A New Series for the Calculation of Steady Laminar Boundary Layer Flows

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Dedicated to ALBERT BETZ on his 70th birthday

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SUMMARY

A new and general method for solving problems of plane and steady laminar boundary layer flows in incompressible fluids with arbitrary outer pressure distribution is developed. This method is based on the introduction of the dimensionless quantities

$$\xi = \frac{1}{\nu} \int_0^x U(x) dx, \quad \eta = \frac{U(x)y}{\left\{ 2\nu \int_0^x U(x) dx \right\}^{\frac{1}{2}}}$$

as new independent spatial variables. (x, y are the usual boundary layer coordinates, $U(x)$ the given outer velocity distribution, ν the kinematic viscosity.) The solution of the boundary layer problem is then given as a power series in ξ with coefficient functions depending on η . This series is a formally exact solution of the boundary layer problem.

The new series solution has the following qualities:

1. Whereas the original variables x, y have the significance only of Cartesian coordinates, the influence of wall curvature being neglected in boundary layer theory, the new coordinates are adjusted to the data of the special problem in any case of application. The new variables represent a logical development of former efforts in the field of boundary-layer flow calculation. (*Cf.* §6.)
2. An important feature, as compared with other series solutions known for some special cases is that the leading term of the new series satisfies exactly the outer boundary condition at all cross-sections along the wall. Therefore, the succeeding terms give corrections only in the inner part of the boundary layer. Accordingly, taking also No. 1 into account, the zero order term by itself gives a good approximation for the boundary layer flow

from the forward singular point $x = 0$ (e.g. forward stagnation point, leading edge of a flat plate, etc.) downstream to cross-sections surprisingly far away (e.g. around a circular cylinder within 4 % error from the forward stagnation point over 60 % of the way to the separation point; cf. §16).

3. The new series is not only applicable to all cases of cylinders with rounded noses, cases to which the well-known BLASIUS series is also applicable, but may also be applied to cylinders with a forward vertex line (arbitrary forward wedge angle), including e.g. any problem of flow along a flat plate with arbitrary outer pressure distribution (wedge angle zero). (Cf. § 4.)
4. The well-known "similar" boundary layer flows, exact solutions of the boundary layer equations, appear as those special cases in which the solution depends only on η . In these cases the series reduces to the zero order term, and the variable η specializes to the well known variable of the "wedge" flows. (In the general case the zero order term depends also only on η , but now η is the more general variable given above and reduces to the special variable of wedge flows only in the limit as $x \rightarrow 0$.)
5. The transformation to the new independent spatial variables given above leads to a formulation of the boundary layer problem in which the given data of any special application appear explicitly only in one coefficient term. This coefficient, which we shall call the *principal function* of the boundary layer problem, is of fundamental importance for the structure of the series solution. This principal function is defined by

$$\beta(\xi) = 2U'(x) \frac{\int_0^x U(x) dx}{U^2(x)}.$$

It reduces to the HARTREE parameter of wedge flows for the special cases $U(x) = \text{const} \cdot x^m$. In the general case, $\beta(0) = \beta_0$ determines the forward wedge angle of the cylinder, this angle being $\pi\beta_0$. (Examples: $\beta_0 = 0$, flat plate at zero incidence; $\beta_0 = 1$, cylinders with rounded noses.)

6. The coefficient functions of the new series can be expressed as linear combinations of *universal functions*, "universal" in the sense that these functions, for any fixed value of β_0 (determining the geometry of the cylinder at the forward line $x = 0$), are independent of the given data of the problem. For each value of β_0 they can therefore be tabulated once and for all up to a sufficiently high order. This is a main feature of the new series. Since the major part of the work can thus be done once and for all, the application of the new series in any special case is only a question of a few elementary evaluations (cf. §11) and is thus at least as simple as the application of any non-rigorous and unreliable method of the POHLHAUSEN type, at present preferred in engineering applications for reasons of simplicity.

Tables of the universal functions have been tabulated to some extent for $\beta_0 = 0$ and $\beta_0 = 1$ (cf. §10), also including for $\beta_0 = 1$ universal functions of the cross-flow component for cases of yawed cylinders.

A number of examples have been evaluated, and the results are described in Part III of this memoir. These examples show the success and also the limitations of the method. For "Case A" of flows along flat plates ($\beta_0 = 0$), convergence of the new series proves to be very satisfactory; the first five terms of the series provide a good approximation from the leading edge to a point immediately ahead of the separation point. The separation point can then easily be determined by a short extrapolation. This is true not only for the cases studied in this paper but also for many further cases we have evaluated.

Applications in "Case B" of cylinders with rounded noses ($\beta_0 = 0$), to which the BLASIUS series is also applicable, show no improvement of convergence compared with the BLASIUS series. Here the main achievement is that mentioned in No. 2 above. The zero order approximation will prove of considerable advantage for certain theoretical investigations where separation of variables in a large domain from $x = 0$ downstream is desired.

Further examples will be published soon in separate papers prepared with the cooperation of H. WITTING.

§1. Introduction: statement of the problem and method of attack.

The well known BLASIUS series for the calculation of laminar boundary layer flows has proved to converge rather slowly. Although, at least for the case of symmetrical plane flows, a large number of universal functions for the coefficients of the BLASIUS series have been evaluated, so that a high order of approximation may be attempted, that series is used generally only to procure a good approximation from the forward stagnation point to some cross-section downstream so as to provide initial values for one of the standard difference procedures, the further flow downstream then being calculated step by step with a far more complicated but reliable method.

Two points seem to be of importance for this failure of the BLASIUS series: (a) The outer boundary condition, asserting that the velocity component parallel to the wall assumes the prescribed values of the outer velocity $U(x)$ asymptotically with increasing distance from the wall, is satisfied by expanding $U(x)$ in a power series in x and then adjusting each term of the BLASIUS series so as to assume the value of the corresponding term of the outer velocity series asymptotically with increasing wall distance. If $U(x)$ does not happen to be a polynomial of low degree, the outer boundary condition will be satisfied with sufficient accuracy only if an appropriately large number of terms of the BLASIUS series is taken into account, and the rapidity of convergence of the BLASIUS series will depend to a considerable extent on the convergence of the power series for $U(x)$. (b) The coordinates used in applying the BLASIUS series method— x , the distance along the wall from the forward stagnation point, and y , the distance along the normal to the wall—have the significance only of Cartesian coordinates, since the influence of wall curvature is neglected in boundary layer theory. These coordinates, therefore, are not specially adjusted to the data of the particular problem in question.

With the research reported in this paper it was my aim to achieve the following qualities:

- (1) A new series should be found such that the first term would satisfy the outer boundary condition exactly and for all values of x . All further terms, therefore, should vanish at the outer edge of the boundary layer (and, of course, also at the wall), thus supplying correction for the first term only in the interior of the boundary layer.
- (2) The choice of the independent variables for the new series should be such that the first term would approximate the solution well from the forward stagnation point as far as possible downstream. The corrections given by further terms in the inner points of the boundary layer would thus begin to carry weight only relatively far downstream. Also, it could at least be hoped that only rather far downstream would convergence weaken to an extent that terms of very high order need be considered.
- (3) The new series should retain the one essential advantage of the BLASIUS series: Calculation of the coefficient functions can be reduced to the evaluation of linear combinations of universal functions, which can be tabulated once and for all. Application of the series thus becomes very simple, in fact so simple that, if convergence is sufficiently good, this series would provide a mathematically rigorous method requiring at least no more labor, if not less, than the parameter methods of the POHLHAUSEN type still being used to a large extent because of their simplicity, even though they are not rigorous and lead in general only to an unreliable rough overall approximation and in fact often not even that.
- (4) The new series should be general enough so that it can be applied not only to those outer pressure distributions for which the BLASIUS series offers at least a formal solution (flows around cylinders with forward stagnation line on rounded noses) or for which known special series (HOWARTH'S series for flow along a flat plate with retarding pressure gradient varying linearly with the distance x from the forward edge, and a recent generalization by I. TANI for plates with outer velocity distributions $U(x) = u_0 + u_n x^n$) are valid; it should be general enough to enable treatment of a much more extensive class of outer velocities including the cases mentioned. This would make it possible to solve a large number of problems of practical interest for which hitherto no exact solution of the boundary value problem in question has been possible.

Concerning the rapidity of convergence, there is a limit to any hopes in this direction. However good the convergence of a power series with respect to a suitable variable from the forward line onwards to some point far downstream, present knowledge indicates that if there exists a point of separation it will generally be a singularity of the flow and hence make an end to convergence.

Therefore the best we can hope for is convergence that does not become too slow for practical applications until immediately in front of the separation point. Then the separation point may be found by a short and sufficiently accurate extrapolation, or, more accurately, by one or two steps of a difference method.

Our new series in the new independent variables ξ , η entirely fulfills the requirements (1), (3), and (4). The new variables represent a logical development of former efforts in calculation of boundary layer flows (*cf.* §6).

The well known wedge flows, the *similar solutions*, which are exact integrals of the boundary layer equations, appear as those special cases when the new series reduces to its first term, all further terms vanishing identically. The solutions then depend on only one of our variables, which in turn specializes to the well known variable of the similar solutions.

Concerning the requirement (2) above, the examples of applications of the new series described in Part III and in further papers soon to be published show that in most cases of retarding pressure gradients along flat plates the approximation given by the new series, truncated after the fifth order term, is sufficiently accurate for practical purposes until immediately ahead of the separation point, and, therefore, this point can be determined with satisfactory accuracy by a short extrapolation. In cases of flow around cylinders with rounded noses, cases for which also the BLASIUS series solution is valid, no improvement of convergence in cross-sections behind the point of minimum pressure, as compared with convergence of the BLASIUS series, has been achieved. But, as expected, the first term of the new series gives a good approximation to the exact solution very much further downstream from the forward stagnation point than does the first term of the BLASIUS series. Behind the point of minimum pressure the convergence of the new series slows down rapidly. It seems that convergence is best in cases of monotone (retarding or accelerating) outer velocities.

Our intention in giving some examples of applications in Part III is to convey a general impression of progress and limitations. Not all examples evaluated could be included in this memoir. The first two refer to problems that have not been solved rigorously before. They are only the first examples of the class of problems, the solution of which has now become accessible.

Application of the new series consists only in a few elementary calculations (*cf.* §11), allowing any further applications to be worked out with the aid of tables in a relatively short time. To the extent that tables of the universal functions are now available, applications are even easier than evaluations utilizing the non-rigorous and accordingly unreliable parameter methods (momentum and energy integral methods) preferred until now for their simplicity. Even in cases where the fifth order truncated series approximation should turn out to be insufficient far downstream, it should be applied as far as approximation is satisfactory to check non-rigorous methods.

Extension of the theory to flows of compressible fluids is possible and will

be investigated in the near future. The extension to boundary layer flows with arbitrary continuous suction distribution along the wall is also possible. A report on this extension will be published soon.

PART I: STUDY OF A BOUNDARY LAYER TRANSFORMATION

§2. Transformation of the boundary layer equations.

The differential equations of plane and steady laminar boundary layer flows are

$$(1) \quad \begin{aligned} uu_x + vu_y &= U(x)U'(x) + \nu u_{yy}, \\ u_x + v_y &= 0. \end{aligned}$$

Here x denotes the wall curve length, y is the distance from the wall along the normal, u and v are the x and y components of the boundary layer velocity, ν the constant kinematic viscosity, and $U(x)$ the prescribed outer potential flow velocity parallel to the wall at the edge of the boundary layer. The subscripts indicate partial derivatives, and we write $U'(x)$ for dU/dx .

By introduction of the stream function $\psi(x, y)$, defined by

$$(2) \quad u = \psi_y, \quad v = -\psi_x \quad \text{with} \quad \psi(x, 0) = 0,$$

the equation of continuity $(1)_2$ is satisfied. $(1)_1$ then reads

$$(3) \quad \psi_y \psi_{xy} - \psi_x \psi_{yy} = UU' + \nu \psi_{yyy}.$$

The condition that there be no slip along the wall and the asymptotic condition far from the wall are

$$(4) \quad \begin{aligned} u(x, y) = v(x, y) &= 0 \quad \text{for} \quad y = 0, \\ u(x, y) &\rightarrow U(x) \quad \text{for} \quad y \rightarrow \infty. \end{aligned} \quad (0 \leq x \leq x_1)$$

In the following the point $x = 0, y = 0$ will always be the point of the contour denoting the forward line of the cylinder where the oncoming flow divides. This can be the forward stagnation point of a blunt-nosed contour or of a contour with wedge-like nose, or it may be the leading edge of a flat plate or pointed contour. $x = x_1$ designates only that cross-section downstream up to which, in a given case, we may wish to investigate the boundary layer. In general $x = x_1, y = 0$ will be a separation point of the boundary layer flow.

Since $\psi(x, 0) = 0$, the boundary conditions (4) are equivalent to

$$(5) \quad \begin{aligned} \psi(x, y) = \psi_y(x, y) &= 0 \quad \text{for} \quad y = 0, \\ \psi_y(x, y) &\rightarrow U(x) \quad \text{for} \quad y \rightarrow \infty. \end{aligned} \quad (0 \leq x \leq x_1)$$

For uniqueness of the solution (*cf.* H. GÖRTLER [1], also an interesting theorem of W. RHEINOLDT [2], and, under more general assumptions, [3]), the velocity profile $u(x, y)$ at an initial cross-section must be prescribed. As our initial cross-

section in the following investigations will be the cross-section of the singular forward point $x = 0$ of the flow and not a regular cross-section, prescription of a velocity profile will be replaced by the prescribed behavior of the stream function at the singular point $x = 0, y = 0$.

We introduce the following preliminary dimensionless independent and dependent variables:

$$(6) \quad \begin{aligned} \xi &= \frac{1}{\nu} \int_0^x U(x) dx \\ \eta &= \frac{1}{\nu} U(x)y \end{aligned}$$

$$(7) \quad \begin{aligned} \hat{\psi}(\xi, \eta) &= \frac{1}{\nu} \psi(x, y) \\ \hat{u}(\xi, \eta) &= \hat{\psi}_\eta = \frac{u(x, y)}{U(x)} \\ \hat{v}(\xi, \eta) &= -\hat{\psi}_\xi = \frac{\{U(x)v(x, y) + U'(x)yu(x, y)\}}{U^2(x)}. \end{aligned}$$

As it is assumed that $U(x) > 0$ for $x > 0$, the variable ξ is a monotone increasing function of x . The transformation (6) is a (1—1) mapping when $U(x) > 0$. Because of its physical meaning, $U(x)$ vanishes at most at the singular point $x = 0$ (namely, in those cases when this point is a stagnation point).

Inverting (7) yields

$$(8) \quad \begin{aligned} \frac{u(x, y)}{U(x)} &= \hat{u}(\xi, \eta) \\ \frac{v(x, y)}{U(x)} &= \hat{v}(\xi, \eta) - \hat{\beta}(\xi) \eta \hat{u}(\xi, \eta), \end{aligned}$$

where

$$(9) \quad \hat{\beta}(\xi) = \frac{\nu U'(x)}{U^2(x)}.$$

With the new variables the boundary layer equation (3) and the boundary conditions (5) read:

$$(10) \quad \hat{\psi}_\eta \hat{\psi}_{\xi\eta} - \hat{\psi}_\xi \hat{\psi}_{\eta\eta} = \hat{\beta}(\xi)(1 - \hat{\psi}_\eta^2) + \hat{\psi}_{\eta\eta\eta}$$

$$(11) \quad \begin{aligned} \hat{\psi}(\xi, \eta) = \hat{\psi}_\eta(\xi, \eta) &= 0 \quad \text{for} \quad \eta = 0 \\ \hat{\psi}_\eta(\xi, \eta) &\rightarrow 1 \quad \text{for} \quad \eta \rightarrow \infty. \end{aligned} \quad 0 \leq \xi \leq \xi_1 = \xi(x_1),$$

We now choose as final independent and dependent variables:

$$(12) \quad \begin{aligned} \xi &= \xi \\ \eta &= \frac{\hat{\eta}}{\sqrt{2\xi}} \end{aligned}$$

and

$$(13) \quad F(\xi, \eta) = \frac{\hat{\psi}(\xi, \hat{\eta})}{\sqrt{2\xi}}.$$

Expressing these directly in the original quantities, we have:

$$(14) \quad \begin{aligned} \xi &= \frac{1}{\nu} \int_0^x U(x) dx \\ \eta &= \frac{U(x)y}{\left\{2\nu \int_0^x U(x) dx\right\}^{\frac{1}{2}}} \end{aligned}$$

and

$$(15) \quad \begin{aligned} F(\xi, \eta) &= \frac{\psi(x, y)}{\left\{2\nu \int_0^x U(x) dx\right\}^{\frac{1}{2}}} \\ (F(\xi, 0) &= 0 \quad \text{for } 0 < \xi \leq \xi_1). \end{aligned}$$

Furthermore, we put

$$(16) \quad \beta(\xi) = 2\xi\hat{\beta}(\xi).$$

Then we have

$$(17) \quad \begin{aligned} \psi(x, y) &= \nu\sqrt{2\xi} F(\xi, \eta) \\ \frac{u(x, y)}{U(x)} &= F_\eta(\xi, \eta) \\ \frac{v(x, y)}{U(x)} &= -\frac{1}{\sqrt{2\xi}} \{F(\xi, \eta) + 2\xi F_\xi(\xi, \eta) + (\beta(\xi) - 1)\eta F_\eta(\xi, \eta)\}. \end{aligned}$$

According to (10), (11) and (12), (13), the boundary layer equation (3) and the boundary conditions (5) finally read:

$$(18) \quad F_{\eta\eta\eta} + FF_{\eta\eta} + \beta(\xi)\{1 - F_\eta^2\} = 2\xi\{F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta}\},$$

$$(19) \quad \begin{aligned} F(\xi, 0) &= F_\eta(\xi, 0) = 0 \\ \lim_{\eta \rightarrow \infty} F_\eta(\xi, \eta) &= 1. \end{aligned} \quad (0 < \xi \leq \xi_1)$$

We mention now that the data of any special problem appear in (18) and (19) only in the function $\beta(\xi)$. We have

$$(20) \quad \beta(\xi) = 2\xi \frac{d}{d\xi} \ln \left(\frac{U(x)}{U_0} \right),$$

where, on the right hand side, x is to be understood as the function of ξ defined uniquely by (14)₁. The constant velocity U_0 is introduced here only for

dimensional correctness and can be chosen as the unit of velocity or in any other way suitable. $\beta(\xi)$ is, for any given outer velocity distribution, a single-valued function of x :

$$(21) \quad \beta(\xi) = \frac{2U'(x) \int_0^x U(x) dx}{U^2(x)}$$

or

$$(22) \quad \beta(\xi) = \frac{2\xi(x)\xi''(x)}{\xi'^2(x)}.$$

This function plays an important role in the analysis and will be called the *principal function* of the boundary layer problem. It is only in the form of $\beta(\xi)$ that the boundary layer problems (18), (19) differ from case to case. (If two problems are similar in the sense of REYNOLDS similarity they have the same principal function $\beta(\xi)$. (Cf. §3.))

Apart from the local value of the outer velocity and the outer velocity gradient at a position x , the history of the flow upstream from this cross-section in $\int_0^x U(x)dx$ determines the form of the principal function $\beta(\xi)$. In what follows, the significance of the principal function will be demonstrated from various points of view. (The function $\hat{\beta}(\xi)$ plays the corresponding role for (10), (11).)

