On the Propagation of Weak Discontinuities in Perfectly Plastic Solids

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1. Introduction. The concept of the singular surface which was used extensively by Hadamard¹ at the beginning of this century in his researches on the propagation of waves has recently been applied by J. L. ERICKSEN and the present writer² to derive conditions for the existence of singular or characteristic surfaces in the von Mises and the Prandtl-Reuss theories of plasticity.³ These surfaces are associated with a *weak* type of discontinuity of the solution functions of the problem in that only discontinuities in the derivatives of the functions occur, the functions themselves being continuous across the surfaces in question. Surfaces over which the actual functions are discontinuous represent *shocks* or *strong* discontinuities.

In the following paper⁴ we have investigated the existence of *singular surfaces of order 1*, also referred to as *wave surfaces*, in the plane stress problem for perfectly plastic solids, under the assumption of the von Mises and the Prandtl-Reuss equations of plastic flow. The interesting feature of this work is that the additional conditions due to the requirement of plane stress enable us, in general, to arrive at an *algebraic* determination of the direction of propagation of the singular or wave surface, *i.e.*, its unit normal vector, as well as the actual speed of propa-

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¹ See, J. HADAMARD, *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique*, Paris, 1903.

² J. L. ERICKSEN, *Singular surfaces in plasticity*, J. Math. and Physics **34** (1955) pp. 74–79. Also, T. Y. THOMAS, *On the characteristic surfaces of the von Mises plasticity equations*, J. Rat. Mech. and Anal. **1** (1952) pp. 343–357; *Singular surfaces and flow lines in the theory of plasticity*, *ibid.* **2** (1953) pp. 339–381; and *Characteristic surfaces in the Prandtl-Reuss plasticity theory*, *ibid.* **5** (1956) pp. 251–262. A discussion of discontinuities in plastic flow with applications and numerous references has been given by W. PRAGER, *Discontinuous fields of plastic stress and flow*, Proc., 2nd U.S. National Congress of Appl. Mech., Ann Arbor, 1954, pp. 21–32; see also, *The theory of plasticity: A survey of recent achievements*, Proc. Inst. Mech. Engrs. **169** (1955) pp. 41–57.

³ For a general discussion and application of the von Mises and Prandtl-Reuss theories of plasticity see, W. PRAGER and P. G. HODGE, JR., *Theory of Perfectly Plastic Solids*, J. Wiley and Sons, New York, 1951.

⁴ This investigation was conducted at the request of the Naval Research Laboratory and has been supported by the Office of Naval Research under Contract Nonr-908(09), NR 041 037, with Indiana University.

gation of this surface. General formulæ for these quantities are given in §8, §10, and §11. No mention will be made here of the main results obtained since all such results can readily be observed from the italicized statements in the following sections. In §12 we have briefly discussed the application of our results to the special case of simple tension which is of interest in connection with the usual tension tests on flat rectangular bars.

Attention has been limited to wave surfaces whose speed of propagation is different from zero except for the digression in §9 which was made in order to orient our work more closely with earlier work on wave surfaces.² This limitation has caused us to omit some new and more or less interesting results but was felt to be desirable from the standpoint of the exposition.

We have also investigated the possibility of wave propagation, under the assumption of plane stress, for theories of plastic flow which involve a generalization of the von Mises and the Prandtl-Reuss equations. These more general theories lead, in the case of simple tension, to plane waves which make angles with the direction of tension different from those deduced in §12. However this work is not presented here since we have deemed it advisable to concentrate attention on results obtainable from theories which have been established by workers in the field of plasticity, *e.g.*, the von Mises and the Prandtl-Reuss theories of plastic flow.

2. Plane stress. Denoting by $\sigma_{\alpha\beta}$ the components of the symmetric stress tensor relative to a system of rectangular coordinates x_1, x_2, x_3 the stress is said to be *plane* if the coordinate system can be chosen so that

$$(2.1) \quad \sigma_{\alpha\beta} = \phi_{\alpha\beta}(x_1, x_2, t); \quad \sigma_{\alpha 3} = 0,$$

for values of the indices $\alpha, \beta = 1, 2, 3$. The first set of equations (2.1) indicates, of course, that the components $\sigma_{\alpha\beta}$ are functions only of the two coordinates x_1, x_2 and the time t .

On account of the equations (2.1) we have

$$(2.2) \quad \sigma_{\alpha\beta,3} = 0; \quad \sigma_{\alpha 3,\beta} = 0, \quad (\alpha, \beta = 1, 2, 3),$$

where the comma denotes partial differentiation with respect to the coordinates. Some of the main consequences of the assumption of plane stress will be obtained directly from (2.2), rather than (2.1), in the following discussion.

3. Canonical coordinates. Let h_i for $i = 1, 2, 3$ denote a set of three mutually perpendicular unit vectors giving the principal directions determined by the tensor σ at a point P . Now construct a rectangular coordinate system y with origin at P such that the y_i -axis lies along the direction h_i . Then, if $h_{i\alpha}$ are the components of the vector h_i in the underlying x coordinate system, it is easily seen that the relation between the x and y coordinates is given by

$$(3.1) \quad x_\alpha = p_\alpha + h_{i\alpha}y_i,$$

where the p_α are the coordinates of the point P relative to the x system and the subscript i is summed over the values 1, 2, 3. The y 's will be called *canonical coordinates* and the transformation (3.1) to these coordinates will be called a *canonical transformation* for the purpose of convenient reference.

It is well known that the above components $h_{i\alpha}$ are determined by the set of equations

$$(3.2) \quad (\sigma_{\alpha\beta} - \sigma_i \delta_{\alpha\beta}) h_{i\beta} = \sigma_{\alpha\beta} h_{i\beta} - \sigma_i h_{i\alpha} = 0,$$

where the σ_i are the principal values of the tensor σ ; no summation on the index i is involved in (3.2). The principal values σ_i are given as solutions of the determinantal equation

$$(3.3) \quad |\sigma_{\alpha\beta} - \sigma_i \delta_{\alpha\beta}| = -\sigma_i \begin{vmatrix} (\sigma_{11} - \sigma_i) & \sigma_{12} \\ \sigma_{21} & (\sigma_{22} - \sigma_i) \end{vmatrix} = 0,$$

which can be written as the product of the two factors, as shown, on account of the conditions (2.1).

Let us now choose the indices so that the second order determinant in (3.3) vanishes for $i = 1, 2$. In other words by equating this determinant to zero the principal values σ_1 and σ_2 are determined.⁵ *The remaining principal value σ_3 is then determined by the vanishing of the first factor in (3.3), i.e., $\sigma_3 = 0$.*

Restricting the free indices α and i in (3.2) to the values 1, 2 we see that the summation on the index β in these equations can also be restricted to the range 1, 2 on account of the conditions (2.1). These restricted equations (3.2) will determine the components $h_{i\beta}$ for $i, \beta = 1, 2$; moreover the two *incomplete* vectors defined by these components $h_{i\beta}$ can be supposed to satisfy the condition of perpendicularity as is well known from the theory of equations of the type (3.2). Hence taking $h_{i3} = 0$ for $i = 1, 2$ we will thus have a set of quantities $h_{i\beta}$ with $i = 1, 2, \beta = 1, 2, 3$ which constitute the components of two perpendicular vectors in the x_1, x_2 -plane and which are seen to satisfy the equations (3.2) in which the indices α, β have the full range of values. Finally, taking $i = 3$ in (3.2), we see that the equations are satisfied by $h_{3\beta} = \delta_{3\beta}$ since $\sigma_3 = 0$ by the above result. We have thus arrived at a set of three mutually perpendicular vectors h_i with components $h_{i\alpha}$ satisfying the equations (3.2).

