On Certain Partial Differential Equations and Their Singularities

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1. Introduction. We investigate the partial differential equation

$$(1.1) \quad \Delta\psi + \alpha(x, y) \frac{\partial\psi}{\partial x} + \beta(x, y) \frac{\partial\psi}{\partial y} + \gamma(x, y)\psi = 0,$$

whose coefficients α , β , and γ are analytic functions of the real variables x and y . As is known, for every linear partial differential equation it is possible to generate various systems of particular solutions; see *e.g.* BERGMAN [1]. It is of interest to obtain systems of independent particular solutions having special characteristic properties, for instance, solutions which satisfy also an ordinary differential equation with rational or algebraic coefficients. In the present paper we shall determine such systems of solutions for a certain class of partial differential equations (1.1). In order to do so we first continue the functions α , β , and γ to complex values of the variables. We write $z_1 = x + iX$, $z_2 = y + iY$ and then introduce the variables $z = z_1 + iz_2$, $z^* = z_1 - iz_2$. For complex values of z_1 and z_2 the variables z and z^* are independent of each other while for real values, *i.e.* for $z_1 = x$, $z_2 = y$, the variable z^* is conjugate complex to z , *i.e.*, $z^* = \bar{z}$. Using the notations

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial z_1} - i \frac{\partial}{\partial z_2} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial z_1} + i \frac{\partial}{\partial z_2} \right),$$

(see *e.g.* BERGMAN [1]) we write (1.1) in the form

$$(1.2) \quad w_{zz^*} + a(z, z^*)w_z + b(z, z^*)w_{z^*} + c(z, z^*)w = 0, \quad w_{zz^*} \equiv \frac{\partial^2 w}{\partial z \partial z^*} \quad \text{etc.},$$

or

$$(1.3) \quad L(u) \equiv u_{zz^*} + B(z, z^*)u_{z^*} + C(z, z^*)u = 0,$$

where $w = u \exp \{ - \int_0^{z^*} a(z, \xi^*) d\xi^* \}$ and

$$B = b - \int_0^{z^*} a_z(z, \xi^*) d\xi^*, \quad C = c - a_z - ab.$$

Without loss of generality we consider (1.3) instead of (1.2). In order to obtain independent particular solutions of (1.3) which satisfy also an ordinary differential equation, we represent them in the form

$$(1.4) \quad u(z, z^*) = P(f) \equiv \int_{-1}^1 E(z, z^*, t) f\left\{ \frac{z}{2} (1 - t^2) \right\} \frac{dt}{\sqrt{1 - t^2}}.$$

$P(f)$ is called the *Bergman operator* (see BERGMAN [1, 2]). $E(z, z^*, t)$ is denoted as the *generating function* and $f(z)$ as the *associated function* of this operator. $f(z)$ is an analytic function of a complex variable, regular at the origin, and is independent of the coefficients of (1.3). Bergman operators transform analytic functions into (complex) particular solutions of the given partial differential equation (1.3).

We choose E in such a manner that the solutions of (1.3) generated by E satisfy also an ordinary differential equation. The "generating functions of exponential type",

$$(1.5) \quad E = e^{Q(z, z^*, t)}, \quad Q(z, z^*, t) = \sum_{\mu=0}^m q_{\mu}(z, z^*) t^{\mu},$$

possess this property as we shall see in the following. In a recent paper [3] it was proved that (1.5) generates solutions of a large and interesting class of partial differential equations (1.3). The following necessary and sufficient conditions for the coefficients B and C of (1.3) in order that the solutions (1.4) of (1.3) can be generated by a generating function (1.5) were obtained:

Theorem 1. (See [3]). (a) *If the coefficients B and C of the partial differential equation (1.3) can be represented in the form*

$$(1.6) \quad B = -\frac{\partial q_0}{\partial z} - \frac{q_2}{z},$$

$$(1.7) \quad C = -\frac{q_1}{2z} \frac{\partial q_1}{\partial z^*},$$

where

$$(1.8) \quad q_0 = q_0(z),$$

$$(1.9) \quad q_1(z, z^*) = \sum_{\nu=0}^{\sigma(m)} a_{\nu} z^{\nu+\frac{1}{2}}, \quad a_0 = a_0(z^*), \quad a_{\nu} = \text{const.} \quad (\nu = 1, 2, \dots, \sigma(m)),$$

$$(1.10) \quad q_2(z) = \sum_{\nu=1}^{\tau(m)} d_\nu z^\nu, \quad d_\nu = \text{const.} \quad (\nu = 1, 2, \dots, \tau(m)),$$

$$\sigma(m) = \left[\frac{m-1}{2} \right], \quad \tau(m) = \left[\frac{m}{2} \right],$$

then the solutions $u(z, z^*)$ of (1.3) can be generated by (1.5), and the other coefficients q_μ , $2 < \mu \leq m$ of $Q(z, z^*, t)$ are given by the following expressions:

$$(1.11) \quad q_{2\mu+1} = \frac{(-2)^\mu}{3 \cdot 5 \cdots (2\mu+1)} \sum_{\nu=\mu}^{\sigma(m)} \nu(\nu-1) \cdots (\nu-\mu+1) a_\nu z^{\nu+\frac{1}{2}},$$

