*On the Completeness of the Boussinesq-Papkovich Stress Functions**

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1. Introduction.

In the absence of body forces, the displacement equations of equilibrium in the linear theory of homogeneous and isotropic elastic media, appear as

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\nu} \nabla \nabla \cdot \mathbf{u} = 0, \quad (1.1)^1$$

where $\mathbf{u}$ is the displacement field, while ν denotes Poisson's ratio.

PAPKOVICH [1],² [2] in 1932 established the general solution of (1.1) in the form

$$\mathbf{u} = \nabla(\varphi + \mathbf{r} \cdot \boldsymbol{\psi}) - 4(1-\nu)\boldsymbol{\psi}, \quad (1.2)$$

where

$$\nabla^2 \varphi = 0, \quad \nabla^2 \boldsymbol{\psi} = 0. \quad (1.3)$$

Here $\mathbf{r}$ is the position vector of a field point, referred to the origin O . Substitution into (1.1) at once confirms that the vector field (1.2) satisfies the equilibrium equations provided that the generating stress functions (displacement potentials) φ , $\boldsymbol{\psi}$ are sufficiently smooth and meet (1.3). Conversely, every sufficiently regular solution of (1.1) admits the representation (1.2), (1.3), as shown by PAPKOVICH (*loc. cit.*). A more direct proof of the completeness of this representation is due to MINDLIN [3].³

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¹Throughout this paper bold face letters designate vectors; the symbols “ $\cdot$ ” and “ $\times$ ” indicate scalar and vector multiplication of two vectors, respectively. ∇ is the usual del-operator.

²Numbers in brackets refer to the bibliography at the end of the paper.

³See also SOKOLNIKOFF [4], p. 329 *et seq.*, where the same argument is presented. MINDLIN [5] recently extended his proof to the case where the body forces are included; in this instance the Laplace equations (1.3) need to be replaced by appropriate Poisson equations.

The solution (1.2), (1.3) was independently discovered by NEUBER [6] in 1934. Elements of this general solution were known to BOUSSINESQ [7].⁴ Its application to special problems has yielded a variety of technically significant three-dimensional solutions, as illustrated by the work of NEUBER [8]. It should be emphasized that *all* of the stress functions employed in the classical theory of elasticity (*e.g.*, those appropriate to the plane problem, the problem of pure torsion, and the problem characterized by torsionless axisymmetry) are related to, and derivable from, the Boussinesq-Papkovich stress functions in (1.2), (1.3).⁵

The general solution of (1.1) under discussion involves *four* scalar harmonic stress functions, *i.e.*, φ and the three Cartesian scalar components of the vector potential ψ . Having established the completeness of (1.2), (1.3), PAPKOVICH [1], [2], through an erroneous argument, is led to claim that φ may be taken as zero without loss in completeness—a claim which is reflected in the title of [2].⁶ NEUBER [8], assuming the completeness of (1.2), (1.3), asserts that φ or any one of the components of ψ may be set equal to zero without impairing the generality of the solution. In his proof of this statement he takes for granted, in two instances, the existence of a function satisfying a pair of simultaneous partial differential equations. NEUBER's argument is, therefore, inconclusive, as pointed out previously by SOKOLNIKOFF [4], p. 331. The statement has since been repeated (without proof) at various places in the literature.

SLOBODYANSKY [10], returns to the question as to whether the number of harmonic stress functions in (1.2), (1.3) may be reduced to *three*. He claims to show that φ may be taken as zero without loss in completeness, if the region occupied by the medium is bounded and simply connected or is the exterior of an arbitrary closed surface. Moreover, since his argument fails when 4ν is an integer, he concludes that, in this event, the reduction is not possible—a conclusion which does not follow.

