Linear Parabolic Differential Equations Containing a Small Parameter

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1. We consider the first boundary value problem for the linear parabolic partial differential equation

$$(1.1) \quad L_\epsilon(u) \equiv \epsilon u_{xx} + a(x, y)u_x - b(x, y)u_y + c(x, y)u = d(x, y)$$

where $\epsilon > 0$ is a parameter and $b(x, y) > 0$. Let R be the open region in the x, y -plane bounded above by the line $y = y_2$ and below by the concave-upward curve S . If the data is sufficiently smooth, there exists for each $\epsilon > 0$ a unique solution $u = u(x, y, \epsilon)$ in R of the boundary value problem

$$L_\epsilon(u) = d, u = \varphi \text{ on } S,$$

where φ is a given function. We are interested in the behavior of $u(x, y, \epsilon)$ as $\epsilon \rightarrow 0+$; in particular, in its relation to the *reduced equation*

$$(1.2) \quad L_0(v) \equiv a(x, y)v_x - b(x, y)v_y + c(x, y)v = d(x, y).$$

Indeed, if φ and S are such that there exists throughout $\bar{R}$ a solution $v = v(x, y)$ of the initial value problem

$$L_0(v) = d, v = \varphi \text{ on } S,$$

which is of class C^2 in R , then it is easy to show that

$$(1.3) \quad u(x, y, \epsilon) = v(x, y) + O(\epsilon)$$

uniformly in $\bar{R}$ as $\epsilon \rightarrow 0+$.*

* If A is an open set, we will denote by $\bar{A}$ the closure of A .

The general situation is more complicated. Suppose, *e.g.*, that for some subarc I of S the initial value problem

$$L_0(v) = d, v = \varphi \text{ on } I,$$

has the solution $v^* = v^*(x, y)$ in the closed region $Q \subset \bar{R}$ bounded by subarcs I and J of S and the curves γ_1 and γ_2 (see Fig. 1). In general, $v^* \neq \varphi$ on J , and

FIG. 1

hence a relationship like (1.3) cannot hold uniformly in Q . However, we show under suitable hypotheses that

$$(1.4) \quad u(x, y, \epsilon) = v^*(x, y) + z(x, y, \epsilon) + O(\epsilon)$$

uniformly in Q as $\epsilon \rightarrow 0+$, where $z = \varphi - v^*$ on J , $z = e^{-f(x,y)/\epsilon} h(x, y)$, $f > 0$ near J , and $z = O(e^{-\delta/\epsilon})$, $\delta > 0$ away from J . Since z is important only in the neighborhood of the boundary at J , it is called the *boundary layer* term.

Our result, which will be formulated more precisely below, is analogous to the result of LEVINSON [3] for the elliptic equation

$$\epsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y).$$

2. Let D be an open region in the x, y -plane, and let $\bar{S}_2$ and $\bar{S}_3$ be two arcs in D which are defined for $y_1 \leq y \leq y_2$ and which do not meet for $y > y_1$. Denote by $\bar{R} \subset D$ the closed region bounded above by a segment $\bar{S}_4$ of $y = y_2$, below by a segment $\bar{S}_1$ of $y = y_1$, and laterally by the $\bar{S}_i$ ($i = 2, 3$). Let s denote arc length on $\bar{R}$ — $R = S_1 + \bar{S}_2 + \bar{S}_3 + S_4$ measured positively in the counter-clockwise direction. We consider in $\bar{R}$ the boundary value problem

$$(2.1) \quad L_\epsilon(u) = d(x, y), u = \varphi_i \text{ on } \bar{S}_i \quad (i = 1, 2, 3),$$

where the φ_i are given functions of s on the $\bar{S}_i$. We will assume

- (I) $a(x, y)$, $b(x, y)$, $c(x, y)$, and $d(x, y)$ are of class C^4 in D .
- (II) $\epsilon > 0$ and $b(x, y) > m > 0$ in D .
- (III) $\bar{S}_2$ and $\bar{S}_3$ are of class C^4 with respect to s and are nowhere horizontal.
- (IV) The φ_i ($i = 1, 2, 3$) are of class C^4 with respect to s and $\varphi_i = \varphi_1$ at $\bar{S}_i \cap \bar{S}_1$ ($j = 2, 3$).

It is known that, under these assumptions, there exists in $\bar{R}$ for each $\epsilon > 0$ a unique regular solution $u = u(x, y, \epsilon)$ of the boundary value problem (2.1).* (See Lemma 3.1 below.)

