On Solutions of Riccati's Equation as Functions of the Initial Values

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Introduction. Subject to mild restrictions on a , b , and c , the general Riccati equation

$$\frac{du}{dy} = a(y) + 2b(y)u(y) + c(y)u^2(y)$$

has a solution through any given point of the (u, y) plane. In particular, one may consider that solution which satisfies the initial condition

$$u = 0 \quad \text{when} \quad y = x$$

or indeed

$$u = f(x) \quad \text{when} \quad y = x$$

where $f(x)$ is given.

The function u so obtained depends on x , naturally, as well as on y , and our notation emphasizes this feature. In view of the x -dependence it is appropriate to write

$$u = u(x, y)$$

rather than simply $u = u(y)$. We have, then,

$$u_y \equiv \frac{\partial u}{\partial y} = a(y) + 2b(y)u(x, y) + c(y)u^2(x, y)$$
$$u(x, x) = f(x).$$

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This is the situation with which this paper is concerned. The solution $u(x, y)$ is studied as a function of both variables; the role of x is not subordinate to that of y . It will be found that introduction of two auxiliary functions associated with $u(x, y)$ yields a symmetric set of differential equations, in which x is the independent variable. The same expressions satisfy certain nonlinear functional equations which are connected with, and illuminate, a class of problems in electromagnetic theory.

The associated functions. Define $u(x, y)$, $v(x, y)$ and $w(x, y)$ by the system

$$(1a) \quad u_y = a(y) + 2b(y)u(x, y) + c(y)u^2(x, y), \quad u(x, x) = 0$$

$$(1b) \quad v_y = b(y) + c(y)u(x, y), \quad v(x, x) = 0$$

$$(1c) \quad w_y = c(y) \exp [2v(x, y)], \quad w(x, x) = 0.$$

Thus, v and w are given in terms of u by one and two quadratures, respectively. For much of our work it suffices merely to have u, v, w, a, b, c well-defined complex-valued functions in the region for which the results are established and to have (1) valid at every interior point. It seems necessary to restrict the region, however; and for convenience we introduce the following definition:

Definition. An admissible region R in the (x, y) plane is an open convex set symmetrical about the line $y = x$.

The intent is to ensure the validity of (1) for each fixed x_0 and all y in the set, and to ensure that the initial condition is also available. Less stringent restrictions on the region would suffice, but the matter lies to one side of our main objective.

That v and w are introduced appropriately is suggested by the following theorems, which form the chief content of the present note:

Theorem I. Let u, v, w , be well-defined functions satisfying (1) in an admissible region R of the (x, y) plane. Suppose $v(x, y)$ is a continuous function of x for each fixed y , $(x, y) \in R$. Then the partial derivatives with respect to x exist and satisfy

$$(2a) \quad -w_x = c(x) + 2b(x)w(x, y) + a(x)w^2(x, y)$$

$$(2b) \quad -v_x = b(x) + a(x)w(x, y)$$

$$(2c) \quad -u_x = a(x) \exp [2v(x, y)]$$

at every point of R .

When we say that a relation is valid in R , we mean it is valid when all the points (x, y) , (x, z) , $\dots$ which enter in the conclusion are interior to R . This convention is adopted, for brevity, throughout the present paper.

Theorem II. Let u, v , and w be well-defined functions satisfying (1) in an admissible region R . Then the following functional equations are valid in R :

$$(3a) \quad \frac{u(x, z) - u(y, z)}{u(x, y)} = e^{-v(x, y) + v(x, z) + v(y, z)}$$

$$(3b) \quad 1 - u(x, y)w(y, z) = e^{v(x, y) - v(x, z) + v(y, z)}$$

$$(3c) \quad \frac{w(x, z) - w(x, y)}{w(y, z)} = e^{v(x, y) + v(x, z) - v(y, z)}.$$

Here, and in all analogous cases throughout this paper, the relation actually established is the one which arises after clearing of fractions. In case $u(x, y) = 0$ (for example) Equation (3a) means $u(x, z) = u(y, z)$. The form given makes the structure more transparent and symmetrical.

Proof of the first theorem. Since it is easy to do so, we prefer to give a self-contained discussion, independent of the known theorems concerning differential equations with a parameter. Theorem I is a simple consequence of Theorem II, as is now to be proved.