In view of the above determinations we can now state the following result.

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⁵ Either of the two roots of the above second order determinantal equation may be selected as the value of σ_1 after which the other root will be assigned as the value of σ_2 . How this selection is made is of no particular importance in our discussion. However, for definiteness, it will be supposed that the selection of the values of σ_1 and σ_2 is made in such a way that the associated principal directions h_1 and h_2 , which lie in the x_1, x_2 -plane, will have the same orientation as the x_1 - and x_2 -coordinate axes. Then the Jacobian determinant of the canonical transformation (3.5) will be equal to 1 and the transformation will have the property of preserving the orientation of coordinate axes.

In the plane stress problem as defined in §2 the principal value $\sigma_3 = 0$ and the coefficients $h_{i\alpha}$ in the canonical transformation (3.1) satisfy the conditions

$$(3.4) \quad h_{13} = h_{23} = h_{31} = h_{32} = 0; \quad h_{33} = 1.$$

Hence the relations (3.1) decompose into

$$(3.5) \quad x_\alpha = p_\alpha + h_{i\alpha}y_i, \quad (\alpha, i = 1, 2); \quad x_3 = p_3 + y_3,$$

in which the p 's and h 's are constants and the subscript i is summed over the values 1, 2 only. It may also be observed from the considerations here that, apart from the above special determinations, the principal values σ_i and the components $h_{i\alpha}$ will, in general, vary with the point P but will depend, at any time t , only on the two coordinates x_1, x_2 of this point.

Now suppose that the components $\sigma_{\alpha\beta} \rightarrow \tau_{\alpha\beta}$ as a result of the canonical transformation (3.1). Specifically this means that

$$\tau_{ii} = \sigma_{\alpha\beta} h_{i\alpha} h_{i\beta}.$$

It follows from the first condition (2.1) and the form (3.5) of the canonical transformation that the $\tau_{\alpha\beta}$ can depend on y_1, y_2 and t alone. Also from the second condition (2.1) and (3.4) we see that $\tau_{3i} = 0$. In other words the plane stress conditions (2.1) are invariant under the canonical transformation. On account of this result the special forms assumed by tensorial relations, as a consequence of the conditions (2.1), will be unaltered if these relations are referred to the canonical coordinates y . When such relations are evaluated at the origin of the y system a simplified or more compact form of the relations is usually obtained. We shall apply this procedure to advantage in the following discussion.

4. Kinematical and dynamical conditions of compatibility. A moving surface $\Sigma(t)$ will be said to be *singular of order 1* for the problem under consideration if the density ρ , the velocity components v_α and the stress components $\sigma_{\alpha\beta}$ are continuous across $\Sigma(t)$ while at least one of their first partial derivatives, with respect to the space coordinates, is discontinuous at points of $\Sigma(t)$. Denoting by the bracket [ ] the difference in the values of a quantity on the two sides of $\Sigma(t)$ in the usual manner, it can be shown that, for a singular surface $\Sigma(t)$ of order 1, we shall have relations of the form

$$(4.1) \quad [\rho, \alpha] = \zeta v_\alpha; \quad \left[\frac{\partial \rho}{\partial t} \right] = -G\zeta,$$

$$(4.2) \quad [v_{\alpha, \beta}] = \lambda_\alpha v_\beta; \quad \left[\frac{\partial v_\alpha}{\partial t} \right] = -G\lambda_\alpha,$$

$$(4.3) \quad [\sigma_{\alpha\beta, \gamma}] = \xi_{\alpha\beta} v_\gamma; \quad \left[\frac{\partial \sigma_{\alpha\beta}}{\partial t} \right] = -G\xi_{\alpha\beta},$$

where the ζ, λ_α and the $\xi_{\alpha\beta}$ are quantities defined over $\Sigma(t)$, the v_α are the components of the unit normal vector to the surface and G denotes the velocity

of the surface, relative to the coordinate system, in the direction specified by the vector ν . Relations of the type (4.1), (4.2), and (4.3) have been called *kinematical conditions of compatibility* by HADAMARD.¹

Dynamical conditions of compatibility are obtained from the dynamical relations which involve, for the problem under consideration, such equations as

$$(4.4) \quad \frac{d\rho}{dt} + \rho v_{\alpha, \alpha} = 0 \quad (\text{equation of continuity}),$$

$$(4.5) \quad \sigma_{\alpha\beta, \beta} = \rho \frac{dv_{\alpha}}{dt} \quad (\text{equations of motion}),$$

$$(4.6) \quad v_{\alpha, \alpha} = 0 \quad (\text{equation of incompressibility}),$$

$$(4.7) \quad \sigma_{\alpha\beta}^* \sigma_{\alpha\beta}^* = K \quad (\text{quadratic yield condition}),$$

where the $\sigma_{\alpha\beta}^*$ are the components of the deviator of the stress tensor σ and $K > 0$ is a material constant. From (4.1), $\dots$, (4.6) we can now deduce

$$(4.8) \quad (v_n - G)\zeta = 0; \quad \lambda_i \nu_i = 0,$$

$$(4.9) \quad \xi_{\alpha\beta} \nu_{\beta} = \rho(v_n - G)\lambda_{\alpha},$$

where v_n denotes the component of the velocity in the direction of the normal ν to the surface $\Sigma(t)$. To obtain a corresponding relation from (4.7) we must first differentiate this equation. Differentiating (4.7) with respect to the space coordinates we have

$$\sigma_{\alpha\beta}^* \sigma_{\alpha\beta, \gamma}^* = 0.$$

Then from these relations and (4.3) we obtain

$$(4.10) \quad \sigma_{\alpha\beta}^* \xi_{\alpha\beta} = 0.$$

Differentiation of (4.7) with respect to the time t will lead to no additional condition.

The relations (4.8), (4.9), and (4.10) are part of the dynamical conditions of compatibility. Other such conditions will be obtained in §6 and §7 from the stress-strain relations. In addition there will be certain special relations arising from the plane stress condition (2.1). These latter relations will be considered in the following section.

5. Compatibility conditions arising from the assumption of plane stress. The quantity s defined as the difference $G - v_n$ will be called the *speed of propagation* of the singular or wave surface $\Sigma(t)$. As so defined the speed s is the relative velocity of propagation of the surface $\Sigma(t)$ and the velocity of the moving particles along the direction of the normal ν to this surface. *It will be assumed (except in §9) that the speed s does not vanish at points of the wave surface $\Sigma(t)$.*

From the above assumption $s \neq 0$ and the first equation (4.8) it follows that

$\zeta = 0$. However not all the quantities $\xi_{\alpha\beta}$ can vanish at any point of $\Sigma(t)$. In fact, if $\xi_{\alpha\beta} = 0$ for $\alpha, \beta = 1, 2, 3$ at a point P of $\Sigma(t)$ then, since $s \neq 0$ at P , we must have all $\lambda_\alpha = 0$ at P from equations (4.9). But this contradicts the strict assumption that $\Sigma(t)$ is a singular surface of order 1 (see §4).