The characteristics of the reduced equation (1.2) are the solutions of the ordinary differential equations

$$(2.2) \quad \frac{dx}{dt} = -a(x, y), \quad \frac{dy}{dt} = b(x, y).$$

In view of (I) and (II), there exists a unique solution of (2.2) through every point of D which can be continued as far as the boundary of D and which is continuous with respect to its initial values. From (II) it follows that there are no critical points or limit cycles of (2.2) in D . Hence any solution of (2.2) which starts in a compact subregion of D cannot remain in that subregion for all t .

Let I be a closed subarc of S_i ($i = 1, 2, 3$) such that no characteristic of (1.2) is tangent to I and such that every characteristic which starts on I enters R , for increasing t , and first leaves R via a closed subarc $J \subset S_2 + S_3 + \bar{S}_4 - I$. In addition, let no characteristic starting on I be tangent to J . We call the closed region bounded by I , J , and the characteristics joining their end points a *regular multilateral* M . Clearly $M \subset \bar{R}$.

If $\bar{R} - R$ is given by $x = x(s)$, $y = y(s)$, then the quantity

$$(2.3) \quad \Delta = -a \frac{dy}{ds} - b \frac{dx}{ds}, \quad (x, y) \text{ on } \bar{R} - R,$$

is related to the angle between the normal to $\bar{R} - R$ exterior to R and the characteristics of (1.2) through (x, y) . In particular, we have

$$(2.4) \quad \Delta < 0 \text{ on } I, \Delta > 0 \text{ on } J.$$

If a characteristic of (1.2) is tangent to $\bar{R} - R$ at any point, then $\Delta = 0$ there. Thus, *every point in $\bar{R}$ which does not lie on a characteristic of (1.2) which is tangent to $\bar{R} - R$ or on the characteristics through $\bar{S}_1 \cap \bar{S}_i$ ($i = 2, 3$) can be included in a regular multilateral.*

We prove

* We say that u is a *regular solution* of $L_\epsilon(u) = d$ in $\bar{R}$ if u satisfies the differential equation in R and is continuous in $\bar{R}$, and if u_x , u_y , and u_{xx} are continuous in $R + S_4$.

Theorem 1. *Let (I)—(IV) hold. Then in any regular multilateral M the solution of (2.1) can be written in the form*

$$(2.5) \quad u(x, y, \epsilon) = v(x, y) + z_2(x, y, \epsilon) + z_3(x, y, \epsilon) + w(x, y, \epsilon),$$

where v is the solution of (1.2) satisfying the given boundary values of u on I , $w = 0$ on $M \cap \{S_1 + \bar{S}_2 + \bar{S}_3\}$, and $w = O(\epsilon)$ uniformly in M as $\epsilon \rightarrow 0+$. If $J \cap \bar{S}_i = 0$ ($i = 2, 3$), then $z_i \equiv 0$. If $J \cap \bar{S}_i \neq 0$ ($i = 2, 3$), then

$$(2.6) \quad z_i(x, y, \epsilon) = e^{-f_i(x, y)/\epsilon} h_i(x, y)$$

at points of M on or near $J \cap \bar{S}_i$, where $f_i = 0$ on $J \cap \bar{S}_i$, $f_i > 0$ off $J \cap \bar{S}_i$, $h_i = u - v$ on $J \cap \bar{S}_i$, and h_i and f_i are of class C^2 . Moreover, there exists a $\delta > 0$ such that $z_i = O(e^{-\delta/\epsilon})$ uniformly in the part of M where (2.6) is no longer valid, and $z_i = 0$ on I .

The construction of the boundary layer term z_i of (2.5) is similar to that of the corresponding term in [3]; this is carried out briefly in §4. The proof of the theorem depends on the estimation of a regular solution of $L_\epsilon(u) = O(\epsilon)$ in the interior of a regular multilateral $\hat{M}$, which is known to vanish on $\hat{M} \cap \{S_1 + \bar{S}_2 + \bar{S}_3\}$ but which is only known to be bounded on the characteristic boundaries of $\hat{M}$ (§5). The required estimate is obtained in §6 by means of a suitable comparison function. In [3] the corresponding estimate is obtained by means of the so-called characteristic integral inequality. We note that the present method can be applied in the elliptic case and does, in fact, yield a sharper error estimate under the hypotheses of [3].

