On Linear Difference-Differential Equations with Constant Coefficients

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1. Background.

Throughout this investigation we are concerned with functions of a complex variable defined on a logarithmic Riemann surface with the origin and infinity as branch points. We denote by z^+ and z^- the points related to z by

$$\log z^+ = \log z + 2\pi i, \quad \log z^- = \log z - 2\pi i.$$

Our aim is the study of the solutions of the equation

$$(1) \quad D[F(z^+) - F(z)] + i[F(z^+) + F(z)] = 0, \quad D = \frac{d}{dz}.$$

This equation arises in the theory of surface waves of an incompressible irrotational liquid, adjacent to a segment of the negative real axis where its motion is restricted to be tangential, and to a segment of the positive real axis where it is free to move in all directions, subject only to the restraining force of gravity. Without reference to the hydrodynamical background, let G be an open connected set of the lower half H_- of a z -plane, s_p and s_n open segments of the positive and negative real axes, respectively, and assume that G contains those points of neighborhoods of s_p and s_n which lie in H_- , making G "adjacent" to s_p and s_n . Let u be a harmonic function in G , continuous upon approach to s_p and s_n , and assume, with $z = x + iy$, that u satisfies

$$\frac{\partial u}{\partial y} = 0 \quad \text{on } s_n, \quad \frac{\partial u}{\partial y} = u \quad \text{on } s_p.$$

It is then easy to verify ([1], p. 262) that the regular function $F(z)$, whose real part is u and whose imaginary part is properly adjusted, satisfies (1) in a domain $\tilde{G}$ obtained from G thus: Let G^* be the mirror image of G on s_n ; then $\tilde{G}$ consists of all those points of the logarithmic Riemann surface whose base points belong to $s_p + G + s_n + G^*$. Conversely, however, not every solution of (1) in $\tilde{G}$ is obtained in this simple manner, in fact no solution of (1) whose imaginary part is not a constant on s_n .

Notice that a single-valued function in the ordinary z -plane, *i.e.*, a function $F(z)$ for which identically

$$(2) \quad F(z^+) = F(z),$$

which satisfies (1) in $\tilde{G}$ vanishes identically there.

In [2], a special solution $\chi^R(z)$ of (1) was constructed for the case $G = H_-$, s_p and s_n the positive and negative axes. As this solution plays a fundamental role in the present investigation, we recall those of its properties which we utilize here. (See also [6], [7].)

Put

$$I(z) = I_0(z) = 2\chi^R(z).$$

Then

$$\lim I(z) = 1, \quad \text{as } z \rightarrow 0 \quad \text{while} \quad 0 \geq \arg z \geq -2\pi,$$

$$(3) \quad I'(z) + \frac{1}{\pi} \log z \quad \text{remains bounded as } z \rightarrow 0, \quad 0 \geq \arg z \geq -2\pi.$$

In view of (1), both of these properties hold for arbitrary approach to zero provided $\arg z$ remains bounded. For the equation may be used to extend $I(z)$ analytically into all sheets of the logarithmic plane as a solution of an ordinary differential equation of first order, and the above estimates follow immediately in a successive sheet if they hold in a previous one and conversely.

The behavior at ∞ of $I(z)$ is as follows. There are certain constants A and B , whose exact values do not matter here, such that $\epsilon > 0$ and

$$(4) \quad \begin{aligned} I(z) - Ae^{-i\epsilon} - Bz^{-\frac{1}{2}} &= O(z^{-1-\epsilon}) & \text{in } \frac{1}{2}\pi \geq \arg z \geq -\frac{1}{2}\pi, \\ I(z) - Bz^{\frac{1}{2}} &= O(z^{-1-\epsilon}) & \text{in } -\frac{1}{2}\pi \geq \arg z \geq -\frac{3}{2}\pi, \\ I(z) - \overline{A}e^{i\epsilon} - Bz^{-\frac{1}{2}} &= O(z^{-1-\epsilon}) & \text{in } -\frac{3}{2}\pi \geq \arg z \geq -\frac{5}{2}\pi. \end{aligned}$$

Furthermore

$$(5) \quad \begin{aligned} (D + i)I(z) &\text{ real for } \arg z = 0, \\ I(z) &\text{ real for } \arg z = -\pi. \end{aligned}$$

2. Solutions continuous at the origin.

We start our investigation with the consideration of those solutions of (1) which are regular in a full neighborhood $0 < |z| < r$ of the origin of the logarithmic surface and tend to a limit as $z \rightarrow 0$ for bounded $\arg z$. Let $f(z)$ and $g(z)$ be two such solutions of (1).

THEOREM 1: *The convolution*

$$\int_0^s f(z - t)g(te^{-\pi i}) dt = \varphi(z)$$

is single-valued.

Proof:

$$\begin{aligned}
 2[\varphi(z^+) - \varphi(z)] &= 2 \int_0^z [f(z-t)^+ g(te^{\pi i}) - f(z-t) g(te^{-\pi i})] \\
 &= \int_0^z \{ [f(z-t)^+ - f(z-t)] [g(te^{\pi i}) + g(te^{-\pi i})] \\
 &\quad + [f(z-t)^+ + f(z-t)] [g(te^{\pi i}) - g(te^{-\pi i})] \} dt \\
 &= \int_0^z \{ [f(z-t)^+ - f(z-t)] \cdot -i \frac{d}{dt} [g(te^{\pi i}) - g(te^{-\pi i})] \\
 &\quad - i \frac{d}{dt} [f(z-t)^+ - f(z-t)] \cdot [g(te^{\pi i}) - g(te^{-\pi i})] \} dt \\
 &= -i \int_0^z \frac{d}{dt} \{ [f(z-t)^+ - f(z-t)] [g(te^{\pi i}) - g(te^{-\pi i})] \} dt \\
 &= 0.
 \end{aligned}$$

Thus $\varphi(z)$ is single-valued.

