On a Complementary Energy Principle

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In the mathematical theory of elasticity, two important principles have been widely used, namely the Theorem of Minimum Potential Energy, a minimal principle for strains, and Castigliano's Principle, a minimal principle for stresses occasionally referred to as a complementary energy principle [1]. On the other hand, numerous minimal principles have been developed in the mathematical theory of plasticity due to the need of numerous stress-strain relations. We shall consider here a stress-strain relation for homogeneous, isotropic, incompressible, strain hardening materials exhibiting a gradual transition from the elastic to the plastic state, the so-called stress theory of plastic flow [2]. In differential form the stress-strain relations for loading and unloading respectively are

$$ds_{ij} = 2G_0 de_{ij} - p(E)e_{ij} dE \quad \text{for } dE > 0 \quad (1a)$$

$$ds_{ij} = 2G_0 de_{ij} \quad \text{for } dE < 0 \quad (1b)$$

where the differential of stress deviation ds_{ij} and the differential of strain deviation de_{ij} are defined in terms of the differentials of stress tensor $d\sigma_{ij}$ and strain tensor $d\epsilon_{ij}$ by $ds_{ij} = d\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}$ and $de_{ij} = d\epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}$ —repeated indices indicate summation in accordance with the summation convention of tensor calculus, and δ_{ij} is the Kronecker delta. G_0 is the shear modulus in the elastic range, $p(E)$ is a positive definite function, dependent on the material under consideration, of the strain invariant E defined as $E = \frac{2}{3}e_{ij}e_{ij}$, and dE is the loading-unloading criterion. For incompressible materials $\epsilon_{kk} = 0$ so that $e_{ij} = \epsilon_{ij}$. By means of stress-strain relations (1) a minimal principle for the differential of strain deviation has been developed [3]. The purpose of this paper is to introduce a minimal principle for the differential of stress deviation, *i.e.* a complementary energy principle. This pair of minimal principles for the *stress theory of plastic flow* can be regarded as analogues to the HODGE-PRAGER Principle [4] and the accompanying one established by GREENBERG [5] for the so-called *strain theory of plastic flow*.

The stress-strain relations (1) have the differential of stress deviation as a function of the strain deviation and the differential of strain deviation. For our purposes inverted forms must be obtained so that the differential of strain

deviation is given as a function of the strain deviation and the differential of stress deviation. A contracted product of Eq. (1a) with e_{ij} gives

$$\begin{aligned} e_{ij} ds_{ij} &= 2G_0 e_{ij} de_{ij} - p(E) e_{ij} e_{ij} dE \\ &= \frac{3}{2} \{G_0 - Ep(E)\} dE \end{aligned}$$

so that

$$dE = \frac{2e_{ij} ds_{ij}}{3\{G_0 - Ep(E)\}}. \quad (2)$$

Substituting this expression of dE into Eq. (1a) and solving for de_{ij} gives

$$de_{ij} = \frac{ds_{ij}}{2G_0} + Fe_{ij}e_{mn} ds_{mn} \quad \text{for loading} \quad (3a)$$

$$de_{ij} = \frac{ds_{ij}}{2G_0} \quad \text{for unloading} \quad (3b)$$

where $F = \frac{1}{3}p(E) \{G_0^2 - G_0Ep(E)\}^{-1}$. The loading-unloading criterion given by the right hand member of Eq. (2) can be simplified by taking into account the irreversible character of plastic deformation, *i.e.* $de_{ij}^p ds_{ij} \geq 0$, where de_{ij}^p is a differential of plastic strain tensor. For incompressible materials this condition becomes $de_{ij}^p ds_{ij} \geq 0$. Since de_{ij}^p is given by $Fe_{ij}e_{mn}ds_{mn}$ in Eq. (3a), $F(e_{ij}ds_{ij})^2 \geq 0$ or F is positive definite. Hence $\{G_0 - Ep(E)\} > 0$ in Eq. (2) because $p(E) \geq 0$ and $G_0 > 0$. The loading-unloading criterion therefore reduces to $e_{ij}ds_{ij}$; Eq. (3a) is to be used when $e_{ij}ds_{ij} > 0$ and Eq. (3b) when $e_{ij}ds_{ij} < 0$.