It has recently been brought to our attention that S. L. KAMENOMOSTSKAYA [2] has also dealt with (2.1) by means of the same type of comparison function, but only in the case in which the lateral boundaries $\bar{S}_i$ ($i = 2, 3$) of R are characteristics of the reduced equation. She shows that

$$u(x, y, \epsilon) = v(x, y) + z(x, y, \epsilon) + w(x, y, \epsilon),$$

where $z = \varphi_i - v$ on $\bar{S}_i$ ($i = 2, 3$), $|z| \leq Ke^{-g_i/\sqrt{\epsilon}} + \sqrt{\epsilon}r$, $g_i > 0$ near $\bar{S}_i$ ($i = 2, 3$), and $|z| \leq Ke^{-\delta/\sqrt{\epsilon}} + \sqrt{\epsilon}r$, $\delta > 0$ elsewhere.

3. In this section we collect a number of results to which we refer in the remainder of this paper. Let R be any region of the general type described at the beginning of §2 and consider the equation

$$(3.1) \quad \Lambda_\epsilon(u) \equiv \epsilon u_{xx} + \alpha(x, y)u_x - \beta(x, y)u_y + \gamma(x, y)u = \delta(x, y)$$

in R . Then we have:

Lemma 3.1. *Suppose that*

(i) $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$, and $\delta(x, y)$ are of class C^1 in D .

- (ii) $\epsilon > 0$ and $\beta(x, y) > m > 0$ in D .
- (iii) $\bar{S}_2$ and $\bar{S}_3$ are piecewise smooth and nowhere horizontal.
- (iv) Functions φ_i of class C^1 are given on $\bar{S}_i$ ($i = 1, 2, 3$), where $\varphi_i = \varphi_1$ at $\bar{S}_i \cap \bar{S}_1$ ($j = 2, 3$).

Then, for fixed ϵ , (3.1) has a unique regular solution $u = u(x, y, \epsilon)$ in $\bar{R}$, which is equal to φ_i on $\bar{S}_i$ ($i = 1, 2, 3$).

Lemma 3.2. If u is a regular solution of $\Lambda_\epsilon(u) = 0$ in $\bar{R}$, where $\epsilon > 0$, $\beta > 0$, $\gamma < 0$, and α, β, γ are bounded in $\bar{R}$, then u cannot take on a positive maximum or a negative minimum in $R + S_4$.

Lemma 3.3. Let u be a regular solution of (3.1) in $\bar{R}$, where $\epsilon > 0$, $\beta > 0$, and α, β , and γ are bounded in $\bar{R}$. If $|u| \leq n$ on $S_1 + \bar{S}_2 + \bar{S}_3$ and $|\delta| \leq n$ in $\bar{R}$, then there exists a constant $K > 0$, independent of ϵ , such that $|u| \leq Kn$ in $\bar{R}$.

Lemma 3.1 is simply a restatement of some of the results of GEVREY; the proof can be found in [1, chapter II]. To prove Lemmas 3.2 and 3.3, put $u = e^{kv}v$, where k is a positive constant such that $\gamma - k\beta \leq -\mu < 0$ in $\bar{R}$ for some $\mu > 0$. Now v satisfies

$$\epsilon v_{xx} + \alpha v_x - \beta v_y + (\gamma - k\beta)v = e^{-kv}\delta.$$

If $|\delta| \leq n$ in $\bar{R}$ then v cannot have a positive maximum or a negative minimum in $R + S_4$ exceeding $n\mu^{-1}e^{-kv_1}$ in magnitude. Suppose, e.g., that v has such a positive maximum at a point P on S_4 . Then at P

$$(3.2) \quad 0 \geq \epsilon v_{xx} - \beta v_y = \delta e^{-kv_2} + (k\beta - \gamma)v.$$

But, if $v > n\mu^{-1}e^{-kv_1}$ at P , then the right hand side of (3.2) is greater than $e^{-kv_2}(\delta + ne^{k(v_2-v_1)}) > 0$, which is impossible. A similar result holds for P in R and for a negative minimum. This proves Lemma 3.2 since, in that case, $n = k = 0$. Now, if $|u| \leq n$ on $S_1 + \bar{S}_2 + \bar{S}_3$ then $|v| \leq ne^{-kv_1}$. It follows from the continuity of v in $\bar{R}$ that $|v| \leq \max\{1, \mu^{-1}\}ne^{-kv_1}$ there. Thus $|u| \leq \max\{1, \mu^{-1}\}ne^{k(v-v_1)} \leq Kn$ in $\bar{R}$.

