Bounds for Solutions of a System of Partial Differential Equations

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1. Introduction.*

The deep insight into the theory of harmonic functions obtained from the connection between them and analytic functions of a complex variable stimulates us to extend this approach for the study of other linear partial differential equations. The question of generalization of these methods to the case of functions of two variables satisfying linear partial differential equations is one of the problems which arises in this connection. Equations in two variables and with entire coefficients have been investigated from this point of view in a number of papers (see [1, 2, 3, 4]). In particular, in this case various integral operators transforming analytic functions of a complex variable into solutions of these equations have been defined, and various properties of solutions of equations mentioned above have been established. These investigations showed that there exist *various* operators generating solutions of the same equation and that for various purposes it is convenient to use different operators.

It is natural to attempt to extend this approach to differential equations and systems of differential equations in three and more variables. The attempts in this direction show that a still greater variety of operators appear and that the research requires not merely a technical improvement in those used in the case of two variables. In many instances we have to introduce and develop completely new procedures, having no analogy in the cases of the differential equation in two variables.

The theory of systems of partial differential equations which can be connected with analytic functions of two complex variables by using conveniently chosen operators seems of particular interest in this respect. In a recent paper [5], SCHIFFER and the author derived an integral operator transforming certain

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combinations of analytic functions of two complex variables into functions of four real variables satisfying the system

$$(1) \quad \begin{aligned} \tilde{L}_k(\tilde{\psi}) &= \frac{1}{4}\Delta_k\tilde{\psi} + \tilde{F}_k(x_k, y_k)\tilde{\psi} = 0, \quad k = 1, 2, \\ \tilde{\psi} &= \tilde{\psi}(x_1, y_1, x_2, y_2), \quad \Delta_k\tilde{\psi} \equiv \frac{\partial^2\tilde{\psi}}{\partial x_k^2} + \frac{\partial^2\tilde{\psi}}{\partial y_k^2}. \end{aligned}$$

The theory of functions of two complex variables in many instances uses procedures which have no analogy in the case of one variable. For instance, in the case of two complex variables we consider domains whose boundary consists of segments of analytic hypersurfaces; these we call *domains with a distinguished boundary surface*. The intersections of these hypersurfaces form a two-dimensional manifold, called the *distinguished boundary surface* of the domain, which plays a role to some degree similar to that of the boundary curve in the case of one variable.

In this case, the (closed) domain consists of (1) the interior J^4 , (2) the distinguished boundary surface B^2 , and (3) the complementary part of the three-dimensional boundary b^3 . The behavior of the functions in the complementary part b^3 of the boundary shows features which cannot be compared with the behavior of functions of one complex variable, either on the boundary curve or in the interior of the domain.

It is of interest to investigate what properties of the behavior in b^3 of a solution of the system (1) can be used to obtain bounds for the solution $\tilde{\psi}$, bounds which are essentially independent of the functions F_k .

As is shown in the corollary (see p. 1001), some results of this type can be immediately generalized to the case of a system of two non-linear equations

$$(2) \quad \begin{aligned} \frac{1}{4}\Delta_k\tilde{\Psi} + \tilde{F}_k(x_k, y_k)\tilde{\Psi} &= P_k(\tilde{\Psi}, x_1, y_1, x_2, y_2), \quad k = 1, 2, \\ \tilde{\Psi} &\equiv \tilde{\Psi}(x_1, y_1, x_2, y_2), \end{aligned}$$

where P_k , considered as a function of $\tilde{\Psi}$ is entire and non-negative for $|x_k|^2 + |y_k|^2 \leq 1$, $k = 1, 2$, and $\tilde{F}_k(x_k, y_k) \leq 0$ there.

