On Some Transformation Semigroups

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Introduction

Certain considerations in the theory of partial differential equations suggested to the author the study of some semigroups of projective transformations or transformations of the Moebius group. The results obtained form the contents of this paper.

A transformation τ of the projective group in n dimensions will be called a projective translation if it is similar, within the projective group, to an ordinary translation, *i.e.*, if there exists a projective transformation π such that $\tau^* = \pi^{-1}\tau\pi$ represents an ordinary translation. It can be described also in the following way: There exists a unique point p and a unique hyperplane l through p such that all points on l and all hyperplanes through p remain fixed under τ . (In case of an ordinary translation l coincides with the infinite hyperplane and p with a point on it.) We shall speak of a (p, l) -translation. The (p, l) -translations with given p and l form a one-parameter group.

If D is an open convex domain of the n -dimensional projective space* we may associate with it a semigroup of projective transformations in the following way: Let p be a boundary point of D and l a hyperplane of support of D through p . It is easily seen that a (p, l) -translation transforms D either into a subdomain of D or into a domain containing D . Transformations of the first kind will be called *projective translations of D* . Now we form all composites of projective translations of D and their limits. *They form a semigroup $\mathfrak{P}_D$* .

Similar concepts can be introduced in the Moebius geometry. A transformation τ of the Moebius group in n dimensions will be called a *Moebius translation* if it is similar, within the Moebius group, to an ordinary translation, *i.e.*, if there exists a Moebius transformation μ such that $\mu^* = \mu^{-1}\tau\mu$ is an ordinary

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* Such a domain is convex if through any boundary point passes a hyperplane of support, *i.e.*, a hyperplane that does not penetrate into the interior of D .

translation. It can be described also in the following way: There exists a unique line element e such that all circles containing e remain fixed under τ . We shall call it an e -translation. The e -translations with a given e form a one-parameter group.

Let D be an open domain of the n -dimensional Moebius space bounded by a Jordan hypersurface of class C_1 . Assume further that each circle containing an arbitrary line element e orthogonal to the boundary surface intersects D in one connected segment. It is easily seen that an e -translation transforms D either into a subdomain of D or into a domain containing D . A transformation of the first kind will be called a *Moebius translation* of D . Forming all composites of Moebius translations of D and their limits, we obtain a *semigroup* $\mathfrak{M}_D$ of *Moebius transformations transforming D into itself*.

For illustration of these concepts let us consider in this introduction only the one-dimensional case, where the projective and Moebius group coincide. Since $\mathfrak{P}_D$ is covariantly connected with D under the projective group there are only two essentially different possibilities for D : either D is the projective line from which one point is deleted or D is a segment of the projective line with two boundary points. After introduction of a suitable coordinate system we may identify D in the first case with the x -axis with the point ∞ as the only boundary point and in the second case with the half line $x > 0$ with 0 and ∞ as the boundary points. In the first case all ordinary translations $x' = x + a$ ($-\infty < a < \infty$) form the group $\mathfrak{P}_D$. In the second case there are two types of translations corresponding to the two boundary points 0 and ∞ :

$$\begin{aligned} x' &= x + a & (a \geq 0) \\ x' &= \frac{x}{1 + bx} & (b \geq 0). \end{aligned}$$

The process of composition of transformations of these two types corresponds evidently to the process of formation of a *continued fraction with positive elements*. One sees, therefore, that the study of the semigroups $\mathfrak{P}_D$ and $\mathfrak{M}_D$ may be considered as a generalization of the theory of continued fractions to higher dimensions.

In the first section of this paper distortion theorems for the semigroup $\mathfrak{P}_D$ for arbitrary domains D and $\mathfrak{M}_D$ for spherical domains are derived. In the second section finer distortion theorems for $\mathfrak{M}_D$ are obtained, but only for circles in the plane. Section 3 contains some group-theoretical considerations giving a group-theoretical characterization of some of the semigroups $\mathfrak{P}_D$ and $\mathfrak{M}_D$.

