Minimum Principles of Piecewise Linear Isotropic Plasticity

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Summary

A theory of plasticity is considered in which the yield function is a piecewise linear function of the stresses, the flow law is based on the plastic potential, and the strain-hardening is isotropic and linear. First, the flow law at a "corner" of the yield function is investigated in detail. Next, it is shown that the flow law at all points can be piecewise integrated to obtain some of the advantages of a deformation theory. Finally, two classical minimum principles of elasticity are extended to include certain cases of piecewise linear plasticity.

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1. *Introduction*

It is generally agreed that the behavior of an elastic solid under small strains is completely specified by the assumption of Hooke's law and the determination of the elastic constants. Since this foundation of elasticity is so generally recognized, research workers in elasticity have been able to concentrate on methods and techniques for solving problems.

In the field of plasticity, much more is needed to specify the material behavior, and there is no general agreement as to how this should be done. The primary ingredients must certainly include a yield condition, a stress-strain or flow law, a description of the polyaxial nature of strain-hardening, and a description of the rate of hardening. All of these ingredients may, in general, be non-linear and anisotropic. Much experimental work has been done and much more is needed to determine adequately the plastic properties of various materials.

However, the evidence now available indicates that an accurate description of these properties would lead to mathematical problems which are completely intractable. Therefore, much of the effort in plasticity research has gone into the development of simplified theories which can be used to solve problems. Even though admittedly inexact, these theories may lead to solutions which are of practical value.

The present paper is concerned with a certain simplified theory which may be described as "piecewise linear." As implied by its name, this theory retains some of the characteristics of a linear theory, but certain complications are introduced at the "corners" where different "pieces" come together.

The basic assumptions of the theory, together with an abbreviated history, will be presented in the next section. Section 3 will then go into a detailed description of the behavior at the corners. Next, it will be shown that although the theory lies in the general category of "flow" theories, it can be "piecewise integrated" and hence retain some of the advantages of a "deformation" theory. Sections 5 and 6 will be concerned with the establishment and proof of two minimum principles which are analogous to the minimum complementary energy and minimum potential energy theorems of elasticity. Finally, Section 7 will discuss some of the conclusions and limitations of the results.

2. Basic Theory

The behavior of an isotropic elastic-plastic material is partially described by a yield function f with the following properties. If f is less than a certain value f_0 , or if f is less than its previous maximum value, or if f is decreasing, then changes in strain are entirely elastic. On the other hand, if f is equal to or greater than both f_0 and its previous maximum, and if f is non-decreasing, then part of the change in strain is plastic. In general, the function f may depend upon any of the physical variables present such as stress, strain, strain history, stress gradient, *etc.* However, it will be assumed in the following that f depends only upon the stresses.

In discussing plastic behavior and yield functions, it is convenient to use a geometrical terminology. To this end, the stress components are taken as Cartesian coordinates in a suitably dimensioned stress space. The equation $f = f_0$ may then be represented as a surface in that space. DRUCKER [1] has shown that this surface must be convex. Similarly, other level surfaces of f may be visualized. The requirements for plastic straining then state that the "stress point" whose coordinates are the corresponding stress components must lie on or outside both of the surfaces $f = f_0$ and $f = f_{\max}$ and must not be moving in. Thus, in

FIG. 1 LOADING AND UNLOADING: MISES YIELD CONDITION

Fig. 1, the segments AB , DE , and GH represent plastic straining, while OA , BCD , and EFG represent purely elastic behavior.

The present paper will be concerned with yield functions which are piecewise linear. Geometrically, the level surfaces of such functions will be plane sections. Figure 2 shows a typical example of a piecewise linear yield function.

FIG. 2 TRESCA YIELD CONDITION

Historically, it is interesting to note that one of the earliest proposed yield conditions was piecewise linear. In 1868 TRESCA [2] suggested that a material was initially elastic as long as the maximum shear stress did not exceed the yield stress k in pure shear. In terms of principle stresses TRESCA's yield function may be written

$$f = \max [|\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|, |\sigma_1 - \sigma_2|] - 2k. \quad (2.1)$$

The level curves of Fig. 2 may be associated with Eq. (2.1) for the particular case $\sigma_3 = 0$.

A non-linear yield function was proposed by MISES [3] in 1913. This differed from the Tresca condition in that the elastic criterion was the octahedral (rather than the maximum) shearing stress. Thus

$$f = (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2 - 8k^2. \quad (2.2)$$

The level curves of Fig. 1 will fit Eq. (2.2) if $\sigma_3 = 0$.

In addition to a yield function, it is necessary to specify a flow rule or stress-strain relation in order to describe the behavior of a plastic material. The first flow rules to be used were due to MISES [3], PRANDTL [4], and REUSS [5], and specified that the plastic strain-rate tensor be proportional to the stress deviation tensor. While this rule seemed to lend itself naturally to the Mises yield function, it led to certain difficulties when applied with the Tresca yield function, so that the Tresca yield function gradually dropped out of favor.

In 1928, MISES [6] proposed the "plastic potential" flow rule, which required that a certain strain-rate vector be orthogonal to the surface $f = \text{const.}$, where f is the yield function. Thus the flow rule and yield function were intimately connected. It was shown that the Mises and Prandtl-Reuss flow rules were related to the Mises yield function by this orthogonality relationship.

