Analytic Continuation in Elasticity

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1. Introduction.

Of concern here are the equations of static equilibrium for an isotropic elastic body. This system of partial differential equations of the second order governs the three components of the displacement vector. In the first part of this paper formulæ are developed which express the displacements and the stresses in terms of three harmonic functions. (The proof is restricted to certain cylindrical regions.) These formulæ are suited for the study of reflection principles in elasticity.

The reflection principle of SCHWARZ applies to a function which is harmonic in a region and which vanishes on a plane boundary surface of the region. The function can be continued analytically across the surface by considering the reflection of a point in this plane and ascribing the same absolute value of the function at the reflected point but the opposite sign. This principle together with the representation formulæ mentioned above are combined to yield analytic continuation of solutions of the equations of elasticity. It is assumed, of course, that one of the boundary surfaces of the elastic body is planar. Continuation formulæ are developed for the case of a fixed planar boundary and for the case of a free planar boundary. A boundary is termed "fixed" if the displacement vanishes there and a boundary is termed "free" if the surface tractions vanish there. Boundary conditions involving a mixture of such restrictions are also treated.

This paper may be regarded as a sequel to a previous paper concerning analytic continuation of biharmonic functions [2]. The techniques employed here are patterned after those developed in that paper. In addition to the references cited in the previous paper the works of DIAZ & LUDFORD [1], HUBER [6], and JOHN [7] are here noted. As in the previous paper the boundary conditions are expressed as suitable limiting relations as the boundary is approached from the interior. This procedure avoids *a priori* hypotheses on the existence of derivatives at boundary points.

2. Representation of a state of stress with harmonic functions.

Of concern in this paper is the following system of partial differential equations:

$$\begin{aligned}
 & \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial f}{\partial x} \\
 & \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial f}{\partial y} \\
 & \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{\partial f}{\partial z} \\
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -cf.
 \end{aligned}
 \tag{1}$$

Here c denotes a constant. These equations may be expressed in more condensed notation by use of the Laplacian operator Δ and by use of the notation $f_1 = \partial f / \partial x$, etc. Equations (1) can be interpreted as the displacement equations of equilibrium of an isotropic elastic body [9, p. 68] in the absence of body forces. Then x , y , and z denote rectangular coordinates and u , v , and w denote the components of the infinitesimal displacement vector at the point (x, y, z) in the body. The function $-cf$ is the dilatation. The constant c can be expressed as $c = 1 - 2\sigma$ where σ is Poisson's ratio or as $c = \mu / (\lambda + \mu)$ where λ and μ are the Lamé constants.

If $c = 0$ equations (1) become the conditions for the slow steady flow of a viscous fluid [9, p. 77]; then u , v , and w denote the components of the velocity vector and f is the pressure divided by the coefficient of viscosity. In the preceding paper [2] the case $c = 0$ was treated and the present paper does not add anything essential to this case.

Equations (1) are assumed to hold in an open set E . More precisely, it is supposed that all the derivatives appearing in equations (1) exist and are continuous in E . For the moment it is assumed that u , v , and w have third derivatives. Then equations (1) give

$$f_{11} + f_{22} + f_{33} = \Delta(u_1 + v_2 + w_3) = -c\Delta f$$

or $(1 + c)\Delta f = 0$. Thus if c is not -1 the function f is harmonic in the set E . It is desirable to establish this result without assuming the existence of third derivatives. To this end let $\rho(x, y, z)$ be a function which satisfies the following conditions: ρ has derivatives of all orders, $\rho \geq 0$, $\rho = 0$ for $x^2 + y^2 + z^2 \geq b^2$, and $\iiint \rho dx dy dz = 1$. Following FRIEDRICHS [5] such a function is termed a mollifier. Given a function u , the mollified function $\bar{u}$ is defined by

$$\bar{u}(x', y', z') = \iiint \rho(x' - x, y' - y, z' - z)u(x, y, z) dx dy dz.$$

It is clear that $\bar{u}$ has derivatives of all orders. Likewise mollified functions $\bar{v}$, $\bar{w}$, and $\bar{f}$ are introduced. Let E_1 be the set of points (x', y', z') such that the closed sphere of radius b and center (x', y', z') is contained in E . Multiply equations (1) by $\rho(x' - x, y' - y, z' - z)$ and integrate over all space. Integrating the various terms of these relations by parts shows that $\bar{u}$, $\bar{v}$, $\bar{w}$, and $\bar{f}$ satisfy equations (1) in the set E_1 . By the proof given above it is seen that $\bar{f}$ is harmonic in E_1 if $c \neq -1$.