Boundary layer theory sometimes employs *shape factors* having the form $\delta^2 U'(x)/\nu$, where δ is a measure for the thickness of the boundary layer. For this latter, the *displacement thickness* δ^* and the *momentum thickness* ϑ recommend themselves. In terms of our independent variables ξ, η we obtain:

$$(23) \quad \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \frac{\left\{2\nu \int_0^x U dx\right\}^{\frac{1}{2}}}{U(x)} \eta_0(x) \quad \text{with} \quad \eta_0(x) = \lim_{\eta \rightarrow \infty} \{\eta - F(\xi, \eta)\}$$

$$\vartheta = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy = \frac{\left\{2\nu \int_0^x U dx\right\}^{\frac{1}{2}}}{U(x)} \int_0^\infty F_\eta (1 - F_\eta) d\eta.$$

If we use the displacement thickness δ^* for the shape factor mentioned above, we obtain

$$(24) \quad \beta(\xi) = \frac{1}{\eta_0^2(x)} \frac{\delta^{*2} U'(x)}{\nu}.$$

The shape factor

$$\frac{1}{\nu} \delta^{*2} U' = \frac{-\frac{dp}{dx}}{\frac{\mu U}{\delta^*} / \delta^*}$$

can be interpreted as ratio of the pressure forces to the viscous forces.

§3. Some useful relations.

According to (18) and because of $F = F_\eta = 0$ for $\eta = 0$, we have

$$(25) \quad F_{\eta\eta}(\xi, 0) \equiv -\beta(\xi).$$

Likewise, after differentiation of (18) with respect to η , we find

$$(26) \quad F_{\eta\eta\eta}(\xi, 0) \equiv 0.$$

Adopting the usual terms, we call (25) and (26) the first and second *conditions of compatibility at the wall* (in German: "Grenzschichtbindung"). Further conditions of compatibility at the wall can be obtained by differentiating (18) repeatedly with respect to η , then putting $\eta = 0$.

For the local shearing stress at the wall, defined by

$$(27) \quad \tau(x) = \mu u_y(x, 0),$$

where $\mu = \rho\nu$ is the absolute viscosity, ρ being the density, we have

$$(28) \quad \frac{\tau(x)}{\rho U_\infty^2} = \frac{U^2(x)}{U_\infty^2} \frac{1}{\sqrt{2\xi}} F_{\eta\eta}(\xi, 0).$$

Here U_∞ is a constant velocity and will usually denote the free stream velocity at infinity.

Normally $U(x)$ will be given. Then, to write out the differential equation (18) one must first calculate the principal function $\beta(\xi)$ as a function of ξ by using (21) and (14)₁. In §4 we will consider the practical evaluation of $\beta(\xi)$. Because of the important role of the principal function $\beta(\xi)$ it is of interest to consider also the inverse problem, to calculate for a given principal function $\beta(\xi)$ the corresponding outer velocity $U(x)$. We may restrict ourselves to functions $\beta(\xi)$ single-valued and continuous in $0 \leq \xi \leq \xi_1$. (In (21) $U(x)$ is positive for positive values of x and may there be considered as a single-valued continuous function with a continuous first derivative. At the forward point $x = 0$, we may assume, with no loss of generality for practical cases, that $U(x)$ behaves like x^m . Then, according to (21), the principal function exists at $\xi = 0$ if $m \neq -1$.)

Now, according to (14),

$$(29) \quad \nu \xi'(x) = U(x).$$

Defining for any x_0 in $0 \leq x_0 \leq x_1$

$$(30) \quad \xi(x_0) = \xi_0, \quad U(x_0) = U_0,$$

we have

$$(31) \quad \nu \xi'(x) = U_0 g(\xi),$$

where

$$(32) \quad g(\xi) = \exp \left\{ \frac{1}{2} \int_{\xi_0}^{\xi} \frac{\beta(\xi)}{\xi} d\xi \right\}$$

for any ξ_0 , ξ in $0 \leq \xi_0, \xi \leq \xi_1$. (Since $|\beta(0)| < \infty$, the function $g(\xi)$ exists when $\xi_0 = 0$.) We therefore have

$$(33) \quad U(x(\xi)) = U_0 g(\xi).$$

For the case of a given function $\beta(\xi)$ we still need the relation $\xi(x)$ which leads to the independent variable x in (33). Now, from (31) we find

$$(34) \quad x(\xi) - x_0 = \frac{\nu}{U_0} \int_{\xi_0}^{\xi} \frac{d\xi}{g(\xi)}$$

as a known function if $\beta(\xi)$ is given. From this, the inverse function can be determined. In §4 we will consider its practical evaluation. The correspondence $\xi_0 = \xi(x_0)$ remains free. With $\xi(x)$ available one can determine $U(x)$ by substituting in (33), but it may be easier to use (29) directly.

We have introduced the auxiliary function $g(\xi)$ not only for this purpose but also because it will prove useful in applications in other stages of calculation.

For investigations based on a given principal function we note:

$$(35) \quad \begin{aligned} u(x, y) &= U_0 g(\xi) F_\eta(\xi, \eta) \\ x - x_0 &= \frac{\nu}{U_0} \int_{\xi_0}^{\xi} \frac{d\xi}{g(\xi)} \\ y &= \frac{\nu}{U_0} \frac{\sqrt{2\xi}}{g(\xi)} \eta. \end{aligned}$$

For a given principal function $\beta(\xi)$ as a function of ξ , the quantities $u(x, y)/U_0$, $(x - x_0)U_0/\nu$ and yU_0/ν are uniquely determined according to (35). This means that for a given principal function the quantities of the flow plane are determined only to within REYNOLDS similitude.

As already stated above, in normal cases of application the outer velocity $U(x)$ is a given function of x . In cases in which $U(x)$ is found by measuring the pressure distribution along the wall, the corresponding distribution of $U(x)$ is usually approximated by a polynomial. If $U(x)$ is calculated on the basis of potential flow around the body in question, it will usually be available as an infinite series. For such cases, in the following section we give formulæ sufficient for calculating the corresponding principal function and also the inverse formulæ.

§4. Series expansions of the outer velocity $U(x)$ and of the principal function $\beta(\xi)$, and their relations.

Let us first assume that the outer velocity $U(x)$ is an analytic function at $x = 0$ and that the power series

$$(36) \quad U(x) = \sum_{k=0}^{\infty} u_k x^k$$

converges for all values of x of interest here ($0 \leq x \leq x_1$).

In the case of flow past a contour such that the point $x = 0$, where the oncoming flow is divided by the contour, is a cuspidal point, we have

$$(36A) \quad U(x) = u_0 + u_1x + u_2x^2 + u_3x^3 + \cdots \quad \text{with } u_0 \neq 0.$$

This class of outer flows includes as special cases flows along flat plates with arbitrary pressure distribution.

If the point $x = 0$ of the contour is rounded, we have a stagnation point at $x = 0$, in the vicinity of which $U(x)$ is proportional to x . In such cases

$$(36B) \quad U(x) = u_1x + u_2x^2 + u_3x^3 + \cdots \quad \text{with } u_1 \neq 0.$$

In this class of flows, those special cases in which $U(x)$ is an antisymmetric function of x will be of particular interest, for it will be possible to treat these cases more simply than the general case (36B), to be considered later. In applications functions $U(x)$ with $u_{2k} = 0$ for $k = 0, 1, 2, \dots$ usually correspond to problems in which a parallel flow with constant velocity coming from infinity flows around a body symmetric to the direction of the flow at infinity. We therefore call the case $u_{2k} = 0$, $k = 0, 1, 2, \dots$ the *symmetric case*.

The two very extensive classes of boundary layer problems related to outer velocities of the form (36A) or (36B) are of main interest in this memoir and will be investigated thoroughly, but also we shall examine certain generalizations. These extensions will permit study of flows in which the contour has a salient point at $x = 0$ and is therefore of wedge-like form in the vicinity of the forward point. The two cases (36A) and (36B) will then be special cases in which the wedge angle at the forward edge is 0° or 180° , respectively.

Case A: $u_0 \neq 0$

If we substitute (36A) in equation (14) defining ξ we find

$$(37) \quad \xi(x) = \frac{1}{\nu} \sum_{k=0}^{\infty} \frac{u_k x^{k+1}}{k+1}.$$

Since $u_0 \neq 0$, the theorem on reversion of power series guarantees for $x(\xi)$ an expansion of the form

$$(38) \quad x(\xi) = c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots.$$

By introducing (36A) into (21) we obtain

$$(39) \quad \begin{aligned} \beta(\xi(x)) &= 2 \frac{u_0 u_1 x + (2u_0 u_2 + \frac{1}{2}u_1^2)x^2 + \cdots}{u_0^2 + 2u_0 u_1 x + \cdots} \\ &= 2 \frac{u_1}{u_0} x + \cdots. \end{aligned}$$

Taking (38) into account, according to the WEIERSTRASS double series theorem we obtain for the principal function $\beta(\xi)$ a power series of the form

$$\begin{aligned}
 \beta(\xi) &= \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3 + \cdots \\
 (40) \qquad &= \sum_{k=0}^{\infty} \beta_k \xi^k \quad \text{with} \quad \beta_0 = 0.
 \end{aligned}$$

With this insight the calculation of $\beta(\xi)$ is reduced to the calculation of the coefficients β_k ($k = 1, 2, 3, \dots$) and this, in turn, can easily be done, at least in principle. Writing the equation (21) for $\beta(\xi)$ as

$$(41) \qquad 2U'(x) \int_0^x U(x) dx = U^2(x) \sum_{k=1}^{\infty} \beta_k \left\{ \frac{1}{\nu} \int_0^x U(x) dx \right\}^k,$$

we introduce $U(x) = \sum_{k=0}^{\infty} u_k x^k$ into this equation and compare coefficients on the left and right hand sides, thus obtaining a recursive system of equations for $\beta_1, \beta_2, \beta_3, \dots$.

To solve the inverse problem (given $\beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k$ with $\beta_0 = 0$, determine $U(x) = \sum_{k=0}^{\infty} u_k x^k$ with $u_0 \neq 0$) the easiest way is now to use the expressions just found for β_k as function of $u_0, u_1, \dots, u_k$ and solve for the u_k . This proves to be very simple.

For these calculations it is useful to introduce the abbreviations

$$(42) \qquad a_j = \frac{\nu^j u_j}{u_0^{j+1}} \qquad (j = 1, 2, 3, \dots).$$

The coefficient β_k is then a polynomial in $a_1, a_2, \dots, a_k$ with the general form:

$$(43) \qquad \beta_k = \sum_i B_i a_1^{c_{i1}} a_2^{c_{i2}} \cdots a_k^{c_{ik}} \quad \text{with} \quad c_{i1} + 2c_{i2} + \cdots + kc_{ik} = k,$$

where the B_i are universal constants.

For future applications we give these formulæ for the orders $k \leq 5$. In principle, there is no difficulty in continuing up to any order; however, the formulæ will soon become rather complicated. For the explicit applications to be described in Part II we need only the coefficients here exhibited:

$$\begin{aligned}
 \beta_0 &= 0 \\
 \beta_1 &= 2a_1 \\
 (44) \quad \beta_2 &= 4\{-a_1^2 + a_2\} \\
 \beta_3 &= 2\{4a_1^3 - 7a_1a_2 + 3a_3\} \\
 \beta_4 &= \frac{1}{3}\{-48a_1^4 + 118a_1^2a_2 - 66a_1a_3 - 28a_2^2 + 24a_4\} \\
 \beta_5 &= \frac{1}{6}\{192a_1^5 - 605a_1^3a_2 + 375a_1^2a_3 + 320a_1a_2^2 - 192a_1a_4 - 150a_2a_3 + 60a_5\}.
 \end{aligned}$$

For the inverse problem, where $\beta(\xi)$ with $\beta(0) = \beta_0 = 0$ is given and $U(x)$ with $U(0) = u_0 \neq 0$ is to be determined, we know, according to the remark following (35), that the quantities of the flow plane can be determined only up to REYNOLDS similarity. One obtains uniquely the dimensionless auxiliary coefficients a_j , which, according to definition (42), are

$$(45) \quad a_i = \frac{u_i L^i}{u_0} \text{Re}^{-i} \quad \text{with} \quad \text{Re} = \frac{u_0 L}{\nu},$$

L being a characteristic length for the problem in question.

By solving the equations (44) recursively for $a_1, a_2, \dots$ one finds:

$$(46) \quad \begin{aligned} a_1 &= \frac{1}{2}\beta_1 \\ a_2 &= \frac{1}{4}\{\beta_1^2 + \beta_2\} \\ a_3 &= \frac{1}{24}\{3\beta_1^3 + 7\beta_1\beta_2 + 4\beta_3\} \\ a_4 &= \frac{1}{96}\{6\beta_1^4 + 23\beta_1^2\beta_2 + 22\beta_1\beta_3 + 7\beta_2^2 + 12\beta_4\} \\ a_5 &= \frac{1}{960}\{30\beta_1^5 + 163\beta_1^3\beta_2 + 202\beta_1^2\beta_3 + 127\beta_1\beta_2^2 \\ &\quad + 192\beta_1\beta_4 + 100\beta_2\beta_3 + 96\beta_5\}. \end{aligned}$$

From these values of the auxiliary coefficients a_i the coefficients u_i of the outer velocity $U(x)$ are found as

$$(47) \quad u_0 \neq 0 \text{ arbitrary}, \quad u_i = \frac{u_0 a_i \text{Re}^i}{L^i} \quad \left(\text{Re} = \frac{u_0 L}{\nu} \right).$$

If one uses dimensionless quantities in the flow plane, equation (45) suggests that $u_0 = U(0)$ should be introduced as characteristic reference velocity. If one does this from the beginning, one does not need the somewhat clumsy auxiliary coefficients a_i . Accordingly, in order to facilitate the application of our formulæ the following remarks will be of use.

We set

$$(48) \quad \begin{aligned} \bar{x} &= \frac{x}{L} \\ \bar{U}(\bar{x}) &= \frac{U(x)}{u_0} \\ \bar{u}_k &= \frac{u_k L^k}{u_0} \quad (k = 0, 1, 2, \dots), \end{aligned}$$

where the definition of $\bar{u}_k$ suggests itself because $\bar{u}_k \bar{x}^k = u_k x^k$. (We note that $\bar{u}_0 = 1$ and, furthermore, that $\bar{u}_k = a_k \text{Re}^k$.) It is then also practical to set

$$(49) \quad \begin{aligned} \bar{\xi} &= \frac{\xi}{\text{Re}} \quad \left(\text{Re} = \frac{u_0 L}{\nu} \right) \\ \bar{\beta}_k &= \beta_k \text{Re}^k \quad (k = 0, 1, 2, \dots) \end{aligned}$$

$(\bar{\beta}_k \bar{\xi}^k = \beta_k \xi^k)$. From the formulæ (43), viz.

$$(43) \quad \beta_k = \sum_i B_i a_1^{c_{i1}} a_2^{c_{i2}} \dots a_k^{c_{ik}},$$

it follows that

$$(50) \quad \bar{\beta}_k = \sum_i B_i \bar{u}_1^{c_{i1}} \bar{u}_2^{c_{i2}} \cdots \bar{u}_k^{c_{ik}}$$

with the same coefficients B_i and exponents c_{ij} as in (43). *Therefore we note for future applications: The formulæ (44) and (46) remain valid if β_k is replaced by $\bar{\beta}_k$ and, at the same time, a_k is replaced by $\bar{u}_k$.* In our applications in Part III we shall use the barred dimensionless quantities.

It can easily be seen that the extension of the formulæ (44) or (46) to higher orders soon becomes complicated. Even the next formula, that for β_6 , has 11 terms:

$$(51) \quad \beta_6 = \frac{1}{3^6} \{ -1920a_1^6 + 7365a_1^4a_2 - 4815a_1^3a_3 - 6190a_1^2a_2^2 + 2844a_1^2a_4 \\ + 4470a_1a_2a_3 - 1320a_1a_5 + 640a_2^3 - 984a_2a_4 - 450a_3^2 + 360a_6 \}.$$

Case B: $u_0 = 0$, $u_1 \neq 0$

In this case of flows around contours with forward stagnation point $x = 0$ the series

$$(52) \quad \xi(x) = \frac{1}{\nu} \sum_{k=1}^{\infty} \frac{u_k x^{k+1}}{k+1}$$

begins with the quadratic term. From this we have for the inverse function $x(\xi)$ a series in general of the form

$$(53) \quad x(\xi) = c_3 \xi^{\frac{1}{3}} + c_1 \xi + c_3 \xi^{\frac{2}{3}} + \cdots$$

(Because of the monotone increase of x with ξ , only one of the two possible series can be valid; the two series differ in the sign of the terms with non-integral exponents.) Accordingly, for the principal function $\beta(\xi)$ we obtain an expansion of the following kind:

$$(54) \quad \beta(\xi) = \sum_{k=0}^{\infty} \beta_{\frac{1}{3}k} \xi^{\frac{1}{3}k}.$$

In the symmetrical case, where $u_{2k} = 0$ ($k = 0, 1, 2, \dots$),

$$(55) \quad U(x) = \sum_{k=0}^{\infty} u_{2k+1} x^{2k+1}.$$

Since $u_1 \neq 0$ we now have:

$$(56) \quad \beta(\xi(x)) = \frac{u_1^2 x^2 + \cdots}{u_1^2 x^2 + \cdots} = 1 + d_2 x^2 + d_4 x^4 + \cdots$$

With

$$(57) \quad \nu \xi(x) = \frac{1}{2} u_1 x^2 + \frac{1}{4} u_3 x^4 + \cdots$$

the inverse function has the form

$$(58) \quad x(\xi) = \sqrt{\xi} \{c_1 + c_3\xi + \dots\}.$$

Substituting this in (56) leads to a power series for the principal function

$$(59) \quad \begin{aligned} \beta(\xi) &= 1 + \beta_1\xi + \beta_2\xi^2 + \dots \\ &= \sum_{k=0}^{\infty} \beta_k \xi^k \quad \text{with} \quad \beta_0 = 1. \end{aligned}$$

We therefore consider the symmetrical case first.