However, an additional problem remained to be solved before the plastic potential flow rule could be applied to a piecewise linear yield condition such as Tresca's. Indeed, at first glance there seems to be a most undesirable lack of uniqueness in that if the strain-rate vector is given orthogonal to a side of the yield function, it may be associated with any point on that side, while if the stresses correspond to a corner of the yield function, orthogonality is not defined. This problem was partially solved in 1953 by KORTER [7], and an example for a strain-hardening material was given by PRAGER in 1954 [8]. Meanwhile, HILL [9] and DRUCKER [1] had given convincing reasons for accepting the plastic-potential law as the only consistent relation between the yield function and the flow rule.

For a perfectly plastic material, the yield function and flow rule completely specify the plastic behavior, but for a strain-hardening material the details of hardening must also be given. In the following, it will be assumed at all times that the strain-hardening is isotropic.¹ Thus, if the yield function has a given initial shape and center, it will always maintain the same shape and center although it will change in scale. With this assumption, a single test in, say, simple tension completely determines the strain-hardening characteristics of the material.

¹ Some recent papers by PRAGER [10, 11] are concerned with a particular type of non-isotropic hardening which accounts for the so-called Bauschinger effect.

3. Flow Law at a Corner

So that the following discussion may be as general as possible, generalized stresses and strain rates will be used. PRAGER [12] has shown that if generalized stresses are denoted by Q_i , then the corresponding generalized strain-rates q'_i must be chosen so that the rate of dissipation of energy is proportional to²

$$D = Q_i q'_i . \quad (3.1)$$

For example, in a beam problem the only generalized stress is the bending moment M , and the corresponding strain-rate is the rate of curvature ϑ' . It should be noted that since shear deformations are neglected, the shear force is a reaction and not a generalized stress. Similarly, in cylindrical shells with axial symmetry the stresses are the axial and circumferential direct forces N_x and N_φ and the axial moment M_x , while the corresponding strain rates are the middle surface extension rates e'_x and e'_φ and the curvature rate κ'_x . For general three dimensional stress problems, of course, the stresses are σ_x , σ_y , σ_z , τ_{yz} , τ_{zx} , τ_{xy} , and the corresponding strain rates are ϵ'_x , ϵ'_y , ϵ'_z , γ'_{yz} , γ'_{zx} , γ'_{xy} .

It is assumed that the total strain-rates may be decomposed into an elastic and a plastic part:

$$q'_i = e'_i + p'_i . \quad (3.2)$$

The elastic part will be governed by the differentiated form of Hooke's law and hence will present no singularities at the corners. Therefore, only plastic strains need be considered in the present analysis. The plastic potential flow rule, applied to a portion of the yield function with a uniquely determined normal, states [1] that

$$p'_i = F(\partial f / \partial Q_i) f' . \quad (3.3)$$

Here f is the yield function, defined so that $f < 0$ corresponds to initial elastic behavior, and F is the hardening function.

In the particular case of a piecewise linear yield function, f may be written as the combination of several linear functions f_λ defined by

$$f_\lambda = a_{\lambda i} Q_i - 1 . \quad (3.4)$$

² Throughout this paper Latin subscripts will run from 1 to n , where n is the number of generalized stress coordinates necessary to specify the problem. A prime (') will be used to indicate differentiation with respect to time. The summation convention governing repeated Latin subscripts will be used so that

$$Q_i q'_i = \sum_{i=1}^{i=n} Q_i q'_i .$$

In discussing such a function, it will prove convenient to define the "stress intensity" Q by

$$Q = \max_{\lambda} (a_{\lambda i} Q_i). \quad (3.5)$$

The conditions for plastic flow may then be stated as

$$Q \geq 1, \quad Q \geq Q_{\max}, \quad Q' \geq 0. \quad (3.6)$$

If there is a unique value of λ for which the equality (3.5) holds, then the stress point is said to be on face λ of the yield polyhedron (or in plastic regime λ). In view of Eqs. (3.3), (3.4), and (3.5), the plastic strain rates are then given by³

$$p'_i = F(\partial f_{\lambda} / \partial Q_i) f' = F a_{\lambda i} a_{\lambda i} Q'_i. \quad (3.7)$$

As a first example of a corner, the problem of two intersecting sides in a two-dimensional stress problem will be considered. The two intersecting sides may be designated as

$$\begin{aligned} \text{face } A: \quad f &= a_1 Q_1 + a_2 Q_2 - 1, \\ \text{face } B: \quad f &= b_1 Q_1 + b_2 Q_2 - 1. \end{aligned} \quad (3.8)$$