$$\mu = 1, 2, \dots, \sigma(m),$$

$$(1.12) \quad q_{2\mu} = -\frac{(-2)^\mu}{2 \cdot 4 \cdots 2\mu} \sum_{\nu=\mu}^{\tau(m)} (\nu-1)(\nu-2) \cdots (\nu-\mu+1) d_\nu z^\nu,$$

$$\mu = 2, 3, \dots, \tau(m).$$

(b) The same holds if B can be represented in the form (1.6) and

$$(1.13) \quad C = -\frac{1}{2z} \frac{\partial q_2}{\partial z^*}$$

where q_2 is of the form (1.10). However, in this case $q_1 = 0$ while d_1 can be a function of z^* .

(c) Excluding the trivial cases where $C = 0$ and in addition (1.5) is independent of z^* , no coefficients B and C of (1.3) beyond those indicated in (a) and (b) are possible if the generating function E of the operator (1.4) is a function (1.5) of exponential type.

We shall consider the class of partial differential equations (1.3) characterized in Theorem 1 and obtain ordinary differential equations satisfied by the solutions of this class. Then we shall investigate some properties of the singularities of (1.3) and the singularities of the corresponding ordinary differential equations.

In order to obtain these solutions we use the associated functions

$$(1.14) \quad f(z) = z^n, \quad n = 0, 1, \dots$$

in (1.4). The whole procedure can be generalized and then be used also for other classes of partial differential equations (1.3); we refer to [3], where a theorem of BERGMAN [1] useful for this purpose has been indicated. Since we operate with two complex variables z and z^* , we have to consider the functions in a space of four real dimensions whose coordinates were denoted by x, X, y , and Y . In order to obtain the desired ordinary differential equation we consider our functions in two dimensional planes of this space. The form of the ordinary differential equation depends on the choice of these planes. For our purposes it is convenient to consider the planes

$$z_2 = z_2^{(0)} = \text{const.}$$

Points lying in these planes possess the complex coordinates

$$z_1 = x + iX, \quad z_2 = z_2^{(0)}.$$

It is advantageous that the functions considered in this plane become analytic functions of the complex variable z_1 .

We shall prove that every function $U(z_1, z_2) = u(z, z^*)$ represented by (1.4) with a generating function (1.5) of exponential type and with associated functions (1.14) satisfies a linear ordinary differential equation with respect to z_1 whose order is independent of n in (1.14) and whose coefficients depend in a simple manner on n . Theorem 2 asserts the existence of the differential equation and determines its order. Theorem 3 obtains the coefficients of this ordinary differential equation.

It is possible to consider also associated functions possessing poles and to show that the resulting solutions $u(z, z^*)$ satisfy similar ordinary differential equations, but we do not consider this question in the present paper. The special case $u_{zz^*} + \frac{1}{4}u = 0$ was considered in this connection by BERGMAN [4]. (It should be noted that BERGMAN in [1] investigated the functions $u(z, z^*)$ in the planes $z^* = \text{const}$ and showed that these functions satisfy there an ordinary differential equation different from ours.)

2. The derivation of the ordinary differential equation.

Theorem 2. *Let $u(z, z^*)$ be a solution of the partial differential equation (1.3) represented in the form (1.4), where the generating function E is of exponential type (1.5) and the associated function is of the form (1.14). Then $U(z_1, z_2) = u(z, z^*)$ satisfies a linear ordinary differential equation*

$$(2.1) \quad M(U) \equiv \tilde{M}(u) \equiv \sum_{\kappa=0}^k B_{\kappa}(z_1, z_2) U^{(\kappa)} = 0, \quad B_k = 1,$$

where

$$U(z_1, z_2) \equiv u(z, z^*), \quad B_{\kappa}(z_1, z_2) \equiv b_{\kappa}(z, z^*), \quad U^{(\kappa)} \equiv \frac{d^{\kappa} U}{dz_1^{\kappa}}, \quad z_2 = z_2^{(0)} = \text{const.}$$

The order k of equation (2.1) is independent of the exponent n in (1.14) and depends only on the degree m of Q in (1.5). It is always possible to determine an equation (2.1) whose order satisfies

$$k \leq m + 1.$$

Proof: We consider the integrand of (1.4). Due to the choice of the generating function E and of the associated function $f(z)$, it has the form

$$(2.2) \quad V(z_1, z_2) \equiv v(z, z^*) = \left(\frac{z}{2}\right)^n e^{Q(z, z^*, t)} (1 - t^2)^{n-\frac{1}{2}}.$$