Special Case B1: $u_{2k} = 0$ ($k = 0, 1, 2, \dots$), $u_1 \neq 0$

The simple procedure demonstrated in Case A for the calculation of $\beta(\xi)$ for given $U(x)$ or vice versa will now be applied again. Since $u_0 = 0$, $u_1 \neq 0$, instead of the auxiliary coefficients a_i of Case A we introduce

$$(60) \quad b_j = \frac{v^j u_{2j+1}}{u_1^{j+1}} \quad (j = 1, 2, 3, \dots).$$

Then we obtain the following system of equations for the coefficients β_k for orders $k = 0, 1, 2, \dots, 5$:

$$(61) \quad \begin{aligned} \beta_0 &= 1 \\ \beta_1 &= 3b_1 \\ \beta_2 &= -13b_1^2 + \frac{40}{3}b_2 \\ \beta_3 &= 54b_1^3 - 96b_1b_2 + 42b_3 \\ \beta_4 &= -221b_1^4 + \frac{1648}{3}b_1^2b_2 - 312b_1b_3 - \frac{1184}{9}b_2^2 + \frac{576}{5}b_4 \\ \beta_5 &= 898b_1^5 - \frac{8560}{3}b_1^3b_2 + 1790b_1^2b_3 + 1520b_1b_2^2 \\ &\quad - 928b_1b_4 - 720b_2b_3 + \frac{880}{3}b_5. \end{aligned}$$

When these are reverted for the case of a given principal function, we find

$$(62) \quad \begin{aligned} b_1 &= \frac{1}{3}\beta_1 \\ b_2 &= \frac{1}{120} \{13\beta_1^2 + 9\beta_2\} \\ b_3 &= \frac{1}{630} \{22\beta_1^3 + 36\beta_1\beta_2 + 15\beta_3\} \end{aligned}$$

$$b_4 = \frac{1}{362880} \{4069\beta_1^4 + 11034\beta_1^2\beta_2 + 7800\beta_1\beta_3 + 2331\beta_2^2 + 3150\beta_4\}$$

$$b_5 = \frac{1}{4989600} \{17926\beta_1^5 + 69336\beta_1^3\beta_2 + 64140\beta_1^2\beta_3 + 37809\beta_1\beta_2^2$$

$$+ 45675\beta_1\beta_4 + 21870\beta_2\beta_3 + 17010\beta_5\}.$$

From these auxiliary coefficients b_i one obtains the coefficients of $U(x)$ as

$$(63) \quad u_1 \neq 0, \text{ arbitrary}, \quad u_{2k+1} = \frac{u_1 L b_k \text{Re}^k}{L^{2k+1}} \quad \left(\text{Re} = \frac{u_1 L^2}{\nu} \right).$$

When dimensionless quantities are introduced in the flow plane, equation (63) suggests that again any characteristic length L may be chosen as reference length, but that it is then advisable to take $u_1 L = U'(0)L$ as reference velocity. We therefore now define

$$\bar{x} = \frac{x}{L}$$

$$(64) \quad \bar{U}(\bar{x}) = \frac{U(x)}{u_1 L}$$

$$\bar{u}_{2k+1} = \frac{u_{2k+1} L^{2k+1}}{u_1 L} \quad (k = 0, 1, 2, \dots)$$

and

$$(65) \quad \bar{\xi} = \frac{\xi}{\text{Re}} \quad \left(\text{Re} = \frac{u_1 L^2}{\nu} \right)$$

$$\bar{\beta}_k = \beta_k \text{Re}^k.$$

Again we note for the purpose of future applications in Part III: The formulæ (61) and (62) remain valid if we replace the coefficients β_k by $\bar{\beta}_k$ and, at the same time, b_k by $\bar{u}_{2k+1}$.

General Case B: $u_0 = 0, u_1 \neq 0$

If we assume only that $u_0 = 0, u_1 \neq 0$, we have for the principal function a series of the form (54). Again the calculation procedure for the coefficients is as in Case A. It is now advisable to define as auxiliary coefficients

$$(66) \quad c_{\frac{1}{2}j} = \nu^{\frac{1}{2}j} \frac{u_{j+1}}{u_1^{\frac{1}{2}(j+2)}} \quad (j = 1, 2, 3, \dots).$$

The generalization, when compared with (60), is obvious. As expressions in these auxiliary coefficients we obtain the following formulæ for the coefficients $\beta_{\frac{1}{2}k}$ for the orders $k = 0, 1, 2, 3, 4$ needed in Part III of this memoir:

$$\begin{aligned}
(67) \quad & \beta_0 = 1 \\
& \beta_{\frac{1}{2}} = \frac{2\sqrt{2}}{3} c_{\frac{1}{2}} \\
& \beta_1 = -\frac{22}{9} c_{\frac{1}{2}}^2 + 3c_1 \\
& \beta_{\frac{3}{2}} = 2\sqrt{2} \left(\frac{59}{27} c_{\frac{1}{2}}^3 - \frac{27}{6} c_{\frac{1}{2}} c_1 + \frac{12}{5} c_{\frac{1}{2}} \right) \\
& \beta_2 = -\frac{1252}{81} c_{\frac{1}{2}}^4 + 44c_{\frac{1}{2}}^2 c_1 - \frac{144}{5} c_{\frac{1}{2}} c_{\frac{1}{2}} - 13c_1^2 + \frac{40}{3} c_2.
\end{aligned}$$

The inverse formulæ are:

$$\begin{aligned}
(68) \quad & c_{\frac{1}{2}} = \frac{3}{2\sqrt{2}} \beta_{\frac{1}{2}} \\
& c_1 = \frac{1}{12} \{11\beta_{\frac{1}{2}}^2 + 4\beta_1\} \\
& c_{\frac{3}{2}} = \frac{1}{48\sqrt{2}} \{50\beta_{\frac{1}{2}}^3 + 45\beta_{\frac{1}{2}}\beta_1 + 10\beta_{\frac{3}{2}}\} \\
& c_2 = \frac{1}{480} \{274\beta_{\frac{1}{2}}^4 + 421\beta_{\frac{1}{2}}^2\beta_1 + 162\beta_{\frac{1}{2}}\beta_{\frac{3}{2}} + 52\beta_1^2 + 36\beta_2\}.
\end{aligned}$$

From these auxiliary coefficients we obtain the coefficients of $U(x)$ as follows:

$$\begin{aligned}
(69) \quad & u_0 = 0, \quad u_1 \neq 0 \text{ arbitrary}, \quad u_{i+1} = u_1 L c_{\frac{1}{2}i} \frac{\text{Re}^{\frac{1}{2}i}}{L^{i+1}} \\
& \left(j = 1, 2, 3, \dots, \text{Re} = \frac{u_1 L^2}{\nu} \right).
\end{aligned}$$

This is in accordance with (63) for the special Case B1. In the general Case B, as in Case B1, it is also advisable to introduce dimensionless quantities according to (64) and (65), now taken as holding for $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. Again we can state: *The formulæ (67) and (68) remain valid if one replaces $\beta_{\frac{1}{2}k}$ by $\bar{\beta}_{\frac{1}{2}k}$ and, at the same time, $c_{\frac{1}{2}k}$ by $\bar{u}_{k+1}$ ($k = 1, 2, 3, \dots$).*

Generalization of the problem.

Thus far we have found the following correspondences of outer velocity $U(x)$ and principal function $\beta(\xi)$:

$$\begin{aligned}
\text{Case A:} \quad & U(x) = \sum_{k=0}^{\infty} u_k x^k \quad \text{with} \quad u_0 \neq 0 \\
& \beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k \quad \text{with} \quad \underline{\beta_0 = 0}
\end{aligned}$$

(Flow around profiles with cuspidal point $x = 0$ as leading edge, including plates at zero incidence with arbitrary outer velocity distribution.)

$$\text{Case B1:} \quad U(x) = \sum_{k=0}^{\infty} u_k x^k \quad \text{with} \quad u_1 \neq 0, \quad u_{2k} = 0 \quad (k = 1, 2, \dots)$$

$$\beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k \quad \text{with} \quad \underline{\beta_0 = 1}$$

$$\text{Case B:} \quad U(x) = \sum_{k=0}^{\infty} u_k x^k \quad \text{with} \quad u_1 \neq 0, \quad u_0 = 0$$

$$\beta(\xi) = \sum_{k=0}^{\infty} \beta_{\frac{1}{2}k} \xi^{\frac{1}{2}k} \quad \text{with} \quad \underline{\beta_0 = 1}$$

(Flow around profiles with rounded nose; forward line $x = 0$, stagnation line. In Case B1, symmetrical flows.)

The following questions arise: To which class of outer velocity distributions do principal functions of the form

$$(I) \quad \beta(\xi) = \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \dots \quad \text{with any value of } \beta_0$$

correspond? More generally, to which class of outer velocity distributions do principal functions of the form

$$(II) \quad \beta(\xi) = \beta_0 + \beta_{\frac{1}{2}} \xi^{\frac{1}{2}} + \beta_1 \xi + \dots \quad \text{with any } \beta_0$$

correspond?

These questions are of interest in connection with our new series for the calculation of laminar boundary layer flows. *It will be shown in Part II of this paper that the new series can be applied not only to the large classes of boundary layer problems given by Cases A and B above but also to the far more extensive class of boundary layer flows that lead to a principal function of the type (I) or (II).* It is possible to give still further extensions, but this does not lie within the scope of this memoir. On the other hand, there is no simplification of the theory of the new series when the values of β_0 in (I) and (II) are restricted to $\beta_0 = 0$ or $\beta_0 = 1$ as in Cases A and B. Therefore there would be no advantage in disregarding the generalizations (I) and (II) of the Cases A and B considered above.

The answers to both questions can easily be given, since all required relations are already at our disposal.¹ First of all, we have $U(x(\xi))$ according to equation (33) and here we must substitute the principal function according to (I) or (II). After arranging the results in ascending powers of ξ we obtain the following expressions:

¹ The calculations, the course of which I shall outline briefly, have been carried out in detail by Dr. G. HÄMMERLIN. I wish to thank him for his cooperation.

$$\text{Class I:} \quad U(x(\xi)) = U_0 \left(\frac{\xi}{\xi_0} \right)^{\frac{1}{3}\beta_0} \{q_0 + q_1\xi + q_2\xi^2 + \dots\}$$

$$\text{Class II:} \quad U(x(\xi)) = U_0 \left(\frac{\xi}{\xi_0} \right)^{\frac{1}{3}\beta_0} \{q_0 + q_{\frac{1}{3}}\xi^{\frac{1}{3}} + q_1\xi + q_{\frac{2}{3}}\xi^{\frac{2}{3}} + \dots\}.$$

Correspondingly we find $x(\xi)$ according to (34) to be

$$\text{Class I:} \quad x(\xi) = \xi^{1-\frac{1}{3}\beta_0} \{r_0 + r_1\xi + r_2\xi^2 + \dots\}$$

$$\text{Class II:} \quad x(\xi) = \xi^{1-\frac{1}{3}\beta_0} \{r_0 + r_{\frac{1}{3}}\xi^{\frac{1}{3}} + r_1\xi + r_{\frac{2}{3}}\xi^{\frac{2}{3}} + \dots\}.$$

If we restrict attention to rational values of β_0 , the inversion theorem gives us $\xi(x)$. If this is substituted in the expressions given above for $U(x(\xi))$, we find the following answers to our questions:

The outer velocity distributions corresponding to principal functions of Class I are:

$$\begin{aligned} U(x) &= x^m \{s_0 + s_1 x^{m+1} + s_2 x^{2(m+1)} + s_3 x^{3(m+1)} + \dots\} \\ (70) \quad &\text{with } m = \frac{\beta_0}{2 - \beta_0} \quad \left(m + 1 = \frac{2}{2 - \beta_0} \right) \\ &\text{or } \beta_0 = \frac{2m}{m + 1}, \quad m \neq -1, \quad s_0 \neq 0. \end{aligned}$$

Restriction to series with ascending exponents means restriction to $m > -1$ ($-\infty < \beta_0 < 2$). In our applications, in general, we shall restrict ourselves to $0 \leq \beta_0 \leq 1$ ($0 \leq m \leq 1$).

In Case A, $\beta_0 = 0$, therefore $m = 0$; in Case B1, $\beta_0 = 1$, therefore $m = 1$. A further example may be mentioned here: For $\beta_0 = \frac{2}{3}$ we have $m = \frac{1}{2}$ and therefore $U(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} s_k x^{\frac{3}{2}k}$.

For integral values of $m \geq 0$, that is for $\beta_0 = 0, 1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \dots$, and without any restrictions on the remaining coefficients β_k ($k = 1, 2, \dots$), we always have an ordinary power series for $U(x)$.

For $0 \leq m \leq 1$ ($0 \leq \beta_0 \leq 1$), we have outer velocities $U(x)$ the behavior of which, for x sufficiently near $x = 0$, corresponds to that of the well known wedge flows. Therefore we have flows around contours with a vertex at the forward point $x = 0$. The wedge angle is known to be $\beta_0\pi$. In Case A ($\beta_0 = 0$) the point $x = 0$ is a cuspidal point; in Case B1 ($\beta_0 = 1$) the contour has a rounded nose at $x = 0$. For $0 < \beta_0 \leq 1$ the point $x = 0$ is a stagnation point ($0 < m \leq 1$).

If $s_k = 0$ ($k = 1, 2, 3, \dots$), $s_0 \neq 0$, we have the special case $U(x) = s_0 x^m$ leading to the well known similar solutions (wedge flows). We return to this case in the following section, where we show that this case corresponds to a constant principal function $\beta(\xi) = \beta_0$.

We see from the statements above that the theory of the new series (Part II) for principal functions of Class I will be concerned with a very extensive class of boundary layer problems. This class is involved in the more general

Class II of principal functions. For the latter class the answer to our question can be found accordingly:

The outer velocities corresponding to principal functions of the more general Class II are

$$(71) \quad U(x) = x^m \{s_0 + s_{\frac{1}{2}} x^{\frac{1}{2}(m+1)} + s_1 x^{m+1} + s_{\frac{3}{2}} x^{\frac{3}{2}(m+1)} + \dots\},$$

with $m = \frac{\beta_0}{2 - \beta_0}, \quad m \neq -1, \quad s_0 \neq 0.$

For $\beta_0 = 1$ ($m = 1$) we have the previous Case B. For $\beta_0 = 0$ we have $U(x) = u_0 + u_{\frac{1}{2}} x^{\frac{1}{2}} + u_1 x + u_{\frac{3}{2}} x^{\frac{3}{2}} + \dots$. For positive integral values of $\frac{1}{2}(m+1)$ —therefore only for $m = 1, 3, 5, \dots$, i.e. $\beta_0 = 1, \frac{3}{2}, \frac{5}{2}, \dots$, but without any restrictions on the remaining coefficients $\beta_{\frac{1}{2}k}$ ($k = 1, 2, 3, \dots$)—we have an ordinary power series in x for the outer velocity $U(x)$.

We content ourselves here with this formal examination of the outer velocity distributions possible within the classes of principal functions we are considering.

§5. Special case of the constant principal function $\beta(\xi) = \beta_0$ and similar solutions of the boundary layer equations.

With the transformation of §2 we were led to the following form of the boundary layer problem:

$$(18) \quad F_{\eta\eta\eta} + FF_{\eta\eta} + \beta(\xi)\{1 - F_{\eta}^2\} = 2\xi\{F_{\eta}F_{\xi\eta} - F_{\xi}F_{\eta\eta}\},$$

$$(19) \quad F(\xi, 0) = F_{\eta}(\xi, 0) = 0, \quad F_{\eta}(\xi, \eta) \rightarrow 1 \quad \text{for} \quad \eta \rightarrow \infty.$$

We now seek those special boundary layer flows in which F does not depend on ξ :

$$(72) \quad F(\xi, \eta) = F_0(\eta).$$

For such flows the differential equation (18) reduces to an ordinary differential equation; obviously, this equation has solutions satisfying the boundary conditions (19) only if

$$(73) \quad \beta(\xi) = \text{const} = \beta_0;$$

the principal function is constant. Our problem is therefore reduced to the boundary value problem

$$(74) \quad F_0''' + F_0 F_0'' + \beta_0(1 - F_0'^2) = 0,$$

$$F_0(0) = F_0'(0) = 0, \quad F_0'(\infty) = 1.$$

This, however, is the well known boundary value problem of similar solutions studied by FALKNER & SKAN [4], HARTREE [5] and many later authors. (Cf. W. MANGLER [6] for a complete discussion of these flows or, for a shorter treatment, the book of H. SCHLICHTING [7] that recently appeared in an English edition [8].)

Vice versa, it is easy to see that the case of a constant principal function, $\beta(\xi) = \beta_0$, is equivalent to the case of those outer velocities $U(x)$ which allow similar solutions, including the limiting case found by GOLDSTEIN [9]. We give a brief proof. We must show the following: From $\beta(\xi) = \beta_0$ it necessarily follows that $U(x) = s_0 x^m$ for $\beta_0 \neq 2$ and $U(x) = s_0 e^{cx}$ for $\beta_0 = 2$. (It should be mentioned that the case $m = -1$, to which no finite value of β_0 corresponds and for which there is, therefore, no solution of the FALKNER-SKAN equation (74), also leads to a possible similar solution, the flow in a convergent channel. This case is not included in our statement above.)

For $\beta(\xi) = \beta_0$, the function $g(\xi)$ introduced in (32) becomes

$$(75) \quad g(\xi) = \left(\frac{\xi}{\xi_0} \right)^{\frac{1}{2}\beta_0}.$$

Consequently we have, according to (33),

$$(76) \quad U(x(\xi)) = U_0 \left(\frac{\xi}{\xi_0} \right)^{\frac{1}{2}\beta_0} \quad (U_0 = U(x_0) = U(x(\xi_0))),$$

and according to (34),

$$(77) \quad \begin{aligned} x - x_0 &= \frac{\nu}{U_0} \int_{\xi_0}^{\xi} \left(\frac{\xi_0}{\xi} \right)^{\frac{1}{2}\beta_0} d\xi \\ &= \begin{cases} \frac{\nu \xi_0}{(1 - \frac{1}{2}\beta_0) U_0} \left\{ \left(\frac{\xi_0}{\xi} \right)^{\frac{1}{2}\beta_0 - 1} - 1 \right\} & \text{for } \beta_0 \neq 2 \\ \frac{\nu \xi_0}{U_0} \ln \left(\frac{\xi}{\xi_0} \right) & \text{for } \beta_0 = 2. \end{cases} \end{aligned}$$

By a suitable choice of the value x_0 , of no significance here, we can write this result as

$$(78) \quad x = \begin{cases} x_0 \left(\frac{\xi}{\xi_0} \right)^{\frac{1}{2}\beta_0 - 1} & \text{for } \beta_0 \neq 2 \\ x_0 \left\{ 1 + \ln \left(\frac{\xi}{\xi_0} \right) \right\} & \text{for } \beta_0 = 2. \end{cases}$$

The inverse function $\xi(x)$ is therefore

$$(79) \quad \frac{\xi(x)}{\xi_0} = \begin{cases} \left(\frac{x}{x_0} \right)^{2/(2-\beta_0)} & \text{for } \beta_0 \neq 2 \\ e^{(x-x_0)/x_0} & \text{for } \beta_0 = 2. \end{cases}$$

If we substitute this result into (76) we find, as was to be shown,

$$(80) \quad U(x) = \begin{cases} U_0 \left(\frac{x}{x_0} \right)^m & \text{with } m = \frac{\beta_0}{2 - \beta_0}, \text{ for } \beta_0 \neq 2 \\ U_0 e^{(x-x_0)/x_0} & \text{for } \beta_0 = 2. \end{cases}$$

Besides the special case $\beta(\xi) = \text{const.}$, there are many further special cases of the principal function $\beta(\xi)$ that may be worth examination, since, on the one hand, the corresponding distribution of the outer velocity $U(x)$ can be determined more easily than in the general case and, on the other hand, they allow a particularly easy calculation of the boundary layer flow, namely, as special cases of the wide class of principal functions for which the theory of our new series will be valid. *Vice versa*, there are many analytically simple outer velocity distributions $U(x)$ for which one can easily find the corresponding principal function in explicit form, whereupon calculation of the boundary layer flow by means of the new series will be immediately accessible. Here we cannot treat all these possibilities, but in Part III of this memoir we give examples.

§6. Boundary layers around cylinders in yaw.

Since, along with problems of plane boundary layer flows, those of three-dimensional boundary layer flows around yawed cylinders can be treated with relatively small additional calculation work, the theory of our new series will be extended to the corresponding problems of cylinders in yaw. For this purpose we now give the necessary equations.

By $w(x, y)$ we denote the additional component of the boundary layer velocity in the direction of the coordinate z parallel to the axis of the cylinder. Accordingly, we denote by $W_\infty (= \text{const.})$ the additional component of the outer flow around the infinitely long cylinder.

