On the Singularities of Legendre Expansions

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1. In a recent paper [3], G. SZEGÖ proved the following theorem:

Let $P_n(z)$ denote the Legendre polynomial of degree n and let $a_1, a_2, \dots$ be real constants such that $\limsup |a_n|^{1/n} = 1$ for $n \rightarrow \infty$. Let $u(r, \theta)$ denote the harmonic function which, for $0 \leq \theta \leq 2\pi$ and $0 \leq r < 1$, is given by the expansion

$$(1) \quad u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta).$$

Under these assumptions, the point $(1, \theta)$ will be a singular point of the harmonic function $u(r, \theta)$ if, and only if, the point $z = e^{i\theta}$ is a singularity of the analytic function

$$(2) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

In this paper we shall show that this theorem is a limiting case of the following more general result concerned with the singularities of expansions of the type

$$(3) \quad g(t) = \sum_{n=0}^{\infty} a_n P_n(t),$$

where t is a complex variable.

Theorem I: Let $P_n(t)$ denote the Legendre polynomial of degree n and let $a_1, a_2, \dots$ be constants such that

$$(4) \quad \limsup_{n \rightarrow \infty} |a_n|^{1/n} = \zeta < 1.$$

Further, let the analytic function $g(t)$ and $f(z)$ be defined by the expansions

$$(5) \quad g(t) = \sum_{n=0}^{\infty} a_n P_n(t), \quad |t+1| + |t-1| < \frac{1+\zeta^2}{\zeta},$$

$$(6) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < \zeta^{-1},$$

which converge in the regions indicated. Under these assumptions, a point $t = \tau (\tau \neq \pm 1)$, which is reached by analytic continuation along a path C , originating at $t = 1$ and otherwise avoiding the points $t = \pm 1$, will be a singularity of $g(t)$ if, and only if,

$$(7) \quad \tau = \frac{1 + \sigma^2}{2\sigma},$$

where $z = \sigma$ is a singularity of $f(z)$ which is reached by analytic continuation of $f(z)$ along a path C_z originating at $z = 1$ and coinciding with one of the two images of C_t under the conformal mapping

$$(8) \quad t = \frac{1 + z^2}{2z}.$$

2. The proof of theorem I will be based on the following

Lemma: The functions (5) and (6) are connected by the pair of reciprocal integral equations

$$(9) \quad g(t) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(1 - 2zt + z^2)^{\frac{1}{2}}},$$

$$(10) \quad f(z) = \frac{z^2 - 1}{2} \int_{\Gamma} \frac{g(t) dt}{(1 - 2zt + z^2)^{\frac{1}{2}}},$$

where C is the circumference $|z| = (\zeta + \epsilon)^{-1}$, $0 < \epsilon < 1 - \zeta$, and Γ is an arc connecting $t = -1$ with $t = 1$. Γ is restricted to the ellipse E_{ζ} in which the expansion (5) converges and is to be chosen so as to avoid the points at which $1 - 2zt + z^2 = 0$. The branches of the radicals to be taken in (9) and (10) are those which reduce to 1 for $z = 0$.

(9) follows from the classical identity

$$(11) \quad \begin{aligned} z(1 - 2zt + z^2)^{-\frac{1}{2}} &= \left(1 - \frac{2t}{z} + \frac{1}{z^2}\right)^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} P_n(t)z^{-n}, \end{aligned}$$

which is valid for fixed t and sufficiently large $|z|$. The expansion will converge uniformly on the circumference $|z| = (\zeta + \epsilon)^{-1}$ if t is such that the function (11) is regular for $|z| \geq (\zeta + \epsilon)^{-1}$. To show that this is indeed the case we observe that, for fixed t , the function (11) will become singular if z is such as to satisfy (8). But (8) transforms the region $|z| \geq (\zeta + \epsilon)^{-1}$ into the complement of the ellipse E to which t is restricted, and (11) is thus free of singularities in the required region.

For $|z| = (\zeta + \epsilon)^{-1}$, the expansions (6) and (11) converge uniformly and may therefore be used for the computation of the right-hand side of (9). Multiplying out and integrating term by term, we establish (9).

