{1}{2}n-1} + z^{\frac{1}{2}n-s-1}) \left[\sum_{|\mathbf{m}| > 0} e^{-\pi s |\mathbf{m}|^2} \right] dz, \end{aligned}$$

in which the integral on the right is clearly an entire function of s , thus yields

$$(8.16) \quad \begin{aligned} f_n(s) &= \frac{2}{2s-n} \left[\frac{\pi^s}{\Gamma(s)} - \frac{\pi^{\frac{1}{2}n} 2^{2s-n}}{\Gamma(\frac{1}{2}n)} \right] - \frac{\pi^s}{\Gamma(s+1)} \\ &\quad + \frac{\pi^s}{\Gamma(s)} \int_1^{\infty} (z^{s-1} + z^{\frac{1}{2}n-s-1}) \left[\sum_{|\mathbf{m}| > 0} e^{-\pi s |\mathbf{m}|^2} \right] dz \end{aligned}$$

over $\Re[s] > \frac{1}{2}n$. Hence the analytic continuation of $f_n(s)$ exists as an entire function of s ; furthermore

$$f_n(0) = \frac{2}{n} \left(\frac{\pi^{\frac{1}{2}n} 2^{-n}}{\Gamma(\frac{1}{2}n)} \right) - 1 = - \left[F_n(\frac{1}{4}) - \frac{\pi^{\frac{1}{2}n} (\frac{1}{4})^{\frac{1}{2}n}}{\Gamma((\frac{1}{2}n) + 1)} \right] = -B_n(\frac{1}{4}),$$

and condition (7.3) is satisfied by $B_n(x)$. Since the constant terms on the right of (8.11) and (8.12) drop out in condition (7.2), it thus follows from (8.11) and (8.12) that $B_1(x) = \tilde{O}(x^r)$ and $B_2(x) = \tilde{O}(x^r)$ over $x \geq \frac{1}{4}$ for every real r , and

hence also $E_1(\frac{1}{2}\pi) = E_0(\frac{1}{2}\pi) = \frac{1}{48}$ follows from (7.7(IV)) as desired to complete the proof of theorem (8.7).

We remark that the right side of (8.16) easily can be seen to satisfy the estimate needed for application of theorem (7.4). Hence directly, without using (8.11) or (8.12), we find that $B_n(x) = \tilde{O}(x^r)$ over $x \geq \frac{1}{4}$ for every real r and integer $n \geq 1$. Thus, using this result with $n = 3$ in (8.1), (8.2), and (8.4), we obtain the following analogue of theorem (8.7). Clearly formulæ like (8.19) are found similarly for $n > 3$.

Theorem (8.17)

The remainder $B_n(x)$ in (8.1) satisfies for every integer $n \geq 1$

$$(8.18) \quad B_n(x) = \tilde{O}(x^r)$$

over $x \geq \frac{1}{4}$ for every real r . Also for $D = (0, b) \times (0, b) \times (0, b) \subseteq R_3$ the number of eigenvalues $N_\nu(\lambda) = \sum_{\lambda_i \leq \lambda} 1$ satisfies over $\lambda \geq \frac{1}{4}\lambda_1$ for every real r

$$(8.19) \quad N_\nu(\lambda) = \frac{\mu_3(D)}{6\pi^2} \lambda^{\frac{3}{2}} + (-1)^{r+1} \frac{\sigma_3(B)}{16\pi} \lambda + \frac{3b}{4\pi} \lambda^{\frac{1}{2}} + (-1)^{r+1} \frac{1}{8} + \tilde{O}(\lambda^r).$$

9. Discussion of Results

Our principal results are stated in theorems (7.25), (7.27), and (7.29). They give asymptotic series, with logarithmic Gaussian averaged error estimates denoted by $\tilde{O}$, for the number of eigenvalues $N_\nu(\lambda) = \sum_{\lambda_i \leq \lambda} 1$ of the problem (1.1) with Dirichlet or extended Neumann boundary condition according as $\nu = 0$ or 1 . As noted in the introduction, these results, in the form of statements about the Dirichlet series $\int_{\frac{1}{4}\lambda_1}^\infty \lambda^{-s} dN_\nu(\lambda)$, have already been obtained by PLEIJEL using the meson Green function for dimension $n = 2$ or 3 and infinitely differentiable boundary B . In this case we have merely sharpened PLEIJEL's results by considering a variety of weaker smoothness conditions on the boundary (see also GRINBERG [10]). In particular, we have been able to let B have a finite number of corners when $n = 2$, and hence to apply our formulæ to a standard lattice point problem in section 8 and to note the connection with the Epstein zeta function. A student of PLEIJEL has carried through such an analysis also, though by different methods and without publication as yet.

Secondly, we have given the construction of the heat Green function for asymptotic series, which is more convenient than the meson Green function for dimension $n > 3$. The leading term only in the series here is due to MINAKSHISUNDARAM, just as the leading term in PLEIJEL's series from the meson Green function is due to CARLEMAN.