1. Distortion theorems for the semigroups $\mathfrak{P}_D$ and $\mathfrak{M}_D$.

A semigroup $\mathfrak{P}_D$ is always transitive, *i.e.*, any point p of D can be transformed into any other point p' of D by a suitable transformation of $\mathfrak{P}_D$. Indeed, if the line segment of D bounded by p and p' is continued beyond p' the boundary of D will be hit the first time at a point p'' . Let l'' be a hyperplane of support of

D at p'' . Then evidently one of the (p'', l'') -translations of D will transform p into p' . Consider now *all* the transformations of $\mathfrak{P}_D$ mapping p into p' . We are interested in what general assertions can be made about the *linear approximations* of these mappings at the point p . In particular we wish to obtain a bound for the magnification factor of a volume element at p , *i.e.*, after introduction of a coordinate system, a bound for the Jacobian of the mapping.

In order to describe the derivation of the bound in a simple geometric way we first introduce a volume concept in the n -dimensional projective space *relative to a hyperplane* l . The subgroup $\mathfrak{p}_l$ of projective transformations which keep l fixed is similar, within the projective group, to the affine group in n -dimensions. The transformations of the latter change the ordinary volume only by a constant positive factor. We can therefore introduce a definition of volume associated with l such that the transformations of $\mathfrak{p}_l$ change the volume only modulo a positive factor: In order to represent the new volume in a chosen (non-homogeneous) projective coordinate system we introduce the equation of l

$$L \equiv L_0 \equiv a_{00} + \sum_{\sigma=1}^n a_{0\sigma} x_{\sigma} = 0.$$

Any transformation

$$x'_i = \frac{L_i}{L_0} \quad (i = 1, 2, \dots, n)$$

with

$$L_i = a_{i0} + \sum_{\sigma=1}^n a_{i\sigma} x_{\sigma}$$

and a non-zero determinant $a = |a_{\rho\sigma}|$, $\rho, \sigma = 0, \dots, n$ will transform $\mathfrak{p}_l$ into the affine group since l is transformed into the infinite hyperplane. The Jacobian of the mapping is

$$(1) \quad J \begin{pmatrix} x' \\ x \end{pmatrix} = \frac{a}{L^{n+1}},$$

and this shows that a volume element $d\omega_l$ with respect to l is given by*

$$(2) \quad d\omega_l = \frac{d\omega}{|L|^{n+1}},$$

$d\omega$ being the ordinary volume element $dx_1 \cdot dx_2 \cdot \dots \cdot dx_n$.

We now prove

Lemma 1. *Let l be a hyperplane not penetrating into the interior of the domain D . Then the volume of any part of D with respect to l cannot be increased by any transformation of $\mathfrak{P}_D$.*

* The arbitrary multiplicative factor, which may be regarded as corresponding to a change of the unit of volume, can be included in the expression for L .

Proof. It is sufficient to prove the lemma for a translation of D . We may choose the coordinate system so that the translation is an ordinary one in the direction of the positive x_1 -axis. If l contains the direction of the x_1 -axis, L in (2) does not contain the variable x_1 and the volume does not change at all. Otherwise we may assume that the equation of l is $x_1 = 0$ and x_1 positive in D . The volume element is then $d\omega/x_1^{n+1}$, and it is clear that it is decreased by the translation since x_1 increases and the ordinary volume element $d\omega$ remains unchanged.

Now let σ be an arbitrary transformation of $\mathfrak{B}_D$ and $\sigma(p) = p'$. Further let the coordinate system be chosen so that the infinite hyperplane does not penetrate into D . The Jacobian $J(p)$ of the mapping is then positive in D since the translations of D are proper mappings in D and the same holds for any composite of them. Let $L = 0$ be the equation of a hyperplane l not penetrating into D . We may choose the sign of L so that $L > 0$ in D . On account of our lemma we have the inequality

$$\frac{d\omega'}{[L(p')]^{n+1}} \leq \frac{d\omega}{[L(p)]^{n+1}}$$

or

$$(3) \quad 0 < J(p) \leq \left(\frac{L(p')}{L(p)} \right)^{n+1}.$$

This inequality holds for any choice of l . If $p = p'$ it says that $J(p) \leq 1$. If $p' \neq p$ we choose for l a hyperplane of support at the point p'' defined in the beginning of this section. This leads to

Theorem I. *The Jacobian of a transformation σ of $\mathfrak{B}_D$ transforming the point p into p' satisfies the inequality**

$$(4) \quad J(p) \leq \varphi^{n+1}(p, p') \leq 1,$$

$\varphi(p, p')$ being 1 if p and p' coincide and otherwise given by the ratio

$$(5) \quad \varphi(p, p') = \frac{\overline{p'p''}}{pp'}.$$

The inequality is sharp since the equality sign holds if σ is a translation of D transforming p into p' .