THEOREM 2:

$$(6) \quad \int_0^z I(z-t) I(te^{-\pi i}) dt = ie^{-i\pi} - i.$$

Proof: Put $\varphi(z) = \int_0^z I(z-t) I(te^{-\pi i}) dt$. Then, by (3),

$$(7) \quad (D+i)\varphi(z) = \int_0^z [I'(z-t) + iI(z-t)] I(te^{-\pi i}) dt + I(ze^{-\pi i}).$$

Therefore, in view of (5),

$$(8) \quad (D+i)\varphi(z) \text{ real for } \arg z = 0.$$

Furthermore, an integration by parts yields

$$\begin{aligned}
 (D+i)\varphi(z) &= I(z) + \int_0^z I(z-t) [iI(te^{-\pi i}) - I'(te^{-\pi i})] dt \\
 &= I(z) + \int_0^z I(z-t) [-iI(te^{+\pi i}) - I'(te^{+\pi i})] dt.
 \end{aligned}$$

Take $\arg z = -\pi$. Then again, by (5),

$$(8') \quad (D+i)\varphi(z) \text{ real for } \arg z = -\pi.$$

By Theorem 1, $\varphi(z)$ is single-valued; hence $\varphi(z)$ is regular at $z = 0$ as $\lim_{z \rightarrow 0} \varphi(z) = 0$. Thus $(D+i)\varphi(z)$ is regular in the whole z -plane and real on the real axis. Now it follows from the estimates (4) that $\varphi(z)$ remains bounded for $z = z_1 + i$, $|z_1| \rightarrow \infty$ in $0 \geq \arg z_1 \geq -\pi$. This is a direct consequence of the boundedness of the integrals

$$\int_1^z e^{-i(s-t)} t^{-p} dt, \quad p > 0$$

and

$$\int_0^z (z-t)^{-\frac{1}{2}} t^{-\frac{1}{2}} dt = \int_0^1 (1-t)^{-\frac{1}{2}} t^{-\frac{1}{2}} dt = \text{const.}$$

Hence $(D+i)\varphi(z)$ is likewise bounded in $0 \geq \arg z \geq -\pi$, and since $(D+i)\varphi$ is real on the real axis, $(D+i)\varphi(z)$ is bounded in the whole plane. Therefore $(D+i)\varphi(z) = c = \text{const.}$ By (7) we have, for $z \rightarrow 0$, $(D+i)\varphi(z) = c = 1$, and $\varphi(0) = 0$. Now (6) follows by integration.

THEOREM 3: *Every solution $F(z)$ of (1) which is regular in $0 < |z| < r$ and continuous at $z = 0$ for $0 \geq \arg z \geq -2\pi$, can be written as*

$$(9) \quad F(z) = D(D+i) \int_0^z I(z-t)\psi(t) dt$$

where $\psi(t)$ is a single-valued function regular in $|z| < r$ with $\psi(0) = 0$.

Proof: Consider the function

$$(10) \quad \psi(\zeta) = \int_0^\zeta I(t-\zeta)F(t) dt.$$

By Theorem 1, $\psi(\zeta)$ is single-valued. Moreover

$$\begin{aligned} \int_0^z I(z-\zeta)\psi(\zeta) d\zeta &= \int_0^z d\zeta I(z-\zeta) \int_0^\zeta I(-(\zeta-t))F(t) dt \\ &= \int_0^z dt F(t) \int_t^z I(z-\zeta)I(-(\zeta-t)) d\zeta \\ &= \int_0^z dt F(t)(ie^{-i(s-t)} - i), \end{aligned}$$

in view of (6). Now (9) follows.

OBSERVATION: The same proof shows that the integral equation (10) of Volterra type for $F(z)$ has certainly the solution (9) whenever the given $\psi(\zeta)$ is continuously differentiable, ζ and z being considered as real variables.

REMARK: If $F(z)$ is analytically extensible from within $0 < |z| < r$ into a larger domain $\mathfrak{G}'$ of the logarithmic plane, then formula (10) yields the analytic extension of $\psi(\zeta)$ into $\mathfrak{G}'$ along any path in $\mathfrak{G}'$ no two points of which have logarithms differing by an integral multiple of $2\pi i$. This condition permits one first of all to define $I(t-\zeta)$ along this path as a regular function of t and ζ , excluding the endpoint $t = \zeta$. Furthermore, for every interior point ζ of $\mathfrak{G}'$, the integral (10) may in the neighborhood of ζ be defined as a limit of integrals over a path extending only up to $\zeta - \epsilon_n$ where ϵ_n are small constants with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then the integral (10) appears as a bounded limit of analytic functions and is hence itself analytic.

THEOREM 4: Every solution $F(z)$ of (1), regular in $0 < |z| < r$ and continuous at $z = 0$ for $0 \leq \arg z \leq -2\pi$ can be developed into a convergent series in the iterated integrals of $I(z)$

$$(11) \quad F(z) = \sum_0^{\infty} c_n I_n(z)$$

where

$$I_0(z) = I(z), \quad I_{n+1}(z) = \int_0^z I_n(t) dt;$$

the convergence is uniform in every concentric subcircle for bounded $\arg z$.