SOKOLNIKOFF [4], p. 331 *et seq.*, observes correctly that the validity of SLOBODYANSKY's proof is limited to a *region interior or exterior to a sphere*. A second theorem of SLOBODYANSKY [10] asserts that any one of the components of ψ in (1.2), (1.3) may be set equal to zero without affecting the generality of the solution. The corresponding proof, again reproduced in [4], is valid only for a *region interior to a sphere*, despite more sweeping claims made in [10].

In the present paper, we first examine the precise circumstances under which the completeness of the representation (1.2), (1.3), in terms of *four* harmonic

⁴Indeed, the difference between the two particular solutions of (1.1) given by BOUSSINESQ on p. 72 and p. 63 of [7], yields the special form to which (1.2) reduces in the case of rotational symmetry, as will be confirmed later on.

⁵A general bibliography on the subject of stress functions in elasticity theory is beyond the scope of the present paper. In this connection see MARGUERRE [9]. Additional references to the Russian literature are given by SLOBODYANSKY [10].

⁶In a subsequent note [11], PAPKOVICH himself refers to this argument as "artificial," and reverts to the form (1.2).

stress functions, has been established. We then prove that any one of these four functions may be taken as zero without loss in completeness; the proof presented here applies to an *arbitrary convex region*,⁷ if the coordinate system is chosen appropriately and provided 4ν is not an integer. Furthermore, we show with the aid of a counter-example that SLOBODYANSKY's conjecture is correct and that the representation (1.2), (1.3), with $\varphi = 0$, is *incomplete* if 4ν is an integer, *e.g.*, if Poisson's ratio is one-quarter.

Finally, we prove that in the case of problems characterized by torsionless axisymmetry, *two* of the components of ψ in (1.2), (1.3) may be set equal to zero without impairing the completeness of the solution. This last statement may be reached more economically by a direct integration of the displacement equations of equilibrium governing the rotationally symmetric problem.⁸ From the viewpoint of a systematic exposition of the theory, however, it seems desirable to treat the axisymmetric problem as a special case of the general three-dimensional problem.

2. Sufficient conditions for the completeness of the Boussinesq-Papkovich solution. Reduction to three harmonic stress functions.

The completeness proofs given in [1], [2] and [3] rest on the theorem—variously associated with the names of STOKES and HELMHOLTZ—according to which a given vector field may be represented as the sum of the gradient of a scalar field and the curl of another vector field.⁹ The validity of the available completeness proofs for the Boussinesq-Papkovich solution (1.2), (1.3) is, therefore, restricted by the assumptions underlying the proof of the theorem here referred to.¹⁰ We now state the relevant implications of these assumptions.

Let D henceforth be an open region of space, $D + B$ being the closed region with the boundary B . A function $f(x, y, z)$ will be referred to as being of *Class n* in D , if it is continuous and has continuous partial derivatives of order up to and including n , in D . We shall call $f(x, y, z)$ *analytic* in D , if it possesses continuous partial derivatives of all orders at every point of D . The function will be called *harmonic* in D , if it is of Class 2 and satisfies Laplace's equation in D . Finally, we cite the theorem,¹¹ that every function which is harmonic in D , is also analytic in D .

The representation (1.2), (1.3) is complete for a medium occupying D , if $\mathbf{u}$ is continuous in $D + B$ and of Class 3 in D , while D is a regular region of space.¹²

⁷Actually, as will be seen, the theorem holds for a broader class of regions which includes any convex domain.

⁸See, for example, MARGUERRE [9]. On careful examination, the saving is found to be not substantial. Cf. Footnote No. 29.

⁹PAPKOVICH's proof actually starts with a consequence of this theorem. The proof contained in [10], which seemingly avoids the use of the theorem, is untenable.

¹⁰For a proof of the Stokes-Helmholtz resolution, see, for example, PHILLIPS [12], p. 186 *et seq.*

¹¹See KELLOGG [13], p. 220.

¹²The term "regular region of space" is used in the sense of KELLOGG [13].