4. Let $J^* \subset S_i$ ($i = 2, 3$) be any closed subarc such that

$$(4.1) \quad \Delta > 0 \text{ on } J^*.$$

Let $\theta(s)$ be a function of class C^2 defined on J^* which vanishes identically in a neighborhood of each end point of J^* . We consider the boundary value problem

$$(4.2) \quad L_\epsilon(u) = 0, u = \theta(s) \text{ on } J^*, \quad u = 0 \text{ on } S_1 + \bar{S}_2 + \bar{S}_3 - J^*.$$

By Lemma 3.1, there exists a unique regular solution $\tilde{u} = \tilde{u}(x, y, \epsilon)$ of (4.2) in $\bar{R}$. We will show that, near J^* , $\tilde{u}$ has the form

$$(4.3) \quad \tilde{u}(x, y, \epsilon) = e^{-f(x,y)/\epsilon} h(x, y) + \epsilon r(x, y, \epsilon),$$

where $f = 0$, $h = \theta$ on J^* , $f > 0$ off J^* , and $r = O(1)$ uniformly as $\epsilon \rightarrow 0+$.

As in [3], we attempt a formal solution of (4.2) of the form (4.3). Substituting (4.3) in $L_\epsilon(u) = 0$, we find

$$(4.4) \quad \frac{he^{-f/\epsilon}}{\epsilon} \{f_x^2 - af_x + bf_y\} - e^{-f/\epsilon} \{(2f_x - a)h_x + bh_y + (f_{xx} - c)h\} \\ + \epsilon e^{-f/\epsilon} h_{xx} + \epsilon L_\epsilon(r) = 0$$

We determine f and h by the first order equations

$$(4.5) \quad f_x^2 - af_x + bf_y = 0$$

and

$$(4.6) \quad (2f_x - a)h_x + bh_y + (f_{xx} - c)h = 0.$$

In order to identify (4.3) with the actual solution $\tilde{u}$ near J^* we will take $f = 0$ on J^* and $h = \theta(s)$ on J^* . $f \equiv 0$ is clearly one solution of (4.5) which vanishes on J^* . We will show that (4.1) is sufficient for the existence of another solution which is positive in R near J^* .

Put $p = f_x$ and $q = f_y$. Then the characteristic equations of (4.5) are

$$(4.7) \quad \frac{dx}{d\eta} = 2p - a, \quad \frac{dy}{d\eta} = b, \\ \frac{dp}{d\eta} = a_x p - b_x q, \quad \frac{dq}{d\eta} = a_y p - b_y q, \\ \frac{df}{d\eta} = p^2.$$

Since $f = 0$ on J^* , we have $0 = p\dot{x} + q\dot{y}$, where $\dot{}$ denotes d/ds . This, together with (4.5), yields

$$(4.8) \quad p = \frac{a\dot{y} + b\dot{x}}{\dot{y}}, \quad q = -\frac{\dot{x}}{\dot{y}} p \text{ on } J^*.$$

Now, from (4.8) and (4.1),

$$(4.9) \quad \dot{x} \frac{dy}{d\eta} - \dot{y} \frac{dx}{d\eta} = \Delta > 0 \text{ on } J^*.$$

Hence, no characteristic of (4.7) is tangent to J^* . If $d\eta > 0$, then by (4.9) $\dot{x}dy - \dot{y}dx > 0$, i.e., the vector element (dx, dy) makes an acute angle with the

normal to J^* interior to R . Thus a characteristic of (4.7) starting on J^* proceeds into R for increasing η . By (4.8) and (4.1), we have $p \neq 0$ on J^* . Hence $f > 0$ in R near J^* .

The solutions of (4.7) are functions of η and their initial values $x(s, 0) = x(s)$, $y(s, 0) = y(s)$, $f(s, 0) = 0$, and (4.8), i.e., they are functions of η and s . By continuity, (4.9) will continue to hold on S , a little beyond the end points of J^* . It follows that there exists an open set in the x, y -plane which contains J^* where $\partial(x, y)/\partial(s, \eta) \neq 0$ and where $f = f(x, y)$ exists, is of class C^4 , and satisfies (4.5).

Let $\bar{\Sigma}$ be an arc in $\bar{R}$ passing through the end points of J^* and denote by $T \subset R$ the open region bounded by Σ and J^* . We can choose Σ so that x and y are of class C^3 with respect to arc length along it, and so that the normals to it exterior to T are directed into R , and sufficiently close to J^* so that $\bar{T}$ is contained in the open set where f exists and $p^2 > 0$ in $\bar{T}$. By the last equation of (4.7), there exists a $\delta_1 > 0$ such that $f > \delta_1 \eta$ in $\bar{T}$.