Proof: According to (9) we have

$$F(z) = (D^2 + iD) \int_0^z I(z-t) \psi(t) dt.$$

Now

$$\begin{aligned} D \int_0^z I(z-t) \psi(t) dt &= \psi(z) + \int_0^z I'(z-t) \psi(t) dt \\ &= \psi(0)I(z) + \int_0^z I(z-t) \psi'(t) dt. \end{aligned}$$

$$D^2 \int_0^z I(z-t) \psi(t) dt = \psi(0)I'(z) + \psi'(0)I(z) + \int_0^z I(z-t) \psi''(t) dt.$$

Since $\psi(0) = 0$, we find

$$F(z) = \psi'(0)I(z) + \int_0^z I(z-t) [\psi''(t) + i\psi'(t)] dt.$$

Develop $\psi''(t) + i\psi'(t)$ as a power series in t :

$$\psi''(t) + i\psi'(t) = \sum_1^{\infty} c_n \frac{t^{n-1}}{(n-1)!}.$$

Then

$$F(z) = \psi'(0)I(z) + \sum_1^{\infty} c_n \int_0^z I(z-t) \frac{t^{n-1}}{(n-1)!} dt = \sum_0^{\infty} c_n I_n(z),$$

in view of

$$\int_0^z I(z-t) t^{n-1} dt = \int_0^z I(t) (z-t)^{n-1} dt = I_n(z) (n-1)!.$$

Well known facts about power series establish the uniformity of convergence stated in the theorem. Let us recall that all $I_n(z)$, $n = 0, 1, 2, \dots$, are solutions of (1). Theorem 4 is a special case of Theorem 6:5 of [1]. The proof given here is basically different.

3. Solutions vanishing at ∞ .

Our next aim is to establish near $z = \infty$ a situation analogous to that prevailing near $z = 0$. We consider a continuous function $F(z)$ of a real variable $z > r$ for which, as $z \rightarrow \infty$,

$$(12) \quad |z^{\frac{1}{2}}F(z)| \leq \text{const.}, \quad \arg z = 0.$$

This restriction is imposed because of the simplicity which it provides for the following calculations. Put

$$(13) \quad \psi(z) = \int_{\infty}^z I(t - z)F(t) dt.$$

We have certainly $\psi(z) = O(z^{-\frac{1}{2}})$ as $z \rightarrow \infty$, in view of (4); the integral converges absolutely. So then, by (4), does the double integral (in which $\arg(z - \zeta) = -\pi$)

$$\int_{\infty}^z I(z - \zeta) d\zeta \int_{\infty}^{\zeta} I(t - \zeta)F(t) dt = \int_{\infty}^z I(z - \zeta)\psi(\zeta) d\zeta$$

and we have by interchange of the order of integration and by (6)

$$\int_{\infty}^z I(z - \zeta)\psi(\zeta) d\zeta = \int_{\infty}^z F(t) dt (ie^{-i(z-t)} - i).$$

Hence

$$(14) \quad D(D + i) \int_{\infty}^z I(z - \zeta)\psi(\zeta) d\zeta = F(z).$$

Now assume that $F(z)$ is regular for $|z| > r$ in $0 \geq \arg z \geq -2\pi$ and

$$(12') \quad |z^{\frac{1}{2}}F(z)| \leq \text{const.}, \quad 0 \geq \arg z \geq -2\pi, \quad |z| > r.$$

Then $\psi(z)$ is regular in $|z| > r$ for $0 \geq \arg z \geq -2\pi$ and we have on $\arg z = 0$, through deformation of the path of integration,

$$\psi(z^-) = \int_{\infty}^{z^-} I(t - z)F(t) dt = \int_{\infty^-}^{z^-} I(t - z)F(t) dt = \int_{\infty}^z I(t - z)^- F(t^-) dt.$$

Thus if $F(z)$ is also a solution of (1) the reasoning of p. 93 is valid and

$$2[\psi(z) - \psi(z^-)] = -i \int_{\infty}^z \frac{d}{dt} \{ [I(t - z)^- - I(t - z)][F(t^-) - F(t)] \} dt = 0$$

for $\arg z = 0$. But this suffices to establish $\psi(z)$ as a single-valued function of z for $|z| > r$, according to a well known theorem. Notice that $\psi(\infty) = 0$.

Conversely, substitute into (14) for $\psi(\zeta)$ a power series

$$(15) \quad \psi(\zeta) = \sum_1^{\infty} c_n \zeta^{-n}$$

which converges for $|\zeta| > r$. Then, by (4), for $0 \geq \arg z \geq -2\pi$,

$$D \int_{\infty}^z I(z - \zeta) \psi(\zeta) d\zeta = \int_{\infty}^z I(z - \zeta) \psi'(\zeta) d\zeta = O(z^{-\frac{1}{2}}),$$

$$D(D + i) \int_{\infty}^z I(z - \zeta) \psi(\zeta) d\zeta = F(z) = O(z^{-\frac{1}{2}}).$$

THEOREM 5: *If $F(z)$ is a solution of (1), regular for $|z| > r$ and satisfying (12'), then there exists a series (15) converging for $|\zeta| > r$ such that $F(z)$ is given by (14). Conversely, an arbitrary series (15) generates through (14) a solution $F(z)$ of (1) in $|z| > r$ with $F(z) = O(z^{-\frac{1}{2}})$ in $0 \leq \arg z \leq -2\pi$.*