The proof of (10) is based on the identity

$$(12) \quad \frac{1 - z^2}{(1 - 2zt + t^2)} = \sum_{n=0}^{\infty} (2n + 1)P_n(t)z^n$$

which is an elementary consequence of (11). (12) is valid for $t \in E_\zeta$ and $|z| < \zeta$. Since $|P_n(t)| \leq 1$ for $-1 \leq t \leq 1$, the convergence of this expansion is uniform in t in this interval. For values of z such that $|z| < \zeta$ we may thus compute the right-hand side of (10) by substituting the expansions (5) and (12) and integrating over the interval $-1 \leq t \leq 1$. Because of the orthogonality of the Legendre polynomials and the fact that $\int_{-1}^1 P_n^2 dx = 2(2n + 1)^{-1}$, we obtain (10). Evidently, the restriction to values of z such that $|z| < \zeta$ may now be dropped provided care is taken to deform the path of integration in (10) as indicated in the statement of the lemma.

3. Turning now to the proof of Theorem I, we remark that, because of (4), (5) converges uniformly in any closed subdomain of the ellipse E_ζ [2, p. 238], and it therefore represents a regular analytic function $g(t)$ in E_ζ . By the lemma, $g(t)$ has the integral representation (9), which may be used to study the analytic continuation of $g(t)$ along an arc C_t originating in E_ζ . To fix the ideas, we shall assume that C_t starts at the focus $t = 1$ of E_ζ .

We now use an idea which goes back to HADAMARD, who employed it in the proof of his well-known theorem concerning the "multiplication of singularities" [1]. The representation (9) obviously will remain valid if the path of C integration is continuously deformed in such a way that at no time it will be crossed by a singularity of the integrand. These singularities are located at the points $z = \sigma$ at which the function $f(z)$ becomes singular, and at the points z such that

$1 - 2zt + z^2 = 0$. In order to find a deformation of C which will be suitable for determining the analytic continuation of $g(t)$ along C_t , we introduce the arcs C'_z and C''_z ($C'_z + C''_z = C_z$) in the z -plane which are conformal images of C_t as given by the transformation (9). We assume that C_t extends from $t = 1$ to a point $t = t_0$ and that $f(z)$ has no singularities along the corresponding closed arcs C'_z and C''_z between $z = 1$ and the points $z = z'_0$ and $z = z''_0$, respectively ($t(z'_0) = t(z''_0) = t_0$).

We first suppose that the arc C_z does not intersect itself and that it does not contain two points z_1, z_2 such that $z_1 z_2 = 1$. In this case C_t does not intersect itself and there exists a closed domain D_0 such that (a) $f(z)$ is regular in D_0 , and (b) all points of C_z are interior points of D_0 . If we set $t = 1$ in the identity (9), it is clear that the contour C may be continuously deformed into the boundary C' of D_0 without crossing a singularity of the integrand. Hence, (9) may be replaced by

$$(13) \quad g(t) = \frac{1}{2\pi i} \int_{C'} \frac{f(z) dz}{(1 - 2zt + z^2)^{\frac{1}{2}}}$$

for $t = 1$ and, evidently, for all $t \in C_t$ which are between $t = 1$ and the first point of intersection of C_t and the ellipse E_t . If we now let t vary along C_t until it reaches the point $t = t_0$, it follows from the definition of D_0 that in this process the integrand of (13) remains at all times regular. Hence, $g(t)$ may be continued analytically along an arc C_t with the above properties if $f(z)$ may be continued along the arcs C'_z and C''_z which correspond to it by means of the mapping (8). In particular, to each singularity τ of $g(t)$ which is reached along such an arc C_t there must correspond a singularity σ of $f(z)$ such that $1 - 2\sigma\tau + \sigma^2 = 0$.

We now have to remove the restrictions imposed on C_z in order to prevent C_t from intersecting itself. We distinguish two cases: (a) C_z intersects itself; (b) C_z contains two points z_1 and z_2 such that $z_1 z_2 = 1$. In the case (a), we continue $g(t)$ along C_t by means of the procedure just described until we reach a point t_1 which comes after the first, but before the second, passage of the curve through its first point of intersection. The formula (13) is obviously valid for $t = t_1$ if D_0 is such as to exclude the point of intersection. Keeping t_1 fixed, we may now extend D_0 beyond the point of intersection without changing the value of the integral, and it is clear that if t , starting from t_1 , continues to vary along C_t , no singularity of the integrand will cross the boundary of the extended domain D_0 even if t passes through the point of intersection. It follows that such points cannot give rise to singularities of $g(t)$.