Now consider a sequence of mollifiers with the corresponding radii b tending to zero. By continuity it follows that $\bar{f} \rightarrow f$ as $b \rightarrow 0$. Since f is the limit of a sequence of harmonic functions it follows from HARNACK'S convergence theorem that f is harmonic and the proof is complete.

It is possible to show by similar considerations that u , v , and w are biharmonic in E , that is, $\Delta\Delta u = 0$, $\Delta\Delta v = 0$, and $\Delta\Delta w = 0$ at each point of E . These classical relations will be an obvious corollary of Theorem 1 to follow.

It is convenient for the proofs to define the following sets: C^* denotes the open cylinder $(y - y_1)^2 + (z - z_1)^2 < j^2$ and $-2k < x < 2k$. Thus the cylinder is of radius j and altitude $4k$. The center of the cylinder is at the point $(0, y_1, z_1)$. D denotes the disk which is the intersection of C^* with the plane $x = 0$. C denotes the open cylinder which is the intersection of C^* with the half-space $x > 0$. C_1^* denotes a cylinder coaxial to C^* and defined in the same way except for a smaller radius. C_1 denotes the intersection of C_1^* with the half-space $x > 0$ and D_1 is the intersection with the plane $x = 0$.

Lemma 1. *Let u be harmonic in C . Then in C_1 there is a harmonic function v such that $u = \partial v / \partial x$.*

A proof of this simple lemma was given in the preceding paper [2].

Theorem 1. *Let u , v , w , and f satisfy equations (1) in the cylinder C with $c \neq -1$. Then in the cylinder $C_1 \subset C$ there are harmonic functions F , G , and H such that*

$$\begin{aligned} u &= -G_2 - H_3 - (1 + 2c)F + xF_1 \\ v &= G_1 + xF_2 \\ w &= H_1 + xF_3 \\ f &= 2F_1. \end{aligned} \tag{2}$$

Conversely, if F , G , and H are harmonic in a region then equations (2) define a set of functions u , v , w , and f which satisfy equations (1) in that region.

Proof: It has been shown above that f is harmonic in C if the functions u ,

v , w , and f satisfy equations (1) in C and $c \neq -1$. By Lemma 1 there is a harmonic function F such that $f = 2F_1$. It may be assumed that F is defined in a cylinder C_0 of radius intermediate to the radii of C_1 and C .

With F so defined let functions U , V , and W be defined by the relations

$$\begin{aligned} u &= U - (1 + 2c)F + xF_1 \\ v &= V + xF_2 \\ w &= W + xF_3 \\ f &= 2F_1. \end{aligned} \tag{3}$$

Then $\Delta U = \Delta u - 2F_{11} = \Delta u - f_1 = 0$ and $\Delta V = \Delta v - 2F_{12} = \Delta v - f_2 = 0$ and likewise $\Delta W = 0$. Thus U , V , and W are harmonic in C_0 . Moreover

$$U_1 + V_2 + W_3 = 0. \tag{4}$$

This follows from (3) because

$$\begin{aligned} U_1 + V_2 + W_3 &= (u_1 + 2cF_1 - xF_{11}) + (v_2 - xF_{22}) + (w_3 - xF_{33}) \\ &= u_1 + v_2 + w_3 + cf. \end{aligned}$$

Applying Lemma 1 twice more shows that there are harmonic functions g and H in C_1 such that $g_1 = V$ and $H_1 = W$. Let $t = U + g_2 + H_3$. Then $t_1 = U_1 + g_{21} + H_{31} = U_1 + V_2 + W_3 = 0$. Thus t depends only on y and z . Since t is harmonic it may be regarded as the real part of an analytic function of the complex variable $y + iz$ in the region D_1 . This analytic function is the derivative of another analytic function in D_1 whose real part may be denoted by T . It follows that $t = T_2$. Let $G = g - T$, then $G_1 = V$ and $U + G_2 + H_3 = 0$. Substituting the relationships $U = -G_2 - H_3$, $V = G_1$, and $W = H_1$ in (3) gives (2) and the first part of the theorem is proved.

It may be observed that the above proof applies even if the cylinder C is not circular. If, in such a case, the base D were multiply connected it is conceivable that G might be a multiple-valued function.

To prove the second part of the theorem it is supposed that F , G , and H are harmonic in a region R . Then substitution shows that (2) satisfies (1) and the proof is completed.