2. A representation of solutions of (1) in terms of two functions of two complex variables.

As stressed in the introduction, in this paper and in some earlier ones [3, 5] we make an attempt to connect the theory of linear and certain non-linear equations with the theory of functions of two complex variables. In our investigation it is convenient to use complex notation. We shall write

$$\begin{aligned} z_k &= x_k + iy_k, \quad \bar{z}_k = x_k - iy_k, \quad k = 1, 2, \\ \tilde{F}_k(x_k, y_k) &\equiv \tilde{F}_k\left[\frac{1}{2}(z_k + \bar{z}_k), \frac{1}{2i}(z_k - \bar{z}_k)\right] \equiv F_k(z_k, \bar{z}_k) \end{aligned}$$

and

$$\tilde{\psi}(x_1, y_1, x_2, y_2) \equiv \psi(z_1, \bar{z}_1, z_2, \bar{z}_2).$$

The system (1) assumes the form

$$(3) \quad L_k(\psi) \equiv \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_k} + F_k(z_k, \bar{z}_k)\psi = 0, \quad k = 1, 2.$$

If we continue the functions $\tilde{F}_k(x_k, y_k)$, $k = 1, 2$, and $\tilde{\psi}(x_1, y_1, x_2, y_2)$ to complex values of the variables x_1, y_1, x_2, y_2 and write

$$z_k = x_k + iy_k, \quad z_k^* = x_k - iy_k,$$

then $\psi \equiv \psi(z_1, z_1^*, z_2, z_2^*)$ satisfies the system

$$(3a) \quad \frac{\partial^2 \psi}{\partial z_k \partial z_k^*} + F(z_k, z_k^*)\psi = 0, \quad k = 1, 2,$$

where (for $k = 1$ and $k = 2$), z_k and z_k^* are two independent variables.

For later use we need a representation of solutions ψ of (3) in terms of two arbitrary analytic functions of two complex variables.

THEOREM 1. *For every system (3) there exist four functions, say, $T_k(z_k, \bar{z}_k, \xi_k)$ and $P_k(z_k, \bar{z}_k, \xi_k)$, $k = 1, 2$, such that every solution regular at the origin O can be represented in the neighborhood of O in the form*

$$(4) \quad \begin{aligned} \psi(z_1, \bar{z}_1, z_2, \bar{z}_2) = & \operatorname{Re} \left\{ g_1(z_1, z_2) + \int_{\xi_1=0}^{z_1} T_1(z_1, \bar{z}_1, \xi_1) g_1(\xi_1, z_2) d\xi_1 \right. \\ & + \int_{\xi_2=0}^{z_2} T_2(z_2, \bar{z}_2, \xi_2) g_1(z_1, \xi_2) d\xi_2 \\ & + \int_{\xi_1=0}^{z_1} \int_{\xi_2=0}^{z_2} \left[\prod_{k=1}^2 T_k(z_k, \bar{z}_k, \xi_k) \right] g_1(\xi_1, \xi_2) d\xi_1 d\xi_2 + g_2(z_1, \bar{z}_2) \\ & + \int_{\xi_1=0}^{z_1} P_1(z_1, \bar{z}_1, \xi_1) g_2(\xi_1, \bar{z}_2) d\xi_1 + \int_{\xi_2=0}^{z_2} P_2(z_2, \bar{z}_2, \xi_2) g_2(z_1, \xi_2) d\xi_2 \\ & \left. + \int_{\xi_1=0}^{z_1} \int_{\xi_2=0}^{z_2} \left[\prod_{k=1}^2 P_k(z_k, \bar{z}_k, \xi_k) \right] g_2(\xi_1, \xi_2) d\xi_1 d\xi_2 \right\}. \end{aligned}$$

Here g_1 and g_2 are arbitrary analytic functions of z_1, z_2 and $z_1, \bar{z}_2$, respectively, which are regular at the origin.