On either face the strain rate vector is uniquely defined by Eqs. (3.7):

$$\begin{aligned} \text{on face } A, \quad p'_1 &= a_1 F Q', \quad p'_2 = a_2 F Q', \\ \text{on face } B, \quad p'_1 &= b_1 F Q', \quad p'_2 = b_2 F Q', \end{aligned} \quad (3.9)$$

but at the point of intersection the vector is not yet defined. In extending the definition, it is desirable that the singular case be obtainable as the limit of a regular case and that any physical interpretation which can be applied to the sides also holds at the corner. The limiting case is easily obtained by "slicing off" a piece of the corner at an arbitrary angle Ω (Fig. 3). The equation of face C , in the limiting case where ϵ tends to zero, is

$$f = [(a_1 Q_1 + a_2 Q_2 - 1) \cos \omega + (b_1 Q_1 + b_2 Q_2 - 1) \sin \omega] (\cos \omega + \sin \omega)^{-1}, \quad (3.10)$$

where $\omega = \Omega\pi/2\alpha$, and Ω and α are defined in Fig. 3. The corresponding strain rates are given by

$$p'_i = F Q' (a_i \cos \omega + b_i \sin \omega) (\cos \omega + \sin \omega)^{-1}, \quad i = 1, 2. \quad (3.11)$$

Equations (3.11) may also be obtained as the limiting case of a circular arc at the point C [13].

³ Greek subscripts refer to different faces of the yield polyhedron and not to coordinates. The summation convention does not apply to Greek subscripts, and where summations are intended they will be explicitly stated.

FIG. 3 FLOW RULE AT A CORNER

The additional unknown ω which appears in Eqs. (3.11) is consistent with a solution of the problem, since there is one additional equation relating the stresses at a corner. If it is desired to eliminate ω between Eqs. (3.11) this is readily done and the result may be shown to be equivalent to the vanishing of the determinant

$$\begin{vmatrix} a_1 & b_1 & p'_1 \\ a_2 & b_2 & p'_2 \\ 1 & 1 & FQ' \end{vmatrix}. \quad (3.12)$$

Next, it will be shown that the plastic dissipation function is the same at the sides and at the corner. Thus, Eqs. (3.11) would also result if the dissipation function were prescribed. At the face A ,

$$\begin{aligned} D^p &= Q_1 a_1 FQ' + Q_2 a_2 FQ' \\ &= FQ'(Q_1 a_1 + Q_2 a_2) = FQQ', \end{aligned} \quad (3.12a)$$

and the same result is obtained at face B . Further, at the corner AB

$$\begin{aligned}
 D^p &= FQ'[a_1 \cos \omega + b_1 \sin \omega]Q_1 + (a_2 \cos \omega + b_2 \sin \omega)Q_2](\cos \omega + \sin \omega)^{-1} \\
 &= FQ'[\cos \omega(a_1Q_1 + a_2Q_2) + \sin \omega(b_1Q_1 + b_2Q_2)](\cos \omega + \sin \omega)^{-1} \quad (3.12b) \\
 &= FQQ'
 \end{aligned}$$

as before.

The most general type of corner is the intersection of Λ linear surfaces in an n -dimensional stress space. In this case the individual surfaces are given by Eqs. (3.4) and at the point of intersection all Λ of Eqs. (3.4) must hold simultaneously. Any "normal" to this singular point must be normal to at least one of the family of planes

$$f = \frac{\sum_{\lambda=1}^{\Lambda} (a_{\lambda i} Q_i - 1) \cos \omega_{\lambda}}{\sum_{\lambda=1}^{\Lambda} \cos \omega_{\lambda}}. \quad (3.13)$$

Therefore, at the singular point the strain-rate vector can be written

$$p'_i = FQ' \frac{\sum_{\lambda=1}^{\Lambda} a_{\lambda i} \cos \omega_{\lambda}}{\sum_{\lambda=1}^{\Lambda} \cos \omega_{\lambda}}. \quad (3.14)$$

It can easily be shown that each of the determinants

$$\begin{vmatrix}
 a_{1i} & a_{2i} & \cdots & a_{\Lambda i} & p'_i \\
 a_{1j} & a_{2j} & \cdots & a_{\Lambda j} & p'_j \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 a_{1r} & a_{2r} & \cdots & a_{\Lambda r} & p'_r \\
 1 & 1 & \cdots & 1 & FQ'
 \end{vmatrix} \quad (3.15)$$

must vanish at the Λ -multiple intersection. Further, in this case also the dissipation function is

$$\begin{aligned}
 D^p &= FQ' \frac{\sum_{\lambda=1}^{\Lambda} a_{\lambda i} \cos \omega_{\lambda} Q_i}{\sum_{\lambda=1}^{\Lambda} \cos \omega_{\lambda}} \\
 &= FQ' \frac{Q \sum_{\lambda=1}^{\Lambda} \cos \omega_{\lambda}}{\sum_{\lambda=1}^{\Lambda} \cos \omega_{\lambda}} = FQQ'.
 \end{aligned} \quad (3.16)$$

4. Integration of the Flow Law

In generalized coordinates, Hooke's law may be written

$$e_i = B_{ij}Q_j \quad (4.1)$$

where B_{ij} is a positive definite symmetric tensor whose components may always be expressed in terms of Young's modulus and Poisson's ratio. Therefore, the total strain-rates may be written

$$q'_i = e'_i + p'_i = B_{ij}Q'_j + p'_i \quad (4.2)$$

where p'_i are given by the appropriate equations in the preceding section.

The integration of Eqs. (4.2) is trivial in the case where the particle never undergoes plastic straining, the result being simply Eqs. (4.1).