The relation (3a) yields

$$(4) \quad \frac{u(x, z) - u(y, z)}{x - y} = \frac{u(x, y)}{x - y} e^{-v(x, y) + v(x, z) + v(y, z)}$$

and the definition of partial derivative combines with (1a) to give

$$\lim u(x, y)/(x - y) = -u_x(x, x) = -a(x)$$

since $u(x, x) = 0$. Hence, by the assumed continuity of $v(x, y)$, we may take the limit in (4) as $y \rightarrow x$ to obtain

$$(5) \quad u_x(x, z) = -a(x)e^{2v(x, z)}.$$

This is the relation (2c).

Turning to the second relation, we suppose $u(x_0, y) \neq 0$ for some choice of x_0 at a given value, y . Since u is continuous as a function of y , there is actually a neighborhood of y throughout which $u(x_0, y) \neq 0$. Equation (3a) takes the form

$$(6) \quad e^{v(y, z)} = e^{v(x, y) - v(x, z)} \frac{u(x, z) - u(y, z)}{u(x, y)}$$

at $x = x_0$. The left side is nonzero for v well-defined, and hence the right side is also nonzero. Taking the logarithm we see that $v(y, z)$ is differentiable with respect to y .

To carry out the differentiation we use an abbreviated notation, writing

$$u_{12} = u(x, y), \quad u_{13} = u(x, z), \quad u_{23} = u(y, z)$$

and similarly for v and w . In this notation (6) becomes

$$u_{12}e^{v_{23}} = e^{v_{12} - v_{13}}(u_{13} - u_{23})$$

after multiplication by u_{12} . Differentiating with respect to y gives

$$e^{v_{23}}[a + 2bu_{12} + cu_{12}^2 + u_{12}v_x(y, z)] = e^{v_{12}-v_{13}}[ae^{2v_{23}} + (u_{13} - u_{23})(b + cu_{12})]$$

with a, b, c evaluated at y . We have made use of (1) and (5). By (3a) the right side becomes

$$ae^{v_{12}-v_{13}+2v_{23}} + (b + cu_{12})u_{12}e^{v_{23}}.$$

Dividing by $u_{12}e^{v_{23}}$ and simplifying gives

$$v_x(y, z) = a(e^{v_{12}-v_{13}+v_{23}} - 1)/u_{12} + b,$$

and (3b) now gives (2b) from this.

If $u(x, y_0) \equiv 0$ in x at the given value $y = y_0$, the choice $y = y_0, z = x$ in (3a) and (3b) gives, respectively,

$$(7) \quad 0 = u(y_0, x)$$

$$(8) \quad -v(x, y_0) = v(y_0, x)$$

identically in x . Equation (7) shows that $u_x(y_0, x) = 0$ so that

$$(9) \quad a(x) \equiv 0$$

by (2c). Equation (8) gives

$$-v_x(x, y_0) = v_y(y_0, x) = b(x) + c(x)u(y_0, x)$$

which is seen by (7) and (9) to be equivalent to (2b).

Turning to the relation (2a), we see from (3b) that $w(y, z)$ is differentiable with respect to the first argument, y , if $u(x_0, y) \neq 0$ for some x_0 . In these circumstances (3b) gives

$$\begin{aligned} -u_{12}w_y(y, z) - w_{23}[a + 2bu_{12} + cu_{12}^2] &= e^{v_{12}-v_{13}+v_{23}}[b + cu_{12} - b - aw_{23}] \\ &= (1 - u_{12}w_{23})[cu_{12} - aw_{23}] \end{aligned}$$

when we use (1a), (1b), (2b), and (3b), evaluating a, b, c at y . The reader may verify that this reduces to (2a) with (y, x) replacing (x, y) .

If $u(x, y) \equiv 0$ at the given value $y = y_0$, setting $z = x, y = y_0$ in (3c) gives

$$(10) \quad -w(x, y_0) = w(y_0, x)e^{v(x, y_0)-v(y_0, x)} = w(y_0, x)e^{-2v(y_0, x)}$$

by (8). Differentiation with respect to x by means of (1c), (1b), and (7) gives a relation which (10) shows to be equivalent to (2a), since $a(x) = 0$.