As is known, the boundary layer equations (1) and the boundary conditions (4) remain unaltered in the case of a yawed cylinder. On this fact is based the *independence principle* (PRANDTL, GÖRTLER, STRUMINSKY, R. T. JONES, SEARS; cf. the list of references in [10], to which the reference [11] should be added), according to which the components $u(x, y)$ and $v(x, y)$ are independent of $w(x, y)$ and can therefore be calculated as in the two-dimensional case. Subsequently, $w(x, y)$ must be calculated as solution of the linear boundary value problem

$$(81) \quad \begin{aligned} &uw_x + vw_y = vw_{yy} \\ &w(x, 0) = 0, \quad w(x, y) \rightarrow W_\infty \quad \text{for } y \rightarrow \infty. \end{aligned}$$

We define as dimensionless cross-flow component

$$(82) \quad W(\xi, \eta) = \frac{w(x, y)}{W_\infty},$$

at the same time introducing our variables (14). Transformation of (81) leads to

$$(83) \quad \begin{aligned} &W_{\eta\eta} + (F + 2\xi F_\xi)W_\eta = 2\xi F_\eta W_\xi, \\ &W(\xi, 0) = 0, \quad W(\xi, \eta) \rightarrow 1 \quad \text{for } \eta \rightarrow \infty. \end{aligned}$$

From (83), because of $F(\xi, 0) = F_\eta(\xi, 0) = 0$, we obtain for $\eta = 0$ the first condition of compatibility at the wall for the cross-flow component:

$$(84) \quad W_{\eta\eta}(\xi, 0) \equiv 0.$$

Differentiating (83)₁ with respect to η and subsequently taking $\eta = 0$ leads to the second condition of compatibility at the wall:

$$(85) \quad W_{\eta\eta\eta}(\xi, 0) \equiv 0.$$

In the special case of wedge flows (principal function $\beta(\xi) = \beta_0$, cf. §5), we have $F(\xi, \eta) = F_0(\eta)$ and, in order to obtain a similar boundary layer flow also in the case of a yawed cylinder, we must require that $W(\xi, \eta) = W_0(\eta)$ also be a function of η alone. Then (83) reduces to

$$(86) \quad W_0'' + F_0 W_0' = 0 \quad \text{with} \quad W_0(0) = 0, \quad W_0(\infty) = 1.$$

The solution can be found simply by quadrature, as is well known. J. C. COOKE [12] has tabulated $W_0(\eta)$ for a number of values of the FALKNER-SKAN parameter β_0 . For the case $\beta_0 = 1$ tables of $W_0(\eta)$ and $W_0'(\eta)$ have been given with improved accuracy in [10] (published also in the English edition [8] of SCHLICHTING'S book). The tables calculated in connection with the present research and described in §10 include the most accurate tables of $W_0(\eta)$ and its first and second derivatives for $\beta_0 = 1$ now in existence, along with a very accurate re-calculation of $F_0(\eta)$ and its first three derivatives for $\beta_0 = 0$ and $b_0 = 1$.

§7. Related transformations in the literature.

In concluding Part I, we give a brief survey of related transformations that we have found in studying other publications. Here, completeness is not our aim. The point is to show that our choice of variables ξ, η finally revealed itself to us as a logical continuation of former efforts in boundary layer calculation toward the greatest practicality.

In order to introduce coordinates more appropriate than the purely geometrical orthogonal coordinates used elsewhere, BOUSSINESQ, SOUTHWELL & SQUIRE, PIERCY & WINNY, PRESTON, and others (cf. the references in [13] and [14]) in papers on the velocity and temperature fields in viscous fluids have employed as new coordinate lines the orthogonal system of equipotentials and stream-lines of the potential flow around a given contour, and upon these coordinates as independent variables the calculation of the flow of a viscous fluid around the same contour is based. Following other authors, we shall call this choice of new independent variables the BOUSSINESQ *transformation*. For our present purpose we denote by α the velocity potential and by γ the stream function of the potential flow.

It is easily seen that the variables α, γ are closely related to the preliminary variables ξ, η introduced in §2. If, in the relatively thin zone near the wall to which, in flow of a viscous fluid at high REYNOLDS numbers, the boundary layer is confined, the potential flow velocity is replaced by its value $U(x)$ directly at the wall, by this slight alteration one has an approximation to the potential

by $\int_0^x U(x)dx$ and to the stream function by $U(x)y$. Except for the factor $1/\nu$, these values are exactly the variables ξ and η in (6). If one chooses the BOUSSINESQ variables as a basis for the calculation of the flow of a viscous fluid, then it is necessary to solve beforehand the corresponding potential flow problem in order to determine the new variables needed. In general this approach will not be practical. By approximating the BOUSSINESQ variables in the manner above one only needs to know $U(x)$, and this is the knowledge of the outer flow that is always required in boundary layer theory. Obviously, to introduce α and γ first and then to approximate these variables in the above sense would again be impractical. The practical way is, of course, to introduce from the beginning the variables ξ and η by definition according to (6) as expressions given with the prescribed outer velocity $U(x)$. This we have done. (The factor $1/\nu$ in (6) is of no importance here. It was chosen to make ξ and η dimensionless, thus giving η the meaning of a local REYNOLDS number and ξ that of a REYNOLDS number accounting for the history of the flow upstream.)

In a number of papers [15, 16, 17, 18, 19] MEKSYN has used the potential α and the stream function γ (with velocity at infinity normed to 1) as initial variables, then passing on to α and γ ($U_\infty/2\nu\alpha$)¹ as final variables. Again, if one approximates α and γ in the manner described above one obtains from MEKSYN's final variables our final variables ξ and η defined in (14). Against the MEKSYN variables there are the same objections as against the BOUSSINESQ variables. For the purpose of boundary layer calculation it is again more practical to introduce from the beginning the variables ξ and η .

It may be mentioned, by the way, that we first noticed a connection between the investigations of MEKSYN and our own results when we discovered, by chance, that the differential equation to which MEKSYN was led by his transformation is—formally—exactly the same as our differential equation (18) for $F(\xi, \eta)$. Whilst on the one hand the difference of approach lies in the fact that the treatment of MEKSYN's equation makes it necessary to calculate or approximate the potential flow around the same body in order to determine the new coordinates before attacking the boundary layer problem, the formal accord of MEKSYN's final differential equation with our equation (18) shows, on the other hand, that his less practical choice of coordinates does not offer any advantage compared with our choice of coordinates in leading to a more simple final boundary value problem.

A suggestion made by MANGLER but not published comes nearer to our own approach. This suggestion has been described by TOLLMIEEN [20] in a survey in which he points out that for the investigation of the asymptotic transition of the velocity profiles in the boundary layer to the outer potential flow velocity $U(x)$ it is profitable to introduce the independent variables

$$(87) \quad \xi^* = \frac{1}{U_0 L} \int_0^x U(x) dx, \quad \eta^* = \frac{\psi(x, y)}{\left\{ 2\nu \int_0^x U(x) dx \right\}^{\frac{1}{2}}}$$

where U_0 and L are a constant reference velocity and length. Here $\psi(x, y)$ is

the stream function of the boundary layer flow. It was the aim of that suggestion to give a correct representation of the asymptotic behavior of the boundary layer velocity profiles at the outer edge of the boundary layer with the aid of a series expansion in powers of ξ^* . Although the variables (87) and the series come very close to our own approach, an objection similar to that against the variables of BOUSSINESQ and MEKSYN must be made against η^* as far as the general problem of boundary layer calculation is concerned. The variable η^* involves the stream function of the boundary layer and is therefore to be obtained only with the solution of the boundary layer problem. For evaluation of boundary layer flows the use of η^* would be less practical than that of our variable η .

TOLLMEN [21] and recently BETZ [22] in their investigations of the transition of the boundary-layer velocity profiles into the outer potential flow velocity have altered the variable η^* in (87). Taking into account that $\psi(x, y) \rightarrow U(x)(1 - \delta^*)$ when approaching the outer edge of the boundary layer, δ^* being the displacement thickness of the boundary layer, they use the variables

$$(88) \quad \xi = \nu \int_0^x U(x) dx, \quad \tilde{\eta} = U(x)(y - \delta^*).$$

The boundary layer equation can then be simplified for the region near the outer edge of the boundary layer by linearization, leading to the linear heat transfer differential equation for $U(U - u)$ as a function of the variables (88) in the sense of a good approximation sufficiently far away from the wall. The analogy to heat conduction suggests going on from (88) to the independent variables (*cf. e.g.* equation (10) in [22])

$$(89) \quad \xi = \xi = \nu \int_0^x U(x) dx, \quad \tilde{\eta} = \frac{\tilde{\eta}}{2\sqrt{\xi}}.$$

On account of (23), these variables are connected with our variables ξ and η in the following way:

$$(90) \quad \xi = \nu^2 \xi, \quad \tilde{\eta} = \frac{1}{\sqrt{2}} \{ \eta - \eta_0(x) \} \quad (\eta_0(x) = \lim_{\eta \rightarrow \infty} \{ \eta - F(\xi, \eta) \}).$$

The advantage of the variables ξ , $\tilde{\eta}$ in the investigation of the asymptotic transition of the boundary layer profiles to the outer velocity $U(x)$ lies in MÄGLER's unpublished suggestion that in the series expansion in powers of ξ the first term of the series gives the correct asymptotic behavior of the solution at the outer edge of the boundary layer. This essential advantage remains if we use our variables ξ , η and expand the solution in a power series in ξ . This is, of course, an important advantage for a method aiming at a careful evaluation of boundary layer flow. Our new series is a power series in ξ . For a general method for the calculation of boundary layer flows it would be less practical to use the variables (90), useful as they are in the investigation of asymptotic behavior, for $\eta_0(x)$ is known only when the solution is known.

A further paper by BETZ must now be mentioned [23]. It is the aim of that paper to represent the velocity profile at any fixed cross-section of the boundary layer by suitable series expansions in functions proportional to the distance from the wall, y , at this cross-section. The distance from the wall is made dimensionless by a reference length $L(x)$:

$$(91) \quad \eta = \frac{y}{L(x)}.$$

In order to approximate the velocity profile at the cross-section x , expansions of the form

$$(92) \quad \frac{u(x, y)}{U(x)} = 1 - e^{-(b_1 \eta + b_2 \eta^2 + \dots)}$$

are investigated. As reference length L it is found practical to use

$$(93) \quad L = \frac{\left\{ 2\nu \int_0^x U(x) dx \right\}^{\frac{1}{2}}}{U(x)}.$$

Then the degree of approximation achievable with four terms in the exponent on the right hand side of (92) is studied for the special case of the well known velocity profiles of the similar wedge flows. The dimensionless measure for the wall distance η in the case that $U(x) = \text{const} \cdot x^m$, becomes identical with the one variable appearing in similar solutions.

As in BETZ's investigation, the only interest lies in the approximation of single velocity profiles, especially by expressions of the type (92); only the one variable η for the dimensionless distance from the wall appears. This variable is identical with our variable η .

The structure of the variable η is indicated by the structure of the one variable on which the exact solutions of the boundary layer problem, the similar solutions, depend. Again the variable for these special cases is suggested by dimensional reasoning, as was the case for the first exact solutions of boundary layer theory found, the flat plate flow and the pure stagnation point flow. This variable has the structure $y \sqrt{U(x)/\nu x}$. A measure of distance from the wall suitable for quite general boundary layer investigations should become identical, in the special case when $U(x) = s_0 x^m$, with this one variable of the corresponding similar solutions. Necessarily this has the advantage that, when two variables ξ, η are used for the general investigation, the exact similar solutions appear as those special cases in which u/U depends only on the one variable η used as a measure of the distance from the wall. As should already be clear from §5, this is the case for our variable η . We have, for $U(x) = s_0 x^m$,

$$(94) \quad \eta = \frac{U(x)y}{\left\{ 2\nu \int_0^x U(x) dx \right\}^{\frac{1}{2}}} = y \left\{ \frac{s_0(m+1)x^{m-1}}{2\nu} \right\}^{\frac{1}{2}} = y \left\{ \frac{m+1}{2} \frac{U(x)}{\nu x} \right\}^{\frac{1}{2}},$$

the well known dimensionless wall distance measure of the similar boundary layer solutions.

For outer velocities $U(x) = s_0 x^m + \dots$ ($m > -1$) of the general form (70) and (71) treated in this memoir, our general variable η for $x \rightarrow 0$ has the limiting value (94) of the special variable of similar solutions; likewise the solution $F(\xi, \eta)$ has as limit for $\xi \rightarrow 0$ the function $F_0(\eta)$ of the similar solutions.

The explanations given above should show sufficiently how, *vice versa*, the generalization of the variable (94) of similar solutions to a suitable dimensionless measure for the distance from the wall for general boundary layer flows suggests itself. Our variables ξ, η are only in the line of logical development of earlier attempts, although, I must confess, I did not notice this till after completing the theory of the new series.

It may be mentioned here that the introduction of ξ, η as independent variables for the new series method presented in this paper was first described by the author in 1952 [24].

PART II: THE NEW SERIES

§8. Boundary layer series corresponding to principal functions of Class I.

By the transformation discussed in Part I of this paper we were led, in §2, to the following boundary value problem for the calculation of the boundary layer:

$$(18) \quad F_{\eta\eta} + FF_{\eta\eta} + \beta(\xi)\{1 - F_{\eta}^2\} = 2\xi\{F_{\eta}F_{\xi\eta} - F_{\xi}F_{\eta\eta}\},$$

$$(19) \quad F(\xi, 0) = F_{\eta}(\xi, 0) = 0, \quad \lim_{\eta \rightarrow \infty} F_{\eta}(\xi, \eta) = 1 \quad (0 < \xi < \xi_1).$$

The behavior of the solution at the singular point $x = 0$ depends on the behavior of the given outer velocity $U(x)$ at this point or, what is the same thing, on the behavior of the principal function $\beta(\xi)$ at $\xi = 0$. If a regular initial cross-section were given, it would be necessary to prescribe the velocity profile u at this cross-section in order to secure uniqueness of the solution. Instead of this initial velocity distribution at a regular cross-section, at the initial singular cross-section $x = 0$ ($\xi = 0$) of the flow we get a unique determination of the solution by the behavior, just mentioned, of the outer velocity there and by the corresponding form of the series for the solution.

We consider first the Class I of all principal functions for which in the interval $0 \leq \xi < \xi_1$ a convergent power series

$$(I) \quad \beta(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 + \dots$$

exists. ($\xi_1 = \xi(x_1)$ denotes any suitable cross-section not further downstream than that of any possible separation point of the boundary layer.) We know (§4) that to such principal functions correspond outer velocity distributions of the form

$$(70) \quad U(x) = x^m \{s_0 + s_1 x^{m+1} + s_2 x^{2(m+1)} + \dots\}, \quad m \neq -1$$

$$\left(m = \frac{\beta_0}{2 - \beta_0}, \beta_0 = \frac{2m}{m+1} \right),$$

uniquely except for REYNOLDS similarity. We must restrict the values of m to $-1 < m < \infty$, and our future applications will be concerned mainly with $0 \leq m \leq 1$ (vertex angle $\beta_0 \pi$ at the leading edge $x = 0$ in $0 \leq \beta_0 \pi \leq \pi$). Applications in Part III are restricted to $m = 0$ ($\beta_0 = 0$) and $m = 1$ ($\beta_0 = 1$), our Cases A and B1 of §4. In Case A ($\beta_0 = 0$) the contour has a cuspidal point in $x = 0$. This case includes the problems of flow along flat plates with any outer pressure distribution. In Case B1 ($\beta_0 = 1$, symmetrical sub-case of Case B) $x = 0$ is a stagnation point as in all cases with $m > 0$, here with vertex angle π (rounded nose). For these two cases the calculation of the coefficients β_k for given u_k and *vice versa* has been prepared for applications in our formulæ (44), (46), (47) and (61), (62), (63) respectively.

Introduction of the power series (I) for $\beta(\xi)$ in the differential equation (18) makes the calculation of the solution $F(\xi, \eta)$ possible as a power series in ξ with coefficients depending on η :

$$(95) \quad F(\xi, \eta) = \sum_{k=0}^{\infty} F_k(\eta) \xi^k \quad \left(\xi = \frac{1}{\nu} \int_0^x U dx, \eta = \frac{Uy}{\sqrt{2\xi}} \right).$$

From the differential equation (18) we obtain, by comparison of coefficients of equal powers of ξ , the following recursive system of ordinary differential equations, of which all but the first are linear:

$$F_0''' + F_0 F_0'' + \beta_0(1 - F_0'^2) = 0,$$

$$F_k''' + F_0 F_k'' - 2(\beta_0 + k) F_0' F_k' + (2k + 1) F_0'' F_k = R_{k-1}$$

$$(k = 1, 2, 3, \dots)$$

(96) with

$$R_{k-1} = \beta_k(F_0'^2 - 1) + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} \beta_i F_i' F_{k-i-j}' + F_0' \sum_{j=1}^{k-1} \beta_j F_{k-j}'$$

$$+ \sum_{j=1}^{k-1} \{\beta_0 + 2(k-j)\} F_j' F_{k-j}' - \sum_{j=1}^{k-1} (1 + 2j) F_j F_{k-j}''.$$

From the boundary conditions (19) it follows that

$$(97) \quad F_0(0) = F_0'(0) = 0, \quad F_0'(\infty) = 1;$$

$$F_k(0) = F_k'(0) = 0, \quad F_k'(\infty) = 0 \quad (k = 1, 2, 3, \dots).$$

Comparing this with (74), we see that $F_0(\eta)$ is the function of the similar solution belonging to a wedge angle $\beta_0 \pi$. Contrary to the wedge flow solutions, our variable η , on which F_0 depends, is our general variable (14). Only in the special case $\beta(\xi) = \text{const.} = \beta_0$ ($U(x) = s_0 x^m$) does our general variable specialize

to the variable of the similar solutions (*cf.* §7). In the general case, our variable η approaches the variable of similar flows for sufficiently small x or ξ .

Actually, we should write $F_0(\eta; \beta_0)$ and $F_k(\eta; \beta_0)$, as these functions depend on the choice of β_0 . The F_k also depend on the coefficients $\beta_1, \beta_2, \dots, \beta_k$, but we shall see very soon how the calculation of F_k for any fixed value of β_0 can be reduced to the calculation of functions that do not depend on $\beta_1, \beta_2, \dots, \beta_k$ and are therefore, for any fixed value of β_0 (fixed geometrical character of the contour at $x = 0$), universal functions.

For the velocity $u(x, y)$ we have

$$(98) \quad u(x, y) = U(x) \{ F'_0(\eta) + F'_1(\eta)\xi + F'_2(\eta)\xi^2 + \dots \}.$$

Since $F'_0(\infty) = 1$, $F'_k(\infty) = 0$ for $k = 1, 2, 3, \dots$ the first requirement mentioned in our introduction (§1) is fulfilled: The first term of the new series satisfies the outer boundary condition exactly. All further terms of the series give corrections only in the inner parts of the boundary layer. Concerning the second requirement stated in §1, we see that our variable η approaches the variable of wedge flow as $x \rightarrow 0$, and accordingly the velocity field in the boundary layer approaches that of a wedge flow sufficiently near to the leading edge $x = 0$. But further downstream our variable generally differs more and more from that of the wedge flow and, as we can hope according to §7, is better adapted to the flow along the contour in question in a given case. It can therefore be expected that the zero order approximation $u(x, y) \approx U(x)F'_0(\eta)$ will be good not only in the immediate vicinity of the leading edge but even rather far downstream before the corrections in the inner part of the boundary layer gain weight.