Similar considerations lead to a sharp bound for the Jacobian of the transformations of the semigroup $\mathfrak{M}_D$ but only for *spherical* domains D . Again one can introduce a volume concept in the n -dimensional Moebius space *relative to a point* q . The subgroup $\mathfrak{m}_q$ of the Moebius transformations which keep q fixed is similar, within the Moebius group, to the similarity group of the Euclidean

* Although J and φ are not projective invariants, the inequality is projectively invariant since J/φ^{n+1} is, as one can easily see, a projective invariant.

n -space, whose transformations change the ordinary volume only by a constant positive factor. There exists, therefore, a definition of volume such that the volume is changed by the transformations of $\mathfrak{M}_a$ only by a positive factor. A simple computation, which may be left to the reader, gives the following expression for its volume element at p

$$(6) \quad d\omega_a = \frac{d\omega}{[\overline{pq}]^{2n}} = \frac{dx_1 \cdot dx_2 \cdot \dots \cdot dx_n}{\left[\sum_{i=1}^n (x_i - y_i)^2 \right]^n}^*,$$

the x_i being the coordinates of p and the y_i the coordinates of q . In a fashion completely parallel to the consideration in case of $\mathfrak{P}_D$ the reader will obtain

Lemma 2. *Let q be a point outside of the spherical domain D . Then the volume of any part of D with respect to q cannot be increased by application of any transformation of $\mathfrak{M}_D$.*

From Lemma 2 one then concludes

Theorem II. *Let D be a spherical domain of the Moebius n -space not containing the point infinity. Then the Jacobian of a transformation σ of $\mathfrak{M}_D$ transforming the point p into p' satisfies the inequality*

$$(7) \quad 0 < J(p) \leq \psi(p, p') \leq 1,$$

where $\psi(p, p')$ is defined as follows: If $p = p'$ the value of ψ is one. If $p \neq p'$ the circular arc which connects p and p' and is orthogonal to the boundary of D will, if continued beyond p' , hit the boundary in a point p'' . $\psi(p, p')$ is then given by

$$(8) \quad \psi(p, p') = \left(\frac{p''p'}{p''p} \right)^{2n}.$$

We conclude this section with some remarks about the inequalities (4) and (7). The projective transformations transforming a convex domain D into itself and satisfying inequality (4) constitute a semigroup $\mathfrak{P}'_D \supset \mathfrak{P}_D$. Similarly, the Moebius transformations transforming a spherical domain D into itself and satisfying inequality (7) constitute a semigroup $\mathfrak{M}'_D \supset \mathfrak{M}_D$. Indeed (4) says, as we have shown before, the volume of any part of D relative to any hyperplane not penetrating D does not increase under the transformation at hand. This insures the semigroup property. The assertion about $\mathfrak{M}'_D$ can be verified in a similar way.

2. A theorem on the semigroup $\mathfrak{M}_D$ of a circular domain

The first section of this paper obtained a sharp bound for the Jacobian of the transformations of the semigroups under consideration. We may expect

* Again, any constant positive factor may be added.

finer results by answering the following question: Given two contravariant vectors, when can the first be transformed into the second by a suitable transformation of the semigroup under consideration? A complete answer was obtained by the author in case of the semigroup $\mathfrak{M}_D$ of a circular domain for a pair of vectors attached to the *same* point. The result can be described also in the following way: Given a vector ζ attached to a point p of D , we can find all images of ζ by the transformations of $\mathfrak{M}_D$ which keep p fixed.

We start with the general remark that the transformations of any semigroup $\mathfrak{P}_D$ or $\mathfrak{M}_D$ can be generated from infinitesimal transformations. This follows from the fact that the translations of any of these semigroups are imbedded into one-parameter Lie-groups and all transformations of the semigroup either can be composed of translations or are limits of such composites.