For our purpose it is sufficient to prove that the function $V(z_1, z_2)$ satisfies an ordinary differential equation of the form

$$(2.3) \quad M(V) \equiv \tilde{M}(v) = \frac{d}{dt} \{ (1 - t^2)^\lambda F(z, z^*, t) \}, \quad \lambda > 0,$$

where $F(z, z^*, t)$ is a regular function of t for $|t| \leq 1$. If we integrate the equation (2.3) with respect to t from -1 to 1 , then, since the right hand side of (2.3) has the factor $(1 - t^2)^\lambda$, we obtain the ordinary differential equation (2.1). We determine the differential equation (2.3) as follows. We choose the right hand side of (2.3) in the form

$$(2.4) \quad R(t) = \frac{d}{dt} \{ (1 - t^2) P(z, z^*, t) V \},$$

where

$$(2.5) \quad P(t) = \sum_{\lambda=0}^l p_\lambda(z, z^*) t^\lambda$$

denotes a polynomial in t of degree l . We have to determine the number l and the coefficients $p_\lambda(z, z^*)$ in such a manner that (2.3) is satisfied. Let us first determine the order k of (2.3) and the degree l of (2.5). The derivative of the function $V(z_1, z_2)$ with respect to t can be expressed as a product of V and a polynomial in t , namely

$$V_t = \left\{ Q_t - \frac{(2n-1)t}{1-t^2} \right\} V.$$

From (2.4) we obtain therefore the following expression:

$$(2.6) \quad R(t) = \{ (1 - t^2) (P_t + P Q_t) - (2n+1) P t \} V.$$

The derivatives of V with respect to z_1 on the left hand side of (2.3) can also be expressed by products of V and certain polynomials in t . We find

$$(2.7) \quad V' \equiv \frac{dV}{dz_1} = v_z \frac{\partial z}{\partial z_1} + v_{z^*} \frac{\partial z^*}{\partial z_1} = v_z + v_{z^*} = \left(\frac{n}{z} + Q_z + Q_{z^*} \right),$$

$$v = S V, \quad S \equiv \frac{n}{z} + \tilde{Q}',$$

where the primes always denote derivatives with respect to z_1 . Here

$$(2.8) \quad \tilde{Q}'(z_1, z_2, t) = \sum_{\mu=0}^m \tilde{q}'_\mu t^\mu, \quad \tilde{Q}(z_1, z_2, t) \equiv Q(z, z^*, t),$$

$$\tilde{q}_\mu(z_1, z_2) \equiv q_\mu(z, z^*),$$

is a polynomial in t of degree m . Differentiating (2.7); we find for the higher derivatives of V the expressions

$$(2.9) \quad V'' = (S' + S^2)V, \quad V''' = (S'' + 3SS' + S^3)V$$

etc. We see that the degree of the polynomial in t by which V is multiplied becomes larger by m at each differentiation; indeed if we set

$$V^{(\kappa-1)} = TV,$$

then we obtain by differentiation with respect to z_1 the expression

$$V^{(\kappa)} = (T' + TS)V,$$

and the polynomial in parentheses is of degree κm in t . If therefore (2.3) is of the order k , then the highest power of t occurring on the left hand side of (2.3) possesses the exponent km . The degree l of (2.5) must be chosen in such a manner that the highest power of t occurring on the right hand side of (2.3) has also the exponent km . If we substitute the expressions (2.6), (2.7), (2.9), *etc.* into the equation (2.3) then we obtain on each side a common factor V which may be cancelled. The remaining expressions are polynomials in t with the coefficients $b_\kappa(z, z^*)$, $\kappa = 0, 1, \dots, k-1$ and $p_\lambda(z, z^*)$, $\lambda = 0, 1, \dots, l$. The highest power of t on the right hand side is contained in the term $-t^2PQ_t$ of (2.6); the corresponding exponent is $l + m + 1$. If we choose

$$(2.10) \quad l = (k-1)m - 1$$

then we obtain t^{km} as the highest power also on the right hand side of (2.3). In order that (2.3) is satisfied the coefficient of each power of t must be the same on each side of the equation. In this manner we obtain a system of $km + 1$ linear equations from which we can determine the unknown quantities $b_0, b_1, \dots, b_{k-1}$ and also the unknown coefficients p_λ of the polynomial (2.5) provided that the determinant of the system of equations is not identically zero; this case will be considered below. Since the determinant is a function of two complex variables, it vanishes for certain values of z, z^* , but it is sufficient that it does not vanish identically: indeed, in a neighborhood of a point lying on a zero surface of the determinant we can always find points for which the determinant does not vanish. The number of the above mentioned unknowns must be as large as the number of equations; we obtain therefore $k = m + 1$. Hence we can always construct a differential equation (2.1) for the functions $U(z_1, z_2) \equiv u(z, z^*)$ whose order is not larger than $m + 1$. If, however, the determinant vanishes identically, then the order of the ordinary differential equation (2.1) becomes smaller than $m + 1$, but the procedure is in principle the same, provided that the rank of the determinant is different from zero; this holds always as can be seen from (4.4), Section 4. In particular the excep-

tional case of vanishing determinant can occur if the function $Q(z, z^*, t)$ in (1.5) is a polynomial in which some powers of t are missing. Since the degree l of the polynomial (2.5) must be non-negative, we have always $k \geq 2$. It should be noted that for $n = 0$ the function V has singularities for $t = \pm 1$, but the integral over t from -1 to 1 exists also in this case.

3. The determination of the coefficients of the ordinary differential equation (2.1). By Theorem 2 we proved the existence of the ordinary differential equation (2.1) and determined its order. We proceed to the determination of the coefficients $b_\kappa(z, z^*)$ of this equation.