Since the characteristics of (4.6) are precisely those of (4.5), it follows that the solution $h = h(x, y)$ of (4.6) exists in $\bar{T}$, where $h = \theta$ on J^* and h is of class C^2 . Since θ vanishes near the end points of J^* and (4.6) is homogeneous, h is identically zero in two curvilinear triangles in $\bar{T}$, each containing an end point of J^* and bounded by segments of J^* , Σ , and a characteristic of (4.6). Set $z(x, y, \epsilon) = e^{-f/\epsilon} h$ in $\bar{T}$. Then z is of class C^2 in $\bar{T}$. Moreover, there exists a $\delta > 0$ such that z and its derivatives are $O(e^{-\delta/\epsilon})$ uniformly on $\bar{\Sigma}$ for small ϵ .

It is shown in [3, pp. 436–437] that z can be continued into $\bar{R} - \bar{T}$ in such a way that z is of class C^2 in $\bar{R}$, z and its partial derivatives are $O(e^{-\delta/\epsilon})$ for small ϵ at points of $\bar{R} - \bar{T}$ near Σ , and $z \equiv 0$ in the remainder of $\bar{R} - \bar{T}$.

Put $\tilde{u} = z + \epsilon r$. Then, since $\tilde{u}$ is the regular solution of (4.2) in $\bar{R}$ and z is of class C^2 there, r is the regular solution of

$$L_\epsilon(r) = -\frac{1}{\epsilon} L_\epsilon(z), r = 0 \text{ on } S_1 + \bar{S}_2 + \bar{S}_3,$$

in $\bar{R}$. From (4.4), (4.5), and (4.6) we have $L_\epsilon(r) = -e^{-f/\epsilon} h_{xx}$ in $\bar{T}$. Moreover, $L_\epsilon(r) = O(\epsilon^{-1} e^{-\delta/\epsilon})$ uniformly in $\bar{R} - \bar{T}$ for small ϵ . Hence $L_\epsilon(r) = O(1)$ uniformly in $\bar{R}$ for small ϵ . It follows from Lemma 3.3 that $r = O(1)$ uniformly in $\bar{R}$ as $\epsilon \rightarrow 0+$. Thus we have

$$(4.10) \quad \tilde{u}(x, y, \epsilon) = z(x, y, \epsilon) + \epsilon r(x, y, \epsilon),$$

where $z = e^{-f/\epsilon} h$ in $\bar{T}$, $z = O(e^{-\delta/\epsilon})$ uniformly in $\bar{R} - \bar{T}$ as $\epsilon \rightarrow 0+$, and $r = O(1)$ uniformly in $\bar{R}$ as $\epsilon \rightarrow 0+$.

5. Let M be a fixed regular multilateral and let J be given in terms of arc length for $s_1 \leq s \leq s_2$. Each characteristic of (1.2) in M meets J exactly once, hence the coordinate s denotes a unique characteristic in M . Since (2.4) involves

strict inequalities and since no characteristic in M meets $\bar{R} - R - I - J$, we can always extend I and J to obtain an enlarged regular multilateral M_3 which contains the characteristics $s_1 - 3\rho \leq s \leq s_2 + 3\rho$ for some $\rho > 0$. Thus $M = M_0$ can always be imbedded in a family of regular multilaterals M_k ($k = 0, 1, 2, 3$) which contain the characteristics $s_1 - k\rho \leq s \leq s_2 + k\rho$. We will denote by I_k and J_k the non-characteristic boundaries of M_k . Let $v = v(x, y)$ be the solution of the reduced equation (1.2) which takes on the given values of u on I_2 . It is clear that v exists throughout M_2 and is unique there. Moreover, v is of class C^4 .

We now define the term w in (2.5). To this end we distinguish the following three cases: (i) $J \subset S_4$, (ii) $J \subset S_i$ for $i = 2$ or 3 , and (iii) $J \cap \bar{S}_4 \neq \emptyset$, $J \not\subset S_4$. These cases are shown in Figure 2; the arc abc in the figure is a characteristic of

FIG. 2

(1.2) which is tangent to S_2 at b . Note that in each case M can be imbedded in a family of regular multilaterals of the same type.