Harmonic functions are analytic; that is, they may be expanded in a Taylor series. It is seen from Theorem 1 that in the neighborhood of any point every solution of equations (1) may be represented in terms of harmonic functions. This gives a quick proof of the known fact that the solutions of equations (1) are analytic. (Biharmonic functions may be represented in the Almansi form $h + xg$ where h and g are harmonic, so biharmonic functions are analytic.)

Harmonic functions have always played a central role in the theory of elasticity; this fact is brought out by LOVE [8] and by SOKOLNIKOFF [13, Chap. 6].

The representation by harmonic functions given here is closely related to methods of BOUSSINESQ [14], MICHELL [15], PAPCOVITCH [10], and NEUBER [11] and [12]. The literature, however, does not seem to contain a proof or statement of Theorem 1.

3. Fixed boundary conditions.

Let E^* and E be open sets such that E^* contains E . Let ψ be a function (or set of functions) which satisfies a relation Σ at the points of E . Likewise let ψ^* satisfy Σ in E^* . If $\psi^* = \psi$ in E then ψ^* is termed a *continuation* of ψ . In particular this definition is to be applied when ψ is the set of functions u, v, w , and f and Σ denotes equations (1). If Σ denotes $\Delta\psi = 0$ the continuation may be termed a *harmonic continuation*. If Σ denotes $\Delta\Delta\psi = 0$ we speak of a *biharmonic continuation*.

If Σ is the statement " $\psi(x, y, z)$ can be expanded in a Taylor series at the point" then the continuation is spoken of as an *analytic continuation*. (Since three dimensional space is of concern here this will not be confused with analytic continuation of a function $\psi(s)$ of a complex variable s .) It was seen above that the solutions of equations (1) are analytic. Thus any continuation for equations (1) will also be an analytic continuation.

The following theorem concerns the continuation of the solutions of equations (1) at a plane boundary surface. As a matter of convenience this boundary surface is taken to lie in the plane $x = 0$ and it is supposed that the elastic body lies on one side of this plane, that is, $x > 0$. The boundary condition assumed is that the boundary surface is fixed, that is, u, v , and w vanish for $x = 0$.

Theorem 2. *Let E^* be an open set such that if (x, y, z) is a point of E^* so also is $(-x, y, z)$. Let Q be the intersection of E^* with the plane $x = 0$ and let E be the intersection of E^* with the half-space $x > 0$. Suppose that u, v, w , and f satisfy equations (1) in E and that u, v , and w vanish as any point of Q is approached. Then these functions have a continuation which satisfies equations (1) in all of E^* . This continuation satisfies the identities*

$$\begin{aligned}
 u' &= -u + x(2cf + 2u_1 - xf_1)(1 + 2c)^{-1} \\
 v' &= -v + x(-2u_2 + xf_2)(1 + 2c)^{-1} \\
 w' &= -w + x(-2u_3 + xf_3)(1 + 2c)^{-1} \\
 f' &= (f - 2cf - 4u_1 + 2xf_1)(1 + 2c)^{-1}.
 \end{aligned}
 \tag{5}$$

Here u' denotes $u(-x, y, z)$, etc. It is assumed that $c \neq -1$ or $-\frac{1}{2}$.

Proof: A given point of Q is taken to be the center of a cylinder C^* . The radius and altitude of C^* are chosen small enough so that E^* contains C^* . The following two lemmas are taken from the previous paper [2].

Lemma 2. Suppose that h and g are harmonic in C and that $h + xg$ vanishes as any point of D is approached. Then h vanishes as any point of D is approached.

Lemma 3. If f is harmonic (or biharmonic) in C and if f_1 is continuable to C^* then f is continuable to C^* by integration of f_1 . Explicitly

$$f = \int_0^x f_1 dx + f(0, y, z).$$

Here $f(0, y, z)$ is the limit of f as $x \rightarrow 0$.

In C_1 relations (2) of Theorem 1 hold. By Lemma 2 it follows immediately that G_1 and H_1 vanish as a point of D is approached. It follows from the Schwarz reflection principle that G_1 and H_1 can be continued to the cylinder C_1^* as odd functions of x . By Lemma 3 it is seen that G and H can be continued as even functions of x . Again by Lemma 2 the function $G_2 + H_3 + (1 + 2c)F$ vanishes in D_1 so it may be continued as an odd function of x . This shows that F may be continued.