Theorem 3. *If the solutions $u(z, z^*)$ of the partial differential equation (1.3) are generated in the form (1.4) by a generating function (1.5) of exponential type and associated functions $f(z) = z^n$, $n = 0, 1, \dots$, then the coefficients $b_\kappa(z, z^*)$, $\kappa = 0, 1, \dots, k-1$, of the ordinary differential equation $\tilde{M}(u) = 0$ in (2.1) and the coefficients $p_\lambda(z, z^*)$, $\lambda = 0, 1, \dots, l$, of the polynomial (2.5) are solutions of the following system of linear equations:*

$$(3.1) \quad \begin{aligned} c_{k,\kappa} + \delta_{0\kappa} b_0 + \sum_{\sigma=1}^{k-1} c_{\sigma,\kappa} b_\sigma + (2n + \kappa) p_{\kappa-1} - (\kappa + 1) p_{\kappa+1} \\ + \sum_{\lambda=0}^{\kappa} \{(\kappa - \lambda - 1) q_{\kappa-\lambda-1} - (\kappa - \lambda + 1) q_{\kappa-\lambda+1}\} p_\lambda = 0, \end{aligned}$$

provided that the determinant of this system does not vanish identically. Here the quantities are given recursively as follows:

$$(3.2) \quad c_{1,\kappa} = \tilde{q}'_\kappa + \delta_{0\kappa} \frac{n}{z}, \quad \tilde{q}_\kappa(z_1, z_2) \equiv q_\kappa(z, z^*), \quad \kappa = 0, 1, \dots, m,$$

$$(3.3) \quad \begin{aligned} c_{\sigma,\kappa} = c'_{\sigma-1,\kappa} + \sum_{\lambda=0}^{\kappa} c_{\sigma-1,\kappa-\lambda} c_{1,\lambda}, \quad \kappa = 0, 1, \dots, \sigma m, \\ \sigma = 2, 3, \dots, k, \end{aligned}$$

where the prime denotes the derivative with respect to z_1 , and

$$\delta_{0\kappa} = \begin{cases} 0 & (\kappa \neq 0) \\ 1 & (\kappa = 0) \end{cases}$$

is the Kronecker symbol. Remark: For the case of vanishing determinant compare the end of Section 2.

Proof: As it was pointed out in the proof of Theorem 2, each derivative of V can be expressed as a product of V and a polynomial in t . Thus we write

$$(3.4) \quad V^{(\sigma)} = \left\{ \sum_{\kappa=0}^{\sigma m} c_{\sigma,\kappa} t^\kappa \right\} V.$$

The degree σm of the polynomial in this expression was determined in the proof of Theorem 2. For $\sigma = 1$ we obtain in consequence of (2.7) from (3.4) the formula (3.2). We now proceed recursively. If we replace σ by $\sigma - 1$ in (3.4) we have

$$V^{(\sigma-1)} = \left\{ \sum_{\kappa=0}^{(\sigma-1)m} c_{\sigma-1,\kappa} t^\kappa \right\} V.$$

Differentiating this expression with respect to z_1 and using (3.2) we find

$$(3.5) \quad V^{(\sigma)} = \left\{ \sum_{\kappa=0}^{(\sigma-1)m} c'_{\sigma-1,\kappa} t^\kappa + \left(\sum_{\kappa=0}^{(\sigma-1)m} c_{\sigma-1,\kappa} t^\kappa \right) \left(\sum_{\mu=0}^m c_{1,\mu} t^\mu \right) \right\} V.$$

Comparing (3.5) with (3.4) we obtain the recurrence formula (3.3). In consequence of (1.5) and (2.5) the expression (2.6) can be written in the form

$$(3.6) \quad R(z, z^*, t) = \left\{ (1 - t^2) \left[\sum_{\lambda=0}^l \lambda p_\lambda t^{\lambda-1} + \left(\sum_{\lambda=0}^l p_\lambda t^\lambda \right) \left(\sum_{\mu=0}^m \mu q_\mu t^{\mu-1} \right) \right] \right. \\ \left. - (2n + 1) \sum_{\lambda=0}^l p_\lambda t^{\lambda+1} \right\} V.$$

In order that the differential equation (2.3) be satisfied the coefficients of each power of t must vanish. If we substitute the expressions (3.5) and (3.6) into (2.3) (see also (2.1)) we obtain the system (3.1). If the determinant of this system does not vanish identically the coefficients $b_\kappa(z, z^*)$, $\kappa = 0, 1, \dots, k - 1$, of (2.3) and the coefficients $p_\lambda(z, z^*)$, $\lambda = 0, 1, \dots, l$, of the polynomial (2.5) can be determined from (3.1). The order k of the differential equation (2.1) equals $m + 1$, see Theorem 2. (Concerning the case of identically vanishing determinant see Section 2.)

Theorem 3 corresponds to a similar theorem by NIELSEN [5], who considered, however, ordinary differential equations with respect to z .