We can now observe from equations (2.2) and (4.3) that

$$(5.1) \quad \xi_{\alpha\beta}\nu_3 = 0, \quad \xi_{\alpha 3}\nu_\beta = 0.$$

Hence we must have $\nu_3 = 0$, since not all of the quantities $\xi_{\alpha\beta}$ can vanish at any point of $\Sigma(t)$ by the above result, and $\xi_{\alpha 3} = 0$ over the surface $\Sigma(t)$. Also $\lambda_3 = 0$ as we see by taking $\alpha = 3$ in the equations (4.9) and by using the relations $\xi_{3\beta} = 0$ and the above assumption that the speed of propagation s is different from zero. For easy reference we express these results by writing

$$(5.2) \quad \xi_{\alpha 3} = 0; \quad \nu_3 = 0; \quad \lambda_3 = 0.$$

In view of the condition $\nu_3 = 0$ we see that the equation of the wave surface $\Sigma(t)$ must be of the form $\phi(x_1, x_2, t) = 0$ where the function ϕ is independent of the x_3 -coordinate.

Since the equation (4.9) for $\alpha = 3$ is satisfied identically on account of (5.2) we need consider only the two equations (4.9) corresponding to $\alpha = 1, 2$. Expanding we have

$$(5.3) \quad \xi_{11}\nu_1 + \xi_{12}\nu_2 = \rho(v_n - G)\lambda_1,$$

$$(5.4) \quad \xi_{21}\nu_1 + \xi_{22}\nu_2 = \rho(v_n - G)\lambda_2.$$

Combining (5.3) and (5.4) with (4.10) we can, in general, solve these equations for the three quantities ξ_{11} , ξ_{12} , and ξ_{22} . To obtain this solution let us transform (4.10) to a system of canonical coordinates and evaluate it at the origin of this system. Thus (4.10) becomes

$$(5.5) \quad \sigma^*\xi_{11} + \sigma^*\xi_{22} = 0,$$

when we denote by σ^* the principal values of the tensor σ^* . In thus deriving (5.5) we have made use of the first of the relations (5.2) and the fact that these relations are invariant under the canonical transformation (see §3). Similarly the equations (5.3) and (5.4) retain their form under the canonical transformation. Hence (5.3) and (5.4) can be considered to be valid at the origin of the canonical system.

The determinant Δ of the coefficients of the variables ξ_{11} , ξ_{12} , and ξ_{22} in (5.3), (5.4), and (5.5) is given by

$$(5.6) \quad \Delta = \sigma^*\nu_2^2 + \sigma^*\nu_1^2.$$

If $\Delta \neq 0$ at points of the wave surface $\Sigma(t)$ we can solve equations (5.3), (5.4), and (5.5) to obtain

$$(5.7) \quad \xi_{11} = \frac{1}{\Delta} [\rho(v_n - G)(\lambda_1\nu_1 - \lambda_2\nu_2)\sigma^*],$$

$$(5.8) \quad \xi_{12} = \frac{1}{\Delta} [\rho(v_n - G)(\sigma_1^* \lambda_1 \nu_2 + \sigma_2^* \lambda_2 \nu_1)],$$

$$(5.9) \quad \xi_{22} = \frac{1}{\Delta} [\rho(v_n - G)(\lambda_2 \nu_2 - \lambda_1 \nu_1) \sigma_1^*],$$

at the origin of a system of canonical coordinates. The remaining components $\xi_{\alpha\beta}$ vanish by the first of the relations (5.2). However we shall find that the condition $\Delta = 0$ can occur on $\Sigma(t)$ so that the above determinations of the quantities ξ_{11} , ξ_{12} , and ξ_{22} will not always apply.

6. The von Mises equations. Let us first assume that the plastic flow is governed by the stress-strain relations of von Mises which can be written in the form

$$(6.1) \quad \epsilon_{\alpha\beta} = \frac{\sigma_{ij}^* \epsilon_{ij}}{K} \sigma_{\alpha\beta}^*,$$

where $\epsilon_{\alpha\beta}$ is used to denote the components of the rate of strain tensor ϵ . In this connection we observe that there is no distinction between the tensor ϵ and its deviator ϵ^* in view of the condition of incompressibility (4.6). Hence from the relations (4.2) and (6.1) we can deduce

$$(6.2) \quad \lambda_\alpha \nu_\beta + \lambda_\beta \nu_\alpha - \frac{2}{K} \sigma_{ij}^* \lambda_i \nu_j \sigma_{\alpha\beta}^* = 0.$$

Denoting the left members of the above equations (6.2) by $J_{\alpha\beta}$ we readily observe that $J_{\alpha\alpha} = 0$ when use is made of the identity $\sigma_{\alpha\alpha}^* = 0$. Hence there can be at most five independent relations in the set (6.2) and we see, in fact, that it is permissible to omit the relation (6.2) corresponding to $\alpha = \beta = 1$. But, using (5.2), we can decrease still further the number of independent equations (6.2). Thus, since $\sigma_{\alpha\beta}^* = 0$ for $\alpha \neq \beta$ at the origin of the canonical system, it follows that $J_{\alpha\beta}$ vanishes for $\alpha = 1, 2$ when expressed in canonical coordinates y_α and evaluated at $y = 0$. Hence the set (6.2) can be reduced to the three equations corresponding to

$$(6.3) \quad J_{12} = 0, \quad J_{22} = 0, \quad J_{33} = 0$$

at the origin of the canonical system. But at the origin of this system the equations (6.3) can be written

$$(6.4) \quad \lambda_1 \nu_2 + \lambda_2 \nu_1 = 0,$$

$$(6.5) \quad (\sigma_1^* \lambda_1 \nu_1 + \sigma_2^* \lambda_2 \nu_2) \sigma_3^* = 0,$$

$$(6.6) \quad (K - \sigma_2^* \sigma_2^*) \lambda_2 \nu_2 - \sigma_1^* \sigma_2^* \lambda_1 \nu_1 = 0.$$

7. The Prandtl-Reuss equations. We now consider the Prandtl-Reuss equations which can be written in the form

$$(7.1) \quad \frac{D\sigma_{\alpha\beta}^*}{Dt} = 2\mu \left(\epsilon_{\alpha\beta} - \frac{\sigma_{ij}^* \epsilon_{ij}}{K} \sigma_{\alpha\beta}^* \right),$$

where the quantities $D\sigma_{\alpha\beta}^*/Dt$ are the components of the absolute time derivative⁶ of the tensor σ^* and μ is a new material constant. Results analogous to the results in §6 can now be deduced on the basis of these equations. However the formal derivation of the compatibility conditions associated with the system (7.1) need not be undertaken here since the conditions in question have already been found by us in the previous article *Characteristic surfaces in the Prandtl-Reuss plasticity theory*², where it has been shown that

$$(7.2) \quad (v_n - G)(\xi_{\alpha\beta} - \frac{1}{3}\xi_{ii}\delta_{\alpha\beta}) + \frac{1}{2}(\sigma_{\alpha\mu}^*\nu_\beta + \sigma_{\beta\mu}^*\nu_\alpha)\lambda_\mu \\ - \frac{1}{2}(\sigma_{\alpha\mu}^*\lambda_\beta + \sigma_{\beta\mu}^*\lambda_\alpha)\nu_\mu - \mu(\lambda_\alpha\nu_\beta + \lambda_\beta\nu_\alpha) + \frac{2\mu}{K}\sigma_{ij}^*\lambda_i\nu_j\sigma_{\alpha\beta}^* = 0.$$

Corresponding to the procedure in §6 we first observe that the relation obtained from (7.2) by putting $\alpha = \beta$, and summing on repeated indices, is satisfied identically. Similarly, identical relations are obtained from (7.2), at the origin of a system of canonical coordinates, by taking $\alpha = 1, 2$ and $\beta = 3$; this follows when use is made of the equations (5.2). Hence there are at most three independent equations in the set (7.2) and these can be written in the form

$$(7.3) \quad (v_n - G)\xi_{12} + \frac{1}{2}(\sigma_1^* - \sigma_2^*)(\nu_2\lambda_1 - \nu_1\lambda_2) - \mu(\nu_1\lambda_2 + \nu_2\lambda_1) = 0,$$

$$(7.4) \quad -\frac{1}{3}(\nu_n - G)\xi_{ii} + \frac{2\mu}{K}(\sigma_1^*\lambda_1\nu_1 + \sigma_2^*\lambda_2\nu_2)\sigma_3^* = 0,$$

$$(7.5) \quad (v_n - G)(\xi_{22} - \frac{1}{3}\xi_{ii}) - 2\mu\lambda_2\nu_2 + \frac{2\mu}{K}(\sigma_1^*\nu_1\lambda_1 + \sigma_2^*\nu_2\lambda_2)\sigma_2^* = 0,$$

at the origin of a system of canonical coordinates.