Thus F , G , and H can be continued to C_1^* as harmonic functions. By the second part of Theorem 1 it follows that equations (2) define a continuation of equations (1) to C_1^* . It is now desired to show that this continuation satisfies relations (5) in C_1^* .

Let F' denote $F(-x, y, z)$, etc. Then

$$\begin{aligned} G'_2 + H'_3 + (1 + 2c)F' &= -G_2 - H_3 - (1 + 2c)F, \\ (1 + 2c)F' &= -(1 + 2c)F - 2G_2 - 2H_3 \\ &= (1 + 2c)F + 2u - 2xF_1, \end{aligned}$$

or

$$(6) \quad (1 + 2c)(F' - F) = 2u - xf.$$

Then

$$v' + v = G'_1 + G_1 - x(F'_2 - F_2) = -x(F'_2 - F_2) = (-2xu_2 + x^2f_2)(1 + 2c)^{-1}.$$

This gives a continuation formula for v in C_1^* which is the same as that given in (5). The continuation formula for w follows by symmetry. Differentiating (6) with respect to x gives $(1 + 2c)(f' + f) = -4u_1 + 2f + 2xf_1$. This proves the continuation formula for f in C_1^* . It follows from (2) that

$$u' + u = -x(F'_1 - F_1) = -x(f' - f)/2 = x(2cf + 2u_1 - xf_1)(1 + 2c)^{-1}.$$

This proves the continuation formula for u in C_1^* . Thus it has been shown that u , v , w , and f have a continuation in a neighborhood of any point of Q and that this continuation satisfies relations (5).

Let E' be the reflection of E in the plane $x = 0$. If (x, y, z) is a point of E let equation (5) define u, v, w , and f in E' . By differentiation it is readily verified that the set of functions so defined satisfies equations (1) in E' . Thereby a continuation is defined in $E + E'$. It follows from the preceding paragraph that u, v, w , and f can be defined on Q so as to extend this continuation to $E^* = E + E' + Q$.

It remains to show that relations (5) are an identity in E^* . Let ψ denote the set of functions u, v, w , and f and let L denote the operation defined by the right side of equations (5). Thus (5) is written

$$\psi(-x, y, z) = L\psi(x, y, z).$$

This holds by definition if (x, y, z) is a point of E . The problem is to prove this for (x, y, z) a point of E' . To this end let ψ be a set of functions with second derivatives. It is not required that ψ satisfy equations (1). Let $\phi(-x, y, z) = L\psi(x, y, z)$. Then a straightforward but lengthy calculation gives the identity $\psi(-x, y, z) = L\phi(x, y, z)$. This completes the proof of Theorem 1.

The operation L can be used to generate functions which satisfy the fixed boundary conditions. Thus let ψ have continuous second derivatives in E^* and let $\phi(-x, y, z) = L\psi(x, y, z)$. Then

$$L[\psi(x, y, z) + \phi(x, y, z)] = \phi(-x, y, z) + \psi(-x, y, z).$$

This shows that $\psi + \phi$ satisfies relations (5). It is apparent that relations (5) imply the fixed boundary conditions. These considerations prove the following statement. *Given a set of functions ψ which satisfy equations (1) in E^* then the set $\psi + (L\psi)'$ satisfies (1) and also the fixed boundary conditions in Q .*

4. Free boundary conditions.

The stress tensor S_{ij} is now introduced in order to analyze other types of boundary conditions. By definition

$$\begin{aligned} S_{11} &= u_1 + \frac{1}{2}(c-1)f, & S_{22} &= v_2 + \frac{1}{2}(c-1)f, \\ S_{12} &= \frac{1}{2}(u_2 + v_1), & S_{23} &= \frac{1}{2}(v_3 + w_2), \\ S_{13} &= \frac{1}{2}(u_3 + w_1), & S_{33} &= w_3 + \frac{1}{2}(c-1)f. \end{aligned}$$

The other components are derived by symmetry, $S_{ij} = S_{ji}$. Actually these relations give relative stress; to obtain the actual stress multiply S_{ij} by 2μ where μ is the shear modulus.