We recall that the boundary B may exhibit corners and edges and that D need not be bounded or simply connected. If D is not bounded, it suffices to assume that

$$\mathbf{u} = O(r^{-2}) \quad \text{as } r \rightarrow \infty, \quad (2.1)^{13}$$

where $r = |\mathbf{r}|$ is the distance from the origin.

The differentiability requirement on $\mathbf{u}$ in D assures that the stress functions φ and ψ are of Class 2 in D ; in view of (1.3), this implies further that φ and ψ are harmonic, and thus analytic, in D . It should be emphasized that the preceding restrictions on $\mathbf{u}$ are merely *sufficient* conditions for the completeness of (1.2), (1.3). Indeed, the regularity condition (2.1) for unbounded domains is unduly stringent for the purpose of applications.¹⁴ We observe that a given displacement field does not uniquely determine its generating stress functions φ and ψ . In fact, it is precisely this lack of uniqueness which, under certain circumstances, permits the elimination of one of the scalar stress functions entering (1.2). We now turn to the question as to when such a reduction is permissible without loss in completeness, and deal with it in two separate theorems: the first is concerned with the possibility of setting $\varphi = 0$, the second with the possibility of equating to zero one of the scalar components of ψ in (1.2).

Let D be an open region of space which is "star-shaped" with respect to one of its points Q , i.e., every straight-line segment connecting Q and another point of D , lies wholly in D .¹⁵ Let $\mathbf{u}(P)$ be defined for all points P in D and admit the representation,

$$\mathbf{u} = \nabla\varphi, \quad \nabla^2\varphi = 0, \quad (2.2)$$

where $\varphi(P)$ is of Class 2 in D . Let the origin $\mathbf{r} = 0$ be chosen at Q and 4ν be not an integer. Then $\mathbf{u}(P)$ also admits the representation

$$\mathbf{u} = \nabla(\mathbf{r} \cdot \psi) - 4(1 - \nu)\psi, \quad \nabla^2\psi = 0, \quad (2.3)$$

where $\psi(P)$ is analytic in D .

As a preliminary to the proof, we establish *necessary* conditions for the existence of the alternative representation (2.3). Setting, for convenience,

$$4(1 - \nu) = \alpha \quad (2.4)$$

and equating the right members of the first of (2.2), (2.3), we have,

$$\nabla(\mathbf{r} \cdot \psi) - \alpha\psi = \nabla\varphi. \quad (2.5)$$

¹³This means that $r^2|\mathbf{u}|$ remains bounded as $r \rightarrow \infty$.

¹⁴In this connection it is interesting to recall that BLUMENTHAL [14] effected a resolution of a vector field $\mathbf{v}$ into solenoidal and irrotational components on the assumption that $\mathbf{v}$ is analytic in the entire space and vanishes at infinity (no particular order of vanishing being prescribed). He did not, however, establish scalar and vector potentials for the respective component fields.

¹⁵Note that D is star-shaped with respect to all of its points if it is convex. Cf. Footnote No. 7.

Operating on (2.5) with the curl, we find that $\psi(P)$ is irrotational, whence there exists a scalar field $\Phi(P)$, analytic¹⁶ in D , such that

$$\psi = \nabla \Phi. \quad (2.6)$$

Substitution of (2.6) in (2.5) at once yields

$$\mathbf{r} \cdot \nabla \Phi - \alpha \Phi = \varphi + c_1, \quad (2.7)$$

where c_1 is an arbitrary constant. Furthermore, (2.6) and the second of (2.3) imply

$$\nabla^2 \Phi = c_2, \quad (2.8)$$

c_2 being another arbitrary constant.

On the other hand, the existence of a function $\Phi(P)$, which is analytic in D and satisfies (2.7), (2.8), say for $c_1 = c_2 = 0$, is also *sufficient* for the existence of the alternative representation (2.3). To see this, substitute for φ from (2.7) into the first of (2.2) and *define* $\psi(P)$ through (2.6). This leads to the first of (2.3). Moreover, by virtue of (2.6) and the second of (2.2), ψ is harmonic in D .