4. The determinant of the system (3.1). We shall now derive an explicit expression for the determinant of (3.1), which we denote by K . It has in principle a relatively simple structure. We write the unknowns in the following order:

$$b_0, b_1, \dots, b_{k-1}, p_0, p_1, \dots, p_l.$$

By Theorem 3 we have

$$k = m + 1 \quad \text{and} \quad l = m^2 - 1.$$

Hence K is of order $m^2 + m + 1$. The number of the unknowns $b_\sigma(z, z^*)$ and $p_\lambda(z, z^*)$ equals the number of equations in (3.1). As a consequence of (3.1) and (3.3), in the κ^{th} row, if $\kappa \leq \sigma m$, the quantities $b_\sigma(z, z^*)$ $\sigma = 0, 1, \dots, k - 1$, have the coefficients $c_{\sigma,\kappa}$ and the coefficients 0 if $\kappa > \sigma m$. The quantities $p_\lambda(z, z^*)$ have different coefficients in the κ^{th} row, depending on the difference between κ and λ . From (3.1) we obtain for these coefficients the following values

	Coefficient of p in the κ^{th} row	λ
	0	$\geq \kappa + 2$
(4.1)	$-\lambda$	$\kappa + 1$
	$-q_1$	κ
	$2n + \lambda - 2q_2$	$\kappa - 1$
	$(\nu - 1)q_{\nu-1} - (\nu + 1)q_{\nu+1}$	$\kappa - \nu, \quad m + 1 \geq \nu \geq 2$
	0	$< \kappa - m - 1$

where $q_\mu = 0$ for $\mu > m$. The matrix (K) corresponding to the determinant K therefore has the form

$$(4.2) \quad (K) = ((C)(W)),$$

where

$$(4.3) \quad (C) = \begin{bmatrix} (C_0) \\ (C_1) \\ \vdots \\ (C_m) \end{bmatrix}$$

and the matrix (W) has m^2 columns and $m(m + 1) + 1$ rows and is of the form

$$(4.4) \quad (W) = \begin{bmatrix} -q_1 & -1 & & & & & & \\ & \varphi_0 & -q_1 & -2 & & & & \\ & \eta_1 & \varphi_1 & -q_1 & \cdot & \cdot & \cdot & \\ & \eta_2 & \eta_1 & \varphi_2 & \cdot & \cdot & \cdot & -m^2 + 1 \\ & \vdots & \eta_2 & \eta_1 & \cdot & \cdot & \cdot & -q_1 \\ & \vdots & \vdots & \eta_2 & \cdot & \cdot & \cdot & \varphi_{m^2-1} \\ \eta_m & \vdots & \vdots & \cdot & \cdot & \cdot & \cdot & \eta_1 \\ & \eta_m & \vdots & \cdot & \cdot & \cdot & \cdot & \eta_2 \\ & & \eta_m & \cdot & \cdot & \cdot & \cdot & \vdots \\ & & & \cdot & \cdot & \cdot & \cdot & \vdots \\ & & & & \cdot & \cdot & \cdot & \eta_m \end{bmatrix}.$$

In (4.4)

$$\varphi_\nu = 2n + \nu + 2q_2 + 1, \quad \nu = 0, 1, \dots, m^2 - 1$$

$$\eta_\alpha = \alpha q_\alpha - (\alpha + 2)q_{\alpha+2}, \quad \alpha = 1, 2, \dots, m; \quad q_\mu = 0 \quad \text{if } \mu > m,$$

and all elements which are not indicated explicitly or by dots are equal zero.
In (4.3)

$$(C_0) = (1 \ c_{1,0} \ c_{2,0} \ \dots \ c_{m,0})$$

$$(C_\sigma) = ((0)_{[\sigma, m]} \ (D_\sigma)), \quad \sigma = 1, 2, \dots, m$$

where $(0)_{[\sigma, m]}$ denotes a zero matrix of σ columns and m rows, and

$$(D_\sigma) = \begin{bmatrix} c_{\sigma, \lambda} & c_{\sigma+1, \lambda} & \dots & c_{m, \lambda} \\ c_{\sigma, \lambda+1} & c_{\sigma+1, \lambda+1} & \dots & c_{m, \lambda+1} \\ \vdots & \vdots & & \vdots \\ c_{\sigma, \sigma m} & c_{\sigma+1, \sigma m} & \dots & c_{m, \sigma m} \end{bmatrix},$$

$$\lambda \equiv (\sigma - 1)m + 1.$$

The fact that (W) has the form (4.4) is an immediate consequence of (4.1).

To obtain the unknown quantities $b_\kappa(z, z^*)$, $\kappa = 0, 1, \dots, k - 1$, we have to replace the $(\kappa + 1)^{\text{th}}$ column in the determinant K by the column

$$(4.5) \quad \begin{array}{c} -c_{m+1,0} \\ -c_{m+1,1} \\ \vdots \\ -c_{m+1, m(m+1)} \end{array}$$

and then to divide the resulting determinant by the determinant K . (4.5) follows from (4.1).