The fact that $F(z)$ is a solution of (1) for arbitrary z as in (15) requires proof. For $\arg z = 0$ we have

$$F(z) = - \int_z^{\infty} I(z - \zeta) [\psi''(\zeta) + i\psi'(\zeta)] d\zeta.$$

We deform the path of integration into one from z to $\infty e^{\pi i}$, taking care to avoid the origin. The argument $z - \zeta$ in I will then range from 0 to $+\infty$ and leave the point z to the left. Now $F(z^-)$ becomes the same integral with I replaced by I^- and it follows that

$$\frac{d}{dz} [F(z) - F(z^-)] + i[F(z) + F(z^-)] = 0$$

for $\arg z = 0$; but this implies (1) for all z in $|z| > r$.

It is possible to express integrals of the form $\int_z^{\infty} I(z - t) t^{-n} dt$ in terms of sums of sums of $I(z)$ and its derivatives. To show this we introduce a function $J(z^-)$ by

$$(16) \quad J(z^-) = \int_{\infty}^z I(z - t) t^{-1} dt$$

uniquely defined by requiring that $\arg(z - t) = -\pi$ for $\arg z = 0$. We have

$$J(z^-) = \int_0^{-\infty} I(t)(z - t)^{-1} dt,$$

or, in view of (4),

$$J(z^-) = \int_0^{\infty} I(t^-)(z - t)^{-1} dt$$

where the path of integration leaves z to the right for $\arg z = 0$.

Similarly we find

$$(16') \quad J(z) = \int_0^{\infty} I(t)(z - t)^{-1} dt,$$

and consequently

$$(17) \quad J(z) - J(z^-) = 2\pi i I(z).$$

Furthermore, $J(z)$ is a solution of (1).

Now consider the solution $F(z) = I'(z) - iI(z)$ for which $F(z) - F(z^-) = I'(z) - I'(z^-) - iI(z) + iI(z^-) = -2i I(z)$. Therefore $\pi F(z) + J(z)$ is single-valued and a solution of (1), hence vanishes identically. This gives

$$(18) \quad J(z) = -\pi[I'(z) - iI(z)], \quad J(z^-) = -\pi[I'(z) + iI(z)].$$

The form (16') for $J(z)$ is a special case of a function introduced by LEHMAN [3]. Observe that by (4) $J(z)$ is bounded for, say, $0 \leq \arg z \leq -\frac{3}{4}\pi$, provided $|z| > \delta > 0$.

Applying the operator $D + i$ we introduce

$$(19) \quad I_{-1}(z) = -(D + i)J(z) = +\pi[I''(z) + I(z)].$$

Now we see from (4) that $I_{-1}(z)$ has the asymptotic behavior at ∞

$$I_{-1}(z) = \text{const. } z^{-\frac{1}{2}} + O(z^{-1-\epsilon}), \quad \epsilon > 0, \quad 0 \leq \arg z \leq -2\pi.$$

From $I_{-1}(z)$ we derive by successive differentiation solutions $I_{-n}(z)$, $n = 2, 3, \dots$, by the formula

$$(20) \quad I_{-n}(z) = -\frac{d}{dz} I_{-n+1}(z) = (-D)^{n-1}(-D - i)J(z).$$

Just as Theorem 3 leads to the development of Theorem 4, so Theorem 5 leads to the following

THEOREM 6: *If $F(z)$ satisfies (1) and (12'), then*

$$(21) \quad F(z) = \sum_2^{\infty} c_{-n} I_{-n}(z^-)$$

uniformly in $-\pi \leq \arg z \leq \pi$, $|z| > r_1 > r$.

Proof: Since

$$F(z) = \int_{\infty}^* I(z - \zeta) \left(\frac{d}{d\zeta} + i \right) \sum_2^{\infty} c_n' \zeta^{-n} d\zeta$$

for $\arg z = 0$, this relation can be extended into $-\pi \leq \arg z \leq \pi$ by deforming the path of integration to lead from $\infty z^{\frac{1}{2}}$ to z .

Now, similarly, we have by (16) in $-\pi \leq \arg z \leq \pi$

$$I_{-n}(z^-) = j_n \int_{\infty z^{\frac{1}{2}}}^* I(z - \zeta) \left(\frac{d}{d\zeta} + i \right) \zeta^{-n} d\zeta, \quad n = 1, 2, \dots,$$

where j_n is a constant. Theorem 7 follows.

4. Γ -solutions.

The results of Theorems 3 and 5 concern solutions of (1) which are either continuous at $z = 0$ or vanish at $z = \infty$ in certain angular sectors and which

are regular on all of the logarithmic surface S either for $0 < |z| < r$ or for $|z| > r$. What about those solutions which exist only in a ring $R > |z| > r$ of S , or indeed only on the infinitely wound circumference $|z| = r$ of S ? We shall attack the even more general situation: In the ordinary z -plane, the “base”-plane of S , is given a Jordan curve Γ^0 containing the origin in its interior. Assume Γ^0 to have a continuous tangent and denote by Γ the curve of S formed by those points of S whose base point lies on Γ^0 . Let $F(z)$ be defined on Γ and be a solution of (1) on Γ , the derivative $DF(z)$ being understood to refer to tangential increments of z along Γ . We shall call such an $F(z)$ a Γ -solution of (1). It is convenient to assume F to have piecewise continuous derivatives; at discontinuities of F' the equation (1) will be understood to hold for the one-sided derivatives in the obvious way.