In view of these observations, the theorem will have been established if we can exhibit a function $\Phi(P)$, analytic in D , and there satisfying the pair of equations¹⁷

$$\mathbf{r} \cdot \nabla \Phi - \alpha \Phi = \varphi, \quad (2.9)$$

$$\nabla^2 \Phi = 0, \quad (2.10)$$

where α is given by (2.4) and is not an integer. Since the strain-energy density of an isotropic, linear medium is positive definite if and only if $-1 < \nu < 1/2$, we may confine α to the range

$$2 < \alpha < 8. \quad (2.11)$$

Let (r, θ, γ) be spherical coordinates, related to the rectangular Cartesian coordinates (x, y, z) by means of the mapping

$$\left. \begin{aligned} x &= r \sin \theta \cos \gamma, & y &= r \sin \theta \sin \gamma, & z &= r \cos \theta; \\ 0 &\leq r < \infty, & 0 &\leq \theta \leq \pi, & 0 &\leq \gamma < 2\pi. \end{aligned} \right\} \quad (2.12)$$

The function $\varphi(r, \theta, \gamma)$, by hypothesis, is harmonic in D . Hence, if the origin is chosen in D , φ admits the following expansion in terms of solid spherical harmonics

$$\varphi = \sum_{n=0}^{\infty} S_n(\theta, \gamma) r^n, \quad (2.13)^{18}$$

¹⁶The analyticity of Φ follows from that of ψ .

¹⁷NEUBER [8] assumes, without proof, the existence of such a Φ , in the absence of any restrictions on D and ν .

¹⁸See KELLOGG [13], p. 251.

which is uniformly convergent in any spherical neighborhood of the origin that lies wholly in D . The functions $S_n(\theta, \gamma)$ are surface harmonics of degree n .

Next, define a function $F(r, \theta, \gamma)$ through

$$F(r, \theta, \gamma) = \varphi(r, \theta, \gamma) - \sum_{n=0}^{[\alpha+1]} S_n(\theta, \gamma) r^n, \quad (2.14)$$

in which $[\alpha + 1]$ denotes the largest integer not exceeding $\alpha + 1$. We note that $F(P)$ is defined and harmonic throughout D . From (2.13), (2.14) follows

$$r^{-\alpha-1} F(r, \theta, \gamma) \rightarrow 0 \quad \text{as } r \rightarrow 0. \quad (2.15)$$

Equation (2.9) may be written as

$$r \frac{\partial \Phi}{\partial r} - \alpha \Phi = \varphi, \quad (2.16)$$

or, with the aid of (2.14),

$$r^{\alpha+1} \frac{\partial}{\partial r} (r^{-\alpha} \Phi) = F + \sum_{n=0}^{[\alpha+1]} S_n r^n. \quad (2.17)$$

Now consider the function Φ defined by

$$\left. \begin{aligned} \Phi(r, \theta, \gamma) &= G(r, \theta, \gamma) + \sum_{n=0}^{[\alpha+1]} \frac{S_n(\theta, \gamma) r^n}{n - \alpha}, \\ G(r, \theta, \gamma) &= r^\alpha \int_0^r F(\xi, \theta, \gamma) \xi^{-\alpha-1} d\xi. \end{aligned} \right\} \quad (2.18)$$

If, as we suppose, D is star-shaped with respect to the origin $r = 0$, and in view of (2.11), (2.15),¹⁹ Equations (2.18) define $\Phi(P)$ throughout D . Moreover, Φ so defined is of Class 2 in D and satisfies (2.17), and thus (2.9). This last assertion is verified by substitution of (2.18) into (2.17).