Proof. In order to prove (4) we use a representation established in [1, 5], according to which every solution of the system (3) in the neighborhood of the origin can be written in the form

$$\begin{aligned}
 \psi(z_1, \bar{z}_1, z_2, \bar{z}_2) &= \psi_1(z_1, \bar{z}_1, z_2, \bar{z}_2) + \psi_2(z_1, \bar{z}_1, z_2, \bar{z}_2) \\
 (5) \quad &= \operatorname{Re} \left\{ \int_{t_1=-1}^1 \int_{t_2=-1}^1 E_1(z_1, \bar{z}_1, t_1) E_2(z_2, \bar{z}_2, t_2) [f_1(z_1(1-t_1^2), z_2(1-t_2^2)) \right. \\
 &\quad \left. + f_2(z_1(1-t_1^2), \bar{z}_2(1-t_2^2))] \frac{dt_1}{\sqrt{1-t_1^2}} \frac{dt_2}{\sqrt{1-t_2^2}} \right\}.
 \end{aligned}$$

Here $f_k(\zeta_1, \zeta_2)$, $k = 1, 2$, are arbitrary analytic functions which are regular at the origin, and $E_k(z_k, \bar{z}_k, t_k)$, $k = 1, 2$, are entire functions of $z_k, \bar{z}_k$ and regular functions of t_k in $|t_k| \leq 1$ which are determined by (3) in the manner explained in [1, 2, 5]. E_k , when continued to complex values of x_k, y_k , satisfy the relations

$$(6) \quad E_k(z_k, 0, t_k) = E_k(0, z_k^*, t_k) = 1.$$

Let us consider the term

$$\begin{aligned}
 \psi_1(z_1, \bar{z}_1, z_2, \bar{z}_2) &= \operatorname{Re} \left\{ \int_{t_1=-1}^1 \int_{t_2=-1}^1 E_1(z_1, \bar{z}_1, t_1) E_2(z_2, \bar{z}_2, t_2) \right. \\
 (7) \quad &\quad \left. \cdot f_1(z_1(1-t_1^2), z_2(1-t_2^2)) \frac{dt_1}{\sqrt{1-t_1^2}} \frac{dt_2}{\sqrt{1-t_2^2}} \right\}.
 \end{aligned}$$

We continue our functions to complex values of x_k, y_k , $k = 1, 2$. Hence we write z_k^* instead of $\bar{z}_k$, where $z_k^* = x_k - iy_k$. We introduce, for power series developments defined in the neighborhood of the origin, the following operation **S**:

$$(8) \quad \mathbf{S} \left[\sum_{m,n=0}^{\infty} a_{mn} z_k^m \bar{z}_k^{*n} \right] = \frac{1}{2} \left[\sum_{m,n=0}^{\infty} a_{mn} z_k^m z_k^{*n} + \sum_{m,n=0}^{\infty} \bar{a}_{mn} z_k^{*m} z_k^n \right]$$

For real values of x_k, y_k , the operation **S** coincides with “Re”. In this new form “Re” is continued to complex values of x_k, y_k . Repeating the considerations of [2, p. 618], we show that when using the development

$$(9) \quad E_k(z_k, z_k^*, t_k) = 1 + \sum_{n=1}^{\infty} t_k^{2n} z_k^n R_k^{(n)}(z_k, z_k^*)$$

we obtain

$$\begin{aligned}
 \psi_1(z_1, z_1^*, z_2, z_2^*) &= \mathbf{S} \left[g_1(z_1, z_1^*) \right. \\
 &\quad + \sum_{n=1}^{\infty} \frac{\Gamma(2n+1)}{2^{2n} \Gamma(n+1)} R_1^{(n)}(z_1, z_1^*) \\
 &\quad \cdot \int_0^{z_1} \int_0^{z_1^{(1)}} \cdots \int_0^{z_1^{(n-1)}} g_1(z_1^{(n)}, z_2) dz_1^{(n)} \cdots dz_1^{(1)}
 \end{aligned}$$