Concerning requirement No. 4 of §1, no further word is necessary. We shall now show that the third requirement of §1, concerning the reduction to universal functions, is easily fulfilled.

We cannot expect it to be possible to reduce the calculation of the coefficient functions $F_k(\eta)$ to that of linear combinations of universal functions not depending on any of the coefficients β_k including β_0 . The value of β_0 characterizes the geometry of the contour at $x = 0$, and, accordingly, $F_0(\eta; \beta_0)$ characterizes the flow, in the vicinity of $x = 0$, as a flow approaching wedge flow with the same vertex angle $\beta_0\pi$ as x approaches 0. (One could also say: The term of zero order $F_0(\eta; \beta_0)$ at the singular point $x = 0$ plays the role that in cases of a regular initial cross-section is played by the velocity profile prescribed at this cross-section.)

For any fixed value of β_0 , it is possible to reduce the coefficient functions $F_k(\eta; \beta_0; \beta_1, \beta_2, \dots, \beta_k)$ to linear combinations of functions not depending on $\beta_1, \beta_2, \dots, \beta_k$ and therefore for any fixed value of β_0 (*e.g.* $\beta_0 = 0$ for Case A, $\beta_0 = 1$ for Case B1), universal functions that can be tabulated once and for all. In other words: Just as the coefficient functions of the BLASIUS series for flows around cylinders can be reduced to universal functions, while the coefficient functions of the flat plate flow with linear velocity decrease (HOWARTH series) can be reduced to other universal functions, in the much wider range

of outer velocity distributions for which the new series is valid we have a set of universal functions for each value of β_0 . For $\beta_0 = 1$ the situation is equivalent to that of the BLASIUS series in the sense that both series are formally applicable to the same family of outer velocities. In the case $\beta_0 = 0$ our new series and its universal functions are applicable to flows along plates and to contours with forward cuspidal point with any given outer velocity distribution, while the HOWARTH series and its universal functions are applicable only to the case of linear velocity decrease. Other special cases recently studied by TANI each lead to other universal functions. We shall return to these in Part III.

The stated reduction is achieved in the following way. For any fixed value of β_0 we set

$$\begin{aligned}
 F_1 &= \beta_1 f_1 \\
 F_2 &= \beta_1^2 f_{11} + \beta_2 f_2 \\
 (99) \quad F_3 &= \beta_1^3 f_{111} + \beta_1 \beta_2 f_{12} + \beta_3 f_3 \\
 F_4 &= \beta_1^4 f_{1111} + \beta_1^2 \beta_2 f_{112} + \beta_1 \beta_3 f_{13} + \beta_2^2 f_{22} + \beta_4 f_4 \\
 F_5 &= \beta_1^5 f_{11111} + \beta_1^3 \beta_2 f_{1112} + \beta_1^2 \beta_3 f_{113} + \beta_1 \beta_2^2 f_{122} + \beta_1 \beta_4 f_{14} + \beta_2 \beta_3 f_{23} + \beta_5 f_5 \\
 &\dots,
 \end{aligned}$$

where the $f\dots$ are functions of η alone. The manner in which we introduce subscripts may seem clumsy at first sight but it proves to be very practical in applications, as it offers possibilities of checking formulæ. (The traditional way of writing the universal functions of the BLASIUS series is certainly not practical.)

If, as an abbreviation, we define the differential operator

$$(100) \quad L_k[f] = f''' + F_0 f'' - 2(\beta_0 + k)F_0' f' + (2k + 1)f,$$

we have the following differential equations for the functions $f\dots(\eta)$:

$$\underline{k = 1}: \quad L_1[f_1] = F_0'^2 - 1$$

$$\begin{aligned}
 \underline{k = 2}: \quad L_2[f_2] &= F_0'^2 - 1 \\
 L_2[f_{11}] &= 2F_0' f_1' + (\beta_0 + 2)f_1'^2 - 3f_1 f_1''
 \end{aligned}$$

$$\begin{aligned}
 \underline{k = 3}: \quad L_3[f_3] &= F_0'^2 - 1 \\
 L_3[f_{12}] &= 2F_0'(f_1' + f_2') + 2(\beta_0 + 3)f_1' f_2' - 3f_1 f_2'' - 5f_1' f_2' \\
 L_3[f_{111}] &= 2F_0' f_{11}' + f_1'^2 + 2(\beta_0 + 3)f_1' f_{11}' - 3f_1 f_{11}'' - 5f_1' f_{11}'
 \end{aligned}$$

$$\begin{aligned}
 \underline{k = 4}: \quad L_4[f_4] &= F_0'^2 - 1 \\
 L_4[f_{22}] &= 2F_0' f_2' + (\beta_0 + 4)f_2'^2 - 5f_2 f_2'' \\
 L_4[f_{13}] &= 2F_0'(f_1' + f_3') + 2(\beta_0 + 4)f_1' f_3' - 3f_1 f_3'' - 7f_1' f_3'
 \end{aligned}$$

$$\begin{aligned}
L_4[f_{112}] &= 2F'_0(f'_{11} + f'_{12}) + f_1'^2 + 2f'_1f'_2 + 2(\beta_0 + 4)(f'_1f'_{12} + f'_{11}f'_2) \\
&\quad - 3f_1f'_{12} - 5(f_{11}f'_2 + f'_{11}f_2) - 7f'_1f'_{12} \\
L_4[f_{1111}] &= 2F'_0f'_{111} + 2f'_1f'_{11} + (\beta_0 + 4)(2f'_1f'_{111} + f_1'^2) \\
&\quad - 3f_1f'_{111} - 5f_{11}f'_{11} - 7f'_1f'_{111}
\end{aligned}
\tag{101}$$

$$\begin{aligned}
\underline{k = 5}: \quad L_5[f_5] &= F_0'^2 - 1 \\
L_5[f_{23}] &= 2F'_0(f'_2 + f'_3) + 2(\beta_0 + 5)f'_2f'_3 - 5f_2f'_3 - 7f'_2f'_3 \\
L_5[f_{14}] &= 2F'_0(f'_1 + f'_4) + 2(\beta_0 + 5)f'_1f'_4 - 3f_1f'_4 - 9f'_1f'_4 \\
L_5[f_{122}] &= 2F'_0(f'_{12} + f'_{22}) + f_2'^2 + 2f'_1f'_2 + 2(\beta_0 + 5)(f'_1f'_{22} + f'_{12}f'_2) \\
&\quad - 3f_1f'_{22} - 5f_2f'_{12} - 7f'_2f'_{12} - 9f'_1f'_{22} \\
L_5[f_{113}] &= 2F'_0(f'_{11} + f'_{13}) + f_1'^2 + 2f'_1f'_3 + 2(\beta_0 + 5)(f'_1f'_{13} + f'_{11}f'_3) \\
&\quad - 3f_1f'_{13} - 5f_{11}f'_{13} - 7f'_{11}f'_3 - 9f'_1f'_{13} \\
L_5[f_{1112}] &= 2F'_0(f'_{111} + f'_{112}) + 2f'_1f'_{11} + 2f'_1f'_{12} + 2f'_{11}f'_2 \\
&\quad + 2(\beta_0 + 5)(f'_1f'_{112} + f'_{11}f'_{12} + f'_{111}f'_2) - 3f_1f'_{112} \\
&\quad - 5(f_{11}f'_{12} + f'_{111}f_2) - 7(f_{111}f'_2 + f'_{11}f'_2) - 9f'_1f'_{112} \\
L_5[f_{11111}] &= 2F'_0f'_{1111} + f_{11}^2 + 2f'_1f'_{111} + 2(\beta_0 + 5)(f'_1f'_{1111} + f'_{11}f'_{111}) \\
&\quad - 3f_1f'_{111} - 5f_{11}f'_{111} - 7f'_{11}f'_{111} - 9f'_1f'_{111} .
\end{aligned}$$

As boundary conditions we have for all $f \dots$, whatever the subscripts,

$$f \dots(0) = f' \dots(0) = f' \dots(\infty) = 0. \tag{102}$$

The differential equations (101) and the boundary conditions (102) are independent of $\beta_1, \beta_2, \beta_3, \dots$. Therefore, as stated above, for any fixed value of β_0 the solutions $f \dots$ of this recursive system of boundary value problems can be calculated once and for all. Once this main work is carried out to a sufficiently high order k , the remaining numerical work in applying the new series consists only in very few elementary calculations. (*Cf.* the applications in Part III.)

In numerical tables described in §10 the universal functions $f \dots$ and F_0 will be found for $\beta_0 = 0$ and for $\beta_0 = 1$, for $k = 1, 2, 3, 4, 5$. The fact that the differential equations (101) are linear simplifies the numerical solution considerably.

In §6 we have considered, in addition, the case of boundary layer flows around cylinders in yaw. For the cross-flow component $w(x, y) = W_\infty W(\xi, \eta)$ of the boundary layer velocity we obtained the boundary value problem

$$\begin{aligned}
(83) \quad & W_{\eta\eta} + (F + 2\xi F_\xi)W_\eta = 2\xi F_\eta W_\xi, \\
& W(\xi, 0) = 0, \quad W(\xi, \eta) \rightarrow 1 \quad \text{for } \eta \rightarrow \infty.
\end{aligned}$$

Introducing for F the series $\sum_{k=0}^{\infty} F_k \xi^k$ found above, we are able to calculate W as a power series:

$$(103) \quad W(\xi, \eta) = \sum_{k=0}^{\infty} W_k(\eta) \xi^k.$$

From (83) we then obtain for the zero order term of this series

$$(104) \quad W_0'' + F_0 W_0' = 0, \quad W_0(0) = 0, \quad W_0(\infty) = 1.$$

As could be expected according to (86), $W_0(\eta)$ is the well-known function for the cross-flow velocity component of similar flows along wedges in yaw. W_0 can be calculated from F_0 by simple quadrature. The variable η is again our general variable, and only in the case $\beta(\xi) = \text{const} = \beta_0$ is it identical with the special variable of the wedge flow solutions.

For the further coefficient functions $W_k(\eta)$, $k = 1, 2, 3, \dots$, we obtain from (83) the following recursive system of linear second order differential equations:

$$(105) \quad W_k'' + F_0 W_k' - 2k F_0' W_k = \sum_{j=1}^k \{2(k-j) F_j' W_{k-j} - (2j+1) F_j W_{k-j}'\}$$

with $W_k(0) = W_k(\infty) = 0 \quad (k = 1, 2, 3, \dots).$

In order to reduce the calculation of the coefficient functions $W_k(\eta)$ to that of a linear combination of functions, universal for any fixed value of β_0 , we define in full analogy with (99)

$$(106) \quad \begin{aligned} W_1 &= \beta_1 w_1 \\ W_2 &= \beta^2 w_{11} + \beta_2 w_2 \\ W_3 &= \beta_1^3 w_{111} + \beta_1 \beta_2 w_{112} + \beta_3 w_3 \\ W_4 &= \beta_1^4 w_{1111} + \beta_1^2 \beta_2 w_{1112} + \beta_1 \beta_3 w_{113} + \beta_2^2 w_{22} + \beta_4 w_4 \\ W_5 &= \beta_1^5 w_{11111} + \beta_1^3 \beta_2 w_{11112} + \beta_1^2 \beta_3 w_{1113} + \beta_1 \beta_2^2 w_{122} \\ &\quad + \beta_1 \beta_4 w_{14} + \beta_2 \beta_3 w_{23} + \beta_5 w_5 \dots \end{aligned}$$

If we now define the differential operator

$$(107) \quad M_k[w] = w'' + F_0 w' - 2k F_0' w,$$

the differential equations obtained for the functions $w \dots$ are

$$\begin{aligned} \underline{k=1}: \quad M_1[w_1] &= -3W_0' f_1 \\ \underline{k=2}: \quad M_2[w_2] &= -5W_0' f_2 \\ M_2[w_{11}] &= -5W_0' f_{11} + 2f_1' w_1 - 3f_1 w_1' \\ \underline{k=3}: \quad M_3[w_3] &= -7W_0' f_3 \\ M_3[w_{12}] &= -7W_0' f_{12} + 2f_2' w_1 - 3f_1 w_2' + 4f_1' w_2 - 5f_2 w_1' \\ M_3[w_{111}] &= -7W_0' f_{111} + 2f_1' w_{11} - 3f_1 w_{11}' + 4f_1' w_{11} - 5f_{11} w_1' \end{aligned}$$

$$\begin{aligned}
\underline{k = 4}: \quad M_4[w_4] &= -9W'_0 f_4 \\
M_4[w_{22}] &= -9W'_0 f_{22} + 4f'_2 w_2 - 5f_2 w'_2 \\
M_4[w_{13}] &= -9W'_0 f_{13} + 2f'_3 w_1 - 3f_1 w'_3 + 6f'_1 w_3 - 7f_3 w'_1 \\
M_4[w_{112}] &= -9W'_0 f_{112} + 2f'_{12} w_1 - 3f_1 w'_{12} + 4(f'_2 w_{11} + f'_{11} w_2) \\
&\quad - 5(f_2 w'_{11} + f_{11} w'_2) + 6f'_1 w_{12} - 7f_{12} w'_1 \\
M_4[w_{1111}] &= -9W'_0 f_{1111} + 2f'_{111} w_1 - 3f_1 w'_{111} + 4f'_{11} w_{11} \\
&\quad - 5f_{11} w'_{11} + 6f'_1 w_{111} - 7f_{111} w'_1 \\
(108) \quad \underline{k = 5}: \quad M_5[w_5] &= -11W'_0 f_5 \\
M_5[w_{23}] &= -11W'_0 f_{23} + 4f'_3 w_2 - 5f_2 w'_3 + 6f'_2 w_3 - 7f_3 w'_2 \\
M_5[w_{14}] &= -11W'_0 f_{14} + 2f'_4 w_1 - 3f_1 w'_4 + 8f'_1 w_4 - 9f_4 w'_1 \\
M_5[w_{122}] &= -11W'_0 f_{122} + 2f'_{22} w_1 - 3f_1 w'_{22} + 4f'_{12} w_2 - 5f_2 w'_{12} \\
&\quad + 6f'_2 w_{12} - 7f_{12} w'_2 + 8f'_1 w_{22} - 9f_{22} w'_1 \\
M_5[w_{113}] &= -11W'_0 f_{113} + 2f'_{13} w_1 - 3f_1 w'_{13} + 4f'_3 w_{11} - 5f_{11} w'_3 \\
&\quad + 6f'_{11} w_3 - 7f_3 w'_{11} + 8f'_1 w_{13} - 9f_{13} w'_1 \\
M_5[w_{1112}] &= -11W'_0 f_{1112} + 2f'_{112} w_1 - 3f_1 w'_{112} + 4(f'_{12} w_{11} + f'_{11} w_2) \\
&\quad - 5(f_{11} w'_{12} + f_2 w'_{11}) + 6(f'_{11} w_{12} + f'_2 w_{11}) \\
&\quad - 7(f_{111} w'_2 + f_{12} w'_1) + 8f'_1 w_{112} - 9f_{112} w'_1 \\
M_5[w_{11111}] &= -11W'_0 f_{11111} + 2f'_{1111} w_1 - 3f_1 w'_{1111} + 4f'_{111} w_{11} - 5f_{11} w'_{111} \\
&\quad + 6f'_{11} w_{111} - 7f_{111} w'_{11} + 8f'_1 w_{1111} - 9f_{1111} w'_1 .
\end{aligned}$$

The boundary conditions for all $w...$ are independent of the subscripts:

$$(109) \quad w...(0) = w...(\infty) = 0.$$

We have written down the differential equations for the universal functions of orders zero to five because the solutions of these equations have been calculated numerically for the case $\beta_0 = 1$ and will be found in the tables described in §10.

§9. Boundary layer series corresponding to principal functions of the more general Class II.

We now consider the more general class of principal functions given in $0 \leq \xi < \xi_1$ by a convergent series of the form

$$(II) \quad \beta(\xi) = \beta_0 + \beta_1 \xi^{\frac{1}{2}} + \beta_2 \xi + \beta_3 \xi^{\frac{3}{2}} + \dots .$$

The corresponding outer velocity distributions (cf. §4) are

$$(71) \quad U(x) = x^m \{s_0 + s_{\frac{1}{2}} x^{\frac{1}{2}(m+1)} + s_1 x^{m+1} + s_{\frac{3}{2}} x^{\frac{3}{2}(m+1)} + \dots\}$$

with $m = \frac{\beta_0}{2 - \beta_0} \quad \left(\beta_0 = \frac{2m}{m+1} \right) \quad m \neq -1.$

As will be remembered, our first motive for studying this generalization was the fact that principal functions corresponding to the outer velocity distribution

$$(36B) \quad U(x) = u_1 x + u_2 x^2 + u_3 x^3 + \dots \quad (u_1 \neq 0)$$

are of the Class I only if $u_{2k} = 0$, $k = 1, 2, 3, \dots$ (Case B1, symmetric case) but in the general Case B of equation (36B) are of the Class II with $\beta_0 = 1$ ($m = 1$). Symmetric outer potential velocities are found in flows symmetric with respect to an axis (*e.g.* for an axi-symmetric contour with flow at infinity parallel to the axis). But also in cases of non-symmetrical flows it will often be possible to approximate the outer flow with sufficient accuracy by a polynomial or infinite series with $u_{2k} = 0$. As is already well known for the BLASIUS series and is also the case for the new series, symmetric outer velocities lead to far less complicated boundary layer series than does the general case (36B). That is the reason why, in the case of the BLASIUS series, the universal functions have been tabulated to a much higher order in the symmetric case than in the general case (36B). For the same reason our tables for the new series, applicable to (36B), are not carried to as high an order as in the symmetric case. It is our main object here to show how our series can be extended to the non-symmetric case (36B). Contrary to the BLASIUS series, confined to $m = 1$, our new series will be equally applicable to any outer velocity of type (71).