As the next simplest case, it is assumed that a particle is elastic up to $t = t_0$ and that thereafter it remains on face A . For $t > t_0$, it follows from Eqs. (4.2) and (3.7) that

$$q'_i = B_{ij}Q'_j + Fa_iQ'. \quad (4.3)$$

As SANDERS [14] has pointed out, Eqs. (4.3) may be integrated in any case where the hardening function F depends only on the stress intensity Q . Here it will be assumed for simplicity that

$$F = \text{const.} \quad (4.4)$$

although some of the results obtained will be valid in the more general case. The integration of (4.3) then yields

$$q_i = B_{ij}Q_j + a_iFQ + A_i,$$

where A_i is to be determined. However, at $t = t_0$ the strains must be continuous with those given by Eqs. (4.1). Since $B_{ij}Q_j$ is continuous while $Q(t_0) = 1$, it follows that

$$A_i = -a_iF.$$

Therefore, the integrated law may be written

$$q_i = B_{ij}Q_j + a_iF(Q - 1). \quad (4.5)$$

A second simple case is provided when the particle goes directly from the elastic regime into a corner, say the intersection of faces A and B . In this case it is convenient to rewrite the flow law (3.11) in the form

$$p'_i = FQ'(a_i\varphi + b_i\psi), \quad (4.6a)$$

where φ and ψ are not independent but are related by

$$\varphi + \psi = 1. \quad (4.6b)$$

Further, φ and ψ must each be positive. Addition of the elastic strain rates to (4.6) and integration with respect to time then yield

$$q_i = B_{ii}Q_i + F \int_1^Q (a_i\varphi + b_i\psi) dQ. \quad (4.7)$$

The additional unknown φ (or ψ) is accounted for by the fact that at a corner there is one less stress unknown.

Similarly, it is easily shown that where a particle enters the Λ -multiple directly from the elastic state,

$$q_i = B_{ii}Q_i + F \int_1^Q \sum_{\lambda=1}^{\Lambda} \varphi_{\lambda} a_{\lambda i} dQ, \quad (4.8)$$

where

$$\sum_{\lambda=1}^{\Lambda} \varphi_{\lambda} = 1, \quad \varphi_{\lambda} \geq 0. \quad (4.9)$$

The situation becomes somewhat more complex when a particle moves from one plastic regime to another. As an example, a particle may enter plastic regime A at a time t_0 and move to plastic regime B at a later time t_1 . For $t_0 < t < t_1$, then, Eqs. (4.5) will apply, and it will be assumed that the problem can be completely solved for this interval. If this is so, then the value of the stress intensity will be known at $t = t_1$, say

$$Q(t_1) = a_i Q_i = b_i Q_i = P. \quad (4.10)$$

Of course, the time t_1 may vary from one material point to another, and hence P may be a function of position.

For $t > t_1$, it follows from Eqs. (4.3) (with a_i replaced by b_i) that

$$q'_i = B_{ii}Q'_i + F b_i Q'. \quad (4.11)$$

Therefore, integration of (4.12) and application of the initial condition (4.10) furnish the strain

$$q_i = B_{ii}Q_i + F[b_i(Q - P) + a_i(P - 1)], \quad t > t_1. \quad (4.12)$$

In using (4.12) it must be recalled that since the stress point is now in regime B , $Q = b_i Q_i$.

Similarly, if a particle moves from the elastic state to face A at $t = t_0$ and then to intersection AB at $t = t_1$, it may be shown that

$$q_i = B_{ii}Q_i + F[a_i(P - 1) + \int_P^Q (a_i\varphi + b_i\psi)] dQ, \quad \text{for } t > t_1. \quad (4.13)$$

Of particular interest in the following will be the case where after an initial elastic stage a particle first moves from face A_1 to intersection A_1A_2 at the stress intensity P_1 , then moves to the triple intersection $A_1A_2A_3$ at the stress intensity P_2 , and continues in this fashion until it enters the Λ -multiple intersection $A_1A_2 \cdots A_\Lambda$ at the stress intensity $P_{\Lambda-1}$. The flow law for each stage may be written

$$p'_i = F \sum_{\lambda=1}^{\nu} \varphi_{\nu\lambda} a_{\lambda i} Q', \quad (4.14)$$

$$P_{\nu-1} < Q < P_{\nu}, \quad \nu = 1, 2, \cdots, \Lambda.$$

Here the unknown quantities $\varphi_{\nu\lambda}$ must satisfy,

$$\sum_{\lambda=1}^{\Lambda} \varphi_{\nu\lambda} = 1, \quad \varphi_{\nu\lambda} \geq 0, \quad \nu = 1, 2, \cdots, \Lambda, \quad (4.14)$$

and $P_0 = 1$. The corresponding stress-strain law may then be written in the form

$$q_i = B_{ii} Q_i + F \left\{ \sum_{\nu=1}^{\mu-1} \int_{P_{\nu-1}}^{P_{\nu}} \sum_{\lambda=1}^{\nu} \varphi_{\nu\lambda} a_{\lambda i} dQ + \int_{P_{\mu-1}}^Q \sum_{\lambda=1}^{\mu} \varphi_{\mu\lambda} a_{\lambda i} dQ \right\}. \quad (4.15)$$

Equations (4.15) will still remain valid in the limiting case where any of the intervals $P_{\lambda-1}$, P_{λ} shrinks to zero; so that they are valid even if the particle "skips" one or more steps of its assumed history.