$$\begin{aligned}
(10) \quad & + \sum_{m=1}^{\infty} \frac{\Gamma(2m+1)}{2^{2m}\Gamma(m+1)} R_2^{(m)}(z_2, z_2^*) \\
& \cdot \int_0^{z_2} \int_0^{z_2^{(1)}} \cdots \int_0^{z_2^{(m-1)}} g_1(z_1, z_2^{(m)}) dz^{(m)} \cdots dz_2^{(1)} \\
& + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Gamma(2n+1)\Gamma(2m+1)R_1^{(n)}R_2^{(m)}}{2^{2n+2m}\Gamma(n+1)\Gamma(m+1)} \\
& \cdot \int_0^{z_1} \cdots \int_0^{z_1^{(n-1)}} \int_0^{z_2} \cdots \int_0^{z_2^{(m-1)}} g_1(z_1^{(n)}, z_2^{(m)}) dz_1^{(n)} \cdots dz_1^{(1)} dz_2^{(m)} \cdots dz_2^{(1)} \Big]
\end{aligned}$$

REMARK. We note that

$$(11) \quad R^{(n)}(z, z^*) = \int_0^{z^*} P^{(2n)}(z, z^*) dz^*,$$

where $P^{(2n)}$ are functions introduced in (3.11) p. 289 of [4]. (See also [1].) Since $n - 1$ integrations by parts yield

$$\begin{aligned}
(12) \quad & \int_0^{z_1} (z_1 - \zeta_1)^{n-1} g_1(\zeta_1, z_2) d\zeta_1 \\
& = \Gamma(n) \int_0^{z_1} \int_0^{z_1^{(1)}} \cdots \int_0^{z_1^{(n-1)}} g_1(z_1^{(n)}, z_2) dz_1^{(n)} \cdots dz_1^{(1)},
\end{aligned}$$

we get

$$\begin{aligned}
(13) \quad & \psi_1(z_1, z_1^*, z_2, z_2^*) = S \Big[g_1(z_1, z_2) \\
& + \sum_{n=1}^{\infty} \frac{R_1^{(n)}(z_1, z_1^*)}{2^{2n}B(n, n+1)} \int_0^{z_1} (z_1 - \zeta_1)^{n-1} g_1(\zeta_1, z_2) d\zeta_1 \\
& + \sum_{m=1}^{\infty} \frac{R_2^{(m)}(z_2, z_2^*)}{2^{2m}B(m, m+1)} \int_0^{z_2} (z_2 - \zeta_2)^{m-1} g_1(z_1, \zeta_2) d\zeta_2 \\
& + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{R_1^{(n)}(z_1, z_1^*) R_2^{(m)}(z_2, z_2^*)}{2^{2(n+m)}B(n, n+1)B(m, m+1)} \\
& \cdot \int_{\zeta_1=0}^{z_1} \int_{\zeta_2=0}^{z_2} (z_1 - \zeta_1)^{n-1} (z_2 - \zeta_2)^{m-1} g(\zeta_1, \zeta_2) d\zeta_1, d\zeta_2 \Big] \\
& = S \Big[g_1(z_1, z_2) + \int_0^{z_1} T_1(z_1, z_1^*, \zeta_1) g_1(\zeta_1, z_2) d\zeta_1 \\
(13a) \quad & + \int_0^{z_2} T_2(z_2, z_2^*, \zeta_2) g_1(z_1, \zeta_2) d\zeta_2 \\
& + \int_{\zeta_1=0}^{z_1} \int_{\zeta_2=0}^{z_2} \left(\prod_{k=1}^2 T_k(z_k, z_k^*, \zeta_k) \right) g_1(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2 \Big],
\end{aligned}$$

where B denotes the Beta function and

$$(14) \quad T_k(z_k, z_k^*, \zeta_k) = \sum_{n=1}^{\infty} \frac{R_k^{(n)}(z_k, z_k^*)(z_k - \zeta_k)^{n-1}}{2^{2n} B(n, n+1)}.$$

In exactly the same way we transform the series which we obtain for $\psi_2(z_1, z_1^*, z_2, z_2^*)$, and for real values of $x_k, y_k, k = 1, 2$ we obtain (4).