Our aim is the proof of

THEOREM 7: *Every Γ -solution $F(z)$ of (1) can be represented as*

$$F(z) = F_+(z) - F_-(z)$$

where $F_+(z)$ and $F_-(z)$ are the boundary values on Γ of solutions of (1) on S , regular inside and outside, respectively, of Γ , and where $F_+(z)$ is continuous at $z = 0$ for bounded $\arg z$, while $F_-(z)$ is bounded near ∞ in $0 \geq \arg z \geq -2\pi$.

The proof of this theorem requires certain tools which are also instrumental in utilizing Theorem 7 for developing a Γ -solution in terms of $J(z)$ and $I_\nu(z)$, $\nu = 0, \pm 1, \pm 2, \dots$, in case Γ is a circumference about the origin.

LEMMA 1: *For any two solutions $f(z)$, $g(z)$ of (1) with a common domain $\mathfrak{G}$ of regularity, the bilinear invariant*

$$(22) \quad Q[f, g] = 2 \int_{z^+}^z f(t)g(t) dt + i[f(z^+) - f(z)][g(z^+) - g(z)]$$

does not depend on z in $\mathfrak{G}$.

Proof: (Compare [1].) Setting, for brevity's sake,

$$\delta f = f(z^+) - f(z), \quad \sigma f = f(z^+) + f(z),$$

we find

$$\begin{aligned} \frac{d}{dz} Q[f, g] &\equiv -2\delta(fg) + i(\delta f' \delta g + \delta f \delta g') \\ &\equiv -\delta f \sigma g - \sigma f \delta g + i \delta f' \delta g + i \delta f \delta g' \\ &= 0, \end{aligned}$$

by (1).

REMARK: Obviously the lemma remains true if f and g are only Γ -solutions and z varies on Γ .

LEMMA 2:

$$(23) \quad \int_0^z I(z-t)J(te^{-\pi i}) dt = \pi(1 - I(z)).$$

Proof: From (6) we obtain by differentiation

$$\int_0^z I(z-t)I'(te^{-\pi i}) dt = I(z) - e^{-i\pi}$$

and Lemma 2 follows in view of (18) and (6).

LEMMA 3:

$$(24) \quad \int_0^z J(z-t)J(te^{-\pi i}) dt = -\pi J(z) + \pi J(ze^{-\pi i}) + \pi^2 i,$$

$$(25) \quad \int_0^z J(z-t)J(te^{\pi i}) dt = -\pi J(z) + \pi J(ze^{\pi i}) - \pi^2 i.$$

Proof: By (23),

$$\begin{aligned} -\pi \int_0^z [I'(z-t) - iI(z-t)]J(te^{-\pi i}) dt &= -\pi^2(D-i)[1-I(z)] + \pi J(ze^{-\pi i}) \\ &= -\pi J(z) + \pi J(ze^{-\pi i}) + \pi^2 i. \end{aligned}$$

On the other hand, by (17),

$$\begin{aligned} \int_0^z J(z-t)J(te^{\pi i}) dt &= \int_0^z J(z-t)J(te^{-\pi i}) dt + 2\pi i \int_0^z J(z-t)I(te^{\pi i}) dt, \\ \int_0^z J(z-t)I(te^{\pi i}) dt &= \int_0^z I((z-t)e^{\pi i})J(t) dt \\ &= -\int_0^{ze^{\pi i}} I(ze^{\pi i}-t)J(te^{-\pi i}) dt = -\pi[1-I(ze^{\pi i})], \end{aligned}$$

whence

$$\begin{aligned} \int_0^z J(z-t)J(te^{\pi i}) dt &= \pi^2 i - \pi J(z) + \pi J(ze^{-\pi i}) + 2\pi i \cdot -\pi[1-I(ze^{\pi i})] \\ &= -\pi^2 i - \pi J(z) + \pi J(ze^{\pi i}). \end{aligned}$$

LEMMA 4:

$$(26) \quad H(z, \zeta) \equiv \int_0^z J(z-t)J(\zeta-t) dt - \pi J(\zeta-z)$$

satisfies (1) as a function of z as well as a function of ζ , provided $\log \zeta \neq \log z + 2n\pi i$, $n = 0, \pm 1, \pm 2, \dots$, (ζ is not "congruent" to z) and provided $H(z, \zeta)$ is understood to be an analytic extension of the obvious interpretation of $H(z, \zeta)$ for $0 < z < \zeta$, $\arg z = \arg \zeta = 0$. In the passage in (1) from z to z^+ the deformation of the path of integration must not cross a slit from ζ to ∞ nor points congruent to

points of the slit; in the passage in (1) from ζ to ζ^+ , ζ or points congruent to ζ must not cross the path of the integral; moreover it is supposed that $z \neq 0$, $\zeta \neq 0$.

Proof: It is clear that H , as a function of ζ , satisfies (1) since $J(\zeta - t)$ for all t involved is a solution of (1), and the addition of solutions yields a solution of (1). To see (1) for H as a function of z we observe that, by (17),

$$\begin{aligned} \frac{\partial}{\partial z} [H(z, \zeta) - H(z^-, \zeta)] &= \int_0^s \frac{\partial}{\partial z} [J(z - t) - J(z - t)^-] J(\zeta - t) dt \\ &\quad + 2\pi i J(\zeta - z) - \pi \frac{d}{dz} [J(\zeta - z) - J(\zeta - z^-)]. \end{aligned}$$

But $J(\zeta - z) = J(\zeta - z^-)$, since $\zeta - z$ does not change its argument as z changes into z^- . Hence

$$\begin{aligned} \frac{\partial}{\partial z} [H(z, \zeta) - H(z^-, \zeta)] + i[H(z, \zeta) + H(z^-, \zeta)] \\ = 2\pi i J(\zeta - z) - 2i\pi J(\zeta - z) = 0, \end{aligned}$$

q.e.d.