The functional equations. To establish the relation (3a), we consider the function $f(x, y, z)$ defined by

$$f(x, y, z) = (u_{13} - u_{23})e^{-v_{12}-v_{23}}.$$

Since the right-hand side is differentiable with respect to z , by (1a) and (1b), the left-hand side is also. The differentiation yields

$$f_z = e^{-v_{13}-v_{23}}[(u_{13} - u_{23})(-b - cu_{13} - b - cu_{23}) \\ + (a + 2bu_{13} + cu_{13}^2) - (a + 2bu_{23} + cu_{23}^2)]$$

where a, b, c have argument z . The reader may verify that the equation reduces to

$$f_z \equiv 0.$$

This independence of z , and the fact that R is admissible, assure

$$f(x, y, z) = f(x, y, y)$$

for each (x, y) . The initial conditions give

$$f(x, y, y) = (u_{12} - u_{22})e^{-v_{12}-v_{22}} = u_{12}e^{-v_{12}}$$

and thus, (3a) is established.

The derivation makes no use of the initial condition until the very end. Thus, even if $u(x, x)$ and $v(x, x)$ are unrestricted, there is still a functional equation free of a, b, c .

Theorem III. *If u and v are well-defined solutions of the differential equations (1a), (1b) in an admissible region R , then the relation*

$$\frac{u(x, z) - u(y, z)}{u(x, y) - u(y, y)} = \frac{\exp [v(x, z) + v(y, z)]}{\exp [v(x, y) + v(y, y)]}$$

is valid in R for arbitrary initial values $u(y, y), v(y, y)$.

Turning to the relation (3b), we consider the function $g(x, y, z)$ defined by

$$g(x, y, z) = (1 - u_{12}w_{23})e^{v_{13}-v_{23}} - e^{v_{13}}.$$

Differentiation with respect to z is permissible and yields the following, with b and c evaluated at z :

$$g_z = (1 - u_{12}w_{23})e^{v_{13}-v_{23}}(b + cu_{13} - b - cu_{23}) + e^{v_{13}-v_{23}}(-u_{12})(ce^{2v_{23}}) \\ = (g + e^{v_{12}})c(u_{13} - u_{23}) - cu_{12}e^{v_{13}+v_{23}}.$$

Since (3a) is established, we have

$$u_{12}e^{v_{13}+v_{23}} = e^{v_{12}}(u_{13} - u_{23})$$

and hence

$$(11) \quad g_z = g\lambda(z), \\ \lambda(z) = c(z)[u(x, z) - u(y, z)].$$

A simple lemma asserts uniqueness of the solution g in (11):

Lemma. *Let g satisfy $dg/dz = \lambda(z)g(z)$ for $a < z < b$, and suppose $g(y) = 0$ for some value of y on (a, b) . If $\lambda(z)$ is a derivative on (a, b) , then $g(z) \equiv 0$.*

The following proof, due to R. ARENS, is even shorter than the author's. With $\Lambda'(z) = \lambda(z)$, define

$$s(z) = e^{-\Lambda(z)} g(z).$$

Then

$$s' = e^{-\Lambda} g' - e^{-\Lambda} g \Lambda' = 0.$$

Since $s = 0$ at $z = y$, it follows that $s \equiv 0$. Hence $g \equiv 0$.

Returning to Equation (11) we note that

$$\lambda(z) = v_z(x, z) - v_z(y, z) = \frac{\partial}{\partial z} [v(x, z) - v(y, z)]$$

by (1b). Since $g = 0$ when $z = y$, the lemma ensures $g \equiv 0$; and this is (3b).

The result (3c) might be deduced from Theorem I and symmetry; that is, (3c) is really (3a) for the system (2). Since we have not supposed that $v(x, y)$ is a continuous function of x , however, the validity of Theorem I is not assured, and we proceed differently.