5. Examples. It should be noted that as a consequence of the recurrence relation (3.20) of [3] we obtain for the quantities η_α in (4.4) the simpler expressions

$$(5.1) \quad \eta_\alpha = q_{\alpha, z}$$

For example if $m = 1$ then (4.3) and (4.4) have the form

$$(C) = \begin{bmatrix} 1 & c_{1,0} \\ 0 & c_{1,1} \\ 0 & 0 \end{bmatrix}, \quad (W) = \begin{bmatrix} -q_1 \\ 2n + 1 \\ q_1 \end{bmatrix}.$$

For $m = 2$ we obtain

$$(C) = \begin{bmatrix} 1 & c_{1,0} & c_{2,0} \\ 0 & c_{1,1} & c_{2,1} \\ 0 & c_{1,2} & c_{2,2} \\ 0 & 0 & c_{2,3} \\ 0 & 0 & c_{2,4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(W) = \begin{bmatrix} -q_1 & 1 & 0 & 0 \\ 2n - 2q_2 & -q_1 & -2 & 0 \\ q_{1,z} & 2n - 2q_2 + 1 & -q_1 & -3 \\ q_{2,z} & q_{1,z} & 2n - 2q_2 + 2 & -q_1 \\ 0 & q_{2,z} & q_{1,z} & 2n - 2q_2 + 3 \\ 0 & 0 & q_{2,z} & q_{1,z} \\ 0 & 0 & 0 & q_{2,z} \end{bmatrix}.$$

The very simple example

$$u_{zz^*} + \frac{1}{4}u = 0$$

see [3, Section 3.6], corresponds to the case $m = 1$ if we use the simplest generating function which exists in this case:

$$E = e^{i\sqrt{zz^*}t}.$$

As a consequence of Theorem 2 we obtain a linear ordinary differential equation (2.1) of second order. From Theorem 3 follows that the coefficients of this equation are

$$(5.2) \quad \begin{aligned} b_0(z, z^*) &\equiv \tilde{b}_0(z_1, z_2) = -\frac{1}{z_1^2 + z_2^2} \left(n^2 + in \frac{z_2}{z_1} + z_1^2 \right), \\ b_1(z, z^*) &\equiv \tilde{b}_1(z_1, z_2) = \frac{1}{z_1^2 + z_2^2} \left(z_1 - \frac{z_2^2}{z_1} + 2in z_2 \right). \end{aligned}$$

For $n = 0$ this ordinary differential equation is equivalent to the equation (13*) of [4, p. 396], derived by BERGMAN. The ordinary differential equation (2.1) with coefficients (5.2) is of Fuchs' type in the whole z, z^* -space and possesses regular solutions which are essentially Bessel functions and are solutions of the partial differential equation (1.3), see [3, Section 3.7].

6. Singular coefficients of the partial differential equation (1.3), preliminaries.

Combining the results of the previous sections, we find relations between the coefficients B and C of the given partial differential equation (1.3) and the coefficients $b_0, b_1, \dots, b_{k-1}$ of the corresponding ordinary differential equation (2.1). In this way we are in a position to use theorems about the behavior of the solutions of ordinary differential equations with analytic coefficients in order to obtain theorems about solutions of the partial differential equation (1.3). We shall not discuss this procedure in detail in the present paper since the results follow immediately from these theorems. However, we shall investigate how the singularities in the coefficients of (1.3) are connected with those of the coefficients of (2.1). For simplicity we restrict ourselves again to the reduced form (1.3) of the partial differential equation. Using the relations between the coefficients of (1.2) and those of (1.3) the results obtained in the following can be formulated as results about (1.2). The following investigations are based on the Theorems 1–3. We proceed stepwise starting from the results of Theorem 1.

7. Correspondence between the singularities of the coefficients of the partial differential equation (1.3) and those of the generating function (1.5) of exponential type. If a given partial differential equation (1.3) admits generating functions (1.5) of exponential type then from (1.6), (1.7), and (1.13), respectively, we obtain the corresponding coefficients q_0, q_1 , and q_2 of the polynomial $Q(z, z^*, t)$ of (1.5). After a value of m is chosen the coefficients $q_3, q_4, \dots, q_m$ of (1.5) can be determined from (1.11) and (1.12). The choice of m depends on the form of the coefficients B and C of (1.3). There exists in general for a given partial differential equation (1.3) a number of generating functions of exponential type which can be used. This holds also for the simplest cases of (1.3), see *e.g.* [3, Section 3.6]. In most instances it is also clear which of the two cases (a) and (b) of Theorem 1 must be used for the study of solutions of the partial differential equation (1.3) under consideration. Sometimes however it is possible to use both of those cases. In the following we shall characterize the difference between the cases (a) and (b) of Theorem 1 somewhat more in detail.