REMARK. As shown in [5], ((3.9) p. 85), $g_2(z_1, z_2^*)$ can be chosen in such a way that

$$(15) \quad g_2(0, z_2^*) = g_2(z_1, 0) = 0.$$

In the following we shall always assume that g_2 satisfies (15).

COROLLARY. For the "associate functions" g_1 and g_2 of a real solution $\psi(z_1, \bar{z}_1, z_2, \bar{z}_2)$ we obtain the formulæ

$$(16) \quad g_1(z_1, z_2) + \overline{g_1}(0, 0) = 2\psi(z_1, 0, z_2, 0),$$

$$(17) \quad g_2(z_1, z_2^*) = 2\psi(z_1, 0, 0, z_2^*) - [g_1(z_1, 0) + \overline{g_1}(0, z_2^*)].$$

Proof. Since the relations (6) hold for $E_k(z_k, z_k^*, t_k)$ and since g_1 and g_2 can be determined in such a way that relation (15) holds for g_2 , continuing (5) to complex values of $x_k, y_k, k = 1, 2$ [see (7) and the analogous formula for ψ_2], and substituting $z_1^* = z_2^* = 0$ into (10) and $z_1^* = z_2 = 0$ into the analogous formula, we obtain (16) and (17).

In the following section, we shall consider solutions of (3a) for which $g_1(0, 0)$ is real. We shall denote them as N -solutions.

3. Bounds for solutions ψ of (3) in terms of analytic continuation of ψ .

In the following we shall determine bounds for N -solutions of (3) which are regular in the bicylinder $[|z_1| \leq 1, |z_2| \leq 1]$. In this connection it is useful to introduce for the functions $|T_k|$ and $|P_k|, k = 1, 2$ [see (4)], upper bounds which are independent of ζ_k . Since T_k and P_k are regular in $|z_1| \leq 1, |z_2| \leq 1, 0 \leq |\zeta_k| \leq |z_k|$, such upper bounds exist:

$$(18) \quad |T_k(z_k, z_k^*, \zeta_k)| \leq \tilde{T}_k(|z_k|, |z_k^*|), \quad |P_k(z_k, z_k^*, \zeta_k)| \leq \tilde{P}_k(|z_k|, |z_k^*|), \quad k = 1, 2,$$

where $\tilde{T}_k$ and $\tilde{P}_k$ are suitably chosen functions.

THEOREM 2. Suppose that the N -solution $\psi(z_1, \bar{z}_1, z_2, \bar{z}_2)$ of the system (3) is regular in $[|z_1| \leq 1] \times [|z_2| \leq 1]$. When continued to complex values of $x_k, y_k, k = 1, 2$, ψ (then a solution of (3a)) possesses the property that $\psi(z_1, 0, \exp(i\varphi_2), 0)$ and $\psi(z_1, 0, 0, \exp(-i\varphi_2))$ for every $\varphi_2, 0 \leq \varphi_2 \leq 2\pi$, are univalent functions of z_1 in $|z_1| \leq 1$, and satisfy the relations

$$(19) \quad |\psi(0, 0, \exp(i\varphi_2), 0)| \leq \alpha, \quad |\psi(0, 0, 0, \exp(-i\varphi_2))| \leq \alpha,$$

$$(20) \quad \left| \frac{\partial \psi(z_1, 0, \exp(i\varphi_2), 0)}{\partial z_1} \right|_{z_1=0} \leq \beta, \quad \left| \frac{\partial \psi(z_1, 0, 0, \exp(-i\varphi_2))}{\partial z_1} \right|_{z_1=0} \leq \beta,$$

where α and β are conveniently chosen constants. Then in $[|z_1| \leq 1] \times [|z_2| \leq 1]$