We shall now determine the infinitesimal translations of $\mathfrak{M}_D$ for a circular domain D . Since $\mathfrak{M}_D$ is covariantly connected with D under the group of the Moebius transformations we may assume that D is the interior of the unit circle $|z| < 1$ of a complex z -plane. A contravariant vector of the z -plane with the components ξ_1 and ξ_2 will be described by the complex number $\zeta = \xi_1 + i\xi_2$. It is well known that an infinitesimal transformation of the Moebius group is represented by a vector field of the form

$$(9) \quad \zeta = a_0 z^2 + a_1 z + a_2$$

with arbitrary complex coefficients a_0, a_1, a_2 . If (9) describes an infinitesimal *translation*, only one point of the z -plane remains fixed, *i.e.*, the quadratic expression (9) has a double zero, and that zero must lie on $|z| = 1$:

$$(9') \quad \zeta = a_0(z - \epsilon)^2 \quad (|\epsilon| = 1).$$

In order to determine the possibilities for the factor a_0 we make use of the remark that on $|z| = 1$ the vector ζ must be orthogonal to the perimeter and directed toward the interior of the unit circle. Clearly this is the case only if a_0 is a *positive* multiple of $1/\epsilon$. Since a constant positive factor can be disregarded in the description of an infinitesimal transformation of a *semigroup*, we arrive at

Lemma 3. *An infinitesimal translation of $\mathfrak{M}_D$ of the unit circle $|z| < 1$ is given by*

$$(10) \quad \zeta = \frac{(z - \epsilon)^2}{\epsilon} \quad (|\epsilon| = 1).$$

A composition of infinitesimal translations of $\mathfrak{M}_D$ is described by a differential equation

$$(11) \quad \frac{dz}{dt} = \frac{(z - \epsilon(t))^2}{\epsilon(t)} \quad (|\epsilon(t)| = 1), \quad 0 \leq t \leq t'.$$

The composed transformation is given by the transition from an initial value z at $t = 0$ to the end value z' at $t = t'$ of a solution of (11). The function $\epsilon(t)$ is

sectionally constant if the transformation is composed of a finite number of translations. More generally, any sectionally continuous function $\epsilon(t)$ will lead to a transformation of $\mathfrak{M}_D$ since the limit of composites of translations of D also belongs to $\mathfrak{M}_D$.

From (11) one can obtain the variation of a vector during the composition process:

$$\frac{d\zeta}{dt} = \frac{\partial \frac{(z - \epsilon(t))^2}{\epsilon(t)}}{\partial z} \cdot \zeta$$

or

$$(12) \quad \frac{d\zeta}{dt} = 2 \frac{z - \epsilon(t)}{\epsilon(t)} \cdot \zeta.$$

The following considerations are based on the observation that differential equation (12) is closely connected with Levi-Civita's parallelism of a Riemann metric: We introduce into $|z| < 1$ the Poincaré model of the Bolyai-Lobachevski geometry. Its line element is given by

$$(13) \quad ds = \frac{|dz|}{1 - |z|^2}.$$

From (13) one derives that the Levi-civita shift of a vector η along a piecewise continuously differentiable curve is governed by the equation

$$(12') \quad \frac{d\eta}{dt} = -\frac{2\bar{z}}{1 - |z|^2} \eta \frac{dz}{dt}.$$

From (12) and (12') we derive for the quotient ϵ/η , which is invariant under any proper Moebius transformation, if we shift η along a solution of (11), the differential equation

$$\frac{d \log \frac{\zeta}{\eta}}{dt} = 2 \frac{z - \epsilon(t)}{\epsilon(t)} + 2 \frac{\bar{z}}{1 - |z|^2} \frac{dz}{dt},$$

or

$$\frac{d \log \frac{\zeta}{\eta}}{dt} = 2 \frac{z - \epsilon(t)}{\epsilon(t)} + 2 \frac{\bar{z}}{1 - |z|^2} \frac{(z - \epsilon(t))^2}{\epsilon(t)},$$

and, after a simple computation,

$$\frac{d \log \frac{\zeta}{\eta}}{dt} = -2 \frac{|z - \epsilon(t)|^2}{1 - |z|^2}.$$