Let $h(x, y, z)$ be defined by

$$h(x, y, z) = (w_{13} - w_{12})e^{v_{23}-v_{13}} - e^{v_{12}}w_{23}.$$

Differentiating as before we find, with $b = b(z)$, $c = c(z)$,

$$\begin{aligned} h_z &= (w_{13} - w_{12})e^{v_{23}-v_{13}}(b + cu_{23} - b - cu_{13}) + e^{v_{23}-v_{13}}ce^{2v_{13}} - e^{v_{12}}ce^{2v_{23}} \\ (12) \quad h_z &= (h + e^{v_{12}}w_{23})c(u_{23} - u_{13}) + ce^{v_{23}+v_{13}} - ce^{v_{12}+2v_{23}}. \end{aligned}$$

Now, by (3a) and (3b), respectively,

$$\begin{aligned} w_{23}e^{v_{12}}(u_{23} - u_{13}) &= -u_{12}w_{23}e^{v_{12}+v_{23}} \\ &= (e^{v_{12}-v_{13}+v_{23}} - 1)e^{v_{12}+v_{23}}. \end{aligned}$$

Hence the terms free of h in (12) cancel, to give an equation equivalent to (11). As before, we deduce $h \equiv 0$, and this establishes (3c).

Theorems I and II show that the running variable and the initial value enter in a highly symmetric fashion. This feature is further illustrated by the following corollary, which follows by setting $z = x$ in (3a) and (3b):

Corollary: *Let u , v and w be solutions of (1) in an admissible region R . Then*

$$(13) \quad \frac{u(x, y)}{u(y, x)} = -\frac{\exp v(x, y)}{\exp v(y, x)} = \frac{w(x, y)}{w(y, x)}.$$

The converse. We have obtained the functional equation as a consequence of the Riccati equation. It is natural to inquire whether one can proceed in the opposite direction. If u , v , and w are differentiable functions satisfying (3) and satisfying appropriate initial conditions, are they necessarily solutions of some system (1)? That such is the case is shown by the following result, where it is seen also that differentiability on the line $y = x$, with respect to y only, suffices.

Theorem IV. *In an admissible region R , let u , v , and w be continuous solutions of the functional equations (3) and satisfy the additional conditions*

$$\begin{aligned} u(x, x) &= v(x, x) = w(x, x) = 0 \\ u_y(x, x), \quad v_y(x, x), \quad w_y(x, x) &\text{ exist.} \end{aligned}$$

Then there are functions a , b , and c such that the differential systems (1) and (2) are valid in R .

For proof we define a , b , and c by

$$a = u_y(x, x), \quad b = v_y(x, x), \quad c = w_y(x, x).$$

The relation (3a) assumes the form (4), whence we deduce as before that (2c) holds. In just the same way the validity of (3c) ensures that of (1c).

To obtain (1b) we multiply (3b) by $\exp(-v_{23})$ and subtract 1 to find

$$e^{v_{12}-v_{13}} - 1 = e^{-v_{23}} - 1 - e^{-v_{23}} u_{12} w_{23}.$$

This is multiplied by $\exp(v_{13})/(z - y)$ to give

$$\frac{e^{v_{12}} - e^{v_{13}}}{z - y} = e^{v_{12}} \frac{e^{-v_{23}} - 1}{z - y} - e^{v_{12}-v_{23}} u_{12} \frac{w_{23}}{z - y}.$$

With y fixed we let $z \rightarrow y$ to obtain

$$\lim_{z \rightarrow y} \frac{e^{v_{12}} - e^{v_{13}}}{z - y} = e^{v_{12}} \frac{\partial}{\partial z} e^{-v_{23}} \Big|_{z=y} - e^{v_{12}} u_{12} \frac{\partial}{\partial z} w_{23} \Big|_{z=y}.$$

This shows that $\exp(v)$ is differentiable, hence v is (since v is continuous). Moreover we have

$$-\frac{\partial}{\partial y} e^{v_{12}} = -e^{v_{12}} v_y(x, y) = e^{v_{12}} [-b(y)] - e^{v_{12}} u_{12} c(y),$$

which reduces to (1b). The reader may verify that a similar argument gives (2b).

For the remaining relations we use the corollary to Theorem II to find

$$u(x, y) = -u(y, x) e^{v(x, y) - v(y, x)}.$$

The relations previously obtained show that the right-hand side is differentiable, and enable us to calculate the derivative

$$u_y(x, y) = -u_{12} e^{v_{12}-v_{21}} (b + cu_{12} + b + aw_{21}) + e^{v_{12}-v_{21}} a e^{2v_{21}}$$

where $a = a(y)$, $b = b(y)$. Since

$$u_{21}e^{v_{12}-v_{21}} = -u_{12},$$

and since $z = x$ in (3b) gives

$$u_{12}w_{21} + e^{v_{12}+v_{21}} = 1,$$

the equation is equivalent to (1a). It was already shown that (2a) is a consequence of (1) and (3) (or one can argue from symmetry). This establishes Theorem IV.