$$(21) \quad |\psi(z_1, \bar{z}_1, z_2, \bar{z}_2)| \leq [4\alpha + 2\beta |z_1| (1 - |z_1|)^{-2}] \left[\prod_{k=1}^2 (1 + \tilde{T}_k(|z_k|, |z_k|) |z_k|) + 2 \prod_{k=1}^2 (1 + \tilde{P}_k(|z_k|, |z_k|) |z_k|) \right]$$

where $\tilde{T}_k$ and $\tilde{P}_k$ are functions introduced in (18); they depend only upon the coefficients F_k of the equation (3).*

Proof. According to (5), $\psi = \psi_1 + \psi_2$, where ψ_1 is given by the right hand side of (13a). An analogous expression (with T_k replaced by P_k) holds for ψ_2 . Putting $z_1 = z_2 = 0$ into (16) and using (19) and the fact that $|\psi(0, 0, 0, 0)| \leq \max_{0 \leq \varphi_2 \leq 2\pi} |\psi(0, 0, \exp(i\varphi_2), 0)|$, we obtain

$$(22) \quad |g_1(0, 0)| \leq \alpha.$$

For every $|z_1| \leq 1$, $g_1(z_1, z_2)$ is a regular function of z_2 ; therefore

$$(23) \quad |g(z_1, z_2)| \leq \max_{0 \leq \varphi_2 \leq 2\pi} |g(z_1, \exp(i\varphi_2))|.$$

For every φ_2 , $\psi(z_1, 0, \exp(i\varphi_2), 0)$ is a univalent function of z_1 , $|z_1| \leq 1$; thus by (16) $g_1(z_1, \exp(i\varphi_2))$ is also a univalent function of z_1 , $|z_1| \leq 1$. From Koebe's theorem (see e.g. [6, p. 78]) and (23) we obtain

$$(24) \quad |g_1(z_1, z_2)| \leq \max_{0 \leq \varphi_2 \leq 2\pi} \left[|g_1(0, \exp(i\varphi_2))| + \left(\left| \frac{\partial g_1(z_1, \exp(i\varphi_2))}{\partial z_1} \right|_{z_1=0} \right) \frac{|z_1|}{(1 - |z_1|)^2} \right] \\ \leq \max_{0 \leq \varphi_2 \leq 2\pi} |g_1(0, \exp(i\varphi_2))| + 2\beta |z_1| (1 - |z_1|)^{-2}.$$

From (16) and (22) follows

$$(25) \quad |g_1(0, \exp(i\varphi_2))| \leq 2\alpha + \alpha = 3\alpha;$$

hence for $|z_1| \leq 1$, $|z_2| \leq 1$,

$$(26) \quad |g_1(z_1, z_2)| \leq 3\alpha + 2\beta |z_1| (1 - |z_1|)^{-2}.$$

* These results can easily be extended to any bounded simply connected product domain $B_1^2 \times B_2^2$ by using the transformation $z_k^* = z_k^*(z_k)$, $k = 1, 2$, which maps this domain onto the bicylinder.

Using Koebe's theorem we obtain

$$(27) \quad \begin{aligned} 2|\psi(z_1, 0, 0, z_2^*)| &\leq 2 \max_{0 \leq \varphi_2 \leq 2\pi} |\psi(z_1, 0, 0, \exp(-i\varphi_2))| \\ &\leq 2\alpha + 2\beta |z_1| (1 - |z_1|)^{-2}. \end{aligned}$$