Introducing instead of t the non-Euclidean arc-parameter s , we see from (11) and (13) that

$$\frac{ds}{dt} = \frac{|z - \epsilon(t)|^2}{1 - |z|^2}$$

and, therefore,

$$(14) \quad \frac{d \log \frac{\zeta}{\eta}}{ds} = -2.$$

Integration of (14) leads to

Lemma 4. *Let z'_0 be the image of a point z_0 under the transformation generated by the infinitesimal transformations (11) and $z(t)$ the solution of (11) corresponding to the initial value $z(0) = z_0$. (The image of z_0 is then given by $z'_0 = z(t')$.) Further, let ζ'_0 be the image of a vector ζ_0 attached to z_0 under the generated transformation. Then ζ'_0 differs from the vector obtained from ζ_0 by Levi-Civita's parallel shift along the curve $z = z(t)$ by the factor $e^{-2\lambda}$, λ representing the non-Euclidean length of the curve.*

Further, we need

Lemma 5. *Every prescribed piecewise continuously differentiable curve in $|z| < 1$ starting at $z = z_0$ can be obtained as solution of an equation (11) with the initial value z_0 , and the function $\epsilon(t)$ to be used is uniquely determined.*

Proof. In the expression $\omega = (z - \epsilon)^2/\epsilon$ let the value of z be fixed and let ϵ traverse the perimeter of the unit circle. ω then describes once the perimeter of an ellipse of the ω -plane containing $\omega = 0$ in its interior. This shows that just one choice of ϵ leads to a value of ω representing the tangent vector at a prescribed point of the given curve.

Remark. We could have obtained Lemma 5 directly from the definition of an infinitesimal translation, leading to the following geometrical construction of $\epsilon(t)$: Draw the circular arc orthogonal to the unit circle starting at $z(t)$ in the direction of the tangent vector dz/dt until it hits the perimeter. Its endpoint is the required $\epsilon(t)$.

We now specialize to the case $z_0 = z'_0$. Levi-Civita's parallel shift along a closed curve starting and ending at z_0 will transform a vector ζ_0 at z_0 into a vector obtained from ζ_0 by a simple rotation and, as is well known, any angle of rotation θ can be obtained by a suitable choice of the curve. Given such a curve, we construct the corresponding $\epsilon(t)$ according to Lemma 4, and the transformation generated by (11). By this transformation the vector ζ_0 at z_0 is mapped into $\zeta'_0 = \zeta_0 e^{i\theta} e^{-2\lambda}$ as is seen from Lemma 3. The problem formulated in the beginning of this section is solved if we can answer the following question:

Given θ , what possible values can be obtained for λ if all closed curves containing z_0 and yielding the given θ are taken into consideration?

This question can easily be answered by use of the *Gauss-Bonnet formula* and the *isoperimetric inequality* of the Bolyai-Lobachevski geometry. According to the Gauss-Bonnet formula applied to the metric (13) of the Bolyai-Lobachevski geometry, where the Gaussian curvature is identically (-4) , the integral $\int \kappa ds$ of the geodesic curvature of a closed oriented curve is connected with the algebraic area bounded by the curve by the formula

$$(15) \quad \int \kappa ds - 2\pi = 4A.$$

But from Levi-Civita's theory one knows that the left side of (15) represents, modulo 2π , the angle $-\theta$. We have, therefore,

$$(16) \quad \theta \equiv -4A \quad (\text{modulo } 2\pi).$$

Remark. In the derivation of (16) we assumed implicitly that the curve considered is twice continuously differentiable. By a limiting process it can be shown easily that (16) holds for any piecewise continuously differentiable curve.

We now use the *isoperimetric inequality*: Given A , the value of the perimeter λ achieves its minimum for a circle of area A and only for such a circle. Any larger value of λ can naturally be realized by curves of various shapes. Further, one knows that if for a sequence of curves with the same A the perimeters converge to the minimum value, the curves must converge to a circle of area A .