The relations of Theorem 1 are seen to give the following formulæ for the components of the stress tensor S_{ij} in terms of the harmonic functions F, G , and H :

$$\begin{aligned}
 S_{11} &= -G_{12} - H_{13} - (c + 1)F_1 + xF_{11} \\
 S_{12} &= \frac{1}{2}(G_{11} - G_{22} - H_{23}) - cF_2 + xF_{12} \\
 S_{13} &= \frac{1}{2}(H_{11} - H_{33} - G_{23}) - cF_3 + xF_{13} \\
 S_{22} &= G_{12} + (c - 1)F_1 + xF_{22} \\
 S_{23} &= \frac{1}{2}(G_{13} + H_{12}) + xF_{23} \\
 S_{33} &= H_{13} + (c - 1)F_1 + xF_{33} .
 \end{aligned}
 \tag{7}$$

It is convenient to introduce the notation $p = S_{11}$, $q = S_{12}$, $r = S_{13}$. Again p_1 will signify $\partial p / \partial x$, *etc.* Then p , q , and r satisfy the following relations

$$\Delta p = f_{11} , \quad \Delta q = f_{12} , \quad \Delta r = f_{13} , \quad p_1 + q_2 + r_3 = 0 .
 \tag{8}$$

These may be derived from equations (1) or from equations (7). It is apparent that equations (8) have the same form as equations (1) with $c = 0$ and with p , q , r , and f_1 corresponding to u , v , w , and f , respectively.

If no forces are applied to one of the boundary surfaces of an elastic body then this surface may be said to be a "free boundary". It is desired to find the mathematical conditions which define a free boundary. Of particular concern is the case that one of the boundary surfaces lies in the plane $x = 0$. Now p , q , and r give the x , y , and z components of the surface traction acting across a surface element perpendicular to the x -axis and of unit area. Thus the free boundary conditions are simply that p , q , and r vanish as a point of the boundary is approached.

The problem to be treated now is the analytic continuation of a solution of equations (1) at such a free boundary. The sets E^* , E , and Q are taken as in Theorem 2. Equations (1) are supposed satisfied in E and the boundary conditions are that p , q , and r vanish as a point of Q is approached. It has been seen that p , q , r , and f_1 satisfy equations (8) in E . Since equations (8) have the same form as equations (1) it is seen that Theorem 2 can be applied with $c = 0$. This gives immediately the following formulæ for the continuation of p , q , r , and f_1 to E^* :

$$\begin{aligned}
 p' &= -p + 2xp_1 - x^2f_{11} \\
 q' &= -q - 2xp_2 + x^2f_{12} \\
 r' &= -r - 2xp_3 + x^2f_{13} \\
 f_1' &= f_1 - 4p_1 + 2xf_{11} .
 \end{aligned}
 \tag{9}$$

Let C^* be a cylinder contained in E^* . Then from Lemma 3 it follows that f can be continued to C^* by integrating f_1 . The integration of the last equation in (9) can be carried out explicitly to yield

$$(10) \quad f' = f + 4p - 2xf_1$$

This can be checked by differentiation and noting that f so defined is continuous on D . It is then seen that (10) gives a continuation of f as a harmonic function in E^* .

To find the continuation of u_1 it is noted that $u_1 = p + \frac{1}{2}(1 - c)f$ so $u'_1 = p' + \frac{1}{2}(1 - c)f'$ and substituting from (9) and (10) gives

$$(11) \quad u'_1 = (1 - 2c)p + 2xp_1 + \frac{1}{2}(1 - c)f + (c - 1)xf_1 - x^2f_{11}.$$

This gives a continuation of the biharmonic function u_1 in E^* . Next a continuation formula for v_{11} is sought. Since $2q = u_2 + v_1$ it is seen that $v_{11} = 2q_1 - u_{21}$ and $v'_{11} = 2q'_1 - u'_{21}$. From (9) and (11) we get

$$q'_1 = q_1 + 2p_2 + 2xp_{12} - 2xf_{12} - x^2f_{112},$$

$$u'_{12} = (1 - 2c)p_2 + 2xp_{12} + \frac{1}{2}(1 - c)f_2 + (c - 1)xf_{12} - x^2f_{112}.$$

Combining these relations gives

$$(12) \quad v'_{11} = 2q_1 + (3 + 2c)p_2 + 2xp_{12} + \frac{1}{2}(c - 1)f_2 - (3 + c)xf_{12} - x^2f_{112}.$$

Symmetry gives immediately the following relation for w'_{11}

$$(13) \quad w'_{11} = 2r_1 + (3 + 2c)p_3 + 2xp_{13} + \frac{1}{2}(c - 1)f_3 - (3 + c)xf_{13} - x^2f_{113}.$$

Relations (12) and (13) give a continuation of the biharmonic functions v_{11} and w_{11} to all of E^* .