The various conditions which we have now established will be applied in the following sections to the specific determination of the possible wave or singular surfaces $\Sigma(t)$ of order 1 in the plane stress problem.

8. Wave propagation in the von Mises theory. On account of the equations (5.2) and the fact that $\zeta = 0$, as mentioned in §5, we need consider only those equations which give conditions on the three quantities ξ_{11} , ξ_{12} , ξ_{22} , the two quantities λ_1 , λ_2 , and the two components ν_1 , ν_2 of the unit normal vector ν to the surface $\Sigma(t)$. These equations, which we refer to the origin of a system of canonical coordinates, are the second equation (4.8), the equations (5.3), (5.4), (5.5), and the equations (6.4), (6.5), and (6.6).

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⁶ The absolute time derivative must be used instead of the ordinary time derivative of the components $\sigma_{\alpha\beta}^*$ in order to obtain the correct dynamical form of the Prandtl-Reuss equations. For a brief discussion of this problem and references to earlier work see, T. Y. THOMAS, *On the structure of the stress-strain relations*, Proc. N.A.S. 41 (1955) pp. 716-720; also, *Kinematically preferred coordinate systems*, *ibid.* 41 (1955) pp. 762-770. A formal theory of covariant and absolute time differentiation, which leads to the construction of invariants under the time dependent transformations relating moving rectangular coordinate systems, has been developed in this last reference.

Assuming first that λ_1, λ_2 do not both vanish we see from (6.4) that these λ 's are proportional to $\nu_1, -\nu_2$. Let us express this by writing

$$(8.1) \quad \lambda_1, \lambda_2 \propto \nu_1, -\nu_2.$$

Hence from the second equation (4.8) we have $\nu_1^2 = \nu_2^2$ or $\nu_1 = \pm \nu_2$. Using this relation and the proportionality (8.1) it now follows that (5.6), (6.5), and (6.6) become

$$(8.2) \quad \Delta = (\sigma_1^* + \sigma_2^*)\nu_1^2 = -\sigma_3^*\nu_1^2,$$

$$(8.3) \quad (\sigma_1^* - \sigma_2^*)\sigma_3^* = 0,$$

$$(8.4) \quad (\sigma_1^* - \sigma_2^*)\sigma_2^* + K = 0,$$

respectively. Hence $\sigma_1^* \neq \sigma_2^*$ from (8.4) since $K \neq 0$. Hence $\sigma_3^* = 0$ from (8.3) and hence $\Delta = 0$ on account of (8.2).

Now since $\sigma_3^* = 0$ we must have $\sigma_1^* = -\sigma_2^*$ and hence (5.5) yields $\xi_{11} = \xi_{22}$. Making this substitution in (5.3) and (5.4) these equations become

$$\xi_{11}\nu_1 + \xi_{12}\nu_2 = \rho(v_n - G)\lambda_1, \quad \xi_{11}\nu_2 + \xi_{12}\nu_1 = \rho(v_n - G)\lambda_2.$$

If $\nu_1 = \nu_2$ the left members of these two equations are equal and hence we must have $\lambda_1 = \lambda_2$. Hence, from the proportionality (8.1), we have $\nu_1 = -\nu_2$ and there is a contradiction with the fact that ν is a unit vector. Also if we start with the other possibility that $\nu_1 = -\nu_2$ we are led to a similar contradiction. *It follows that the quantities λ_1 and λ_2 must both vanish on the surface $\Sigma(t)$.*

Taking $\lambda_1 = \lambda_2 = 0$ the compatibility relations under consideration are automatically satisfied with the exception of (5.3), (5.4), and (5.5) which become

$$(8.5) \quad \xi_{11}\nu_1 + \xi_{12}\nu_2 = 0,$$

$$(8.6) \quad \xi_{21}\nu_1 + \xi_{22}\nu_2 = 0,$$

$$(8.7) \quad \sigma_1^*\xi_{11} + \sigma_2^*\xi_{22} = 0.$$

The determinant Δ of the coefficients of the ξ 's in these equations must vanish if the surface $\Sigma(t)$ is to be singular of order 1. Hence from (5.6) we have

$$(8.8) \quad \sigma_1^*\nu_2^2 + \sigma_2^*\nu_1^2 = 0.$$

Combining (8.8) with the condition for ν to be a unit vector it follows that

$$(8.9) \quad \nu_1^2 = \frac{\sigma_1^*}{\sigma_1^* - \sigma_2^*}, \quad \nu_2^2 = \frac{-\sigma_2^*}{\sigma_1^* - \sigma_2^*}.$$

When (8.8) is satisfied or when the components ν_1, ν_2 are given by (8.9) at the origin of a system of canonical coordinates the equations (8.5), (8.6), and (8.7) will admit a non-trivial solution ξ_{11}, ξ_{12} , and ξ_{22} which will permit a discontinuity in the derivatives of the components of the stress tensor σ over the surface $\Sigma(t)$.

The following result has now been proved. *There exists the possibility of a moving singular surface $\Sigma(t)$ of order 1 in the von Mises theory, under the assump-*

tion of plane stress, such that the derivatives of the density and the derivatives of the velocity components are continuous over the surface and discontinuities occur only in the derivatives of the components of the stress tensor. At any point P of the singular surface $\Sigma(t)$ the intersection of the surface and the y_1, y_2 -plane of a system of canonical coordinates having its origin at P will make an angle with the y_1 - and y_2 -axes determined by (8.9). Another way of expressing the latter statement would be to say that the normal to the surface $\Sigma(t)$ lies in the x_1, x_2 -plane and that the direction of the normal makes an angle determined by (8.9) with the two principal directions of the tensor σ which lie in the x_1, x_2 -plane. The speed of propagation s of the surface $\Sigma(t)$ is not determined by the compatibility conditions of the problem.

Remark. Since we do not assume the existence of second derivatives of the functions ρ, v_α , and $\sigma_{\alpha\beta}$ on the wave surface $\Sigma(t)$ the fact that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ does not lead to relations of the form

$$(8.10) \quad [\nu_{\alpha, \beta\gamma}] = \mu_\alpha \nu_\beta \nu_\gamma.$$

The above relations (8.5), (8.6), and (8.7) therefore constitute all of the compatibility conditions which necessarily hold on this wave surface.