From elementary formulas of the Bolyai-Lobachevski geometry one derives that A and λ for a circle are related by the formula

$$\lambda = 2\sqrt{|A|(|A| + \pi)}.$$

This gives finally

Theorem III. *A vector ζ_0 attached to a point z_0 can be mapped by the transformations of $\mathfrak{M}_D$ which keep z_0 fixed into the vectors of the form*

$$(17) \quad \zeta'_0 = \zeta_0 e^{i\theta} e^{-\mu\sqrt{|\theta|(|\theta|+4\pi)}} \quad (|\theta| \leq \pi, \mu \geq 1),$$

and into these only. The quotient $\omega = \zeta'_0/\zeta_0$ is therefore restricted to the closed Jordan domain of the ω -plane bounded by the curve L with the polar equation

$$(18) \quad \rho = e^{-\sqrt{|\theta|(|\theta|+4\pi)}} \quad (|\theta| \leq \pi)$$

*from which only the point $\omega = 0$ is deleted.**

* The point $\omega = 0$ would be included if degenerate mappings transforming $|z| < 1$ into a point had also been admitted into $\mathfrak{M}_D$.

The following small table gives an idea of the shape of L :

θ	ρ
0	1
$\frac{1}{4}\pi$	$e^{-3.238}$
$\frac{1}{2}\pi$	$e^{-4.712}$
$\frac{3}{2}\pi$	$e^{-5.930}$
π	$e^{-7.025}$

One sees that L is rather narrow in the direction normal to the polar axis and comes very close to the origin as θ approaches π . One sees further from (18) that L has a sharp point at $\theta = 0$ and a vertex with a non-zero angle at $\theta = \pi$.

We conclude this section by computing the mappings corresponding to the boundary points of L . For $\theta = 0$ we evidently obtain only the identity mapping. We therefore assume $\theta \neq 0$. We know that we have to determine those equations (11) for which a solution exists describing a circle in $|z| < 1$. Since $\mathfrak{M}_D$ is invariant under the group of motions of the Bolyai-Lobachevski geometry, it is sufficient to consider only the case where the circle is concentric with the unit circle and is positively oriented. The latter condition is equivalent to the assumption $(-\theta) > 0$. Evidently $\epsilon(t)$ must have the form $\epsilon(t) = e^{i(\alpha t + \beta)}$ with a positive α and t is to be restricted to the interval $0 \leq t \leq 2\pi/\alpha$ in order to get the full circle. Since further any rotation around the origin belongs to the group of motions we may assume $\beta = 0$, so that

$$(19) \quad \epsilon(t) = e^{i\alpha t} \quad \left(0 \leq t \leq \frac{2\pi}{\alpha}\right).$$

The differential equation to be solved is

$$(20) \quad \frac{dz}{dt} = \frac{(z - e^{i\alpha t})^2}{e^{i\alpha t}} \quad \left(0 \leq t \leq \frac{2\pi}{\alpha}\right).$$

We introduce the new independent variable

$$w = ze^{-i\alpha t}.$$

Equation (20) is then transformed into

$$(21) \quad \frac{dw}{dt} = (w - 1)^2 - i\alpha w,$$

which can be solved by separation of variables. The zeros of the right-hand side of (21) are

$$(22) \quad w_{1,2} = 1 + \frac{1}{2}i\alpha \pm \sqrt{-(\frac{1}{2}\alpha)^2 + i\alpha}.$$

The values of the square root in (22) lie in the third and first quadrant. Let w_1 and w_2 be the corresponding roots in this order. Evidently $|w_1| < 1$ and $|w_2| > 1$. The integration of (21) shows now that the mapping generated is given by

$$(23) \quad \frac{z' - w_1}{z' - w_2} = \frac{z - w_1}{z - w_2} e^{(w_1 - w_2)(2\pi/\alpha)}$$

and w_1 is the fixed point of the transformation in $|z| < 1$.

We are interested only in those transformations (23) giving $|\theta| \leq \pi$. Since $-\frac{1}{4}\theta$ equals the non-Euclidean area A of our circle, this means that

$$A = \frac{\pi |w_1|^2}{1 - |w_1|^2} \leq \frac{1}{4}\pi$$

or

$$(24) \quad |w_1| \leq \frac{1}{\sqrt{5}}.$$

This is equivalent to a restriction of α which we wish to determine.