It should be emphasized that there is nothing in the nature of a Tauberian theorem conclusion in the $\tilde{O}$ -asymptotic series for $N_\nu(\lambda)$ which we have found. Instead these results are to be viewed as summability statements with the property of regularity, since we have shown that any ordinary O -asymptotic

series for $N_\nu(\lambda)$ must agree term by term with our $\tilde{O}$ -series as far as they both go together. In fact, along these lines, it is not hard to show that if $F(x)$ satisfies the hypotheses of theorem (7.4) for some $r < 0$, and hence $F(x) = \tilde{O}(x^{r'})$ over $x \geq b$ for every $r' > r$, then also $F(x)$ is logarithmically Gauss summable to zero as $x \rightarrow +\infty$ in the sense that

$$(9.1) \quad 0 = \lim_{\rho \rightarrow 0} \left[F(b) + \int_b^\infty e^{-\frac{1}{2}\rho^2 (\ln x)^2} dF(x) \right].$$

Following a theorem of BOCHNER ([1], p. 37, theorem 21), we also see that (9.1) implies that $F(x)$ is logarithmically Abel summable to zero as well.

Something should be said about the possibility of using Tauberian theorems to go from the information found here about $\int_{\frac{1}{2}\lambda}^\infty \lambda^{-s} dN_\nu(\lambda)$, and the corresponding $\tilde{O}$ -asymptotic series for $N_\nu(\lambda)$, to ordinary O -asymptotic series for $N_\nu(\lambda)$. It is of course well known that this can be done for the leading term of the series and that the error estimate on the remainder can be taken of the same order as that of the next $\sigma_n(B)$ term except for the addition of a $\ln \lambda$ factor. Clearly it cannot be possible in general to go indefinitely far down in the asymptotic series, since in the case of dimension $n = 2$ it is known ([2], p. 823, 2) that in (8.1) the error term $B_2(x)$ is not $o(x^{\frac{1}{2}})$ as $x \rightarrow +\infty$, although we have shown it to be $\tilde{O}(x^r)$ over $x \geq \frac{1}{4}$ for every real r . This still leaves the possibility of including the $\sigma_2(B)\lambda^{\frac{1}{2}}$ term in an ordinary O -asymptotic series for $N_\nu(\lambda)$, but suggests that it will be very difficult to do so. Apparently no Tauberian theorems presently known will suffice.

We close by remarking that our $\tilde{O}$ results should suffice for most purposes in applications to physics. For example, it should be possible to use them to give precise mathematical sense to the paper of ROE [15].

The author is indebted to the National Science Foundation for financial support for this research.

REFERENCES

- [1] BOCHNER & CHANDRASEKHARAN, *Fourier Transforms*, Ann. of Math. Study no. 19 (1949).
- [2] BOHR & CRÁMER, *Encykl. Math. Wiss.*, 2, part 3, no. 8.
- [3] BROWNELL, *Pacific J. Math.*, 5 (1955), pp. 483–499; abstract, *Bull. A. M. S.*, 60 (1954), p. 28, no. 24.
- [4] BROWNELL, *Bull. A.M.S.*, 61 (1955), p. 79 and p. 139, abstracts, nos. 149, 227, and 228.
- [5] BROWNELL, *Ann. of Math.*, 54 (1951), pp. 554–594.
- [6] CARLEMAN, *Équations Intégrales Singulières à Noyau Réel et Symétrique*, Uppsala U. Årsskrift (1923).
- [7] COURANT & HILBERT, *Methoden der mathematischen Physik*, vol. 1, Springer, Berlin.
- [8] EPSTEIN, *Math. Ann.*, 56 (1902), pp. 615–644.
- [9] GRAVES, *Real Variables*, McGraw-Hill (1946).
- [10] GRINBERG, *Uspehi Mat. Nauk.*, 8 (1953), no. 6, pp. 97–102.

- [11] KELLOG, *Foundations of Potential Theory*, Dover (1953).
 - [12] MINAKSHISUNDARAM, *Canad. J. Math.*, **1** (1949), pp. 320–327.
 - [13] PLEIJEL, *Ark. Mat.*, **2** (1952), no. 29.
 - [14] PLEIJEL, *Proc. 12th Skand. Math. Cong.* (August 1953).
 - [15] ROE, *J. Acoust. Soc. Amer.*, **13** (1941), pp. 1–7.
 - [16] WEYL, *Bull. A.M.S.*, **56** (1950), pp. 115–139.
 - [17] WEYL, *Rend. Circ. Mat. Palermo*, **39** (1915), pp. 1–49.
- (M) BROWNELL, complete manuscript, freely available at Mathematics Department, University of Washington, Seattle, Washington.

University of Washington
Seattle, Washington