In the case (b), the argument is very much the same, provided the coinciding points of C_t belong to different values of z . The argument fails, however, if the two values of z coincide, i.e., for $t = \pm 1$. These points have therefore a special status, and we cannot exclude the possibility of singularities of $g(t)$ at $t = \pm 1$.

(in sheets of its Riemann surface different from the one we started out from) which are not accounted for by the formula (7).

4. To complete the proof of Theorem I, we now have to show that $f(z)$ may be continued along C_z if the continuation of $g(t)$ along C_t is possible. A comparison of the representations (9) and (10) shows that, except for minor modifications, the argument employed in the preceding section will also serve here. If C'_z is a non-self-intersecting path issuing from $z = 1$ and C_t its image as yielded by the mapping (8), the same considerations show that the integration path Γ can be replaced by a path Γ' with the same terminals, which avoids the singularities of $g(t)$ and the arc C_t (except for the point $t = 1$ and, if C'_z passes through $z = -1$, the point $t = -1$). If C'_z intersects itself, the argument is to be modified in the same manner as above. This completes the proof of Theorem I.

The special status given to the points $t = \pm 1$ in the formulation of Theorem I cannot be avoided. As the following example shows, it may indeed happen that a branch of $g(t)$ has singularities at the points $t = \pm 1$ which do not correspond to singularities of $f(z)$ at the points $z = \pm 1$. We choose a constant a such that $0 < a < 1$ and integrate the identity

$$\frac{1}{\sqrt{1 - 2xt + x^2}} = \sum_{n=0}^{\infty} P_n(t)x^n$$

over x from 0 to a . We obtain

$$(14) \quad g(t) = \log \frac{a - t + \sqrt{1 - 2at + a^2}}{1 - t} = \sum_{n=0}^{\infty} P_n(t) \frac{a^{n+1}}{n+1},$$

an expansion which converges in the interior of the ellipse with the foci ± 1 that passes through the point $\frac{1}{2}(a + a^{-1})$. The corresponding function $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} z^n \frac{a^{n+1}}{n+1} = -\frac{1}{z} \log(1 - az),$$

whose only finite singularity occurs at $z = a^{-1}$. Nevertheless, (14) shows that the branches of $g(t)$ which correspond to the branch of the radical that is negative for positive values of $1 - 2at + a^2$ have logarithmic singularities for $t = \pm 1$.

We finally remark that the result of SZEGÖ, quoted at the beginning of this paper, corresponds to the limiting case of Theorem I which is obtained for $\sigma \rightarrow e^{i\theta}$ (θ real). Since, in this case, $\frac{1}{2}(\sigma + \sigma^{-1})$ reduces to $\cos \theta$, a singularity $(1, \theta)$ of the function (1) will be associated with singularities of the function (2) at either of the points $z = e^{\mp i\theta}$, or possibly at both. If the coefficients of the expansion (2) are real, the singularities of (2) appear in conjugate pairs, and this accounts for the one-to-one correspondence between the singularities of

(1) and (2) established by SZEGÖ's theorem. As shown by the example

$$u(r, \theta) = (1 - re^{i(\alpha+\theta)})^{-\frac{1}{2}}(1 - re^{i(\alpha-\theta)})^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(\cos \theta) e^{in\alpha} r^n,$$

$$f(z) = \sum_{n=0}^{\infty} e^{in\alpha} z^n = (1 - e^{i\alpha} z)^{-1},$$

there need not be such a correspondence if the coefficients are not real.

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REFERENCES

- [1] J. HADAMARD, Théorème sur les séries entières, *Acta Math.*, **22** (1898), pp. 55-64.
- [2] G. SZEGÖ, *Orthogonal polynomials*, vol. **23**, Colloquium publications, Am. Math. Soc., 1939.
- [3] G. SZEGÖ, On the singularities of zonal harmonic expansions, *J. Rational Mech. and Anal.*, **3** (1954), pp. 561-564.