We have to return to the solution of (20) with the initial value $z_0 = w_1$ which describes our circle. From the remark on page 798 we see that the Euclidean distance between $z(t)$ and $e^{i\alpha t}$ remains constant. A simple computation using the construction described in the remark, which may be left to the reader, leads to

$$|z(t) - e^{i\alpha t}|^2 = \frac{(1 - |w_1|^2)^2}{1 + |w_1|^2}$$

or

$$(25') \quad \left| \frac{dz}{dt} \right| = \frac{(1 - |w_1|^2)^2}{1 + |w_1|^2}.$$

Integration of (25') over the interval $(0, 2\pi/\alpha)$ gives

$$2\pi |w_1| = \frac{(1 - |w_1|^2)^2}{1 + |w_1|^2} \cdot \frac{2\pi}{\alpha}$$

or

$$(25) \quad \alpha = \frac{(1 - |w_1|^2)^2}{|w_1| (1 + |w_1|^2)}.$$

From (24) we then conclude that α is restricted to $\alpha \geq \frac{8}{15}\sqrt{5}$. We have arrived finally at

Theorem IV. *The system of extremal transformations of $\mathfrak{M}_D$ yielding a point on the curve (18) consists of those transformations which are similar under the group*

$$z' = \eta \frac{z - a}{1 - \bar{a}z} \quad (|\eta| = 1, |a| < 1)$$

*to one of the transformations (23) with the restriction**

$$(26) \quad |\alpha| \geq \frac{8}{15} \sqrt{5}.$$

We conclude this section with

Theorem V. *The transformation considered in Theorem IV with given angle $\theta \neq 0$ and given fixed point depend on one parameter. The totality of the extremal mappings is therefore 4-dimensional.*

Proof. We may assume $\theta < 0$. Equation (25) shows that θ determines the value of α . We need further the fact that the values w_1 and w_2 are not reciprocal conjugates. Otherwise the equation satisfied by w_1 and w_2

$$(27) \quad (w - 1)^2 - i\alpha w = 0 \quad (\alpha > 0)$$

would not change if w were replaced by $1/\bar{w}$, which substitution leads to the equation

$$(28) \quad (w - 1)^2 + i\alpha w = 0.$$

This would be possible if $w_1 = w_2 = 0$, which is not the case. Therefore, the Moebius transformations which keep $z < 1$ invariant and w_1 fixed transform a mapping (23) into a one-parameter family of extremal mappings keeping w_1 fixed and belonging to the angle θ .

3. Some group theoretical consideration.

Let us consider the semigroup $\mathfrak{P}_D$ of the interior of the unit sphere $\sum_{i=1}^n x_i^2 < 1$ or a projectively equivalent domain. We introduce further the group $\mathfrak{G}$ of the projective transformations keeping D invariant (Klein model of the Bolyai-Lobachevski geometry). One may consider all semigroups $\mathfrak{S}$ of projective transformations sharing with $\mathfrak{P}_D$ the following properties:

- (a) Each transformation of $\mathfrak{S}$ transforms D into itself.
- (b) $\mathfrak{S}$ is invariant under the group $\mathfrak{G}$.
- (c) $\mathfrak{S}$ is completely generated from its infinitesimal transformations.
- (d) The infinitesimal transformations of $\mathfrak{S}$ form a closed system.
- (e) $\mathfrak{S}$ contains transformations not belonging to $\mathfrak{G}$.

We shall prove

* We have to include here the limiting case $\alpha \rightarrow \infty$ yielding the identity mapping.

Theorem VI. $\mathfrak{P}_D$ is the intersections of all semigroups of projective transformations satisfying the five conditions (a)-(e).

Proof. An infinitesimal transformation of the projective group with the components ξ_i has the form

$$(29) \quad \xi_i = L_i + x_i L \quad (i = 1, 2, \dots, n)$$

where the L_i and L are linear expressions in the x_i . Let $\mathfrak{S}$ be a semigroup satisfying the five conditions, and (29) an infinitesimal transformation of $\mathfrak{S}$ which is not an infinitesimal transformation of $\mathfrak{G}$. Such a transformation exists because of condition (e). On the boundary S of D the vector (29) cannot be everywhere tangential to S . Since on account of condition (b) any transform of (29) by the group $\mathfrak{G}$ belongs also to $\mathfrak{S}$, we may assume that we have no tangency at the point e with the coordinates: $x_1 = -1, x_i = 0$ ($j > 1$), i.e., $\xi_1 \neq 0$ at e , and since $\mathfrak{S}$ transforms the unit sphere into itself ξ_1 must be positive at e .