We easily arrive at a contradiction if (8.10) is assumed to hold with $\mu_\alpha = 0$. In fact from $[\nu_{\alpha, \beta\gamma}] = 0$ and the form

$$\sigma_{\alpha\beta}^* = \frac{\sqrt{K} \epsilon_{\alpha\beta}}{\sqrt{\epsilon_{ij} \epsilon_{ij}}}$$

of the von Mises equations it follows that $[\sigma_{\alpha, \beta\gamma}^*] = 0$. Hence

$$(8.11) \quad [\sigma_{\alpha\beta, \gamma}^*] = \xi_{\alpha\beta} \nu_\gamma - \frac{1}{3} \xi_{ii} \nu_\gamma \delta_{\alpha\beta} = 0.$$

But, putting $\beta = \gamma$ in (8.11) and summing on the repeated indices, we find $\xi_{ii} = 0$ when use is made of (8.5) and (8.6). Hence $\xi_{\alpha\beta} = 0$ from (8.11) and this contradicts the requirement that $\Sigma(t)$ is singular of order 1 (see §5).

More generally it follows from (8.10), in which not all μ_α are equal to zero, that $\sigma_\beta^* = 0$ on $\Sigma(t)$. To show this we first set up the relations

$$(8.12) \quad [\sigma_{\alpha\beta, \gamma}^*] = \xi_{\alpha\beta} \nu_\gamma - \frac{1}{3} \xi_{ii} \nu_\gamma \delta_{\alpha\beta}, \quad [\epsilon_{\alpha\beta, \gamma}] = \frac{1}{2} (\mu_\alpha \nu_\beta + \mu_\beta \nu_\alpha) \nu_\gamma.$$

Then, differentiating (6.1) with respect to x_γ and making the substitutions (8.12), we find that

$$(8.13) \quad \frac{1}{2} (\mu_\alpha \nu_\beta + \mu_\beta \nu_\alpha) K = \xi_{ii} \epsilon_{ij} \sigma_{\alpha\beta}^* + \mu_i \nu_j \sigma_{ij}^* \sigma_{\alpha\beta}^* + \sigma_{ij}^* \epsilon_{ij} (\xi_{\alpha\beta} - \frac{1}{3} \xi_{kk} \delta_{\alpha\beta}).$$

If we consider (8.13) at the origin of a system of canonical coordinates, take $\alpha = 1, 2, \beta = 3$, and make use of (5.2), it readily follows that $\mu_3 = 0$. Also if we multiply (8.13) by $\sigma_{\alpha\beta}^*$ and use (4.10) or (8.7) we find $\xi_{ii} \epsilon_{ij} = 0$; hence the first term in the right member of (8.13) can be omitted. Differentiating the equation of incompressibility and using the relations (8.10) we obtain $\mu_\alpha \nu_\alpha = 0$ which is equivalent to the proportionality $\mu_1, \mu_2 \propto \nu_2, -\nu_1$. Now multiply (8.13) by $\nu_\alpha \mu_\beta$ and apply (8.5) and (8.6) to obtain

$$(8.14) \quad \mu_\beta \mu_\beta K = 2(\sigma_{ij}^* \mu_i \nu_j)^2.$$

Hence we find

$$(8.15) \quad K = 2(\sigma_1^* - \sigma_2^*)^2 \nu_1^2 \nu_2^2,$$

when we evaluate (8.14) at the origin of canonical coordinates and use the above proportionality for μ_1 and μ_2 . Finally when we express K as the sum of the squares of the scalars σ_i^* and use the relations (8.9) for ν_1 and ν_2 we can deduce, from (8.15), the condition $\sigma_3^* = 0$ on $\Sigma(t)$ as above stated.

9. A digression. In order to orient our work more closely with earlier investigations on singular or wave surfaces in the von Mises theory² let us assume that not all of the quantities λ_α and also that not all of the $\xi_{\alpha\beta}$ vanish on the wave surface $\Sigma(t)$. In other words we shall require that derivatives of components of the velocity and the stress tensor shall be discontinuous along $\Sigma(t)$. For convenience of reference we rewrite the primary set of dynamical conditions of compatibility on which the discussion depends, namely

$$(9.1) \quad (v_n - G)\zeta = 0, \quad \lambda_i \nu_i = 0, \quad \xi_{\alpha\beta} \nu_\beta = \rho(v_n - G)\lambda_\alpha,$$

$$(9.2) \quad J_{\alpha\beta} = \lambda_\alpha \nu_\beta + \lambda_\beta \nu_\alpha - \frac{2}{K} \sigma_{ij}^* \lambda_i \nu_j \sigma_{\alpha\beta}^* = 0, \quad \sigma_{\alpha\beta}^* \xi_{\alpha\beta} = 0.$$

From the second equation (5.1) we again have $\xi_{\alpha 3} = 0$ for $\alpha = 1, 2, 3$. Also $\nu_3 = 0$ from the first equation (5.1) since not all the $\xi_{\alpha\beta}$ vanish by assumption.

Since the quantity $J_{\alpha\alpha}$ vanishes identically we see, as before, that we can omit the equation $J_{11} = 0$ from the above set of the J -equations in (9.2). Omitting this equation let us now consider the set of equations

$$(9.3) \quad J_{12} = 0, \quad J_{13} = 0, \quad J_{23} = 0, \quad J_{33} = 0.$$

We shall later take account of the equation corresponding to J_{22} which does not appear in the set (9.3).

The matrix M of the coefficients of the quantities $\lambda_1, \lambda_2, \lambda_3$ in the equations (9.3), viewed at the origin of a system of canonical coordinates, is given by

$$\left\| \begin{array}{ccc} \nu_2 & \nu_1 & 0 \\ 0 & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ -\frac{2}{K} \sigma_3^* \sigma_1^* \nu_1 & -\frac{2}{K} \sigma_3^* \sigma_2^* \nu_2 & 0 \end{array} \right\|$$

when use is made of the fact that $\nu_3 = 0$. The rank of M must be less than 3 since we have assumed that not all λ_α are equal to zero. But the upper third order determinant in this matrix is seen to vanish identically. Also the matrix determined by the first two rows of M has rank 2 if $\nu_1 \neq 0$ and the matrix determined by the first and third rows has rank 2 if $\nu_2 \neq 0$. Hence we see that the condition for the equations (9.3) to have a non-trivial solution λ_α is that one or the other of the following two equations be satisfied, namely

$$(9.4) \quad \left| \begin{array}{ccc} \nu_2 & \nu_1 & 0 \\ 0 & 0 & \nu_1 \\ -\sigma_3^* \sigma_1^* \nu_1 & -\sigma_3^* \sigma_2^* \nu_2 & 0 \end{array} \right| = 0,$$

or

$$(9.5) \quad \begin{vmatrix} \nu_2 & \nu_1 & 0 \\ 0 & 0 & \nu_2 \\ -\sigma_3^* \sigma_1^* \nu_1 & -\sigma_3^* \sigma_2^* \nu_2 & 0 \end{vmatrix} = 0.$$

Assume first that $\nu_1 \neq 0$ so that (9.4) holds. Expanding the determinant in (9.4) we readily find that the resulting equation can be reduced to

$$(9.6) \quad (\sigma_2^* + \nu_1^2 \sigma_3^*) \sigma_3^* = 0.$$

Now, since the matrix determined by the first two rows of the determinant in (9.4) has rank 2, we have the following proportionality

$$(9.7) \quad \lambda_1, \lambda_2, \lambda_3 \propto \nu_1^2, -\nu_1 \nu_2, 0.$$

Hence $\lambda_3 = 0$. From the second equation (9.1) and the proportionality (9.7) it also follows that $\nu_1^2 = \nu_2^2$, or $\nu_1 = \pm \nu_2$, and hence

$$(9.8) \quad \nu_1 = \frac{1}{\sqrt{2}}, \quad \nu_2 = \pm \frac{1}{\sqrt{2}},$$

since we can assume, without loss of generality, that ν_1 is positive for the determination of the normal direction to possible surfaces $\Sigma(t)$. Hence, since $\nu_1^2 = \frac{1}{2}$, the above equation (9.6) becomes

$$(9.9) \quad (\sigma_1^* - \sigma_2^*) \sigma_3^* = 0.$$

Now consider the relation $J_{22} = 0$, which has been omitted from the above set (9.3), at the origin of a system of canonical coordinates. This relation, when use is made of the proportionality (9.7), is found to reduce to

$$(9.10) \quad (\sigma_1^* - \sigma_2^*) \sigma_2^* + K = 0.$$

Since $K \neq 0$ it follows from (9.10) that $\sigma_1^* \neq \sigma_2^*$. Hence, on account of the relation (9.9), we must have $\sigma_3^* = 0$.