Representation. It is well known [2] that the substitution

$$(14) \quad u = -q_v/(cq)$$

gives a linear equation for q ;

$$(15) \quad q_{vv} - [2b(y) + c'(y)/c(y)]q_v + a(y)c(y)q = 0.$$

If the coefficients are continuous then this equation has a solution manifold of form $\alpha f(x) + \beta g(x)$. Hence,

$$u(x, y) = -\frac{1}{c(y)} \frac{\alpha f'(y) + \beta g'(y)}{\alpha f(y) + \beta g(y)}.$$

The condition $u(x, x) = 0$ determines α/β , whence we obtain the first result in the following theorem:

Theorem V. *Let u , v , and w be solutions of (1) in an admissible region R . Suppose a , b continuous, c continuously differentiable, and $c \neq 0$. Then there exist twice-differentiable functions f and g such that*

$$\begin{aligned} u(x, y) &= \frac{1}{c(y)} \begin{vmatrix} f'(x) & g'(x) \\ f'(y) & g'(y) \end{vmatrix} \div \begin{vmatrix} f(y) & g(y) \\ f'(x) & g'(x) \end{vmatrix} \\ v(x, y) &= \int_x^y b(t) dt + \log \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} - \log \begin{vmatrix} f(y) & g(y) \\ f'(x) & g'(x) \end{vmatrix} \\ w(x, y) &= c(x) \begin{vmatrix} f(x) & g(x) \\ f(y) & g(y) \end{vmatrix} \div \begin{vmatrix} f(y) & g(y) \\ f'(x) & g'(x) \end{vmatrix} \end{aligned}$$

at every point of R where the denominator determinant is not zero.

For the second relation we integrate (1b) in the form

$$v_v = b - q_v/q$$

to find

$$v = \int_x^y b(t) dt - \log q + \text{constant}.$$

Evaluating the constant leads to the desired expression.

For w , we proceed similarly to find

$$w = \int_x^y c(x) \exp \left[2 \int_x^t b(\tau) d\tau \right] \left[\frac{f(t)g'(x) - g(t)f'(x)}{f(x)g'(x) - g(x)f'(x)} \right]^{-2} dt.$$

Now, this is almost the formula for constructing a new solution of a linear equation from an old one. Without detailed calculation, therefore, we know that

$$[f(y)g'(x) - g(y)f'(x)]w$$

satisfies the linear equation (15). Determining the constants from $w(x, x) = 0$ we obtain

$$w(x, y) = h(x) \frac{f(y)g'(x) - g(y)f'(x)}{f(y)g'(x) - g(y)f'(x)}.$$

Computing w , from this and from (1c) at $y = x$ determines $h(x)$, and completes the proof.

With this representation for u, v, w the symmetry between x and y is again in evidence. Moreover the functional equations reduce to simple algebraic identities, as will now be indicated. Substituting the expressions of Theorem V into (3a) and simplifying, we find that a, b, c drop out. In the notation

$$a_k = f, \quad b_k = f', \quad c_k = g, \quad d_k = g'$$

with subscript 1 for x , 2 for y , and 3 for z , the identity (3a) is found to be equivalent to

$$(16) \quad \begin{vmatrix} b_1 & d_1 \\ b_3 & d_3 \end{vmatrix} \begin{vmatrix} a_3 & c_3 \\ b_2 & d_2 \end{vmatrix} - \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} \begin{vmatrix} a_3 & c_3 \\ b_1 & d_1 \end{vmatrix} = \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix} \begin{vmatrix} a_3 & c_3 \\ b_3 & d_3 \end{vmatrix}.$$

That this is an identity is readily verified. The result is reminiscent of the generalized Lagrange formula

$$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$$

though it seems not to be deducible therefrom.

Applying the same process to (3b) we again get an algebraic identity, which the author was surprised to find is the same as (16), not a new one. (That (3c) is the same is clear from symmetry.) From this viewpoint Theorem II is simply a complicated way to write the algebraic fact stated in (16).