We now introduce the projective transformations

$$(30) \quad \begin{aligned} x'_1 &= \frac{x_1 + a}{1 + ax_1} \\ x'_j &= \frac{\sqrt{1 - a^2} x_j}{1 + ax_1} \quad (j > 1), \end{aligned}$$

restricting the parameter a to the interval

$$-1 < a < 1.$$

They form a one-parameter subgroup of $\mathfrak{G}$.

Because of condition (b), a transform of (29) by any of the transformations (30) must also belong to $\mathfrak{S}$. A computation of the components of the transformed infinitesimal transformation gives

$$(31) \quad \begin{aligned} \xi'_1 &= \frac{1 - a^2}{(1 + ax_1)^2} \xi_1 \\ \xi'_j &= \sqrt{1 - a^2} \left\{ \frac{-ax_j}{(1 + ax_1)^2} \xi_1 + \frac{1}{1 + ax_1} \xi_j \right\} \quad (j > 1). \end{aligned}$$

Introducing the primed coordinates into the right-hand side of (31), we are led to

$$(31') \quad \begin{aligned} \xi'_1 &= \frac{(1 - ax'_1)^2}{1 - a^2} \xi_1 \\ \xi'_j &= \frac{1 - ax'_1}{1 - a^2} \{-ax'_j \xi_1 + \sqrt{1 - a^2} \xi_j\} \quad (j > 1). \end{aligned}$$

Multiplication of (31') by the positive factor $(1 - a^2)$ leads again to an infinitesimal transformation of $\mathfrak{S}$. If we then perform the limiting process $a \rightarrow 1$ for

a fixed point (x'_i) , the corresponding point (x_i) converges to the point e , and we obtain

$$\begin{aligned}\lim_{a \rightarrow 1} (1 - a^2) \xi'_1 &= (1 - x'_1)^2 \xi_1(-1, 0, \dots, 0) \\ \lim_{a \rightarrow 1} (1 - a^2) \xi'_j &= -(1 - x'_j) x'_j \xi_1(-1, 0, \dots, 0) \quad (j > 1).\end{aligned}$$

Since $\xi_1(-1, 0, \dots, 0) > 0$ we obtain the result that

$$\begin{aligned}(32) \quad \eta_1 &= (1 - x_1)^2 \\ \eta_j &= -(1 - x_j) x_j \quad (j > 1)\end{aligned}$$

represents an infinitesimal transformation of $\mathfrak{S}$. But (32) is the infinitesimal translation of the unit sphere with the fixed point $e_1(1, 0, 0, \dots, 0)$. The point e_1 can be brought into any other position on the boundary of the unit sphere by a suitable transformation of $\mathfrak{G}$ and the infinitesimal transformation (32) is then mapped into the infinitesimal translation of $\mathfrak{P}_D$ with the image of e_1 as fixed point. Hence we see that $\mathfrak{S}$ must contain the whole $\mathfrak{P}_D$. This proves Theorem VI.

A similar theorem can be proved for the semigroup $\mathfrak{M}_D$ of the unit sphere. Let us consider all semigroups $\mathfrak{S}$ of Moebius transformations satisfying conditions (a')-(e') obtained from (a)-(e) by replacing $\mathfrak{G}$ by the group $\mathfrak{P}$ of Moebius transformations which leave D invariant (Poincaré model of the Bolyai-Lobachevski geometry) and by replacing the total projective group by the total Moebius group. Then the following theorem holds:

Theorem VII. *$\mathfrak{M}_D$ is the intersection of all semigroups of Moebius transformations which satisfy the five conditions (a')-(e').*

The proof is completely parallel to that of Theorem VI. It may, therefore, be left to the reader.

We make the following concluding remarks: The groups $\mathfrak{P}$ and $\mathfrak{G}$ are *similar*. One may ask for all semigroups $\mathfrak{S}$ satisfying the conditions (a)-(e) when the assumption that the transformations belong to an *a priori* given Lie group, such as the projective or Moebius group, is dropped. First of all, one is interested in such semigroups which do not contain proper subsemigroups having the same properties. This problem will be taken up in a later paper.