Since there is no essential distinction between the subscripts 1 and 2 in the above discussion it is clear that if we had assumed $\nu_2 \neq 0$ and used equation (9.5), instead of equation (9.4) with the assumption $\nu_1 \neq 0$, the same results would have been obtained.

No new condition is obtained from the third set of equations (9.1) for $\alpha = 3$ since $\xi_{\alpha 3} = 0$ and $\lambda_3 = 0$. Expanding the other equations of this set we have

$$(9.11) \quad \nu_1 \xi_{11} + \nu_2 \xi_{12} = \rho(v_n - G) \lambda_1,$$

$$(9.12) \quad \nu_1 \xi_{21} + \nu_2 \xi_{22} = \rho(v_n - G) \lambda_2.$$

But $\sigma_1^* = -\sigma_2^*$ since $\sigma_3^* = 0$. Hence the last equation (9.2), when considered at the origin of a system of canonical coordinates, yields $\xi_{11} = \xi_{22}$ and so, from (9.12), we have

$$(9.13) \quad \nu_2 \xi_{11} + \nu_1 \xi_{12} = \rho(v_n - G)\lambda_2.$$

Now consider the following matrix determined by the coefficients in the equations (9.11) and (9.13), namely

$$\begin{vmatrix} \nu_1 & \nu_2 & \rho(v_n - G)\lambda_1 \\ \nu_2 & \nu_1 & \rho(v_n - G)\lambda_2 \end{vmatrix}.$$

Since the second order determinant formed from the elements in the first and second columns of this matrix has the value $\nu_1^2 - \nu_2^2$, and hence vanishes on account of (9.8), the matrix must have rank 1 in order for the equations (9.11) and (9.13) to be consistent. But this means that

$$(9.14) \quad \begin{vmatrix} \nu_1 & \rho(v_n - G)\lambda_1 \\ \nu_2 & \rho(v_n - G)\lambda_2 \end{vmatrix} = \rho(v_n - G)(\lambda_2 \nu_1 - \lambda_1 \nu_2) = 0.$$

Hence, since $\rho \neq 0$, it follows from (9.14) and the proportionality (9.7) that

$$(v_n - G)\nu_1 \nu_2 = 0.$$

But neither of the quantities ν_1 and ν_2 can vanish from (9.8) and hence the speed of propagation $G - v_n$ is equal to zero.

We have now proved the following result with regard to singular surfaces $\Sigma(t)$ of order 1 in the von Mises theory under the assumption of plane stress. *Any singular surface $\Sigma(t)$ of order 1 over which the derivatives of the components of velocity and stress exhibit discontinuities will (a) have speed of propagation $s = 0$, (b) be perpendicular to the x_1, x_2 -plane, and (c) be such that its normal ν at any point P bisects the angle formed by the two principal directions, determined by the stress tensor, in the x_1, x_2 -plane at P .* It may be observed that the first equation (9.1) is satisfied identically since the factor $v_n - G = 0$ and hence a discontinuity in the derivatives of the density ρ is possible. We emphasize, however, that the existence of this singular surface $\Sigma(t)$ requires the stress tensor σ to be such that the invariant $\sigma_3^* = 0$ over the surface.

10. Wave propagation in the Prandtl-Reuss theory. It will be assumed in this section that neither the speed of propagation s nor the determinant Δ vanishes over the singular surface $\Sigma(t)$. We can therefore avail ourselves of the conditions (5.2) and the equations (5.7), (5.8), and (5.9).

If $\lambda_1 = \lambda_2 = 0$ it follows from (5.7), (5.8), and (5.9) that ξ_{11} , ξ_{12} , and ξ_{22} vanish on $\Sigma(t)$. Hence $\Sigma(t)$ cannot be singular of order 1 as postulated (see §4). *Thus any singular surface $\Sigma(t)$ having speed of propagation $s \neq 0$ and over which $\Delta \neq 0$ must be associated with discontinuities in the derivatives of the velocity components as well as discontinuities in the derivatives of the components of the stress tensor. However, the derivatives of the density ρ will be continuous over such surfaces.* This latter statement follows from the first equation (4.8). These facts will apply to the special cases of singular surfaces treated in the remainder of this section and will be considered to apply without further mention.

Since both of the quantities λ_1 and λ_2 cannot vanish the condition given by the second equation (4.8) is equivalent to the proportionality

$$(10.1) \quad \lambda_1, \lambda_2 \propto \nu_2, -\nu_1.$$

Let us now substitute the expressions (5.7), (5.8), and (5.9) for ξ_{11} , ξ_{12} , and ξ_{22} , respectively, into the relations (7.3), (7.4), and (7.5). When this substitution is made and when we avail ourselves of the proportionality (10.1) the resulting equations can be written in the following form

$$(10.2) \quad \frac{\rho(v_n - G)^2}{\mu\Delta} (\sigma_1^* \nu_2^2 - \sigma_2^* \nu_1^2) + \frac{\sigma_1^* - \sigma_2^*}{2\mu} + \nu_1^2 - \nu_2^2 = 0,$$

$$(10.3) \quad \left[\frac{\rho(v_n - G)^2}{3\mu\Delta} + \frac{\sigma_3^*}{K} \right] (\sigma_1^* - \sigma_2^*) \nu_1 \nu_2 = 0,$$

$$(10.4) \quad \left[\frac{\rho(v_n - G)^2}{3\mu\Delta} (\sigma_1^* - \sigma_3^*) - \frac{(\sigma_1^* - \sigma_2^*)\sigma_2^*}{K} - 1 \right] \nu_1 \nu_2 = 0.$$

In treating these relations we shall divide the discussion into several cases according to whether the quantity $\sigma_1^* - \sigma_2^*$ and/or one of the velocity components ν_1 , ν_2 vanishes or does not vanish on the surface $\Sigma(t)$.

Case I. The quantity $\sigma_1^* - \sigma_2^* = 0$ on $\Sigma(t)$. For this case the relation (10.3) is satisfied identically. Also $\sigma_3^* = -2\sigma_1^*$ from the above assumption. Hence $\sigma_1^* \neq 0$ since otherwise we would have a contradiction with the yield condition. Hence, from (5.6), we have $\Delta = \sigma_1^*$ and the equations (10.2) and (10.4) reduce to

$$(10.5) \quad [\rho(v_n - G)^2 - \mu](\nu_1^2 - \nu_2^2) = 0,$$

$$(10.6) \quad [\rho(v_n - G)^2 - \mu]\nu_1 \nu_2 = 0.$$

But the bracket expression in (10.5) and (10.6) cannot be different from zero since this would lead to a contradiction with the fact that ν is a unit vector. Hence⁷

$$(10.7) \quad \rho(v_n - G)^2 = \mu.$$

Hence any wave surface $\Sigma(t)$ with speed of propagation $s \neq 0$ and over which $\Delta \neq 0$ and $\sigma_1^* = \sigma_2^*$ will have its speed $s = G - v_n$ given by (10.7) in the plane stress problem. The unit normal vector ν to this wave surface is not determined by the dynamical conditions of compatibility.