The linear system. We define functions U, V, W, S by

$$(17a) \quad U = ue^{-v}, \quad u = U/V$$

$$(17b) \quad V = e^{-v}, \quad e^v = 1/V$$

$$(17c) \quad W = -we^{-v}, \quad w = -W/V$$

$$(17d) \quad S = e^v - uve^{-v}.$$

The four functions are not independent; in fact,

$$(18) \quad \begin{vmatrix} U & V \\ S & W \end{vmatrix} = -1.$$

The symmetrical behavior noted for the triple

$$(u, v, w)$$

is also found for the quadruple

$$(U, V, W, S)$$

as will now be established.

Theorem VI. *The system (1) is equivalent to the linear systems*

$$\begin{aligned} \begin{pmatrix} U_v \\ V_v \end{pmatrix} &= \begin{pmatrix} b & a \\ -c & -b \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, & \begin{pmatrix} U \\ V \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{when } y = x \\ \begin{pmatrix} W_v \\ S_v \end{pmatrix} &= \begin{pmatrix} -b & -c \\ a & b \end{pmatrix} \begin{pmatrix} W \\ S \end{pmatrix}, & \begin{pmatrix} W \\ S \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{when } y = x. \end{aligned}$$

The proof is simple. Equation (1b) combines with (17b) to give

$$(19) \quad -V_v = bV + cU.$$

Similarly, substituting the result (17a) into (1a) and multiplying by V^2 gives

$$VU_v - UV_v = aV^2 + 2bUV + cU^2$$

which reduces to

$$(20) \quad U_v = aV + bU$$

in view of (19). In a similar manner,

$$\begin{aligned} -VW_v + WV_v &= c && \text{[by (17c) and (1c)]} \\ (21) \quad -W_v &= bW + (c/V)(UW + 1) && \text{[by (19)]} \\ -W_v &= bW + cS && \text{[by (18)].} \end{aligned}$$

Finally, the fact that $S = (1 + UW)/V$ combines with (19)–(21) to give

$$(22) \quad S_v = aW + bS$$

and this completes the proof.

Theorem VII. *In the previous notation let a matrix M be defined by*

$$(23) \quad M = \begin{pmatrix} V & U \\ W & S \end{pmatrix}.$$

Then the system (3) is equivalent to the matrix equation

$$M(x, z) = M(x, y)M(y, z).$$

The initial condition is $M(x, x) = I$, the identity.

We omit the verification, which is simply a matter of elementary algebra. Also, the result is a consequence of a result in [4].

The following theorem, which illuminates this whole discussion, is due to R. ARENS:

Theorem VIII. *The differential system of Theorem VI is equivalent to the matrix equation*

$$M_y = MB$$

where M is as defined by (23) and where

$$(24) \quad B = \begin{pmatrix} -b & a \\ -c & b \end{pmatrix}.$$

The result is verified by inspection. Here $M(x, y)$ is the *matrizant* [6] of the matrix equation $M_y = MB$, and when the coefficients are integrable, the group property of Theorem VII is well known (*ibid.*, p. 218. The assumptions in [6] that M is a vector and that a, b, c are continuous seem to be not essential to the analysis.) From this point of view, the system (3) is the statement of Huygens' principle for the system $M_y = MB$; compare [7]. Alternatively, one may solve for M by use of product integrals [8], [9], when the coefficients are bounded and integrable. The group property is well known in this context also.

An Interpretation. Let $\epsilon(x)$ and $\mu(x)$ be complex-valued functions of the real variable x , defined for $-\infty < x < \infty$. Consider a dielectric slab with plane, parallel faces normal to the x axis, as shown in Fig. 1. Let the slab extend from x to $y > x$, and let its dielectric constant and permeability agree with $\epsilon(x)$ and $\mu(x)$, respectively. Since the slab is thought of as being in free space, the same configuration may be described as an infinite medium having parameters

$$\begin{aligned} \epsilon_0, \quad \mu_0 & \quad (t < x) \\ \epsilon(t), \quad \mu(t) & \quad (x \leq t \leq y) \\ \epsilon_0, \quad \mu_0 & \quad (y < t). \end{aligned}$$