Thus from (17), (26) and (27) follows, for $|z_1| \leq 1$, $|z_2^*| \leq 1$,

$$(28) \quad |g_2(z_1, z_2^*)| \leq 8\alpha + 4\beta |z_1| (1 - |z_1|)^{-2}.$$

We proceed now to the derivation of an upper bound for the right hand side of (4). Using (18) and (26), we conclude that the first four terms on the right hand side of (4) will be less than or equal to

$$(29) \quad \begin{aligned} &3\alpha + 2\beta |z_1| (1 - |z_1|)^{-2} \left[1 + \hat{T}_1(z_1, \bar{z}_1) \left| \int_0^{z_1} d\zeta_1 \right| \right. \\ &\quad \left. + \tilde{T}_2(z_2, \bar{z}_2) \left| \int_0^{z_2} d\zeta_2 \right| + \hat{T}_1(z_1, \bar{z}_1) \hat{T}_2(z_2, \bar{z}_2) \left| \int_{\zeta_1=0}^{z_1} \int_{\zeta_2=0}^{z_2} d\zeta_1 d\zeta_2 \right| \right] \\ &= 3\alpha + 2\beta |z_1| (1 - |z_1|)^{-2} \prod_{k=1}^2 [1 + \hat{T}_k(z_k, \bar{z}_k) |z_k|] \\ &\quad \hat{T}_k(z_1, z_2) \equiv \tilde{T}_k(|z_1|, |z_2|). \end{aligned}$$

Similarly, using (18) and (28), we obtain an upper bound for the four last terms of the right hand side of (4), namely

$$(30) \quad 8\alpha + 4\beta |z_1| (1 - |z_1|)^{-2} \prod_{k=1}^2 [1 + \hat{P}_k(z_k, \bar{z}_k) |z_k|].$$

Combining (29) and (30) yields (21).

If ψ is not an N -solution, in order to obtain the inequality (22) one has to know a bound for a higher derivative of ψ at the origin O . Repeating then the previous considerations one obtains an analogue of Theorem 2 in the general case.

In Theorem 2 we used the fact that the functions $\psi(z_1, 0, \exp(i\varphi_2), 0)$ and $\psi(z_1, 0, 0, \exp(-i\varphi_2))$ are univalent. In a similar way one can obtain analogous theorems if instead of this property one assumes that in $|z_1| \leq 1$ each of the functions ψ mentioned above omit two values, $\alpha_{k1}(\varphi_2)$ and $\alpha_{k2}(\varphi_2)$, $k = 1, 2$, respectively, such that

$$(31) \quad |\alpha_{k1}(\varphi_2) - \alpha_{k2}(\varphi_2)| \geq A, \quad |\alpha_{k1}(\varphi_2)| + |\alpha_{k2}(\varphi_2)| \leq B < \infty,$$

$A > 0$, $B < \infty$ being constants independent of φ_2 .

4. A generalization of results of Section 3 to a class of non-linear partial differential equations.

The results obtained can be generalized almost immediately to the case of certain non-linear partial differential equations.

COROLLARY: Let $\Psi \equiv \Psi(z_1, \bar{z}_1, z_2, \bar{z}_2)$ be a solution of the system

$$(32) \quad \Psi_{z_k \bar{z}_k} + F_k(z_k, \bar{z}_k)\Psi = P_k(\Psi, z_1, \bar{z}_1, z_2, \bar{z}_2), \quad k = 1, 2,$$

where P_k are entire in Ψ for $|z_1| \leq 1, |z_2| \leq 1$, and $F_k(z_k, \bar{z}_k)$ are entire functions such that

$$(33) \quad F_k(z_k, \bar{z}_k) \leq 0 \quad \text{in} \quad |z_1| \leq 1, \quad |z_2| \leq 1.$$

To a solution Ψ of (32) which is defined in $|z_1| \leq 1, |z_2| \leq 1$, we associate a solution ψ of (3) having the same boundary values on the distinguished boundary surface $|z_1| = 1, |z_2| = 1$, i.e., satisfying

$$(34) \quad \begin{aligned} &\Psi(\exp(i\varphi_1), \exp(-i\varphi_1), \exp(i\varphi_2), \exp(-i\varphi_2)) \\ &= \psi(\exp(i\varphi_1), \exp(-i\varphi_1), \exp(i\varphi_2), \exp(-i\varphi_2)). \end{aligned}$$