LEMMA 5: For $0 < z < \zeta$, $\arg z = \arg \zeta = 0$,

$$(27) \quad H(\zeta, z) = \begin{cases} H(z, \zeta) - \pi^2 i \\ H(z, \zeta) + \pi^2 i \end{cases}$$

according to whether z is kept above or below the path of integration in $H(\zeta, z)$.

Proof: We have

$$\begin{aligned} H(z, \zeta) - H(\zeta, z) &= \int_0^s J(\zeta - t) J(z - t) dt - \int_0^\zeta J(\zeta - t) J(z - t) dt \\ &\quad - \pi J(\zeta - z) + \pi J(z - \zeta) \end{aligned}$$

where $\arg(z - t)$ and $\arg(z - \zeta)$ may be assumed to lie between 0 and π in the first case and between 0 and $-\pi$ in the second case. We find

$$\begin{aligned} H(z, \zeta) - H(\zeta, z) &= \int_\zeta^s J(\zeta - t) J(z - t) dt - \pi J(\zeta - z) + \pi J(z - \zeta) \\ &= \int_{\zeta-s}^0 J(\zeta - z - t) J(-t) dt - \pi J(\zeta - z) + \pi J(z - \zeta), \end{aligned}$$

and by (25) in the first case

$$H(z, \zeta) - H(\zeta, z) = \pi^2 i,$$

while in the second case (24) yields

$$H(z, \zeta) - H(\zeta, z) = -\pi^2 i.$$

We now return to the Jordan curve Γ^0 and take an arbitrary point a of Γ^0 and slit the plane by a simple path from 0 to the point ∞ of the real axis which intersects Γ^0 in the single point a . By this procedure the plane becomes one sheet σ^0 of the logarithmic plane and Γ^0 becomes a Jordan arc leading from a^- to a . Let A be a point of Γ^0 between a and a^- . The meaning of $H(z, A)$ is obvious for all z interior to Γ^0 and therefore, if $z \neq A$, also for z on Γ^0 . The meaning of $H(A, z)$ is likewise determined for a point $z (\neq A)$ of Γ^0 as the result of passing to z from points of σ^0 outside of Γ^0 . Now two cases arise: The following points of Γ^0 are arranged in one of two orders as Γ^0 is traversed from a^- to a

$$a^-, A, z, a \quad \text{or} \quad a^-, z, A, a.$$

The reasoning of Lemma 5 applied to the path from 0 to z and thence along Γ^0 to A yields in the first case

$$(28) \quad H(z, A) - H(A, z) = \pi^2 i,$$

in the second case

$$(29) \quad H(z, A) - H(A, z) = -\pi^2 i.$$

These results are now applied in order to evaluate along Γ^0 the invariant $Q[F'(t), H(A, t) - H(t, A)]$

$$\begin{aligned} &= -2 \int_{a^-}^a F'(t) [H(A, t) - H(t, A)] dt \\ &\quad + i[F'(a) - F'(a^-)][H(A, a) - H(a, A) - H(A, a^-) + H(a^-, A)] \\ &= 2\pi^2 i \int_{a^-}^A F'(t) dt - 2\pi^2 i \int_A^a F'(t) dt + 2\pi^2 i [F(a) + F(a^-)]. \end{aligned}$$

Hence

$$(30) \quad Q[F'(t), H(A, t) - H(t, A)] = 4\pi^2 i F(A).$$

Now $H(z, t)$, for t fixed on Γ^0 and z *inside* Γ , is a solution of (1) and is continuous at $z = 0$; so therefore is the function

$$(31) \quad (4\pi^2 i)^{-1} Q[F'(t), H(z, t)] = F_+(z)$$

by addition. Similarly, $H(t, z)$, for t fixed on Γ^0 and z *outside* Γ , is a solution of (1) whence

$$(32) \quad (4\pi^2 i)^{-1} Q[F'(t), H(t, z)] = F_-(z)$$

is likewise a solution of (1). Now (30) yields the equation of Theorem 7

$$(33) \quad F(z) = F_+(z) - F_-(z).$$

The definition (26) of $H(z, \zeta)$ shows that H becomes logarithmically infinite as ζ approaches z or congruent points. Accordingly, (31) and (32) define solu-

tions $F_+(z)$ and $F_-(z)$ which remain continuous on all of Γ . Integration of (1) then shows that $F_+(z)$ and $F_-(z)$ could be defined on all of Γ as continuous solutions of the integrated equation (1). Therefore (33) holds on all of Γ . Of course, $F_-(z)$ will in general not remain bounded as z moves arbitrarily to ∞ on the logarithmic surface.

Notice that the functional Q in (30), (31), (32) which in the above proof was formed for Γ^0 from a^- to a could, without change of value, be extended from an arbitrary point a_1^- of Γ to a_1 such that the arc $a_1^-a_1$ contains A .