8. Singularities in the case (a) of Theorem 1. From (1.10) follows that in the case (a) of Theorem 1 the function q_2/z must be a regular function of z alone, namely a polynomial of the degree $\tau(m) - 1$. In this case in consequence of (1.6), (1.8), and (1.10) the coefficient B of (1.3) is a function of z alone. Singularities of B correspond to those of q_0 . If therefore B is regular in a certain domain, q_0 must be regular in this domain. If B has a pole of first order at a point, say $z = a$, then q_0 must have a logarithmic singularity at a . Poles of B of order $g + 1$ correspond to poles of order g of q_0 . The coefficient C of the partial differential equation (1.3) may be a function of z and z^* of the form indicated by (1.7) and (1.9). Inserting (1.9) into (1.7), we obtain

$$(8.1) \quad C = -\frac{a_{0,z^*}}{2} \sum_{\nu=0}^{\sigma(m)} a_\nu z^\nu.$$

As indicated in (1.9) the coefficients $a_1, a_2, \dots, a_{\sigma(m)}$ of q_1 are constants. C may therefore have singularities considered as a function of z^* which then are singularities of the coefficient $a_0(z^*)$ of q_1 . If $a_0(z^*)$ and therefore also q_1 , considered as a function of z^* for any finite value of z , has a pole of the order r at a point $z^* = a^*$ then C has at the same point a pole of the order $2r + 1$, $r > 0$. In order that C has a pole of first order at a point $z^* = a^*$, by (8.1) a_0 must be of the form

$$(8.2) \quad a_0(z^*) = d\sqrt{\log(z^* - a^*)},$$

where d is a constant.

9. Singularities in the case (b) of Theorem 1. In this case $q_1 = 0$ and consequently from (1.11) $q_3 = q_5 = \dots = 0$. Therefore this case is more special than the case (a). On the other hand we have the advantage that B can depend on z and z^* because in this case q_2 can also depend on z^* . By comparing (1.6) and (1.13) we see that the singularities of B and C are not independent of each other. A pole of q_2 , considered as a function of z^* , of the order s at a point, say $z^* = b^*$, for all values of z corresponds to a pole of B of the same order and to a pole of C of the order $s + 1$ at the point $z^* = b^*$ where B and C are also considered as functions of z^* . In order that C have a pole of first order at $z^* = b^*$, the function q_2 and consequently also B must have a logarithmic singularity at the same point. The coefficient B of (1.3), considered as a function of z , may have additional singularities which then correspond to those of q_0 and are independent of the value of z .

The remarks in Sections 7–9 give us full information concerning the correspondence between singularities of the coefficients B and C of the given partial differential equation (1.3) and those of the coefficients q_0, q_1 , and q_2 of the polynomial $Q(z, z^*, t)$ in (1.5). The other coefficients of $Q(z, z^*, t)$ are always regular functions.

10. Singularities of the ordinary differential equation (2.1) in the case of a generating function (1.5) with regular coefficients q_μ . The next task is to obtain relations between the singularities of the functions q_0, q_1 , and q_2 of (1.5) and those of the coefficients $b_\kappa, \kappa = 0, 1, \dots, k - 1$, of the ordinary differential equation (2.1). In order to do so we use the expressions for the b_κ obtained in Theorem 3. The determinants (cf. also (4.2)–(4.5)) that occur in these expressions can be reduced to diagonal form. We begin with the case when all coefficients q_μ of $Q(z, z^*, t)$ are regular functions of z and z^* ; in Section 11 we shall consider the case when the q_μ have singularities.

Lemma 4. *If the coefficients $q_\mu(z, z^*)$ of the generating function (1.5) of exponential type are regular in a neighborhood N of the origin then*

- (α) *the coefficients B and C of the partial differential equation (1.3) are regular in N ,*
- (β) *the coefficients b_κ of the ordinary differential equation (2.1) have poles at the point $z = 0$ such that this equation is of Fuchs' type at $z = 0$, and*

(γ) the particular solutions (1.4) of (1.3) with a generating function (1.5) and associated functions (1.14) are single valued functions in N .

Proof: The statement (α) is an immediate consequence of Theorem 1 and was considered previously. The statement (β) is a consequence of the fact that according to (3.2):

$$(5.3) \quad c_{1,0} = \tilde{q}'_0 + \frac{n}{z}, \quad \tilde{q}_0(z_1, z_2) \equiv q_0(z, z^*)$$

has a pole of the first order at the point $z = 0$ if $n > 0$. Therefore the other quantities $c_{\sigma,\kappa}$ also have poles at this point if $n > 0$. Using Theorem 3, we find that b_κ , which depends of course on m , has a pole of the order

$$h \equiv m + 1 - \kappa.$$

Consequently the differential equation (2.1) is of Fuchs' type at the point $z = 0$. The statement (γ) can be proved using Theorem 1. The generating function (1.5) can be written in the form

$$(5.4) \quad E = \exp N(z, z^*, t) \exp S(z, z^*, t),$$

where in consequence of (1.10) and (1.12)

$$N(z, z^*, t) = \sum_{\mu=0}^{\lfloor m/2 \rfloor} q_{2\mu}(z, z^*) t^{2\mu}$$

is an even function of t with regular coefficients and

$$S(z, z^*, t) = \sqrt{z} \sum_{\nu=0}^{\lfloor m/2 \rfloor} S_{2\nu+1} t^{2\nu+1},$$

where according to (1.9) and (1.11) the quantities

$$S_{2\nu+1} = z^{-\frac{1}{2}} q_{2\nu+1}$$

are polynomials in z . In the series

$$\exp S(z, z^*, t) = \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!} \left\{ \sqrt{z} \sum_{\nu=0}^{\lfloor m/2 \rfloor} S_{2\nu+1} t^{2\nu+1} \right\}^{\lambda}$$

the square root of z occurs only in the terms with odd powers of t . These terms are multiplied by the even function $(1 - t^2)^{n-1/2}$ and by the terms of the series development of the function $\exp N(z, z^*, t)$, which contains even powers of t only. The products which we obtain in this manner are therefore odd functions in t . Integrating these products with respect to t from -1 to 1 we obtain integrals which have the values zero. This means that the resulting solution has no branch points.