If the $\tilde{P}_k(\tilde{\Psi}, x_1, y_1, x_2, y_2) \equiv P_k(\Psi, z_1, \bar{z}_1, z_2, \bar{z}_2)$, $k = 1, 2$, for all real values of the arguments $\tilde{\Psi}, x_k, y_k$, $k = 1, 2$, are non-negative functions of the arguments, then in $[|z_1| \leq 1, |z_2| \leq 1]$

$$(35) \quad \Psi(z_1, \bar{z}_1, z_2, \bar{z}_2) \leq \psi(z_1, \bar{z}_1, z_2, \bar{z}_2).$$

Proof. For every $\varphi_1^0, 0 \leq \varphi_1^0 \leq 2\pi$, and $|z_2| < 1$ the solution Ψ of the system (32) can be represented in the form

$$(36) \quad \begin{aligned} &\Psi(z_1^0, \bar{z}_1^0, z_2, \bar{z}_2) = \psi(z_1^0, \bar{z}_1^0, z_2, \bar{z}_2) \\ &- 4 \iint_{|\zeta_2| < 1} G_2(z_2, \bar{z}_2, \zeta_2, \bar{\zeta}_2) P_2(\Psi, z_1^0, \bar{z}_1^0, \zeta_2, \bar{\zeta}_2) d\zeta_2 d\bar{\zeta}_2, \end{aligned}$$

where $z_1^0 = \exp(i\varphi_1^0)$, G_2 is the Green's function of the equation $\psi_{z_2 \bar{z}_2} + F_2\psi = 0$, and $\zeta_2 \equiv \xi_2 + i\eta_2$. (See [4] p. 300.) Since we assumed that $F_k(z_k, \bar{z}_k) \leq 0$, the Green's function of the equation $L_2(\psi)$ is non-negative (see e.g. [4] p. 261). According to our assumptions, P_2 is also non-negative and therefore the double integral in (36) is non-negative. Consequently

$$(37) \quad \Psi(\exp(i\varphi_1^0), \exp(-i\varphi_1^0), z_2, \bar{z}_2) \leq \psi(\exp(i\varphi_1^0), \exp(-i\varphi_1^0), z_2, \bar{z}_2), |z_2| \leq 1.$$

We now consider the solution $\Psi(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0)$ in the domain $|z_1| < 1, z_2^0 = \text{const}$. It can be written in the form

$$(38) \quad \begin{aligned} &\Psi(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0) = \psi(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0) + \psi_1(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0) \\ &- 4 \iint_{|\zeta_1| < 1} G_1(z_1, \bar{z}_1, \zeta_1, \bar{\zeta}_1) P_1(\Psi, \zeta_1, \bar{\zeta}_1, z_2^0, \bar{z}_2^0) d\zeta_1 d\bar{\zeta}_1, \quad \zeta_1 = \xi_1 + i\eta_1, \end{aligned}$$

where G_1 is the Green's function of the equation $\psi_{z_1 \bar{z}_1} + F_1\psi = 0$ and $\psi_1(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0)$ is the solution of this equation which on the boundary $|z_1| = 1$

assumes the values* $\Psi(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0) - \psi(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0)$. As we showed before, the values of the difference are non-positive on $|z_1| = 1$. Since $F_1 \leq 0$, see (33), $(\Psi - \psi)$ must be non-positive in $|z_1| < 1$.

The last integral is non-negative, since the Green's function of $\psi_{z_1\bar{z}_1} + F_1\psi = 0$ is non-negative and P_1 is positive by assumption. Therefore for every $|z_2| < 1$, $|z_1| < 1$ the inequality (35) holds.

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*We note that the values $\Psi(z_1, \bar{z}_1, z_2^0, \bar{z}_2^0)$ for $|z_1| = 1$ are known from (36).