Formulae (31) and (32) lead to developments in terms of $I_n(z)$ and $J(z)$, convergent in the neighborhood of 0 and ∞ , respectively, in view of Theorems 4 and 6. In particular, if Γ is a circumference $|z| = R > 0$, the developments for $F_+(z)$ and $F_-(z)$ can be shown to converge on $|z| = R$ toward $F_+(z)$ and $F_-(z)$ whence, in this case, there is a convergent development for the Γ -solution $F(z)$ itself in terms of $I_n(z)$ and $J(z)$.

We start with the remark that by (16') for $|\zeta| > |z|$, $0 \geq \arg z \geq -2\pi$, $0 \geq \arg \zeta \geq -2\pi$,

$$J(\zeta - z) = \int_0^\infty \frac{I(\tau) d\tau}{\zeta - z - \tau} = \sum_0^\infty z^n \int_0^\infty \frac{I(\tau) d\tau}{(\zeta - \tau)^{n+1}}$$

where the path can be chosen so as to satisfy $|\zeta - \tau| > |z|$. Now by (18)

$$\begin{aligned} \int_0^z J(z - t)J(\zeta - t) dt &= \pi \int_0^z (-D_z + i)I(z - t)J(\zeta - t) dt \\ &= \pi \int_0^z (D_t + i)I(z - t)J(\zeta - t) dt \\ &= \pi J(\zeta - z) - \pi I(z)J(\zeta) \\ &\quad + \pi \int_0^z I(z - t)(D_t + i)J(\zeta - t) dt. \end{aligned}$$

Hence, for $|\zeta| > |z|$,

$$H(z, \zeta) = -\pi I(z)J(\zeta) + \pi \sum_0^\infty I_{n+1}(z)n! \int_0^\infty (D_t + i)[I(\tau)(\zeta - \tau)^{-n-1}] d\tau$$

or

$$(34) \quad H(z, \zeta) = -\pi I(z)J(\zeta) - \pi \sum_0^\infty I_{n+1}(z)I_{-n-1}(\zeta).$$

This formula remains true for $|\zeta| = |z| = R$, provided $\zeta \neq z$. The remainder of the sum is, but for the factor π ,

$$\begin{aligned} \rho_N &= \int_0^z dt I(z - t)(D_t + i) \sum_N t^n \int_0^\infty \frac{I(\tau) d\tau}{(\zeta - \tau)^{n+1}} \\ &= \int_0^z dt I(z - t) \int_0^\infty (D_t + i) \left[\frac{I(\tau)t^N}{(\zeta - \tau)^N(\zeta - \tau - t)} \right] d\tau. \end{aligned}$$

We show $\rho_N \rightarrow 0$, under the above circumstances. First we can lead the inner integral from 0 to ∞ over a path on which $|\zeta - \tau| > |z| = R$, except for $\tau = 0$, by moving τ out of the origin along a straight segment in a direction opposite to ζ , up to a point P_ϵ of distance $\epsilon > 0$ from 0, and then keeping $|\zeta - \tau| \geq |z| (1 + \epsilon)$ on the remainder of the path. We write

$$\rho_N = \rho_N^* + \rho_N^\dagger$$

where

$$\rho_N^* = \int_{s(1-\epsilon)}^s dt I(z-t) \int_0^{P_\epsilon} (D_\zeta + i) \left[\frac{I(\tau) t^N}{(\zeta - \tau)^N (\zeta - \tau - t)} \right] d\tau.$$

Since $N t^N (\zeta - \tau)^{-N}$ tends uniformly to zero on the path involved in ρ_N^* , we see that ρ_N^* converges uniformly to zero. As to $\rho_N^\dagger$ we remark that

$$\begin{aligned} \left| \int_0^{P_\epsilon} (D_\zeta + i) \frac{t^N I(\tau) d\tau}{(\zeta - \tau)^N (\zeta - \tau - t)} \right| &= \left| \int_0^{P_\epsilon} t^N I(\tau) (-D_\tau + i) \frac{d\tau}{(\zeta - \tau)^N (\zeta - \tau - t)} \right| \\ &\leq \left| \frac{-I(\tau) t^N}{(\zeta - \tau)^N (\zeta - \tau - t)} \right|_0^{P_\epsilon} + \left| \int_0^{P_\epsilon} \frac{[iI(\tau) + I'(\tau)] t^N d\tau}{(\zeta - \tau)^N (\zeta - \tau - t)} \right| \\ &\leq K |\zeta - t|^{-1}, \end{aligned}$$

where K is a constant which does not depend on ζ, z, t , or N ; use has been made of the boundedness of $\int_0^{P_\epsilon} |I'(\tau)| |d\tau|$. (This follows from $I'(\tau) = \text{const. log } \tau + \text{continuous function.}$) Accordingly

$$(35) \quad |\rho_N^\dagger| \leq \text{const.} \left| \int_{s(1-\epsilon)}^s |\zeta - t|^{-1} dt \right|.$$

Therefore

$$(36) \quad \overline{\lim}_{N \rightarrow \infty} \rho_N \leq \text{const.} \epsilon |\zeta - z|^{-1},$$

with arbitrarily small $\epsilon > 0$, or $\rho_N \rightarrow 0$ as $N \rightarrow \infty$.