Similar results are available if a particle moves from an intersection to a face, from face A to face B to face C , *etc.* Unloading can also be handled by this method, it only being necessary to integrate the elastic equations with appropriate initial conditions at the start of unloading.

5. Minimum Complementary Energy

The minimum principles to be proved in this and the following section are valid only under certain rather restrictive conditions. First of all, it is assumed that each point of the body under consideration either remains elastic or, if it once becomes plastic, remains plastic from then on. Further, it is assumed that if an element becomes plastic on face A it will never entirely leave face A . However, it may progress from face A to intersection AB to multiple intersection ABC , etc. The significance of these restrictions will be discussed in the concluding section of the paper.

The minimum complementary energy principle is concerned with "statically admissible" states of stress. Such a stress state is any stress state which is in internal and external equilibrium with the applied tractions. Strains can be associated with these statically admissible stresses by means of the appropriate stress-strain law, but in general these strains will not be compatible.

The total complementary energy associated with any statically admissible stress state is defined as the difference between the internal energy and the external work done on the prescribed displacements. In evaluating the latter quantity it will be assumed that the entire surface of the body falls into one of two categories: traction or displacement. On a traction surface S_T , either each component of the traction vector $\mathbf{T}$ is prescribed, or else the corresponding component of the displacement is prescribed to be zero. Thus on S_T

$$\delta \mathbf{T} \cdot \mathbf{u} = 0, \quad (5.1)$$

where $\delta \mathbf{T}$ is any variation (large or small) between two statically admissible stress states. Similarly, on a displacement surface S_D ,

$$\mathbf{T} \cdot \delta \mathbf{u} = 0. \quad (5.2)$$

The total complementary energy associated with any statically admissible stress state Q_i^0 is then given by

$$\Pi_c(Q_i^0) = \int_V W_c(Q_i^0) dV - \int_{S_D} \mathbf{T}^0 \cdot \mathbf{u} dS, \quad (5.3a)$$

where

$$W_c(Q_i^0) = \int_0^{Q_i^0} q_i(Q_i^0) dQ_i^0, \quad (5.3b)$$

and $\mathbf{u}$ is the actual boundary displacement vector.

With these definitions, the principle of minimum complementary energy states that among all statically admissible stress fields which satisfy the loading

restrictions stated in the first paragraph of this section, the total complementary energy is minimized by the actual stress field. In other words, the quantity

$$\begin{aligned}\Delta\Pi_e &= \Pi_e(Q_i^0) - \Pi_e(Q_i) \\ &= \int_V [W_e(Q_i^0) - W_e(Q_i)] dV - \int_{S_D} (\mathbf{T}^0 - \mathbf{T}) \cdot \mathbf{u} dS\end{aligned}\quad (5.4)$$

is equal or greater than zero.

Now, it follows from Eq. (5.1) that the second integrand in (5.4) would vanish identically on S_T . Therefore,

$$\begin{aligned}\int_{S_D} (\mathbf{T}^0 - \mathbf{T}) \cdot \mathbf{u} dS &= \int_S (\mathbf{T}^0 - \mathbf{T}) \cdot \mathbf{u} dS \\ &= \int_V (Q_i^0 - Q_i) q_i dV\end{aligned}$$

where the last step is a consequence of the principle of virtual work. Thus

$$\Delta\Pi_e = \int_V [W_e(Q_i^0) - W_e(Q_i) - \Delta V_e] dV \quad (5.5)$$

where

$$\Delta V_e = (Q_i^0 - Q_i) q_i. \quad (5.6)$$

All the quantities in Eqs. (5.5) may now be evaluated from the appropriate stress-strain law from Section 4. Where the body is elastic, it follows from Eq. (4.1) that

$$W_e = \int_{t=0}^{t=t} B_{ij} Q_j dQ_i = \frac{1}{2} B_{ij} Q_i Q_j, \quad (5.7)$$

$$\Delta V_e = B_{ij} Q_j (Q_i^0 - Q_i).$$

Therefore, if the entire body is elastic

$$\begin{aligned}2\Delta\Pi_e &= \int_V B_{ij} (Q_i^0 Q_j^0 - Q_i Q_j - 2Q_i Q_j^0 + 2Q_i Q_j) dV \\ &= \int_V B_{ij} (Q_i^0 - Q_i) (Q_j^0 - Q_j) dV\end{aligned}$$

where the last step follows from the symmetry of B_{ij} . Since B_{ij} is positive definite, $\Delta\Pi_e$ is zero if and only if $Q_i^0 = Q_i$ and is otherwise positive; thus the classical elastic principle is regained.