Formally, all steps are the same as in the preceding paragraph. Inserting the series (II) for the principal function into the differential equation (18) and accordingly introducing for $F(\xi, \eta)$ a series of the form

$$(110) \quad F(\xi, \eta) = \sum_{k=0}^{\infty} F_{\frac{1}{2}k}(\eta) \xi^{\frac{1}{2}k},$$

we find again for the zero order term

$$(111) \quad F_0''' + F_0 F_0'' + \beta_0(1 - F_0'^2) = 0, \quad F_0(0) = F_0'(0) = 0, \quad F_0'(\infty) = 1.$$

As a function of η , the term $F_0(\eta)$ is therefore identical with the function F_0 found in §8. (Of course, the variable η is not the same function of x and y .) For the coefficient functions $F_k(\eta)$, $k = 1, 2, 3, \dots$, we obtain the recursive system

$$F_{\frac{1}{2}k}''' + F_0 F_{\frac{1}{2}k}'' - (k + 2\beta_0) F_0' F_{\frac{1}{2}k}' + (k + 1) F_0'' F_{\frac{1}{2}k} = R_{\frac{1}{2}(k-1)} \quad (k = 1, 2, 3, \dots)$$

with

$$\begin{aligned}
R_{\frac{1}{2}(k-1)} &= \beta_{\frac{1}{2}k}(F_0'^2 - 1) + \sum_{j=1}^{k-1} \sum_{i=1}^{k-j} \beta_{\frac{1}{2}i} F_{\frac{1}{2}i}' F_{\frac{1}{2}(k-i-j)}' \\
(112) \quad &+ F_0' \sum_{j=1}^{k-1} \beta_{\frac{1}{2}j} F_{\frac{1}{2}(k-j)}' + \sum_{j=1}^{k-1} (\beta_0 + k - j) F_{\frac{1}{2}j}' F_{\frac{1}{2}(k-j)}' \\
&- \sum_{j=1}^{k-1} (1 + j) F_{\frac{1}{2}j}'' F_{\frac{1}{2}(k-j)}'
\end{aligned}$$

and the boundary conditions

$$(113) \quad F_{\frac{1}{2}k}(0) = F_{\frac{1}{2}k}'(0) = F_{\frac{1}{2}k}'(\infty) = 0 \quad (k = 1, 2, 3, \dots).$$

The reduction to universal functions is achieved by defining

$$\begin{aligned}
F_{\frac{1}{2}} &= \beta_{\frac{1}{2}} f_{\frac{1}{2}} \\
F_1 &= \beta_{\frac{1}{2}}^2 f_{\frac{1}{2}\frac{1}{2}} + \beta_1 f_1 \\
(114) \quad F_{\frac{3}{2}} &= \beta_{\frac{1}{2}}^3 f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} + \beta_{\frac{1}{2}} \beta_1 f_{\frac{1}{2}1} + \beta_{\frac{1}{2}} f_{\frac{3}{2}} \\
F_2 &= \beta_{\frac{1}{2}}^4 f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} + \beta_{\frac{1}{2}}^2 \beta_1 f_{\frac{1}{2}\frac{1}{2}1} + \beta_{\frac{1}{2}} \beta_{\frac{1}{2}} f_{\frac{1}{2}\frac{3}{2}} + \beta_1^2 f_{11} + \beta_2 f_2 \\
&\dots
\end{aligned}$$

Independently of the subscripts, the boundary conditions are

$$(115) \quad f \dots (0) = f' \dots (0) = f' \dots (\infty).$$

At first sight it may not seem advisable to use the same letter "f" for the universal functions as in the preceding section. Luckily, the universal functions $f \dots$ in (114) with only integral subscripts are identical with the functions denoted in the same way in §8. For them, by inserting (114) in (112) and (113) we find the same boundary value problems as in §8. Therefore, having once calculated for a fixed value of β_0 the universal functions appearing in §8, namely $F_0, f_1, f_2, f_{11}, f_3, f_{12}, f_{111}, \dots$, we may apply these functions also, for the same value of β_0 , to the case of this section. For problems arising from principal functions of Class II not contained in Class I, only the additional functions $f_{\frac{1}{2}}, f_{\frac{1}{2}\frac{1}{2}}, f_{\frac{1}{2}1}, f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}, \dots$ must still be tabulated. For these we find the following differential equations:

$$\begin{aligned}
L_3[f_{\frac{1}{2}}] &= F_0'^2 - 1 \\
L_1[f_{\frac{1}{2}\frac{1}{2}}] &= 2F_0' f_{\frac{1}{2}}' + (1 + \beta_0) f_{\frac{1}{2}}'^2 - 2f_{\frac{1}{2}} f_{\frac{1}{2}}'' \\
L_3[f_{\frac{1}{2}}] &= F_0'^2 - 1 \\
L_3[f_{\frac{1}{2}1}] &= 2F_0'(f_{\frac{1}{2}}' + f_1') + (3 + 2\beta_0) f_{\frac{1}{2}}' f_1' - 2f_{\frac{1}{2}} f_1'' - 3f_{\frac{1}{2}}'' f_1 \\
(116) \quad L_3[f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}] &= 2F_0' f_{\frac{1}{2}\frac{1}{2}}' + f_{\frac{1}{2}}'^2 + (3 + 2\beta_0) f_{\frac{1}{2}}' f_{\frac{1}{2}\frac{1}{2}}' - 2f_{\frac{1}{2}} f_{\frac{1}{2}\frac{1}{2}}'' - 3f_{\frac{1}{2}}'' f_{\frac{1}{2}\frac{1}{2}} \\
L_2[f_{\frac{1}{2}\frac{1}{2}}] &= 2F_0'(f_{\frac{1}{2}}' + f_{\frac{1}{2}}') + 2(2 + \beta_0) f_{\frac{1}{2}}' f_{\frac{1}{2}}' - 2f_{\frac{1}{2}} f_{\frac{1}{2}}'' - 4f_{\frac{1}{2}}'' f_{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
L_2[f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}] &= 2F'_0(f'_{\frac{1}{2}\frac{1}{2}} + f'_{\frac{1}{2}}) + f_{\frac{1}{2}}'^2 + 2f_{\frac{1}{2}}'f'_{\frac{1}{2}} + 2(2 + \beta_0)(f'_{\frac{1}{2}}f'_{\frac{1}{2}} + f_{\frac{1}{2}}'f'_{\frac{1}{2}}) \\
&\quad - 2f_{\frac{1}{2}}f_{\frac{1}{2}}'' - 3(f_{\frac{1}{2}}f_{\frac{1}{2}}'' + f_{\frac{1}{2}}''f_{\frac{1}{2}}) - 4f_{\frac{1}{2}}''f_{\frac{1}{2}} \\
L_2[f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}}] &= 2F'_0f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} + 2f_{\frac{1}{2}}'f_{\frac{1}{2}\frac{1}{2}}' + (2 + \beta_0)(2f_{\frac{1}{2}}'f_{\frac{1}{2}\frac{1}{2}}' + f_{\frac{1}{2}}'^2) \\
&\quad - 2f_{\frac{1}{2}}f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}'' - 3f_{\frac{1}{2}}f_{\frac{1}{2}\frac{1}{2}}'' - 4f_{\frac{1}{2}}''f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} \\
&\quad \dots
\end{aligned}$$

The differential operator L_i is the same as in §8, defined in equation (100), where now $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. We have written down as many equations as were solved in calculating the tables for $\beta_0 = 1$, to be described in §10. With the first five terms of the series (orders 0, $\frac{1}{2}, \dots, 2$), we are at least able to achieve a good approximation in a larger vicinity of the forward stagnation point on non-symmetrical contours.

§10. Remarks concerning the available tables of universal functions.

Extensive tables of universal functions were calculated by Mr. GERARD SALTON at the Computation Laboratory of Harvard University on Mark IV. I wish to thank Mr. SALTON for his most valuable cooperation. Mr. SALTON has given, as an introduction to the tables, a description of the procedure applied to carry out the numerical integration of the boundary value problems in question. These tables cannot be published here but are available upon request.

As the calculation needed to tabulate the universal functions for any fixed value of β_0 up to a sufficiently high order is considerable, it was necessary to restrict these first evaluations to the values of β_0 of foremost interest. As already discussed in §4, the problems of main interest are "Case A" and "Class B1" (principal function of Class I with $\beta_0 = 0$ and 1, respectively) and, to a certain extent, "Case B" (principal function of Class II with $\beta_0 = 1$).

In order to minimize the effect of rounding errors during the evaluation of the solutions of the recursive systems of differential equations for the universal functions of increasing order, fourteen decimal places were carried throughout the computation. The printed tables are the results truncated to six decimal places unrounded. With respect to the question of accuracy, we refer to Mr. SALTON's introduction to the tables. As the tables contain the universal functions and all derivatives appearing in the differential equations they satisfy, direct check of the results is possible.

The step length applied in the numerical integration procedure was 0.01, this procedure carried out from $\eta = 0$ to $\eta = 5.99$ where the prescribed boundary condition at infinity was satisfied by F_0 within $3 \cdot 10^{-13}$. The printed tables give the results for all steps in $0(0.01)5.99$ and are therefore rather bulky. For practical applications of the new series it would be advisable to use an abstract of these tables with η -values $0(0.2)5.8$. These abstracts also are now available on request.*

*The short tables, sufficient for most practical applications, will soon be published as *Bericht der Deutschen Versuchsanstalt für Luftfahrt, No. 34*, by H. GÖRTLER.

The complete tables will be of value where interpolation in the shorter tables is necessary. *E.g.*, if it becomes necessary to continue the results of the series approximation further downstream by a difference procedure, linear interpolation in the short tables 0(0.2)5.8 is not sufficiently accurate for the purposes of a difference procedure. (If one interpolates in the short tables for this purpose, EVERETT'S interpolation formula (error of 3rd order) proves practical since only values of the function and its second differences are needed. At the wall where these differences would otherwise be missing one has the first condition of compatibility at the wall (25).)

In the following we describe briefly the extent to which universal functions have been evaluated.

Case A: $U(x) = u_0 + u_1x + u_2x^2 + u_3x^3 + \dots$ with $u_0 \neq 0$ (flows around contours with cuspidal point at the leading edge $x = 0$, including flows along flat plates at zero incidence with arbitrary outer pressure distribution). The corresponding principal function is of Class I ($\beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k$) with $\beta_0 = 0$. The universal functions were evaluated for the orders 0, 1, 2, 3, 4, and 5. It will be seen in Part III that for most cases of flows with adverse pressure gradient the new series thus truncated as a fifth order approximation gives a sufficiently accurate approximation from the leading edge downstream to a point immediately ahead of the separation point, thus making it possible to predict separation with a fairly secure short extrapolation (or by one or two steps of a difference procedure if higher accuracy is required).

Case B1: $U(x) = u_1x + u_3x^3 + u_5x^5 + \dots$ with $u_1 \neq 0$ (symmetrical sub-case of Case B below for flows around contours with rounded noses at forward stagnation point $x = 0$). The corresponding principal function is again of Class I as in Case A, but now with $\beta_0 = 1$. Therefore the same set of differential equations as in Case A must again be solved, but with a different value of the parameter β_0 . This was carried out to the same extent as in Case A, *i.e.* for the universal functions of orders 0, 1, 2, 3, 4, and 5. As already mentioned above, the results in applications are not as satisfactory as for the same order of approximation in Case A.

In addition, the universal functions of the cross-flow velocity component in cases of flows around cylinders in yaw were also tabulated for this case for the same orders 0 through 5.

Case B: $U(x) = u_1x + u_2x^2 + u_3x^3 + \dots$ with $u_1 \neq 0$ (general case of flows around contours with rounded nose at the forward stagnation point $x = 0$). The corresponding principal function is of Class II ($\beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^{1/k}$). Accordingly, a far greater number of universal functions must be tabulated in this case for an order of approximation equal to that in Cases A and B1. For this first memoir it was necessary to restrict evaluation of the universal functions to the orders 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. At least this gives the possibility of studying a larger neighborhood of the generally non-symmetrical forward stagnation point.

It is necessary to say a word concerning the notations used for the universal functions as printed at the Computation Laboratory of Harvard University. At the time when the tables

were being prepared we had not yet chosen a final notation for these functions. Therefore symbols in use at the time were printed. The symbols at the heads of the tables are, as far as the numbers in front of the point are concerned, the present subscripts of our universal functions multiplied by 2. Behind the point, for the functions F_0 and $f_{\dots}$ the value of β_0 is given (0 or 1). To make things quite clear (since the tables were not sent to Germany for correction of the notation) we give the following dictionary: The notations in the tables are:

0.0 for F_0		0.1 for F_0	
2.0 for f_1		2.1 for f_1	
22.0 for f_{11}	of Case A	22.1 for f_{11}	of Case B1
4.0 for f_2	($\beta_0 = 0$)	4.1 for f_2	($\beta_0 = 1$)
222.0 for f_{111}		222.1 for f_{111}	
<i>etc.</i>		<i>etc.</i>	
		1.1 for $f_{\frac{1}{2}}$	
		11.1 for $f_{\frac{1}{2}\frac{1}{2}}$	additional
		111.1 for $f_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}$	functions of
		12.1 for $f_{\frac{1}{2}1}$	Case B
		3.1 for $f_{\frac{1}{2}}$	($\beta_0 = 1$)
		<i>etc.</i>	

For the universal functions of the cross-flow component in Case B1 the notations in the tables are

0.3 for W_0
2.3 for w_1
22.3 for w_{11}
4.3 for w_2
222.3 for w_{111}
<i>etc.</i>

It is suggested that, before applying the tables, our final notations should be filled in.

We wish to remark that the tables for $F_0(\eta)$ ($\beta_0 = 0$ and $\beta_0 = 1$) offer incidentally a careful recalculation of the well known functions of the BLASIUS flat plate flow and of the pure stagnation line flow. Similarly, as far as we know, the table of $W_0(\eta)$ for $\beta_0 = 1$ is the most accurate one for the cross-flow component of pure stagnation line flow available at present.

PART III: SOME APPLICATIONS OF THE NEW SERIES

§11. Short description of the numerical work necessary for any application of the new series.

In order to facilitate applications of the new series, instructions concerning the elementary numerical work necessary to complete any special application are given here.

We restrict this description to Case A, since the method of procedure is completely analogous in Case B or in its sub-case B1.

Normally, the outer velocity distribution $U(x)$ will be prescribed in problems of boundary layer theory. It should be remarked, however, that applications are quite as simple if the principal function $\beta(\xi)$ is prescribed. This latter possibility offers some interest in theoretical studies.

The prescribed outer velocity in problems belonging to Case A has the form

$$(36A) \quad U(x) = u_0 + u_1x + u_2x^2 + u_3x^3 + \cdots \quad \text{with } u_0 \neq 0$$

or can be expanded in a power series of this kind or approximated by a polynomial in x , as usually is the procedure when $U(x)$ is given by measurements of the pressure distribution. Case A is mainly characterized by $U(0) = u_0 \neq 0$, while Case B is comprised of the cases where $U(0) = 0$, $U'(0) \neq 0$.

Following the suggestions given in §4 we use $U(0) = u_0$ as reference velocity and any suitable length L as reference length, thus introducing the dimensionless quantities

$$(117) \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y \operatorname{Re}^{\frac{1}{2}}}{L} \quad \text{with } \operatorname{Re} = \frac{u_0 L}{\nu},$$

$$\bar{U}(\bar{x}) = \frac{U(x)}{u_0}, \quad \bar{u}(\bar{x}, \bar{y}) = \frac{u(x, y)}{u_0}, \quad \bar{u}_k = \frac{u_k L^k}{u_0},$$

$$\bar{\xi} = \frac{\xi}{\operatorname{Re}}, \quad \bar{\beta}(\bar{\xi}) = \beta(\xi), \quad \bar{\beta}_k = \beta_k \operatorname{Re}^k, \quad \bar{F}_k = F_k \operatorname{Re}^k.$$

1. First one has to calculate the transition from the coordinates $\bar{x}, \bar{y}$ to the coordinates

$$(118) \quad \bar{\xi} = \int_0^{\bar{x}} \bar{U}(\bar{x}) d\bar{x},$$

$$\eta = \bar{\eta} = \bar{g}(\bar{x})\bar{y} \quad \text{where} \quad \bar{g}(\bar{x}) = \frac{\bar{U}(\bar{x})}{\sqrt{2\bar{\xi}(\bar{x})}}.$$

It is useful to evaluate the function $\bar{g}(\bar{x})$ separately, since this function is also needed in other relations.

2. One then calculates the coefficients $\bar{\beta}_k$ ($k = 0, 1, \dots, 5$) according to the simple formulæ (44), using the given data $\bar{u}_k$ of the outer velocity. (Note the remark immediately following equation (50).)

3. Usually the distribution of the wall shearing stress is of main interest. According to (28) we have

$$(119) \quad \left[\frac{\partial \bar{u}}{\partial \bar{y}} \right]_{\bar{y}=0} = \frac{\tau(x) \operatorname{Re}^{\frac{1}{2}}}{\rho u_0^2} = \bar{U}(\bar{x})\bar{g}(\bar{x}) \sum_k \bar{F}_k''(0) \bar{\xi}^k.$$

To evaluate this formula it is only necessary to determine the series coefficients $\bar{F}_k''(0)$. These are, according to (99) and our tables,

$$\begin{aligned} \bar{F}_0''(0) &= 0.469600 \\ \bar{F}_1''(0) &= 1.032361\bar{\beta}_1 \\ \bar{F}_2''(0) &= -0.714746\bar{\beta}_1^2 + 0.908119\bar{\beta}_2 \\ \bar{F}_3''(0) &= 1.103512\bar{\beta}_1^3 - 1.191046\bar{\beta}_1\bar{\beta}_2 + 0.829995\bar{\beta}_3 \end{aligned}$$

$$\begin{aligned}
(120) \quad \bar{F}_4''(0) &= -2.313327\bar{\beta}_1^4 + 2.775762\bar{\beta}_1^2\bar{\beta}_2 - 1.047926\bar{\beta}_1\bar{\beta}_3 \\
&\quad - 0.505493\bar{\beta}_2^2 + 0.774210\bar{\beta}_4 \\
\bar{F}_5''(0) &= 5.600941\bar{\beta}_1^5 - 7.803477\bar{\beta}_1^3\bar{\beta}_2 + 2.450686\bar{\beta}_1^2\bar{\beta}_3 \\
&\quad + 2.355199\bar{\beta}_1\bar{\beta}_2^2 - 0.949480\bar{\beta}_1\bar{\beta}_4 \\
&\quad - 0.899401\bar{\beta}_2\bar{\beta}_3 + 0.731424\bar{\beta}_5,
\end{aligned}$$

and one has only to insert the values of the $\bar{\beta}_k$ found under No. 2 above.

4. If, for example, one now wishes to calculate the displacement thickness of the boundary layer along the wall, according to (23) one has to evaluate the formula

$$(121) \quad \frac{\delta^* \text{Re}^\frac{1}{2}}{L} = \frac{1}{\bar{g}(\bar{x})} \left\{ \lim_{\eta \rightarrow \infty} (\eta - \bar{F}_0) - \sum_k \bar{F}_k(\infty) \bar{\xi}^k \right\}.$$

For this purpose we find in our tables the necessary coefficients for the sum in the brackets. As was the case for (120), usually the high accuracy of our tables will not be required. Although higher accuracy is possible, we give the following values rounded to 4 decimal places:

$$\begin{aligned}
&\lim_{\eta \rightarrow \infty} (\eta - \bar{F}_0(\eta)) = 1.2168 \\
&\bar{F}_1(\infty) = 1.1264\bar{\beta}_1 \\
&\bar{F}_2(\infty) = -1.6745\bar{\beta}_1^2 + 0.8976\bar{\beta}_2 \\
&\bar{F}_3(\infty) = 3.2677\bar{\beta}_1^3 - 2.6783\bar{\beta}_1\bar{\beta}_2 + 0.7642\bar{\beta}_3 \\
(122) \quad &\bar{F}_4(\infty) = -7.5042\bar{\beta}_1^4 + 7.9494\bar{\beta}_1^2\bar{\beta}_2 - 2.2862\bar{\beta}_1\bar{\beta}_3 \\
&\quad - 1.0873\bar{\beta}_2^2 + 0.6743\bar{\beta}_4 \\
&\bar{F}_5(\infty) = 18.9844\bar{\beta}_1^5 - 24.5873\bar{\beta}_1^3\bar{\beta}_2 + 6.8467\bar{\beta}_1^2\bar{\beta}_3 \\
&\quad + 6.5130\bar{\beta}_1\bar{\beta}_2^2 - 2.0215\bar{\beta}_1\bar{\beta}_4 \\
&\quad - 1.8731\bar{\beta}_2\bar{\beta}_3 + 0.6084\bar{\beta}_5.
\end{aligned}$$

5. Similarly, the momentum thickness ϑ can easily be evaluated. According to (23) we have

$$(123) \quad \frac{\vartheta \text{Re}^\frac{1}{2}}{L} = \frac{1}{\bar{g}(\bar{x})} \int_0^\infty F_\eta (1 - F_\eta) d\eta = \sum_{k=0}^\infty \bar{d}_k \bar{\xi}^k.$$

Since the coefficients $\bar{d}_k$ cannot be found directly in our tables but must be obtained by quadratures, here we give the values of these coefficients not only for $\beta_0 = 0$ but also for $\beta_0 = 1$.