Let V represent a bounded, simply-connected, closed region with surface S , and let a differential of the surface stresses dT_i be given over the entire surface such that $dT_i = d\sigma_{ij}n_j$ where n_j is the outer unit normal to S . For a given state of strain deviation e_{ij} throughout V , Eqs. (3) determine the differential of strain deviation provided the differential of stress deviation is known at each point. Our problem consists of evaluating ds_{ij} . To do this we define an admissible set of differential stress deviations ds'_{ij} as one satisfying the equations of equilibrium and the boundary conditions, *i.e.* $d\sigma'_{ij}n_j = dT'_i = dT_i$ on S . The corresponding admissible differentials of strain deviation de'_{ij} are given by

$$de'_{ij} = \frac{ds'_{ij}}{2G_0} + Fe_{ij}e_{mn} ds'_{mn} \quad \text{for } e_{mn} ds'_{mn} > 0 \quad (4a)$$

$$de'_{ij} = \frac{ds'_{ij}}{2G_0} \quad \text{for } e_{mn} ds'_{mn} < 0. \quad (4b)$$

The actual differential of stress deviation ds_{ij} in V is an admissible one whose corresponding de_{ij} satisfies the compatibility equations.

Complementary Energy Principle: For any admissible set ds'_{ij} and corresponding de'_{ij} , the integral $I(ds'_{ij}) = \int_V ds'_{ij} de'_{ij} dV$ is an absolute minimum for the actual differentials of stress and strain deviations occurring in the body, *i.e.*

$$I(ds'_{ii}) - I(ds_{ii}) \geq 0,$$

where $I(ds_{ii}) = \int_V ds_{ii} de_{ii} dV$. Equality holds if and only if $ds'_{ii} = ds_{ii}$ throughout V .

Proof: Using Eqs. (4)

$$I(ds'_{ii}) = \frac{1}{2G_0} \int_V ds'_{ii} ds'_{ii} dV + \int_{V'_+} F(e_{ii} ds'_{ii})^2 dV. \quad (5)$$

Similarly by using Eqs. (3)

$$I(ds_{ii}) = \frac{1}{2G_0} \int_V ds_{ii} ds_{ii} dV + \int_{V_+} F(e_{ii} ds_{ii})^2 dV. \quad (6)$$

The sub-domains of V where loading and unloading occur are indicated by plus and minus subscripts respectively. In general the actual domains of loading and unloading, V_+ and V_- , are not known; however, for admissible states V'_+ indicates the sub-domain where $e_{ii} ds'_{ii} > 0$ and V'_- the sub-domain where $e_{ii} ds'_{ii} < 0$. It is evident that the second integrals in Eqs. (5) and (6) are to be integrated over the loading domains only.

It is necessary to consider the portion of the body where actual loading occurs, *i.e.* $e_{ii} ds_{ii} > 0$, while admissible states indicate unloading, $e_{ii} ds'_{ii} < 0$; this we denote by V_{+-} . In V_{-+} , $e_{ii} ds_{ii} < 0$ while $e_{ii} ds'_{ii} > 0$. It follows that $V'_+ = V_+ + V_{-+} - V_{+-}$ and V_{+-} is a sub-domain of V_+ . The following notation is found to be convenient:

$$\begin{aligned} ds'_{ii} &= \Delta s_{ii} + ds_{ii} \quad \text{where} \quad \Delta s_{ii} = ds'_{ii} - ds_{ii}, \\ d\sigma'_{ii} &= \Delta \sigma_{ii} + d\sigma_{ii} \quad \text{where} \quad \Delta \sigma_{ii} = d\sigma'_{ii} - d\sigma_{ii}, \\ dT'_i &= \Delta T_i + dT_i \quad \text{where} \quad \Delta T_i = dT'_i - dT_i. \end{aligned}$$