If an element is first elastic then plastic on face A , it follows from Eqs. (4.1) and (4.5) that

$$W_c = \int_{t=0}^{t=t} B_{ij} Q_i dQ_j + F \int_{t=t_0}^{t=t} a_i (Q - 1) dQ_i \quad (5.8a)$$

$$= \frac{1}{2} [B_{ij} Q_i Q_j + F(Q - 1)^2],$$

$$\Delta V_c = B_{ij} Q_i (Q_i^0 - Q_i) + F(Q - 1)(a_i Q_i^0 - a_i Q_i). \quad (5.8b)$$

In the evaluation of ΔV_c , it was assumed only that the actual state was on face A and hence Eq. (4.5) was applicable. Further, it follows from (3.5) that

$$Q = a_i Q_i.$$

However, there is no assurance that the statically admissible stress field Q_i^0 will attain its maximum intensity on face A , so that if Q^0 is defined by

$$Q^0 = \max_{\lambda} (a_{\lambda i} Q_i^0), \quad (5.9)$$

then Q^0 and $a_i Q_i^0$ are not necessarily equal. However, it does follow directly from Eq. (5.9) that

$$a_i Q_i^0 \leq Q^0.$$

Therefore, since in the actual plastic state Q is greater than one,

$$\Delta V_c \geq B_{ij} Q_i (Q_i^0 - Q_i) + F(Q - 1)(Q^0 - Q). \quad (5.10)$$

If only the elastic state and face A are being considered then four different regions may be distinguished within the body: V_{ee} , V_{ep} , V_{pe} , and V_{pp} . Here the first subscript on a region identifies the actual stress state as elastic or plastic, while the second subscript does the same for the statically admissible stress state. It then follows from Eqs. (5.5), (5.6), (5.7), (5.8a), and (5.10) that

$$\begin{aligned} 2\Delta\Pi_c &\geq \int_V B_{ij} (Q_i^0 - Q_i)(Q_i^0 - Q_i) dV \\ &\quad + F \int_{V_{ep}} (Q^0 - 1)^2 dV \\ &\quad + F \int_{V_{pe}} [-(Q - 1)^2 - 2(Q - 1)(Q^0 - Q)] dV \\ &\quad + F \int_{V_{pp}} [(Q^0 - 1)^2 - (Q - 1)^2 - 2(Q - 1)(Q^0 - Q)] dV. \end{aligned} \quad (5.11)$$

The first integrand is non-negative since B_{ij} is positive definite. The second integrand is a perfect square and the fourth integrand can be written $(Q^0 - Q)^2$, which is also a square. Finally the third integrand may be written

$$(Q - 1)[(Q - Q^0) + (1 - Q^0)].$$

Since the actual state is plastic in $V_{p\sigma}$, while the admissible state is elastic,

$$Q \geq 1 \geq Q^0,$$

and hence the third integrand is also non-negative. Therefore $\Delta\Pi_e \geq 0$ which proves the theorem for this case.

All loadings permitted in this section are contained in the stress-strain law (4.15). At a typical time t_ρ ,

$$\begin{aligned} W_e = & \frac{1}{2}B_{ii}Q_iQ_i + F\left\{\sum_{\mu=2}^{\rho}\int_{t=t_{\mu-1}}^{t=t_{\mu}}\left[\sum_{\nu=1}^{\mu-1}\int_{t=t_{\nu-1}}^{t=t_{\nu}}\sum_{\lambda=1}^{\nu}\varphi_{\nu\lambda}a_{\lambda i}dQ\right]dQ_i\right. \\ & \left.+ \sum_{\mu=1}^{\rho}\int_{t=t_{\mu-1}}^{t=t_{\mu}}\left[\int_{t=t_{\mu-1}}^{t=t_{\mu}}\sum_{\lambda=1}^{\mu}\varphi_{\mu\lambda}a_{\lambda i}dQ\right]dQ_i\right\}. \end{aligned}$$

If the summation over i is taken first it follows from Eq. (3.5) that $a_{\lambda i}dQ_i = dQ$. The summation over λ then involves only $\varphi_{\nu\lambda}$ and hence can be explicitly evaluated from Eqs. (4.14b). Finally, the remaining summations may be combined with the piecewise integrations to yield

$$\begin{aligned} W_e = & \frac{1}{2}B_{ii}Q_iQ_i + F\int_{t=t_0}^{t=t_{\rho}}\left(\int_0^t dQ\right)dQ \\ = & \frac{1}{2}[B_{ii}Q_iQ_i + F(P_{\rho} - 1)^2]. \end{aligned} \tag{5.12}$$

Therefore, Eq. (5.8a) is valid for this general case.

Similarly,

$$\Delta V_e = B_{ii}Q_i(Q_i^0 - Q_i) + F(Q_i^0 - Q_i)\sum_{\nu=1}^{\rho}\int_{t_{\nu-1}}^{t_{\nu}}\sum_{\lambda=1}^{\nu}\varphi_{\nu\lambda}a_{\lambda i}dQ.$$

Since each of the $\varphi_{\nu\lambda}$ must be non-negative,

$$\Delta V_e \geq B_{ii}Q_i(Q_i^0 - Q_i) + F\sum_{\nu=1}^{\rho}(Q_i^0 - Q_i)\int_{t_{\nu-1}}^{t_{\nu}}\sum_{\lambda=1}^{\nu}\varphi_{\nu\lambda}dQ.$$

Finally, it follows from Eq. (4.15b) that Inequality (5.10) is again valid.