Case (i): We define $w = w(x, y, \epsilon)$ in M_2 by

$$(5.1) \quad u(x, y, \epsilon) = v(x, y) + w(x, y, \epsilon),$$

where u is the unique regular solution of (2.1) in $\bar{R}$.

Case (ii): Let $\theta(s) = u - v$ on J_2 , and let θ be continued over $J_3 - J_2$ so as to be of class C^2 on J_3 and vanish identically in a neighborhood of $s_1 - 3\rho$ and $s_2 + 3\rho$. Let $\tilde{u} = \tilde{u}(x, y, \epsilon)$ be the regular solution of $L_\epsilon(u) = 0$ in $\bar{R}$ which equals θ on J_3 and zero on $S_1 + \bar{S}_2 + \bar{S}_3 - J_3$. We define w in M_2 by

$$(5.2) \quad u(x, y, \epsilon) = v(x, y) + \tilde{u}(x, y, \epsilon) + w(x, y, \epsilon).$$

Case (iii): Let $\bar{R}^* \subset D$ be the closed region bounded below by S_1 , above by a segment of the line $y = y_3 > y_2$, and laterally by the arcs $\bar{S}_i^*$ ($i = 2, 3$) which

are obtained by continuing the $\tilde{S}_i$ above $y = y_2$ in such a way that (III) continues to hold. Now, $J_k \cap \tilde{S}_i \neq \emptyset$ ($i = 2, 3; k = 0, 1, 2, 3$) for at least one value of i . For such an i we have $J_2 \cap \tilde{S}_i \subset S_i^*$ and $\Delta > 0$ on $J_2 \cap \tilde{S}_i$. By continuity, there exists a closed subarc $J_i^* \subset S_i^*$ which coincides with $J_3 \cap \tilde{S}_i$ below $\tilde{S}_4 \cap \tilde{S}_i$, which extends a finite distance above $\tilde{S}_i \cap \tilde{S}_4$, and which is such that $\Delta > 0$ on J_i^* . Let $\theta_i(s) = u - v$ on $J_2 \cap \tilde{S}_i$ and let θ_i be continued over $J_i^* - J_2$ so as to be of class C^2 on J_i^* and vanish identically in a neighborhood of each end point of J_i^* . Denote by $\tilde{u}_i = \tilde{u}_i(x, y, \epsilon)$ the regular solution of $L_\epsilon(u) = 0$ in $\tilde{R}^*$ which equals θ_i on J_i^* and zero on $S_1 + \tilde{S}_2^* + \tilde{S}_3^* - J_i^*$. Note, since $\tilde{R}^* \subset D$ and $\Delta > 0$ on $J_i^* \subset S_i^*$, that the results of §4 can be applied to $\tilde{u}_i$. If $J_k \cap \tilde{S}_i = \emptyset$, we set $\tilde{u}_i \equiv 0$. We define w in M_2 by

$$(5.3) \quad u(x, y, \epsilon) = v(x, y) + \tilde{u}_2(x, y, \epsilon) + \tilde{u}_3(x, y, \epsilon) + w(x, y, \epsilon).$$

In all the above cases we have that w is a regular solution of

$$(5.4) \quad L_\epsilon(w) = -\epsilon v_{xx}$$

in M_2 and $w = 0$ on $M_2 \cap \{S_1 + \tilde{S}_2 + \tilde{S}_3\}$. By Lemma 3.3, u and the $\tilde{u}$'s are uniformly bounded in $\tilde{R}$ as $\epsilon \rightarrow 0+$. Hence, w is uniformly bounded in M_2 as $\epsilon \rightarrow 0+$. We will show in §6 that these facts imply that $w = O(\epsilon)$ uniformly in $M_0 = M$ as $\epsilon \rightarrow 0+$. This, together with (4.10), will complete the proof of Theorem 1.