Case II. The component ν_1 or ν_2 has the value zero on $\Sigma(t)$. Here the equations (10.3) and (10.4) are satisfied identically. Recourse to equation (10.2) now

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⁷ The relation (10.7) without modification and the relations (10.8), (10.9), (11.8), (11.10), and (12.3), when we neglect the term in σ_1^* and/or σ_2^* or W in comparison with μ , give the speed of propagation s equal to $\sqrt{\mu/\rho}$ which is formally identical with the velocity of the so called *equivoluminal waves* in the classical elasticity theory. See, A.E.H. LOVE, *Mathematical Theory of Elasticity*, 4th Ed., New York, Dover Publications, 1944, p. 294.

permits us to state the following result.⁸ According as ν_1 or ν_2 is equal to zero on the wave surface $\Sigma(t)$ the determinant Δ and the speed s are given respectively by the following two equations⁷

$$(10.8) \quad \Delta = \sigma_1^*, \quad \rho(v_n - G)^2 = \mu + \frac{1}{2}(\sigma_2^* - \sigma_1^*),$$

$$(10.9) \quad \Delta = \sigma_2^*, \quad \rho(v_n - G)^2 = \mu + \frac{1}{2}(\sigma_1^* - \sigma_2^*),$$

where it is to be understood that $\sigma_1^* \neq 0$ in (10.8) and $\sigma_2^* \neq 0$ in (10.9) in accordance with the above assumption that $\Delta \neq 0$ on the surface $\Sigma(t)$. In the first instance the moving surface $\Sigma(t)$ is perpendicular, at any of its points P , to the y_2 -axis of the canonical coordinate system with origin at P and in the second instance the surface is perpendicular to the y_1 -axis of this coordinate system.

Case III. The quantities σ_1^* and σ_2^* are distinct and neither of the components ν_1 and ν_2 vanishes on the surface $\Sigma(t)$. From equation (10.3) we now have

$$(10.10) \quad \rho(v_n - G)^2 = \frac{-3\mu\Delta\sigma_3^*}{K}.$$

If we eliminate the speed s from equation (10.4) by the substitution (10.10) we find that the resulting equation is satisfied identically when use is made of the yield condition. However elimination of s from (10.2) by means of (10.10) leads to an equation for the determination of the components ν_1 and ν_2 of the unit normal to $\Sigma(t)$. We shall not give the complete details of this calculation but note merely that we can first deduce the intermediate relation

$$(10.11) \quad (2K - 3\sigma_3^*\sigma_3^*)\nu_1^2 = K + 3\sigma_1^*\sigma_3^* - \frac{(\sigma_1^* - \sigma_2^*)K}{2\mu}.$$

It can then be shown that

$$(10.12) \quad 2K - 3\sigma_3^*\sigma_3^* = (\sigma_2^* - \sigma_1^*)^2,$$

$$(10.13) \quad K + 3\sigma_1^*\sigma_3^* = (\sigma_2^* - \sigma_1^*)(\sigma_2^* - \sigma_3^*).$$

Making the substitutions (10.12) and (10.13) the equation (10.11) then leads to the following two equations

$$(10.14) \quad \nu_1^2 = \frac{\sigma_2^* - \sigma_3^*}{\sigma_2^* - \sigma_1^*} + \frac{K}{2\mu(\sigma_2^* - \sigma_1^*)},$$

$$(10.15) \quad \nu_2^2 = \frac{\sigma_1^* - \sigma_3^*}{\sigma_1^* - \sigma_2^*} + \frac{K}{2\mu(\sigma_1^* - \sigma_2^*)}.$$

When we eliminate the quantities ν_1^2 and ν_2^2 from (5.6) by means of (10.14) and (10.15) the expression so obtained for the determinant Δ can be shown to reduce to

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⁸ It is understood here that the plane stress condition (2.1) applies, even though not explicitly mentioned in connection with the statement of this result.

$$(10.16) \quad \Delta = -2\sigma_3^* + \frac{K}{2\mu}.$$

Hence, eliminating Δ from the relation (10.10), we obtain

$$(10.17) \quad \rho(v_n - G)^2 = \frac{6\mu(\sigma_3^*)^2}{K} - \frac{3\sigma_3^*}{2}.$$

The case under consideration represents the general situation and the results now established for this case will be emphasized by the following italicized statement.⁸ *If the determinant Δ , the speed s , the difference $\sigma_1^* - \sigma_2^*$, and the components ν_1 and ν_2 do not vanish over the wave surface $\Sigma(t)$ then, at any point P of the surface, Δ and s are given by (10.16) and (10.17), respectively, and ν_1 and ν_2 are given by (10.14) and (10.15) with reference to the system of canonical coordinates having its origin at P .*

11. Special case for which Δ vanishes on the singular surface. The assumption $\Delta \neq 0$ on the wave surface $\Sigma(t)$ was made in the preceding section. In this section we consider the possibility that $\Delta = 0$ over surfaces $\Sigma(t)$ in the Prandtl-Reuss theory. Since we shall assume the speed of propagation s to be different from zero the derivatives of the density ρ will be continuous on the wave surfaces $\Sigma(t)$ from the first equation (4.8) and we can also avail ourselves of the conditions (5.2). However the equations (5.7), (5.8), and (5.9) cannot now be used but must be replaced by the equations (5.3), (5.4), and (5.5). Our discussion will be based on the following two mutually exclusive cases.

Case I. The quantities λ_1 and λ_2 are both zero on $\Sigma(t)$. Under this condition on the λ 's and the condition $s \neq 0$ the equations (7.3), (7.4), and (7.5) reduce to

$$(11.1) \quad \xi_{12} = 0, \quad \xi_{ii} = 0, \quad \xi_{22} - \frac{1}{3}\xi_{ii} = 0.$$

From the second and third equations (11.1) we immediately obtain $\xi_{11} = 0$ and $\xi_{22} = 0$ on $\Sigma(t)$. Hence, since $\xi_{12} = 0$ also from the first equation (11.1), we have a contradiction with the assumption that $\Sigma(t)$ is a singular surface of order 1 (see §4). *In other words there is no wave surface $\Sigma(t)$ which satisfies the conditions of this case.*

Case II. The quantities λ_1 and λ_2 are not both zero on $\Sigma(t)$. Let us now consider the matrix formed from the coefficients of the quantities ξ_{11} , ξ_{12} , and ξ_{22} in the equations (5.3), (5.4), (5.5) and the right members of these equations, namely

$$\begin{vmatrix} \nu_1 & \nu_2 & 0 & \rho(v_n - G)\lambda_1 \\ 0 & \nu_1 & \nu_2 & \rho(v_n - G)\lambda_2 \\ \sigma_1^* & 0 & \sigma_2^* & 0 \end{vmatrix}.$$