Since the proof given previously depended only upon (5.8a) and (5.10) it is immediately applicable to the general case also. Therefore $\Delta\Pi_e \geq 0$ for the most general type of loading considered in the present section, and the theorem has been proved.

6. Minimum Potential Energy

The strain energy of an element is defined by

$$W = \int Q_i dq_i, \quad (6.1)$$

and the potential energy is given by

$$\Pi(q_i) = \int_V W(q_i) dV - \int_{s_r} \mathbf{T} \cdot \mathbf{u} dS. \quad (6.2)$$

The principle of minimum potential energy then states that among all “geometrically admissible” displacement fields the potential energy is minimized by the actual field. Here a “geometrically admissible” displacement field $\mathbf{u}^*$ is any set of sufficiently continuous displacements which satisfy all displacement boundary conditions. In other words,

$$\Delta\Pi = \int_V [W(q_i^*) - W(q_i)] dV - \int_{s_r} [\mathbf{T} \cdot \mathbf{u}^* - \mathbf{T} \cdot \mathbf{u}] dS \quad (6.3)$$

is equal or greater than zero with equality if and only if $\mathbf{u}^* = \mathbf{u}$.

In applying this principle, it is necessary to start from an assumed set of displacements $\mathbf{u}^*$ and compute the corresponding strains q_i^* . The stress-strain laws in Section 4 must then be inverted and solved for the stresses, and only after this is done can it be determined which stress-strain law is correct. In the following it will be assumed that this can be done and the resulting “geometrically admissible” stresses will be denoted by Q_i^* . Further, it will be assumed that both the actual and geometrically admissible stresses satisfy the restrictions in the first paragraph of Section 5.

As in Section 5, the surface integral may be taken over the entire surface and the principle of virtual work applied to reduce it to a volume integral. Thus

$$\Delta\Pi = \int_V \{[W(q_i^*) - V(q_i^*)] - [W(q_i) - V(q_i)]\} dV \quad (6.4)$$

where

$$V(q_i^*) = Q_i q_i^*, \quad V(q_i) = Q_i q_i. \quad (6.5)$$

The plastic part of the strain energy is easily evaluated in a form which is valid for all loading histories as follows:

$$\begin{aligned} W^p &= \int Q_i dp_i = \int Q_i p_i' dt = \int D^p dt \\ &= \int FQQ' dt = (F/2)(Q^2 - 1), \end{aligned}$$

where Eqs. (3.1), (3.13b), (3.14), and (3.17) have been used. Thus the complete strain energy is

$$W(q_i) = \frac{1}{2}[B_{ij}Q_iQ_j + F(Q^2 - 1)]. \quad (6.5)$$

Similarly, for the geometrically admissible state,

$$W(q_i^*) = \frac{1}{2}[B_{ij}Q_i^*Q_j^* + F(Q^{*2} - 1)]. \quad (6.6)$$

All of the plastic cases to be considered may be regarded as special cases of Eq. (4.16). Thus

$$\begin{aligned} V(q_i^*) = B_{ij}Q_iQ_j^* + FQ_i \sum_{\nu=1}^{\mu-1} \int_{P_{\nu-1}^*}^{P_\nu^*} \sum_{\lambda=1}^{\mu} \varphi_{\nu\lambda}^* a_{\lambda i} dQ^* \\ + FQ_i \int_{P_{\mu-1}^*}^{Q^*} \sum_{\lambda=1}^{\mu} \varphi_{\nu\lambda}^* a_{\lambda i} dQ^*. \end{aligned} \quad (6.7)$$

The choice of stress strain law used in Eq. (6.7) depended upon the history of the geometrically admissible state, not upon the actual state. Therefore, Eq. (3.5) cannot be used directly. However, in direct analogy to the procedure of the preceding section, the inequality

$$Q_i a_{\lambda i} \leq Q \quad (6.8)$$

where Q is defined by (3.5) may be used. Summation over λ then eliminates $\varphi_{\nu\lambda}^*$, since the starred quantities must also satisfy (4.15b). Finally, the integration and remaining summation yield

$$V(q_i^*) \geq B_{ij}Q_iQ_j^* + FQ(Q^* - 1). \quad (6.9)$$

For the actual state, the equality sign will be valid in (6.8) for each λ and hence

$$V(q_i) = B_{ij}Q_iQ_j + FQ(Q - 1). \quad (6.10)$$

The substitution of Eqs. (6.5), (6.6), (6.9), and (6.10) into Eq. (6.4) furnishes

$$\begin{aligned} 2\Delta\Pi \geq \int_V B_{ij}(Q_i^*Q_j^* - 2Q_iQ_j^* + Q_iQ_j) dV \\ + F \int_{V_{ep}} (Q^* - 1)(Q^* + 1 - 2Q) dV + \int_{V_{ps}} (Q - 1)^2 dV \\ + \int_{V_{pp}} [(Q^* - 1)(Q^* + 1 - 2Q) + (Q - 1)^2] dV. \end{aligned} \quad (6.11)$$

As in Section 5, the first integrand is non-negative since B_{ij} is symmetric and positive definite; the integrand in V_{ep} is non-negative since $Q^* \geq 1 \geq Q$; and the remaining integrands are both perfect squares. Therefore, $\Delta\Pi \geq 0$ and the principle has been proved.