6. The partial differential equation

$$(6.1) \quad a(x, y)\sigma_x - b(x, y)\sigma_y = 0$$

has solutions $\sigma = \sigma(x, y)$ of class C^4 in M_2 and the curves $\sigma = \text{constant}$ are the characteristics of the reduced equation (1.2). Let $\sigma = \sigma_k^{(1)}$ and $\sigma = \sigma_k^{(2)}$ be the characteristics which bound M_k ($k = 0, 1, 2$), where, e.g., $\sigma_2^{(1)} < \sigma_1^{(1)} < \sigma_0^{(1)} < \sigma_0^{(2)} < \sigma_1^{(2)} < \sigma_2^{(2)}$. We will prove

Theorem 2. *Let (I)—(III) hold and let $w = w(x, y, \epsilon)$ be a regular solution of $L_\epsilon(u) = \epsilon g(x, y)$ in M_2 , where g is of class C^1 there. If w satisfies*

$$(6.2) \quad w = 0 \text{ on } M_2 \cap \{S_1 + \tilde{S}_2 + \tilde{S}_3\}, |w| < B \text{ on } \sigma = \sigma_2^{(1)} \text{ and } \sigma = \sigma_2^{(2)} \text{ in } M_2,$$

where B is independent of ϵ , then

$$(6.3) \quad w = O(\epsilon) \text{ uniformly in } M_0 \text{ as } \epsilon \rightarrow 0+.$$

Since we are only interested in the order of magnitude of w , we can assume without loss of generality that $c(x, y) < 0$ in $\tilde{R}$. Let $\tilde{M}_k$ be the closed region obtained by excluding the part of M_k above the line $y = y^*$, where y^* is the largest constant such that the line $y = y^*$ has at least one point in common

with M_{k-1} . If $J_k \cap \bar{S}_4 \neq 0$, then $\hat{M}_k \equiv M_k$; if $J_k \cap \bar{S}_4 = 0$, then $M_{k-1} \subset \hat{M}_k \subset M_k$. The latter case is shown in Figure 3, where $M_k = abcdefghia$, $\hat{M}_k = abcefgghia$, and $M_{k-1} = bcghb$.

FIG. 3

We will need the following:

Lemma 6.1. *Let (I)—(III) hold. If $W = W(x, y, \epsilon)$ is a regular solution of $L_\epsilon(W) = 0$ in $\hat{M}_k$ which satisfies*

$W = 0$ on $\hat{M}_k \cap \{S_1 + \bar{S}_2 + \bar{S}_3\}$, $|W| < K\epsilon^n$ on $\sigma = \sigma_k^{(1)}$ and $\sigma = \sigma_k^{(2)}$ in $\hat{M}_k$, where K is independent of ϵ , then

$$W = O(\epsilon^{n+\frac{1}{2}}) \text{ uniformly in } M_{k-1} \text{ as } \epsilon \rightarrow 0+.$$

Proof of Lemma 6.1: Let Γ_k denote the boundary of $\hat{M}_k$ below $y = y^*$. (In Figure 3, $\Gamma_k = cbaihgf$.) Suppose that we have in $\hat{M}_k$ a regular solution $V = V(x, y, \epsilon)$ of $L_\epsilon(V) = 0$ such that

$$(6.4a) \quad V \geq K\epsilon^n \text{ on } \sigma = \sigma_k^{(1)} \text{ and } \sigma = \sigma_k^{(2)} \text{ in } \hat{M}_k,$$

$$(6.4b) \quad V \geq 0 \text{ on } \Gamma_k \cap \{S_1 + \bar{S}_2 + \bar{S}_3\},$$

$$(6.4c) \quad V = O(\epsilon^{n+\frac{1}{2}}) \text{ uniformly in } M_{k-1} \text{ as } \epsilon \rightarrow 0+.$$

By Lemma 3.2, $V \geq 0$ in $\hat{M}_k$. Consider $P = V - W$. Clearly $L_\epsilon(P) = 0$ and $P \geq 0$ on Γ_k . It follows from Lemma 3.2 that $V \geq W$ in $\hat{M}_k$. Similarly, by considering $V + W$, we find $W \geq -V$ in $\hat{M}_k$. In view of (6.4c) this proves the Lemma once a suitable V is exhibited.

Consider the function

$$(6.5) \quad V(x, y, \epsilon) = \sum_{\alpha=1}^2 \exp \{-\gamma_k^{(\alpha)}\} h^{(\alpha)}(x, y, \epsilon) + \sqrt{\epsilon} Z(x, y, \epsilon),$$

where $\gamma_k^{(\alpha)} = (\sigma - \sigma_k^{(\alpha)})^2 / \sqrt{\epsilon}$. By a simple computation, taking into account (6.1), we find

$$(6.6) \quad \begin{aligned} L_\epsilon(V) = \sum_{\alpha=1}^2 \exp \{-\gamma_k^{(\alpha)}\} [ah_x^{(\alpha)} - bh_y^{(\alpha)} + \{c + 4(\sigma - \sigma_k^{(\alpha)})^2 \sigma_x^2\} h^{(\alpha)}] \\ + \sqrt{\epsilon} G(x, y, \epsilon) + \sqrt{\epsilon} L_\epsilon(z), \end{aligned}$$