$\beta_0 = 0$:

$$\begin{aligned}
 \bar{d}_0 &= 0.4696 \\
 \bar{d}_1 &= -0.2180\bar{\beta}_1 \\
 \bar{d}_2 &= 0.1259\bar{\beta}_1^2 - 0.1556\bar{\beta}_2 \\
 \bar{d}_3 &= -0.0995\bar{\beta}_1^3 + 0.1723\bar{\beta}_1\bar{\beta}_2 - 0.1223\bar{\beta}_3 \\
 (124) \quad \bar{d}_4 &= 0.1169\bar{\beta}_1^4 - 0.2082\bar{\beta}_1^2\bar{\beta}_2 + 0.1314\bar{\beta}_1\bar{\beta}_3 \\
 &\quad + 0.0609\bar{\beta}_2^2 - 0.1014\bar{\beta}_4 \\
 \bar{d}_5 &= -0.1831\bar{\beta}_1^5 + 0.3378\bar{\beta}_1^3\bar{\beta}_2 - 0.1606\bar{\beta}_1^2\bar{\beta}_3 \\
 &\quad - 0.1493\bar{\beta}_1\bar{\beta}_2^2 + 0.1064\bar{\beta}_1\bar{\beta}_4 \\
 &\quad + 0.0946\bar{\beta}_2\bar{\beta}_3 - 0.0868\bar{\beta}_5
 \end{aligned}$$

$\beta_0 = 1$:

$$\begin{aligned}
 \bar{d}_0 &= 0.2923 \\
 \bar{d}_1 &= -0.0704\bar{\beta}_1 \\
 \bar{d}_2 &= 0.0172\bar{\beta}_1^2 - 0.0529\bar{\beta}_2 \\
 \bar{d}_3 &= -0.0044\bar{\beta}_1^3 + 0.0260\bar{\beta}_1\bar{\beta}_2 - 0.0446\bar{\beta}_3 \\
 (125) \quad \bar{d}_4 &= 0.0012\bar{\beta}_1^4 - 0.0098\bar{\beta}_1^2\bar{\beta}_2 + 0.0209\bar{\beta}_1\bar{\beta}_3 \\
 &\quad + 0.0100\bar{\beta}_2^2 - 0.0388\bar{\beta}_4 \\
 \bar{d}_5 &= -0.0004\bar{\beta}_1^5 + 0.0035\bar{\beta}_1^3\bar{\beta}_2 - 0.0078\bar{\beta}_1^2\bar{\beta}_3 \\
 &\quad - 0.0074\bar{\beta}_1\bar{\beta}_2^2 + 0.0175\bar{\beta}_1\bar{\beta}_4 \\
 &\quad + 0.0164\bar{\beta}_2\bar{\beta}_3 - 0.0344\bar{\beta}_5
 \end{aligned}$$

6. For details of the boundary layer flow one usually evaluates the velocity profiles in several cross-sections $\bar{x}$:

$$(126) \quad \bar{u}(\bar{x}, \bar{y}) = \bar{U}(\bar{x}) \sum_k \bar{F}'_k(\eta) \bar{\xi}^k.$$

For this purpose we need the tables of $F_k(\eta)$ at a sufficient number of wall distances η or $\bar{y}$.

All in all, the numerical work necessary to apply the series is minimal and elementary. The reason is, of course, that the main part of the work has already been done, once and for all, in supplying the tables of the universal functions. This is the great advantage of the series method, thus leading to a rigorous procedure for solving boundary layer problems for arbitrary outer pressure distributions that can certainly compare in simplicity of application to any of the non-rigorous and unreliable methods of the POHLHAUSEN type.

To our last sentence we should have added:—so long as the convergence of the series is sufficiently rapid to yield a good approximation with the six orders $k = 0, 1, \dots, 5$ for which the universal functions are available. As a rigorous estimate of the remainder of the series is not possible, the rapidity of convergence and the degree of our 6-term approximation can be tested only by the rate at which the terms grow small with increasing order, and by the satisfying of necessary conditions such as the first condition of compatibility at the wall (25). The situation is, of course, quite the same in the case of the older series of BLASIUS-HOWARTH type. In most cases the graphs of the wall shearing stress, the displacement thickness, *etc.*, together with the numerical comparison of partial sums for increasing available orders, will be quite sufficient to determine the cross-section with increasing $\bar{x}$ or $\bar{\xi}$ where the approximation begins to fail to show sufficient accuracy. Furthermore, the fulfillment of the first condition of compatibility at the wall will often offer to some degree a method of checking the results. According to (25), for a range of increasing values of $\bar{\xi}$ we must investigate the accuracy of the approximation

$$(127) \quad \sum_{k=0}^5 \bar{F}_k''''(0) \bar{\xi}^k \approx -\bar{\beta}(\bar{\xi}) \quad \text{or} \quad \bar{U}(\bar{x}) \bar{g}^2(\bar{x}) \sum_{k=0}^5 \bar{F}_k''''(0) \bar{\xi}^k \approx \frac{1}{\rho \mu_0^2} \frac{dp}{d\bar{x}},$$

where p is the given outer pressure distribution. Therefore $(1/\rho \mu_0^2) dp/d\bar{x} = -\bar{U}(\bar{x}) \bar{U}'(\bar{x})$. Often in cases of application a truncated series will satisfy this condition exactly and therefore offer no possibility of checking. It can easily be stated when this will be the case. For the BLASIUS series, *cf.* [25]. For our series it follows from (127) and from $\bar{F}_k''''(0) = -\bar{\beta}_k$ that the fifth order approximation offered by our truncated series will satisfy the first condition of compatibility at the wall if and only if the principal function $\bar{\beta}(\bar{\xi})$ is a polynomial in $\bar{\xi}$ of degree not higher than 5.

In those cases of adverse or favorable pressure gradient where a continuation of the solution downstream past a cross-section in which the truncated series begins to fail as an approximation is desired, it is recommended that one apply a difference procedure (*e.g.* [26], [27], the latter being a modification of the former by H. WITTING, especially suited to studies of the immediate vicinity of separation points). The difference method has proved to be a reliable tool. It is also often a better method of *checking* than (127). If one starts a continuation calculation using the difference procedure and the initial velocity profiles used are a poor approximation, then this will usually show up in the first step of the difference method.

A. Applications of the series for Case A.

§12. Flow along a plane wall in a divergent channel.

As outer potential flow we consider the flow emerging from a source located at the point $x = -L$, $y = 0$ in plane Cartesian coordinates x , y . In our dimensionless coordinates (117) this is the point $\bar{x} = -1$, $\bar{y} = 0$. The positive

x axis, $x \geq 0$, $y = 0$, is assumed to be a flat semi-infinite plate (Fig. 1). Therefore the leading edge of the flat plate lies at a distance L from the source. This plate may represent one of the two non-parallel plane walls of a divergent channel. We are mainly interested in calculating that point of the wall where the boundary layer separates from the wall and purely divergent flow in the channel becomes impossible. But we wish also to obtain full details of the boundary layer flow from the leading edge downstream to the separation point, since accurate results for the velocity profiles should be of interest for a future investigation of the stability of this boundary layer flow.

As is well known (cf. S. GOLDSTEIN (ed.) [28], pp. 105-110, where further references can also be found), the less realistic limiting case $L = 0$, where the channel walls intersect at the point where the source is located, has been investigated with some success as a special case of HAMEL flows. In this case, after

FIG. 1. Flat plate in diverging flow.

introducing polar coordinates with origin in the location of the source, separation of variables is possible. This is not possible if the leading edge of the plane wall is at a finite distance $L > 0$ from the source. On the other hand, since the assumption $L = 0$ does not seem very satisfactory from the standpoint of actual physical situations of interest, here it is assumed that $L > 0$.

The velocity of the retarded outer flow is

$$(128) \quad \bar{U}(\bar{x}) = \frac{1}{1 + \bar{x}}, \quad \bar{x} > -1 \quad \left(U(x) = \frac{u_0 L}{L + x} \right).$$

The corresponding principal function $\bar{\beta}(\bar{\xi})$ can easily be calculated explicitly. One finds the simple expression

$$(129) \quad \bar{\beta}(\bar{\xi}) = -2\bar{\xi} \quad \left(\beta(\xi) = -\frac{2\xi}{\text{Re}} \text{ with } \text{Re} = \frac{u_0 L}{\nu} \right).$$

Therefore we have $\bar{\beta}_1 = -2$, $\bar{\beta}_k = 0$ for $k = 0$ and $k = 2, 3, 4, \dots$.

As the principal function is a polynomial of degree not exceeding five, the first condition of compatibility at the wall is satisfied exactly for all $\bar{x} \geq 0$ by our new series truncated after the fifth order term. (Cf. §11.)

The terms of the new series simplify considerably in the case of (129) with only $\bar{\beta}_1 \neq 0$. Of the universal functions only F_0 , f_1 , f_{11} , f_{111} , f_{1111} , f_{11111} are needed.

The truncated series with the six terms available in our tables gives good accuracy (the accuracy of a good graph) for values of $\bar{x}$ from the leading edge $\bar{x} = 0$ downstream to about $\bar{x} = 0.15$. With a short extrapolation the separation point is found at $\bar{x} = 0.160$ with a possible error of $\pm 1\%$.

Although the terms of the series decrease satisfactorily with increasing order for all $\bar{x}$ until immediately ahead of the separation point, the evaluation downstream from the separation point indicates that the separation point sets a limit to the convergence of the series. We should expect this.

With L as the distance of the leading edge from the source (or virtual source) we may state the following: *Separation occurs approximately at a distance $x = L/6$ from the leading edge.*

Fig. 2 shows the resulting distribution of the local shearing stress at the wall. The dotted extension from $\bar{x} = 0.15$ onwards shows the result given by the

FIG. 2. Local shearing stress along the plate. Separation at $\bar{x}_A = 0.160$.

Unbroken line: Result of new series.

Dotted extension: Result of new series where accuracy of 5th order approximation is no longer sufficient.

Dash-dot extension: Result of extrapolation by difference method.

Dashed line: Local wall shearing stress for BLASIUS flat plate ($U = u_0$), for comparison.

truncated series where convergence is no longer satisfactory. The extension with dots and dashes is the result of our extrapolation, found by applying our difference procedure [27], starting at $\bar{x} = 0.140$ and taking two steps downstream of size $\Delta\bar{x} = 0.010$. This result of short extrapolation will prove to be characteristic for the course of the wall shearing stress immediately ahead of a separation point as compared with the result given by the truncated series: The curve turns down to the $\bar{x}$ axis. Since all the experience we have gained in many examples indicates this same typical form for the curve immediately ahead of the separation point, the application of the difference procedure seems dispensable within requirements of practical accuracy, as one can guess the course of this short extrapolation within the accuracy of a graph.

For comparison, the dashed line in Fig. 2 shows the local wall shearing stress for the flat plate with constant outer velocity $u = u_0$ (BLASIUS solution). For this case we have

$$\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} = \frac{F_0''(0)}{\sqrt{2\bar{x}}} = \frac{0.469600}{\sqrt{2\bar{x}}}$$

according to our tables, and this is equal to the well known result $0.33206/\sqrt{\bar{x}}$ for this case.

The evaluation of the displacement thickness and the momentum thickness according to (121), (122) and (123), (124) requires just as little work as that of the shearing stress. The result is shown in Fig. 3 in comparison with the corresponding quantities along the BLASIUS plate (dashes). For the ratio δ^*/ϑ , cf. Fig.13 below.

The velocity profiles have been calculated for some cross-sections. Results are demonstrated for eight velocity profiles $\bar{u}(\bar{x}, \bar{y})$ in Fig. 4. (The relation between our series coefficient variable η and the dimensionless Cartesian coordinate $\bar{y}$ is now $\eta = \bar{y}/(1 + \bar{x}) (2 \cdot \ln(1 + \bar{x}))^{\frac{1}{2}}$.)

§13. Flow along a flat plate of finite depth in a convergent channel.

We now assume a potential flow sink at the point $x = +L, y = 0$ ($\bar{x} = 1, \bar{y} = 0$) of the Cartesian coordinates x, y . The interval $0 \leq \bar{x} \leq \bar{x}_1 < 1, \bar{y} = 0$ may now represent a flat plate of finite depth $x = \bar{x}_1 L$, where L is the distance of the *leading* edge of the flat plate from the sink further downstream behind the plate (Fig. 5).

The velocity of the accelerated outer flow is

$$(130) \quad \bar{U}(\bar{x}) = \frac{1}{1 - \bar{x}}, \quad -\infty \leq \bar{x} < 1 \quad \left(U(x) = \frac{u_0 L}{L - x} \right),$$

and the corresponding principal function is easily found to be

$$(131) \quad \bar{\beta}(\bar{\xi}) = +2\bar{\xi} \quad \left(\beta(\xi) = \frac{2\xi}{\text{Re}} \text{ with } \text{Re} = \frac{u_0 L}{\nu} \right).$$

FIG. 3. Displacement thickness δ^* and momentum thickness ϑ (unbroken lines). The dotted extensions are the results given by the truncated new series (5th order approximation) where this approximation becomes unsatisfactory and the difference method was applied to get the more accurate results. For comparison, again the corresponding quantities, δ_B^* and ϑ_B , for the BLASIUS flat plate are shown (dash lines). ($Re = u_0 L / \nu$)

Comparison with the problem of the preceding §12 shows that only the sign of the principal function has changed.

Again and for the same reason as in §12, the first condition of compatibility along the wall is satisfied exactly by our series truncated after the fifth order term.

With respect to the question how far downstream from the leading edge $\bar{x} = 0$ the truncated series will offer a satisfactory approximation, a rough estimate can be given from the results of §12. As now the variable $\bar{\xi}$ grows more

FIG. 4. Velocity profiles of the boundary layer as functions of $\bar{y} = y \operatorname{Re}^{1/2}/L$ ($\operatorname{Re} = u_0 L/\nu$) at cross-sections $\bar{x} = 0.02(0.02)0.16$. For the cross-section $\bar{x} = 0.16$ the dotted line is the no longer sufficiently accurate result given by the 5th order truncated series approximation, whereas the unbroken line is the more accurate result found by applying the difference procedure.

rapidly with increasing $\bar{x}$, the approximation given by the truncated series will fail at a cross-section a little upstream compared with $\bar{x} = 0.15$ of §12.

The evaluation of the distribution of the wall shearing stress shows that a satisfactory approximation is found for all cross-sections from the leading edge downstream to about $\bar{x} = 0.13$. We are therefore able to calculate the complete

FIG. 5. Flat plate in converging flow.

boundary layer flow along plates with depth up to roughly $L/8$, where L is the distance from the *leading* edge to the location of the sink (or apparent sink) of the convergent channel flow.

Fig. 6 presents the shearing stress along the plate (unbroken line) compared with the shearing stress along the BLASIUS flat plate with constant outer velocity $U = u_0$. Fig. 7 gives the corresponding comparison for the displacement thickness and the momentum thickness. (For ratio δ^*/ϑ , cf. Fig. 13.) In Fig. 8 some velocity profiles are shown.

§14. New series solution for the HOWARTH flat plate flow with outer velocity

$$U(x) = u_0 (1 - x/L).$$

We now assume the dimensionless outer flow velocity

$$(132) \quad \bar{U}(\bar{x}) = 1 - \bar{x}.$$

FIG. 6. Local wall shearing stress.

Unbroken line: Result of truncated series as far as 5th order approximation is satisfactory within the accuracy of the graph.

Dotted extension: Result of 5th order approximation where accuracy is no longer satisfactory.

Dashed line: For comparison, local wall shearing stress along the BLASIUS flat plate with $U = u_0$.

FIG. 7. Corresponding distribution of displacement thickness δ^* and momentum thickness ϑ , compared with δ_B^* and ϑ_B along the BLASIUS flat plate. ($Re = u_0 L/\nu$.)

This case of flow along a flat plate with retarding pressure gradient varying linearly with the distance x from the leading edge $x = 0$ was first solved rigorously by L. HOWARTH with the aid of a series of the BLASIUS type (power series with respect to the distance x along the wall from the forward edge). This series gives the velocity u in the boundary layer in the form [29]

$$(133) \quad \frac{u(\bar{x}, \bar{\eta})}{u_0} = \frac{1}{2} \{ h'_0(\bar{\eta}) - 8\bar{x}h'_1(\bar{\eta}) + (8\bar{x})^2 h'_2(\bar{\eta}) - \dots \} \quad \text{with} \quad \bar{\eta} = \frac{\bar{y}}{2\sqrt{\bar{x}}}.$$

Therefore $\frac{1}{2} h'_0(\bar{\eta}) = F'_0(\eta)$, with $\eta = \sqrt{2}\bar{\eta}$, is the universal function of the BLASIUS flat plate flow. The outer boundary condition $u/u_0 \rightarrow 1 - \bar{x}$ is satisfied by requiring $h'_0(\infty) = 2$, $h'_1(\infty) = \frac{1}{4}$, $h'_k(\infty) = 0$ for $k = 2, 3, \dots$. The outer boundary condition is therefore satisfied exactly by the first two terms of the series, while the higher order terms $k = 2, 3, \dots$ of the series only give correc-

FIG. 8. Velocity profiles for the cross-sections $\bar{x} = 0.02(0.02)0.12$.

tions in the inner parts of the boundary layer. Thanks to the low order of the outer velocity polynomial, the series (133), compared with the new series, therefore shows an advantage similar to that given by the first term $\bar{U}(\bar{x})F'_0(\eta)$ of our series. A recent extension of the HOWARTH-BLASIUS series to outer velocities of the form $\bar{U}(\bar{x}) = 1 - \bar{x}^n$ ($n = 1, 2, 3, \dots$) is due to I. TANI [30], [31]. The application of our new series to this class of flows will be given in a paper soon to be published.

HOWARTH tabulated the functions $h_k(\eta)$ and their first and second derivatives for $k = 0, 1, 2, \dots, 6$. In order to obtain still better results for the local wall shearing stress, he also evaluated $h_k''(0)$ for $k = 7$ and 8. He applied his series from the forward edge $\bar{x} = 0$ downstream till $\bar{x} = 0.0875$. From this cross-section downstream he utilized a special continuation method developed for this special purpose, leading to a separation point located at $\bar{x} = 0.120$. (Cf. Fig. 9 and the enlarged representation of the vicinity of the separation point in Fig. 10. Curve H or H_8 is the 8th order approximation given by the HOWARTH series, while Hc is the result of the HOWARTH continuation process.)

The HOWARTH series as well as our new series give the local wall shearing stress as a monotone decreasing series. The rapidity of convergence of the new series is demonstrated in Fig. 11, where graphs of the partial sums for orders 0, 1, 2, 3, 4, and 5 are given. It can be seen from this figure that the 6th and higher order terms will effect only small corrections immediately ahead of the separation point.

As must be expected, the zero order term of the new series, since it satisfies

FIG. 9. Local wall shearing stress.

- S new series, 5th order approximation.
- H HOWARTH series, 8th order approximation.
- Hc result of HOWARTH's continuation method.
- Sc result of continuation by applying the difference method.

the outer boundary condition for all $\bar{x}$, gives an approximation from the forward edge further downstream than the zero order term of the HOWARTH series. From our numerical data we have gained the impression that the new series converges more rapidly than the HOWARTH series for growing values of $\bar{x}$ up to about $\bar{x} = 0.05$. HOWARTH's 5th order approximation for the wall shearing stress, 5th order with respect to $\bar{x}$, and our 5th order approximation with respect to $\bar{\xi}$ (curves H_5 and S_5 in Fig. 10) coincide until far downstream and still agree well in the vicinity of the separation point. This indicates that immediately ahead of the separation point the rapidity of convergence of both series will be about equal. (HOWARTH's evaluations mentioned above allow an

FIG. 10. Local wall shearing stress. Enlargement of Fig. 9 in the neighborhood of the separation point.