11. Correspondence between the singularities of the coefficients of the ordinary differential equation (2.1) and those of the coefficients q_μ of the generating function (1.5). We consider now the case where the coefficients q_0 , q_1 , and q_2 of the generating function (1.5) have singularities. From Theorem 3 we obtain

Lemma 5. *The following relations exist between the singularities of the coefficients q_0 , q_1 , and q_2 of the generating function (1.5) of exponential type and the singularities of the coefficients b_κ (which depends of course on m) of the ordinary differential equation (2.1):*

I. *If q_0 (which is a function of z alone) has a pole of the order g at a point, say $z = a$, then b_κ , considered as a function of z , has a pole of the order*

$$(11.1) \quad h(g+1), \quad h \equiv m+1-\kappa, \quad g > 0,$$

at the same point and for all values of z^ . If q_0 has a logarithmic singularity at a point $z = a$, then b_κ has a pole of the order h , and the ordinary differential equation (2.1) is therefore of Fuchs' type at this point.*

II. *If the coefficient q_1 , considered as a function of z^* , has a pole of the order r at a point, say $z^* = b^*$, then b_κ , considered as a function of z^* , has at the same point a pole of the order*

$$(11.2) \quad h(r+1) + (\delta_{0\kappa} + \delta_{2\kappa})r, \quad r > 0, \quad m > 1,$$

and if $m = 1$ then b_0 has a pole of the order $2r+2$ and b_1 has a pole of the first order at $z^ = b^*$; this holds for all values of z . In (11.2), $\delta_{0\kappa}$ and $\delta_{2\kappa}$ are Kronecker symbols.*

III. *If $q_1 = 0$ then q_2 , considered as a function of z^* , can have singularities. If q_2 has a pole of the order s at a point, say $z^* = b^*$, then b_κ , considered as a function of z^* , for sufficiently large values of m has a pole of the order*

$$(11.3) \quad (h+3-\delta_{0\kappa})s+h$$

at the same point.

For example, in case III, (11.3) holds for $0 \leq \kappa \leq 3$ if $m \geq 4$, for $0 \leq \kappa \leq 4$ if $m \geq 6$, for $0 \leq \kappa \leq 5$ if $m \geq 9$, etc. The fact that we obtain exceptional cases in III if the difference between m and κ is small can be easily understood because the order of the poles of the quantities $c_{\kappa,\kappa}$, defined by (3.2) and (3.3), depends on the order of the poles of q_2 in a non-monotonic and somewhat complicated manner.

From Lemmas 4 and 5 we see that in certain cases the ordinary differential equation (2.1) is of Fuchs' type.

12. Correspondence between the singularities of the coefficients of the partial differential equation (1.3) and those of the coefficients of the ordinary differential equation (2.1). Combining Lemma 5 with the results discussed in Sections 7–9, we can omit all statements involving the theory of operators, which has served as a tool for our work. In this way we obtain the following

Theorem 6. *If the coefficients B and C of the partial differential equation (1.3) are of the form (1.6), (1.7), and (1.13), respectively, and if B and C have singularities, then the coefficients b_κ of the ordinary differential equation (2.1) satisfied by the*

particular solutions (1.4) with the generating function (1.5) and associated functions (1.14) also have singularities. Between these two sets of singularities the following relations hold:

(1) If B , considered as a function of z for any finite value of z^* , has a pole of the order p at a point, say $z = a$, then b_κ , considered as a function of z , has at the same point and for all values of z^* a pole of the order

$$(12.1) \quad ph, \quad h \equiv m + 1 - \kappa,$$

where m is defined in (1.5). In particular, if B has a pole of the first order at $z = a$, then the differential equation (2.1) is of Fuchs' type at this point. The regular solution which (2.1) possesses in this case is a particular solution of the partial differential equation (1.3).

(2) If C , considered as a function of z^* for any finite value of z , has a pole of the order $j > 1$ at a point $z^* = b^*$, then b_κ , considered as a function of z^* , has a pole of the order

$$(12.2) \quad \frac{h}{2}(j+1) + \frac{1}{2}(\delta_{0\kappa} + \delta_{2\kappa})(j-1)$$

at the same point. $\delta_{0\kappa}$ and $\delta_{2\kappa}$ are Kronecker symbols.

(According to Theorem 1, case (a), the singularities indicated in (1) and (2) of the present theorem are independent of each other).

(3) If $q_1 = 0$ and B , considered as a function of z^* for any finite value of z , has a pole of the order $v > 1$ at a point $z^* = c^*$, then b_κ , considered as a function of z^* , has a pole of the order

$$(12.3) \quad (n + 3 - \delta_{0\kappa})v + h$$

at the same point. (C , considered as a function of z^* , has a pole of the order $v + 1$ at the point $z^* = c^*$, see Theorem 1, case (b)).

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