where

$$(6.7) \quad \begin{aligned} G(x, y, \epsilon) = \sum_{\alpha=1}^2 \exp \{-\gamma_k^{(\alpha)}\} [\sqrt{\epsilon} h_{xx}^{(\alpha)} - 4(\sigma - \sigma_k^{(\alpha)}) \sigma_x h_x^{(\alpha)} \\ - 2\{\sigma_x^2 + (\sigma - \sigma_k^{(\alpha)}) \sigma_{xx}\} h^{(\alpha)}]. \end{aligned}$$

We take $h^{(\alpha)} = h^{(\alpha)}(x, y, \epsilon)$ to be the solution of the initial value problem

$$(6.8) \quad \begin{aligned} ah_x^{(\alpha)} - bh_y^{(\alpha)} + \{c + 4(\sigma - \sigma_k^{(\alpha)})^2 \sigma_x^2\} h^{(\alpha)} = 0 \\ h^{(\alpha)} = \frac{K\epsilon^n}{N} \text{ on } I_k, \end{aligned}$$

where N is a positive constant to be specified below. The characteristics of (6.8) are precisely those of (1.2). It follows from (2.4) that the unique solution $h^{(\alpha)}$ of (6.8) exists throughout $\hat{M}_k$ and is of class C^3 there. Let t be as in (2.2) and let $t = 0$ correspond to I_k . Along each characteristic, $h^{(\alpha)}$ is continued in $\hat{M}_k$ by means of

$$(6.9) \quad \frac{dh^{(\alpha)}}{h^{(\alpha)}} = \{c + 4(\sigma - \sigma_k^{(\alpha)})^2 \sigma_x^2\} dt.$$

Hence $h^{(\alpha)} > 0$ in $\hat{M}_k$, and $h^{(\alpha)}$ and its partial derivatives are all $O(\epsilon^n)$ uniformly in M_k . On $\sigma = \sigma_k^{(\alpha)}$, we have from (6.9) that $h^{(\alpha)} = K\epsilon^n N^{-1} \exp \{\int_0^t c d\tau\}$. Thus, if N is the minimum of $\exp \{\int_0^t c d\tau\}$ on $\hat{M}_k$ it follows that $0 < N \leq 1$ and $h^{(\alpha)} \geq K\epsilon^n$ on $\sigma = \sigma_k^{(\alpha)}$ in $\hat{M}_k$.

Now, it follows from (6.7) that G is of class C^1 in $\hat{M}_k$ and is $O(\epsilon^n)$ uniformly in $\hat{M}_k$. By Lemma 3.1, there exists in $\hat{M}_k$ a unique regular solution $Z = Z(x, y, \epsilon)$ of the boundary value problem

$$L_\epsilon(Z) = -G, \quad Z = 0 \text{ on } \Gamma_k.$$

Moreover, we have, by Lemma 3.3, that $Z = O(\epsilon^n)$ uniformly in $\hat{M}_k$. With the $h^{(\alpha)}$ and Z thus determined, the function V given by (6.5) is a regular solution of $L_\epsilon(V) = 0$ which satisfies (6.4a, b, c).

Proof of Theorem 2: By Lemma 3.1, there exists in $\hat{M}_2$ a unique regular solution $W_0 = W_0(x, y, \epsilon)$ of the boundary value problem

$$L_\epsilon(W_0) = \epsilon g, \quad W_0 = 0 \text{ on } \Gamma_2,$$

and, by Lemma 3.3, $W_0 = O(\epsilon)$ uniformly in $\hat{M}_2$. Put $w = W_0 + W$. Then W is a regular solution of $L_\epsilon(W) = 0$ in $\hat{M}_2$ which satisfies (6.2) on Γ_2 . It suffices to prove that (6.3) holds for W . By Lemma 6.1, with $n = 0$ and $k = 2$, we have $W = O(\epsilon^{1/2})$ uniformly in M_1 , *i.e.*, there exists a $K > 0$, independent of ϵ , such that $|W| < K\epsilon^{1/2}$ in M_1 . Since $\hat{M}_1 \subseteq M_1$, it follows from Lemma 6.1, with $n = \frac{1}{2}$ and $k = 1$, that $W = O(\epsilon)$ uniformly in M_0 as $\epsilon \rightarrow 0+$.

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