Continuation formulæ for u , v , and w can be found for those points (x, y, z) of E^* such that the line segment joining (x, y, z) and $(-x, y, z)$ is entirely in E^* . Under this hypothesis Lemma 4 applies, hence u can be continued by integrating (11) with respect to x . Likewise, continuation formulæ for v and w are obtained by integrating relations (12) and (13) twice with respect to x . Thus it has been shown that u , v , and w can be continued as biharmonic functions in any cylinder C^* contained in E^* . In C the functions u , v , w , and f satisfy equations (1); it remains to prove that these functions also satisfy equations (1) in C' . This can be verified by direct substitution from formulæ (10), (11), (12), and (13). This follows more rapidly from the fact that biharmonic functions can be expanded in power series. Thus relations (1) are analytic relations between analytic functions so if they hold in part of a region C they must hold in all of the region C^* . This completes the proof of the continuation of the equations of elasticity at a free boundary.

5. Mixed boundary conditions.

In 1902 C. SOMIGLIANA [4, Art. 25, No. 2] studied a class of solutions of the equilibrium equations which have the following simple reflection properties:

$$(14) \quad u' = u, \quad v' = -v, \quad w' = -w.$$

This class was introduced by SOMIGLIANA to develop an analogue of KELVIN's method of images. It is seen that (14) implies that

$$(15) \quad p = v = w = 0 \quad \text{for } x = 0.$$

Relations (15) are termed a mixed boundary condition by SOMMERFELD [9, p. 315]. Here it is to be shown that the *boundary conditions* (15) *imply a continuation satisfying* (14).

The hypotheses of Theorem 2 are assumed to hold except that the boundary condition u , v , and $w \rightarrow 0$ on Q is replaced by

$$(15a) \quad p, v, \text{ and } w \rightarrow 0 \quad \text{on } Q.$$

Attention is first confined to a cylinder C^* contained in E^* . Applying Theorem 1 and Lemma 2 it follows that G_1 and H_1 vanish in D_1 . Thus G_1 and H_1 can be continued as odd functions of x . Likewise from equations (7) it is seen that $G_{12} + H_{13} + (c + 1)F_1$ can be continued as an odd function. Since $c \neq -1$ it follows that F_1 is odd. By Lemma 3 it is seen that F , G , and H are continuable as odd functions. Then equations (2) furnish a continuation to C_1^* and it is seen that relations (14) are satisfied. There is no difficulty in extending (14) to hold in all of E^* .

SOMIGLIANA also introduced a class of solutions with the reflection property

$$(16) \quad u' = -u, \quad v' = v, \quad w' = w.$$

Again it may be verified that

$$(17) \quad u = q = r = 0 \quad \text{for } x = 0.$$

This is another mixed boundary condition. It is now to be shown that *the boundary conditions* (17) *imply a continuation satisfying* (16).

The hypotheses of Theorem 2 are again assumed but the boundary condition is now taken to be

$$(17a) \quad u, q, \text{ and } r \rightarrow 0 \quad \text{on } Q.$$

Proceeding as in the previous case it is seen from (2) that $G_2 + H_3 + (1 + 2c)F$ can be continued as an odd function. By differentiation it follows that $G_{22} + H_{23} + (1 + 2c)F_2$ is also odd. From the second relation in (7) it follows that $G_{11} - G_{22} - H_{23} - 2cF_2$ is continuable as an odd function of x . The sum of odd func-

tions is odd so $G_{11} + F_2$ is odd. A similar argument shows that $H_{11} + F_3$ is odd. Thus $(H_{113} + F_{33}) + (G_{112} + F_{22}) = H_{113} + G_{112} - F_{11}$ is an odd function of x . By differentiation it is seen that $G_{113} + G_{112} + (1 + 2c)F_{11}$ is odd. By addition it results that $(1 + c)F_{11}$ is odd. Applying Lemma 3 it is seen that F_1 is continuable as an even function. It is then apparent from (2) that u is continuable as an odd biharmonic function.

Differentiation of the second relation in (2) gives $v_1 = G_{11} + F_2 + xF_{12}$. Thus v_1 is continuable as an odd biharmonic function. By Lemma 3 v is continuable as an even biharmonic function. A similar argument shows that w is continuable as an even biharmonic function. Use of the analyticity argument given at the end of Section 4 shows that the reflection relations (16) give a continuation which satisfies equations (1) in C_1^* . The continuation to all of E^* presents no difficulty.

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