S and S_5 new series, 5th order approximation.

H_5 HOWARTH series, 5th order approximation, for comparison.

H and H_8 HOWARTH series, 8th order approximation.

H_c result of HOWARTH's continuation method.

S_c result of continuation with the aid of the difference method.

8th order approximation of the shearing stress at the wall (curve H_8 in Fig. 10) but the accuracy of his higher order coefficients $h_k''(0)$ is probably not very good.)

The 5th order approximation in ξ available from the tables for our new series gives satisfactory results for wall shearing stress and velocity profiles until about $\bar{x} = 0.1125$. With the results in this cross-section as initial velocity profile, the continuation of the solution was calculated with the aid of our difference method as in the case in the preceding section. The result for the wall shearing stress is shown in Fig. 9 and Fig. 10 as the unbroken curve S_c . Separation was found at $\bar{x} = 0.126$. Compared with $\bar{x} = 0.120$ found by HOWARTH, the difference is of no practical importance. Still, from the mathematical standpoint, the continuation method applied by HOWARTH is not based on rigorous considerations.* In Table 1 giving velocity profiles at a number of cross-sections we

*I wish to thank Prof. L. HOWARTH for calling my attention to recent work done by D. C. LEIGH (*Proc. Cambridge Phil. Soc.* **51**, 320–332 (1955)). LEIGH, using EDSAC in Cambridge, has recomputed the point of separation applying former results of GOLDSTEIN and JONES. Since his result is $\bar{x} = 0.1198$ in agreement with that of HOWARTH, we have repeated our calculation, starting the difference procedure much further upstream. We believe we can claim that we arrived at our old result within the accuracy of 1%, now finding $\bar{x} = 0.125$. In other words our result depends only weakly on the cross-section where the difference procedure is started. It must, again, be emphasized that the small difference between our result and that of HOWARTH and LEIGH is only of mathematical interest.

FIG. 11. Local wall shearing stress. Approximations of increasing order given by the partial sums of the new series for orders 0, 1, 2, 3, 4, and 5, to demonstrate the rapidity of convergence of the new series.

therefore give also the result obtained with the aid of the difference method at $\bar{x} = 0.125$ (separation at $\bar{x} = 0.126$) in the last column.

It should be added that the first condition of compatibility at the wall is satisfied exactly by the HOWARTH series because of the low degree of the polynomial $\bar{U}(\bar{x})$. But also the new series satisfies this condition for all $\bar{x}$ with very high accuracy (relative deviation below 0.1%).

Fig. 12 shows the displacement thickness δ^* and the momentum thickness ϑ of the boundary layer. Fig. 13 gives a comparison of the ratios δ^*/ϑ for the examples discussed in this and the preceding sections.

FIG. 12. Displacement thickness δ^* and momentum thickness ϑ (unbroken lines). The dotted extensions are the results of the new series 5th order approximation where this approximation is unsatisfactory and the difference method was applied to get the more accurate results. The corresponding quantities δ_B^* and ϑ_B for the BLASIUS flat plate ($U = u_0$) are shown as dashed lines. ($Re = u_0 L / \nu$.)

B. Application of the series for Case B.

§15. Preliminary remarks concerning the accuracy of the tables of universal functions of the BLASIUS series hitherto available.

Case B is the case of outer velocities of the form $U(x) = \sum_{k=0}^{\infty} u_k x^k$, $u_0 = 0$, $u_1 \neq 0$ and of corresponding principal functions of the form $\beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^{\frac{1}{2}k}$, $\beta_0 = 1$. Here, the symmetrical subcase, dealt with in Parts I and II as Case B1, with outer velocities $U(x) = \sum_{k=0}^{\infty} u_{2k+1} x^{2k+1}$ ($u_{2k} = 0$, $u_1 \neq 0$), is of special interest because one obtains for the principal function a power series, $\beta(\xi) = \sum_{k=0}^{\infty} \beta_k \xi^k$ ($\beta_0 = 1$).

Up to now the only solution valid for outer velocities of Class B has been the series due to H. BLASIUS [32]. It was one of our aims to find a new series with properties better suited to the requirements of applications. It will be seen in the following section and from further examples to be published soon that we have succeeded only in part. Compared with the success in application of the new series to outer velocity distributions of Class A reported in the preceding sections, the success in applications to outer velocity distributions of Type B is less satisfactory.

As must be expected, the first term of the new series will prove to give a good

FIG. 13. Ratio δ^*/δ for the examples of this and the preceding sections.

approximation of the solution from the forward stagnation point $x = 0$ to cross-sections much further downstream than the first term of the BLASIUS series. In fact, we believe this to be the main achievement with regard to flows belonging to Case B. The possibility of giving a one-term approximation of the boundary layer flow over a long interval along the wall from the forward stagnation point downstream will be of considerable advantage for theoretical and probably also for practical purposes.

In the further hope that convergence of the new series would prove to be better still further downstream and up to cross-sections immediately ahead of separation points, we have been disappointed. More carefully stated: If one applies the tables at present available for the universal functions of the BLASIUS series,

TABLE 1: Values of $\bar{u}$ for the case $\bar{U} = 1 - \bar{x}$

$\bar{x}$	0.0125	0.0250	0.0375	0.0500	0.0625	0.0750	0.0875	0.1000	0.1125	0.1250	0.1250 diff. method
0	0	0	0	0	0	0	0	0	0	0	0
0.2	0.0880	0.0818	0.0753	0.0685	0.0612	0.0535	0.0452	0.0361	0.0263	0.0154	0.0083
0.4	0.1767	0.1654	0.1534	0.1409	0.1275	0.1132	0.0977	0.0810	0.0627	0.0427	0.0273
0.6	0.2657	0.2502	0.2339	0.2166	0.1981	0.1782	0.1567	0.1333	0.1076	0.0794	0.0571
0.8	0.3541	0.3354	0.3156	0.2946	0.2721	0.2479	0.2215	0.1926	0.1608	0.1256	0.0909
1.0	0.4406	0.4197	0.3975	0.3738	0.3483	0.3208	0.2906	0.2575	0.2209	0.1802	0.1458
1.2	0.5240	0.5016	0.4779	0.4525	0.4251	0.3954	0.3627	0.3266	0.2866	0.2420	0.2028
1.4	0.6026	0.5796	0.5552	0.5290	0.5007	0.4698	0.4358	0.3981	0.3560	0.3091	0.2666
1.6	0.6750	0.6521	0.6277	0.6016	0.5732	0.5422	0.5079	0.4693	0.4272	0.3795	0.3352
1.8	0.7399	0.7176	0.6939	0.6685	0.6409	0.6106	0.5771	0.5398	0.4979	0.4509	0.4065
2.0	0.7965	0.7752	0.7526	0.7284	0.7021	0.6733	0.6414	0.6058	0.5659	0.5210	0.4778
2.2	0.8443	0.8242	0.8030	0.7803	0.7558	0.7289	0.6992	0.6662	0.6291	0.5874	0.5468
2.4	0.8832	0.8646	0.8449	0.8239	0.8013	0.7766	0.7495	0.7194	0.6858	0.6480	0.6112
2.6	0.9139	0.8966	0.8784	0.8591	0.8385	0.8161	0.7917	0.7647	0.7348	0.7013	0.6695
2.8	0.9372	0.9211	0.9043	0.8866	0.8678	0.8476	0.8258	0.8019	0.7756	0.7463	0.7193
3.0	0.9543	0.9392	0.9235	0.9072	0.8900	0.8718	0.8523	0.8312	0.8081	0.7829	0.7611
3.2	0.9663	0.9520	0.9373	0.9221	0.9063	0.8897	0.8721	0.8533	0.8332	0.8113	0.7944
3.4	0.9744	0.9607	0.9467	0.9324	0.9177	0.9023	0.8863	0.8695	0.8516	0.8325	0.8200
3.6	0.9797	0.9665	0.9530	0.9393	0.9253	0.9110	0.8961	0.8807	0.8646	0.8477	0.8387
3.8	0.9830	0.9701	0.9570	0.9437	0.9303	0.9166	0.9026	0.8883	0.8735	0.8581	0.8518
4.0	0.9850	0.9723	0.9594	0.9465	0.9334	0.9202	0.9068	0.8931	0.8792	0.8649	0.8608
4.2	0.9862	0.9735	0.9608	0.9481	0.9352	0.9223	0.9093	0.8961	0.8828	0.8692	0.8666
4.4	0.9869	0.9742	0.9616	0.9490	0.9363	0.9236	0.9108	0.8979	0.8849	0.8718	0.8703
4.6	0.9872	0.9746	0.9621	0.9495	0.9369	0.9243	0.9116	0.8989	0.8861	0.8733	0.8724
4.8	0.9873	0.9748	0.9623	0.9498	0.9372	0.9246	0.9121	0.8994	0.8868	0.8741	0.8736
5.0	0.9874	0.9749	0.9624	0.9499	0.9374	0.9248	0.9123	0.8997	0.8872	0.8746	0.8742
5.2	0.9875	0.9750	0.9625	0.9499	0.9374	0.9249	0.9124	0.8999	0.8873	0.8748	0.8746
5.4	0.9875	0.9750	0.9625	0.9500	0.9375	0.9250	0.9125	0.8999	0.8874	0.8749	0.8748
5.6	0.9875	0.9750	0.9625	0.9500	0.9375	0.9250	0.9125	0.9000	0.8875	0.8750	0.8749
5.8	0.9875	0.9750	0.9625	0.9500	0.9375	0.9250	0.9125	0.9000	0.8875	0.8750	0.8750
6.0	0.9875	0.9750	0.9625	0.9500	0.9375	0.9250	0.9125	0.9000	0.8875	0.8750	0.8750

comparison of results shows no better convergence of the new series far downstream than of the BLASIUS series.

The restriction in our statement is necessary. A critical examination of the probable accuracy of the existing tables of higher order universal functions of the BLASIUS series leads us to question whether these tables can claim any reasonable degree of accuracy. Therefore, although in applying these tables convergence seems to be as good as that of the new series, only the tables of the new series are reliable, being indeed highly accurate.

A discussion of the accuracy of the tables for the universal functions of the BLASIUS series must now follow. The BLASIUS series (*cf. e.g.* [7], [8]) is a power series with respect to arc length x along the wall, measured from the forward stagnation point, with coefficient functions depending on the distance y from the wall. The coefficient functions can be expressed as linear combinations of universal functions depending on the dimensionless variable $y\sqrt{u_1/\nu}$. It is sufficient to restrict the following discussion to the symmetrical case, as orders of tabulated functions are high and compare with the orders available in our tables for the new series only in this Case B1. The traditional notation used for the universal functions is, with increasing order $2k + 1$, $k = 0, 1, 2, \dots$:

$$k = 0: f_1$$

$$k = 1: f_3$$

$$k = 2: g_5, h_5$$

$$k = 3: g_7, h_7, k_7$$

$$k = 4: g_9, h_9, k_9, j_9, q_9.$$

The functions f_1 and f_3 were calculated by K. HIEMENZ [33] in 1911 with not very high accuracy. These functions were recalculated with higher accuracy by L. HOWARTH [34] in 1935. HOWARTH also tabulated g_5 , h_5 , and k_7 . Using HOWARTH's table for f_1 , N. FRÖSSLING [35] in 1940 recalculated with higher accuracy f_3 , g_5 , and h_5 . During the war A. ULRICH used FRÖSSLING's tables and calculated g_7 , h_7 , k_7 , and g_9 , h_9 , k_9 , j_9 , q_9 (*cf.* [36]). These tables were later published in [37]. Along with the tables of the functions mentioned, the authors also gave tables for the first and second derivatives.

The table for f_1 is accurate to 4 decimal places (HOWARTH). Using this table to solve the further recursive system of linear third order differential equations, FRÖSSLING claims accuracy within several units of the 4th decimal place for his results not only for f_3 and its derivatives but also for g_5 and h_5 and their derivatives. Using these tables and carrying 4 decimal places throughout the calculation, ULRICH claims accuracy within one unit of the 3rd decimal place for his tables for g_7 , h_7 , k_7 and g_9 , h_9 , k_9 , j_9 , q_9 . I must leave it to the criticism of any reader who has ever been engaged in the numerical solution of recursive systems of differential equations to decide to what extent these tables are reliable. Taking into account that even if these claims could be upheld,

some of these tables would have only two significant figures, a more careful recalculation of all tables seems indispensable.

As comparison: For the evaluation of the universal functions of the new series 14 decimal places were carried throughout the computation on Mark IV and the results were truncated to six decimal places.

In the following example of application of the new series the results will be compared with those found by application of the BLASIUS series utilizing the above-mentioned tables of universal functions. It must therefore be stressed here that this comparison will always be of only limited value in as far as the results of the BLASIUS series will not be reliable in cross-sections far enough downstream that the higher order terms gain weight, even if the series appears to converge well. This is very unfortunate.*

§16. Boundary layer on the circular cylinder with outer velocity distribution according to HIEMENZ' experimental results.

The experimental pressure distribution obtained by K. HIEMENZ [33] along the contour of a circular cylinder has previously often been used as a basis for the comparison of methods for the approximate solution of the boundary layer equations. In HIEMENZ' experiments the diameter of the cylinder was $2R = 9.75$ cm, the velocity of the oncoming stream was 19.2 cm sec⁻¹, therefore, with $\nu = 0.01$ cm²sec⁻¹ (water), the REYNOLDS number was $Re = U_\infty 2R/\nu = 18720$.

The experimental value found for $U'(0) = u_1$ was 7.151 sec⁻¹. Following HIEMENZ, we choose as reference length $L = 1$ cm. Therefore our reference velocity must be $u_1 L = 7.151$ cm sec⁻¹. According to the polynomial approximation of the outer velocity distribution found by HIEMENZ, we have

$$(134) \quad \bar{U}(\bar{x}) = \bar{x} - 0.006289\bar{x}^3 - 0.00004615\bar{x}^5.$$

Applying the formulæ (61) we find the following values for the coefficients $\bar{\beta}_k$ of the corresponding principal function for $k = 0, 1, \dots, 5$:

$$(135) \quad \begin{aligned} \bar{\beta}_0 &= 1 \\ \bar{\beta}_1 &= -0.018867 \\ \bar{\beta}_2 &= -0.0011295 \\ \bar{\beta}_3 &= -0.000041295 \\ \bar{\beta}_4 &= -0.0000016286 \\ \bar{\beta}_5 &= -0.000000061948. \end{aligned}$$

*During proofreading some results of an investigation by A. N. TIFFORD in which, among other things, the universal functions of the BLASIUS series have been re-evaluated with high accuracy for all odd orders (symmetrical case) from 1 through 11, have been brought to my attention. The work has not yet been published in a journal but it has been distributed in a small number of copies as WADC Techn. Report 53-288, Part 4 (August 1954). A comparison with the tables calculated by ULRICH shows that the latter are, indeed, of very poor accuracy.

Convergence of the new series is sufficiently rapid for a good approximation by the truncated series up to $\bar{x} = 5.5$. Continuation of the results by the difference method was started at $\bar{x} = 5$ as initial cross-section. The results for the velocity distribution in the boundary layer are shown in Fig. 14 representing the variation of $\bar{u}(\bar{x}, \bar{y})$ with $\bar{x}$ at constant distances $\bar{y} = c$ from the wall for several values of c . The unbroken lines are the results of the new series up to $\bar{x} = 5.5$, and from there on downstream the results of continuation by the

FIG. 14. Variation of $\bar{u}(\bar{x}, \bar{y})$ with $\bar{x}$ for constant values of $\bar{y}$.

Unbroken lines: Result of new series 5th order approximation up to $\bar{x} = 5.5$ and of difference method continuation further downstream.

Dotted tangents in $\bar{x} = 0$: Approximation given by first term of BLASIUS series.

Dashed lines: Approximation given by first term of new series.

difference procedure. The tangents to these curves in $\bar{x} = 0$ (dotted lines) show the approximation given by the first term of the BLASIUS series, the dashed curves demonstrate, in comparison, the far better approximation given by the first term of the new series. In another manner, by the graphs of velocity profiles $\bar{u}(\bar{x}, \bar{y})/\bar{U}(\bar{x})$, the same results are presented in Fig. 15. It is seen that the one-term approximation given by the first term of the new series represents a good approximation surprisingly far downstream. Even at $\bar{x} = 4$ the maximum deviation is only about 3 to 4 %. As the central angle, measured in degrees from the forward stagnation point, is $360\bar{x}/9.75\pi = 11.75\bar{x}$, the value $\bar{x} = 4$ is equivalent to the central angle 47° . That is about 60% of the way from the forward stagnation point to the point of separation.

FIG. 15. Velocity profiles $\bar{u}/\bar{U}$ at cross-sections $\bar{x} = 1, 2, 3$, and 4.

Unbroken lines: Results of new series.

Dashed lines: Approximation by first term of Blasius series.

Dotted lines: Approximation by first term of new series.

The first term of the new series gives a satisfactory one-term approximation from the forward stagnation point downstream until $\bar{x} = 4$ (i.e. about 60% of the way to the separation point).

The variation of the wall shearing stress is shown in Fig. 16. Comparison of the result "S" of the new series with that of the continuation by the difference method shows that the truncated series approximation becomes poor near $\bar{x} = 6$, the deviation growing rapidly further downstream. Whereas the approximation given by the truncated BLASIUS series, judged by comparison of the numerical values of terms of increasing order, seems to be good only to $\bar{x} = 6$, the result for the shearing stress distribution remains in satisfactory agreement with difference method continuation further downstream. Continuation curve

FIG. 16. Variation of local shearing stress along wall.

Unbroken line: Result of new series 5th order approximation up to $\bar{x} = 5.5$ and of difference method continuation (D) further downstream.

S: Result of truncated new series where 5th order approximation with respect to $\bar{\xi}$ is unsatisfactory.

B: Result of truncated BLASIUS series where it departs from correct (unbroken line) result and 9th order approximation with respect to $\bar{x}$ is unsatisfactory.

"D" gives separation at $\bar{x} = 6.87$, curve "B" of the truncated BLASIUS series gives separation at $\bar{x} = 7.1$. As an experimental result HIEMENZ obtained separation between $\bar{x} = 6.81$ and $\bar{x} = 6.98$ (*i.e.* for a central angle between 80° and 82°).

Displacement thickness δ^* , momentum thickness ϑ , and ratio δ^*/ϑ are shown in Fig. 17.

The truncated BLASIUS series satisfies exactly the first condition of compatibility at the wall, since in this case $\bar{U}(\bar{x})$ is a polynomial of sufficiently low degree. The truncated new series satisfies this condition to a good approximation where it has been applied. Up to $\bar{x} = 4$ the deviation with reference to

FIG. 17. Displacement thickness δ^* , momentum thickness ϑ , and ratio δ^*/ϑ along wall. Dotted extensions: Results of truncated new series where 5th order approximation is unsatisfactory and difference method of continuation has been applied to get the more accurate results (unbroken lines).

the maximum value of $\bar{U}\bar{U}'$ is below 0.1%, up to $\bar{x} = 5$ below 1.2%, up to $\bar{x} = 6$ below 9%.

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