For ζ approaching z the estimate (36) is no longer integrable in z or ζ . But we can easily improve it for the purpose of forming Q in (31), (32). We retain (36) for $|\zeta - z| \geq \epsilon^{-\frac{1}{2}}$; but for $|\zeta - z| < \epsilon^{-\frac{1}{2}}$ we obtain for all N from (35)

$$\begin{aligned} |\rho_N^\dagger| &\leq \text{const.} \int_0^\epsilon \frac{dt}{(t^2 + |\zeta - z|^2)^{\frac{1}{2}}} \\ &\leq \text{const.} \int_0^{\epsilon/|\zeta - z|} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \\ &\leq \text{const.} \end{aligned}$$

Therefore the functional Q in (31), (32), with the substitution (34), differs for $2 = A$ on the circle Γ^0 from one that converges uniformly in A by an error no larger in absolute value than $\text{const.} \epsilon^{\frac{1}{2}}$ with $\epsilon > 0$ arbitrary, the constant not

depending on A . It follows easily by an application of Cauchy's criterion that (31) and (32) converge *uniformly* in A . Thus we conclude from (30) and (34) that

$$(37) \quad 4\pi i F(A) = cJ(A) + \sum_{-\infty}^{\infty} c_n I_n(A)$$

with

$$(38) \quad \begin{aligned} c_n &= -Q[F'(t), I_{-n}(t)] \operatorname{sgn} n, & n \neq 0, \\ c_0 &= -Q[F'(t), J(t)], & c = Q[F'(t), I_0(t)]. \end{aligned}$$

THEOREM 8: *If Γ is a circumference $|z| = R$, then any Γ -solution $F(A)$ of (1) can be developed in a series (37) with coefficients given by (38), the series converging uniformly in $0 \leq \arg A \leq -2\pi$.*

REMARK: The uniqueness of the development (37) is obviously equivalent to the relations

$$(39) \quad \begin{aligned} Q[I'_n, I_m] &= 0, & n + m \neq 0, \\ Q[I'_n, J] &= 0 = Q[J', I_n], & n \neq 0. \end{aligned}$$

These relations follow easily from the estimates valid in $0 \leq \arg z \leq -2\pi$,

$$\begin{aligned} |I_n(z)| &\leq C_n |z|^n, & n \geq 0, \\ |I_{-n}(z)| &\leq C_{-n} |z|^{-n+\frac{1}{2}}, & n > 0, & |z| \text{ large}, \\ |I_{-n}(z)| &\leq C'_{-n} |z|^{-n}, & n > 0, & |z| \text{ small}, \\ |J(z)| &\leq C' |\log z|, \end{aligned}$$

and the identity

$$Q[f', g] = -Q[f, g']$$

valid for solutions of (1). It is only necessary to consider the invariant Q near $z = 0$ or $z = \infty$. We omit carrying out the details (see also [1], [3], [4]). Notice that a consequence of (38) and of (39) is

$$Q[I'_n, I_{-n}] = -4\pi i \operatorname{sgn} n$$

and

$$Q[I'_0, J] = -Q[J', I_0] = -4\pi i.$$

5. Connection with similar problems.

Solutions of the equation (1) are related to those of other similar equations by simple identities. As an example consider a solution $f(z)$ of (1) which is continuous near $z = 0$ for bounded $\arg z$. The integral

$$g(z) = \int_0^z (z - t)^{\alpha-1} f(t) dt,$$

in which $0 < \alpha < 1$, is evidently a solution of

$$\frac{d}{dz} [g(z^+)e^{-2\pi i\alpha} - g(z)] + i[e^{-2\pi i\alpha}g(z^+) + g(z)] = 0.$$

More general equations of second order, generated by the confluence on the axis of two general first-order linear boundary conditions with constant coefficients for harmonic functions, were considered by MACCAMY [5].

On the other hand the equation (1), with the interpretation

$$(*) \quad z^+ = e^{2\pi i\alpha}z, \quad 0 < \alpha \leq 1,$$

was studied by LEHMAN [3]. Both authors obtain developments of solutions continuous near $z = 0$. In [4], this author gave another proof of Theorem 8, under somewhat stronger assumptions, which, however, also yields the developments in the case (*). It is likely that a large part of the present investigation can be extended to this case, in particular, Theorem 7. This is, however, beyond the scope of this paper.

Still another interpretation of (1) arises if we take

$$(**) \quad z^+ = z + 2hi, \quad h > 0.$$

Then (1) becomes the equation which governs the surface waves of a liquid in a channel of depth h if the acceleration of gravity is equal to 1. This is a simpler situation than the one treated here because there are exponential solutions of the form $e^{\lambda z}$ with

$$\lambda(e^{2ih\lambda} - 1) + i(e^{2ih\lambda} + 1) = 0.$$

Here either $\text{Re } \lambda > 0$ or < 0 or $= 0$; accordingly, $e^{\lambda z}$ vanishes as $z \rightarrow -\infty$ or $z \rightarrow \infty$ or remains bounded for z in the strip $0 \geq \text{Im } z \geq -2h$. Again, developments of solutions of (1) may be considered which are bounded for $z \rightarrow -\infty$ or for $z \rightarrow \infty$ in the strip, and the analogue of Γ -solutions may be studied.

Finally, we wish to remark that for $\psi(0) \neq 0$, $\psi(z)$ single-valued and regular near $z = 0$, the equation with non-constant coefficients

$$\frac{d}{dz} [f(z^+) - f(z)] + i\psi(z)[f(z^+) + f(z)] = 0$$

can be reduced to (1) by the introduction of $\zeta = \int_0^z \psi(t) dt$ as a new independent variable, wherever the map $z \rightarrow \zeta$ is univalent.

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