Thus

$$\begin{aligned} I(ds'_{ii}) &= \frac{1}{2G_0} \int_V (\Delta s_{ii} \Delta s_{ii} + 2\Delta s_{ii} ds_{ii} + ds_{ii} ds_{ii}) dV \\ &\quad + \int_{V_+} F\{(e_{ii} \Delta s_{ii})^2 + 2(e_{ii} \Delta s_{ii})(e_{mn} ds_{mn}) + (e_{ii} ds_{ii})^2\} dV \\ &\quad + \int_{V_{-+}} F(e_{ii} ds'_{ii})^2 dV - \int_{V_{+-}} F(e_{ii} ds'_{ii})^2 dV. \end{aligned} \quad (7)$$

A simplification of this expression is possible by noting that $\int_V \Delta s_{ii} de_{ii} dV = 0$, because

$$\begin{aligned} \int_V \Delta s_{ii} de_{ii} dV &= \int_V \Delta \sigma_{ii} d\epsilon_{ii} dV = \int_V \Delta \sigma_{ii} du_{i,i} dV = \int_V (\Delta \sigma_{ii} du_i)_{,i} dV \\ &= \int_S \Delta \sigma_{ii} du_i n_i dS = \int_S \Delta T_i du_i dS = 0 \end{aligned}$$

due to the incompressibility condition, the strain tensor-displacement relation— $d\epsilon_{ij} = \frac{1}{2}(du_{i,j} + du_{j,i})$ where u_i is the displacement vector and $u_{i,j}$ is $\partial u_i / \partial x_j$, the symmetry of $\Delta\sigma_{ij}$, the equations of equilibrium, the divergence theorem, and the boundary conditions. Hence by Eqs. (3)

$$\int_V \Delta s_{ij} de_{ij} dV = \frac{1}{2G_0} \int_V \Delta s_{ij} ds_{ij} dV + \int_{V_+} F(\Delta s_{ij} e_{ij})(e_{mn} ds_{mn}) dV = 0. \quad (8)$$

In view of Eqs. (6) and (8), Eq. (7) reduces to

$$\begin{aligned} I(ds'_{ij}) - I(ds_{ij}) &= \frac{1}{2G_0} \int_V \Delta s_{ij} \Delta s_{ij} dV + \int_{V_+} F(e_{ij} \Delta s_{ij})^2 dV \\ &\quad + \int_{V_{+-}} F(e_{ij} ds'_{ij})^2 dV - \int_{V_{+-}} F(e_{ij} ds'_{ij})^2 dV. \end{aligned} \quad (9)$$

In order to complete the proof, it remains to be shown that the right member of Eq. (9) is positive definite. In V_{+-} , $(e_{ij} \Delta s_{ij})^2 \geq (e_{ij} ds'_{ij})^2$ due to the fact that $e_{ij} ds'_{ij} < 0$ and $e_{ij} ds_{ij} > 0$. Since $F \geq 0$, it follows that

$$\int_{V_+} F(e_{ij} \Delta s_{ij})^2 dV \geq \int_{V_{+-}} F(e_{ij} \Delta s_{ij})^2 dV \geq \int_{V_{+-}} F(e_{ij} ds'_{ij})^2 dV$$

so that $I(ds'_{ij}) - I(ds_{ij}) \geq 0$. Equality occurs if and only if $ds'_{ij} = ds_{ij}$ which indicates that if a solution exists it is unique.

BIBLIOGRAPHY

- [1] I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1946, pp. 281, 286.
- [2] D. TRIFAN, *A new theory of plastic flow*, Q. Appl. Math. **7**, 201–211 (1949).
- [3] D. TRIFAN, *A minimum principle of plasticity*, Q. Appl. Math. **13**, 337–339 (1955).
- [4] P. HODGE & W. PRAGER, *A variational principle for plastic materials with strain-hardening*, J. Math. and Phys. **27**, 1–10 (1948).
- [5] H. J. GREENBERG, *On the variational principles of plasticity*, Monograph A11–S4, Grad. Div. of Appl. Math., Brown Univ. (March 1949), p. 80.

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