Let $u(x, y)$ be the complex right-hand reflection coefficient for this medium, while $w(x, y)$ is the left-hand reflection coefficient, and $e^{v(x, y)}$ is the transmission coefficient in either direction. If x be fixed, these coefficients satisfy a Riccati system of the form (1), with $a(y) \equiv c(y)$. At normal incidence

$$a = c = (\pi i / \lambda)(\epsilon / \epsilon_0 - \mu / \mu_0), \quad \lambda = \text{wavelength}$$

$$b = (\pi i / \lambda)(\epsilon / \epsilon_0 + \mu / \mu_0).$$

(See [4] or [5].) Evidently, the initial condition is satisfied.

FIG. 1. A dielectric slab associated with a given pair of functions, where ϵ and μ are replaced by ϵ_0 and μ_0 exterior to the medium.

If we hold y fixed and vary x , a similar Riccati equation must also be valid, with u and w interchanged, for the two situations are physically equivalent. An increase of x now corresponds to a decrease in the slab thickness, whereas the thickness increases with y . For this reason one expects the change of sign which makes its appearance in (2).

The composite medium extending from x to z can be regarded as a medium extending from x to y , adjacent to a medium extending from y to z ($x < y < z$). The rule of composition is well known from transmission-line theory [3], [4],—or it may be established directly by considering multiple reflection. The result is

$$u(x, z) = u(y, z) + \frac{e^{2v(y, z)} u(x, y)}{1 - u(x, y) w(y, z)}$$

$$e^{v(x, z)} = \frac{e^{v(x, y)} e^{v(y, z)}}{1 - u(x, y) w(y, z)}$$

$$w(x, z) = w(x, y) + \frac{e^{2v(x, y)} w(y, z)}{1 - u(x, y) w(y, z)}$$

when we recall that $u(x, z)$, $\exp [v(x, z)]$, and $w(x, z)$ are the overall coefficients for the slab extending from x to z . This system is equivalent to (3). The physical interpretation is valid even when ϵ and μ are discontinuous; and that is perhaps the reason why no hypothesis on a , b , c was required in the passage from (1) to (3). (Of course the mere validity of (1) ensures that a , b , c are measurable, and that they cannot have a simple discontinuity.)

The change of variable giving the linear system is also suggested by physical considerations. The matrix M is simply the scattering matrix for a four-terminal network representing the slab; it is the matrix

$$\begin{pmatrix} 1/t & r/t \\ -\rho/t & d/t \end{pmatrix}, \quad d = t^2 - r\rho$$

introduced in [3]. Just as this change of variable linearizes certain Riccati-type difference equations in [3], so here it linearizes the Riccati differential equation (1). The various theorems of [3] concerning energy transfer also have analogies in the present setting which enable us to draw general conclusions concerning the matrix

$$(25) \quad \begin{pmatrix} e^* & u \\ w & e^* \end{pmatrix}.$$

(For instance if $a = c$ and a , b , c are continuous and pure imaginary, the matrix (25) is unitary. This aspect of the subject will be considered in a sequel to the present paper.)

A formal procedure. If the Taylor series

$$\begin{aligned} u(x, y) &= u(x, x) + u_y(x, x)(y - x) + \cdots \\ &= a(x)(y - x) + \cdots \end{aligned}$$

is differentiated formally one obtains

$$-u_x(x, x) = a(x)$$

and similarly for v and w . Equating cross derivatives in (1) now gives the system (2). For instance from

$$\begin{aligned} u_{yx} &= 2(b + cu)u_x \\ &= 2v_y u_x \end{aligned}$$

one obtains, formally,

$$\frac{(u_x)_y}{u_x} = 2v_y$$

or

$$u_x = e^{2v(x, y)} c_0(x) = -a(x) e^{2v(x, y)}$$

when the constant of integration $c_0(x)$ is determined by setting $y = x$. The other equations (2) can be established similarly. This procedure enables us to consider more general equations, such as

$$(26) \quad u_v = a + 3bu + 3cu^2 + du^3.$$

Whether there exists a system analogous to (1), (2) for (26) is an open question, though certain natural generalizations have been proved false. Still less is there any evidence for an extension of (3). Such an extension would be of interest, since (26) is not reducible to a linear equation.

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