The third order determinant formed from the elements of the first three columns vanishes from the condition $\Delta = 0$, i.e.,

$$(11.2) \quad \sigma_1^* \nu_2^2 + \sigma_2^* \nu_1^2 = 0.$$

Now the equations (5.3), (5.4), and (5.5) will have a non-trivial solution ξ if, and only if, the above matrix has rank 2; this follows from the well known theorem on the solutions of systems of linear equations and the fact that the rank of the third order matrix formed from the coefficients of the ξ 's in equations (5.3), (5.4), and (5.5) cannot have rank less than 2 without violating the requirement that ν is a unit vector. Since $\nu_n - G \neq 0$ by assumption the required condition is therefore given by one or the other of the following two determinantal equations, namely

$$\begin{vmatrix} \nu_1 & \nu_2 & \lambda_1 \\ 0 & \nu_1 & \lambda_2 \\ \sigma_1^* & 0 & 0 \end{vmatrix} = 0, \quad \text{if } \nu_1 \neq 0,$$

or

$$\begin{vmatrix} \nu_2 & 0 & \lambda_1 \\ \nu_1 & \nu_2 & \lambda_2 \\ 0 & \sigma_2^* & 0 \end{vmatrix} = 0, \quad \text{if } \nu_2 \neq 0.$$

These conditions reduce to

$$(11.3) \quad (\nu_1 \lambda_1 - \nu_2 \lambda_2) \sigma_1^* = 0, \quad \text{if } \nu_1 \neq 0,$$

$$(11.4) \quad (\nu_1 \lambda_1 - \nu_2 \lambda_2) \sigma_2^* = 0, \quad \text{if } \nu_2 \neq 0.$$

From the second equation (4.8) we now have the proportionality

$$(11.5) \quad \lambda_1, \lambda_2 \propto \nu_2, -\nu_1.$$

Hence (11.3) and (11.4) become

$$(11.6) \quad \nu_2 \sigma_1^* = 0, \quad \text{if } \nu_1 \neq 0; \quad \nu_1 \sigma_2^* = 0, \quad \text{if } \nu_2 \neq 0.$$

But if the first relation in (11.6) applies with $\sigma_1^* = 0$ we see from (11.2) that $\sigma_2^* = 0$ and hence we have a contradiction with the yield condition. Similarly if the second relation (11.6) holds with $\sigma_2^* = 0$ it follows from (11.2) that $\sigma_1^* = 0$ and the contradiction with the yield condition again results. *Hence the requirement for the above matrix to have rank 2 reduces to the condition that $\nu_2 = 0$, if $\nu_1 \neq 0$, and $\nu_1 = 0$, if $\nu_2 \neq 0$.*

If $\nu_1 = 0$ it follows from (11.5) that $\lambda_2 = 0$ and hence $\lambda_1 \neq 0$ from the assumption of this case. Equations (5.3), (5.4), (5.5), and (11.2), in which $\nu_2 = \pm 1$, now yield

$$(11.7) \quad \xi_{12} = \pm \rho(\nu_n - G)\lambda_1, \quad \xi_{22} = 0, \quad \sigma_1^* = 0.$$

Making use of (11.7) the relations (7.3) (7.4), and (7.5) are seen to reduce to the condition $\xi_{11} = 0$ and⁷

$$(11.8) \quad \rho(\nu_n - G)^2 = \mu + \frac{\sigma_2^*}{2}.$$

Correspondingly, if $\nu_2 = 0$, we find that $\lambda_1 = 0$ and⁷

$$(11.9) \quad \xi_{12} = \pm \rho(v_n - G)\lambda_2, \quad \xi_{11} = \xi_{22} = 0, \quad \sigma_2^* = 0,$$

$$(11.10) \quad \rho(v_n - G)^2 = \mu + \frac{\sigma_1^*}{2}.$$

Apart from the above special determinations, *e.g.*, those given by the equations (11.7) and (11.9), the main result now proved is contained in the following statement.⁸ *If there are discontinuities in the derivatives of the components of the velocity and stress over a wave surface $\Sigma(t)$ in the Prandtl-Reuss theory and if, furthermore, the speed of propagation s is different from zero but $\Delta = 0$ on $\Sigma(t)$, then, at any of its points P , the surface $\Sigma(t)$ is perpendicular to the y_2 - or y_1 -axis of a system of canonical coordinates with origin at P and has speed of propagation given by (11.8) or (11.10), respectively. The derivatives of the density ρ will be continuous over this surface $\Sigma(t)$.*

12. Propagation of waves in simple tension. A state of simple tension is said to exist if $\sigma_{22} = W > 0$ and $\sigma_{\alpha\beta} = 0$ otherwise. For these values of the $\sigma_{\alpha\beta}$ we have

$$(12.1) \quad \sigma_{22}^* = \frac{2W}{3}, \quad \sigma_{11}^* = \sigma_{33}^* = -\frac{W}{3}, \quad \sigma_{\alpha\beta}^* = 0, \quad (\alpha \neq \beta).$$

It can also readily be observed that the principal directions determined by the stress tensor can be identified with the coordinate directions and hence from (12.1) we obtain

$$(12.2) \quad \sigma_2^* = \frac{2W}{3}, \quad \sigma_1^* = \sigma_3^* = -\frac{W}{3}.$$

If the material is yielding under the above simple tension then $K = \frac{2}{3}W^2$. Assuming yield to occur for a material of known yield stress W several of the general results of the preceding sections will provide numerical information on the speed and/or direction of propagation of the possible wave surfaces $\Sigma(t)$, in the plane stress problem, which can advance into a region in a state of simple tension. Under the conditions above specified the following results can, in fact, be stated. It is to be understood here that the wave surfaces in question are perpendicular to the x_1, x_2 -plane (see §5).

(α) *There is the possibility in the von Mises theory of two plane wave surfaces, having a speed of propagation different from zero but not determined from the compatibility conditions of the problem, over which discontinuities occur only in the derivatives of the components of the stress and which make a numerical angle of $35^\circ 16'$ with the plane perpendicular to the direction of the tension, i.e., the x_2 -*

direction in this consideration.⁹ This follows from the result at the end of §8 when use is made of the relations (12.2).

(β) *In the Prandtl-Reuss theory a plane wave may arise which is perpendicular to the direction of the tension and which is propagated with a speed given by*⁷

$$(12.3) \quad \rho(v_n - G)^2 = \mu + \frac{1}{2}W.$$

Discontinuities in the derivatives of the components of the stress and also in the derivatives of the components of the velocity but not in the derivatives of the density will occur over this surface. We obtain this result, as well as a corresponding result¹⁰ concerning the propagation of wave surfaces which are planes perpendicular to the x_1 -axis, from the discussion under *Case II* in §10.

The above results (α) and (β) give all the information which can be obtained for the case of simple tension from the italicized statements in the foregoing sections. Thus the wave surface $\Sigma(t)$ in the Prandtl-Reuss theory whose speed of propagation is given by (10.7) cannot be realized in simple tension since we see from (12.2) that one of the conditions for this surface, namely $\sigma_1^* = \sigma_2^*$, is not satisfied; likewise the surface, which is discussed under *Case III* of §10, cannot exist in simple tension since the equation (10.14) gives $v_1^2 > 1$ on account of (12.2). In such cases, however, it is easily seen that a slight modification of the strict state of simple tension, *e.g.*, the addition of a uniform lateral stress, will be sufficient to satisfy the conditions of compatibility required for the existence of the wave surface in question.

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⁹ For the simple tension problem the scalar σ_3^* is different from zero. Hence it follows from the *Remark* in §8 that equations of the form (8.10) are not valid on the plane wave surfaces described in the italicized statement (α). This leads to the conclusion that second derivatives of the velocity components v_α fail to exist on the rear side of these wave surfaces as they advance into the region of simple tension.

¹⁰ By rotating the axes of the underlying x coordinate system about the x_2 -axis this result leads to the possibility of a plane wave surface $\Sigma(t)$ which is perpendicular to any fixed line normal to the direction of the tension.