As in the theory of elasticity, the two minimum principles can be combined to yield upper and lower bounds on the total energy. If Eqs. (5.3) and (6.2) for the actual state are added, the result is

$$\begin{aligned}\Pi(q_i) + \Pi_c(Q_i) &= \int_V \int (Q_i dq_i + q_i dQ_i) dV - \int_{(S_T + S_D)} \mathbf{T} \cdot \mathbf{u} dS \\ &= \int_V Q_i q_i dV - \int_S \mathbf{T} \cdot \mathbf{u} dS = 0,\end{aligned}\tag{6.12}$$

where the last step follows from the principle of virtual work. Therefore, in view of the two minimum principles, the continued inequality

$$\Pi(q_i^*) \geq \Pi(q_i) = -\Pi_c(Q_i) \geq -\Pi_c(Q_i^0)$$

is valid. This is equivalent to the two inequalities

$$\begin{aligned}-\Pi_c(Q_i^0) &\leq \Pi(q_i) \leq \Pi(q_i^*), \\ -\Pi(q_i^*) &\leq \Pi_c(Q_i) \leq \Pi_c(Q_i^0).\end{aligned}\tag{6.13}$$

Therefore, upper and lower bounds on either the potential or total complementary energy may be obtained by considering both a statically admissible and a geometrically admissible state.

7. *Conclusions and Limitations*

The preceding sections have been concerned with the development of a piecewise linear theory of plasticity. It is hoped that such a theory will prove of practical value, despite its limitations. Indeed, before developing the general theory the author [15] has had an occasion to use a specific application of part of it. These limitations are twofold. On the one hand, real materials do not exhibit in general, the piecewise linear properties assumed here, so that the real behavior must be approximated before the theory can be applied. On the other hand, the mathematical difficulties of applying the theory increase tremendously when more than one linear regime is involved.

The development of the theory was carried out in three parts. First, the flow rule at a corner was derived in its full generality for an isotropic yield function. Previously KOITER [7] and SANDERS [14] had discussed corners in general, but had left some ambiguity when it came to specific applications, while PRAGER [8] had considered a specific example without inquiring into the general theory. The results obtained in Section 3 are not strictly limited to piecewise linear yield functions, but remain valid in all cases where there are corners.

Second, the flow law for a piecewise linear material was integrated. In a case where the entire body enters the same plastic regime, the solution of the problem should now be easily obtained. However, if more than one plastic regime is involved, the boundaries between different regions will, in general, be involved in non-linear equations so that the problem is still far from trivial. Finally, if given material points move from one plastic regime to another, additional unknowns are introduced, namely the stress intensities at which the transfer takes place.

The final contribution of the paper has been the extension of the well known minimum principles of elasticity to certain cases of piecewise linear plasticity. The limitation on these principles is serious, and care must be taken that they are not misused. Not only must the assumed admissible state be restricted but also the actual state. The former can be controlled by choice, but there is no way of knowing whether or not the actual state satisfies the restrictions. The result is that the principles can never be used with the mathematical confidence associated with the theory of elasticity.

At least part of this limitation is intrinsic in the theory and cannot be overcome by any mathematical tricks in the proof. This may be shown by considering a problem in which the entire body is actually first in plastic regime *A* and then in plastic regime *B*. It then follows from Eqs. (4.12) and (5.3) that

$$W_c(Q_i) = \frac{1}{2}B_{ij}Q_iQ_j + \frac{1}{2}F[(P-1)(2R-P-1) + (Q-P)^2], \quad (7.1)$$

where, in regime *B*,

$$Q = b_iQ_i, \quad R = a_iQ_i, \quad (7.2)$$

and P is the value of Q when the element moves from A to B . Similarly,

$$W_e(Q_i^0) = \frac{1}{2}B_{ii}Q_i^0Q_i^0 + \frac{1}{2}F[(P^0 - 1)(2R^0 - P^0 - 1) + (Q^0 - P^0)^2], \quad (7.3)$$

and

$$\Delta V_e = B_{ii}Q_i(Q_i^0 - Q_i) + F[(Q^0 - Q)(Q - P) + (R^0 - R)(P - 1)]. \quad (7.4)$$

Substitution of Eqs. (7.1), (7.3), and (7.4) into Eq. (5.8) leads eventually to

$$\begin{aligned} 2\Delta\Pi_e = \int_V [B_{ii}(Q_i - Q_i^0)(Q_i - Q_i^0) + F(Q^0 - Q)^2 \\ + 2F(Q^0 - R^0)(P - P^0)] dV. \end{aligned} \quad (7.5)$$

Since the actual value of the stress intensity is unknown, it is conceivable that the statically admissible stress state would be such that $Q_i^0 = Q_i$, but that $P^0 > P$. In this case $\Delta\Pi_e$ would be negative and the proposed principle would be false.

However, despite this serious limitation, the principles should be of considerable value, since problems in plasticity are even more difficult than problems in elasticity where the minimum principles have been practically indispensable. Indeed, some plastic shell problems have already been solved with the aid of these minimum principles and will be reported upon elsewhere [13].

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