It remains to be shown that $\Phi(P)$ in (2.18) is harmonic in D . To this end it suffices to demonstrate the harmonicity of $G(P)$. With reference to (2.12) recall that the Laplacian operator in spherical coordinates appears as²⁰

$$\left. \begin{aligned} \nabla^2 &= \frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \nabla_*^2 \right], \\ \text{where } \nabla_*^2 &= \operatorname{cosec} \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \operatorname{cosec}^2 \theta \frac{\partial^2}{\partial \gamma^2}. \end{aligned} \right\} \quad (2.19)$$

¹⁹The integral in (2.18) is proper by virtue of (2.15).

²⁰See KELLOGG [13], p. 183.

From the second of (2.18),

$$\left. \begin{aligned} \frac{\partial}{\partial r} \left(r^2 \frac{\partial G}{\partial r} \right) &= r \frac{\partial F}{\partial r} + (\alpha + 1)F - \alpha(\alpha + 1)G, \\ \nabla_*^2 G &= r^\alpha \int_0^r \nabla_*^2 F(\xi, \theta, \gamma) \xi^{-\alpha-1} d\xi, \end{aligned} \right\} \quad (2.20)$$

and, with the aid of two successive integrations by parts applied to the second of (2.18), we reach the identity,

$$\alpha(\alpha + 1)G = -(\alpha + 1)F - r \frac{\partial F}{\partial r} + r^\alpha \int_0^r \frac{\partial}{\partial \xi} \left[\xi^2 \frac{\partial F(\xi, \theta, \gamma)}{\partial \xi} \right] \xi^{-\alpha-1} d\xi. \quad (2.21)$$

Substitution of (2.21) in (2.20), and use of (2.19), now yield,

$$r^2 \nabla^2 G = r^\alpha \int_0^r \nabla^2 F(\xi, \theta, \gamma) \xi^{-\alpha-1} d\xi. \quad (2.22)$$

But $F(P)$ is harmonic in D . Hence, $G(P)$ and thus $\Phi(P)$ are harmonic in D . The function $\Phi(P)$, being harmonic, is analytic in D . This completes the proof of the theorem. As an immediate consequence *we have the statement that φ in (1.2), (1.3) may be taken as zero without loss in completeness, provided D is star-shaped with respect to the origin, and if 4ν is not an integer.*

Inspection of the first of (2.18) reveals that the proof of the theorem fails when $\alpha \equiv 4(1 - \nu)$ is an integer, *i.e.*, when²¹ $\nu = \frac{1}{4}, 0, -\frac{1}{4}, -\frac{3}{4}$. The question remains whether this difficulty is intrinsic or merely representational. We show presently that the alternative representation (2.3) is, in general, impossible when α is an integer.

Thus, let D be a sphere centered at the origin and let α be an integer, $2 < \alpha < 8$. Suppose that φ in (2.2) is given by the solid harmonic

$$\varphi = r^\alpha S_\alpha(\theta, \gamma). \quad (2.23)$$

Applying conditions (2.7), (2.8), which were shown to be *necessary* for the existence of the alternative representation (2.3), to the example under consideration, we find that there would have to exist a $\Phi(P)$ such that

$$\frac{\partial}{\partial r} (r^{-\alpha} \Phi) = r^{-1} S_\alpha(\theta, \gamma) + c_1 \text{ in } D, \quad (2.24)$$

$$\nabla^2 \Phi = c_2 \text{ in } D, \quad (2.25)$$

where c_1, c_2 are constants. The general solution of (2.24) is

$$\Phi = r^\alpha f(\theta, \gamma) + r^\alpha \log r S_\alpha(\theta, \gamma) + c_1 r^{\alpha+1} \quad (2.26)$$

in which f is an arbitrary function.

Since $\varphi(P)$ of (2.23) is analytic in D , the same would have to be true of the

²¹Recall that $-1 < \nu < \frac{1}{2}$.

corresponding $\psi(P)$ in (2.3). It follows from (2.6) that Φ would have to be analytic in D . On the other hand, it is clear from (2.26) that there does not exist an $f(\theta, \gamma)$ for which $\Phi(P)$ is free from a singularity at the origin. Indeed, there does not exist an $f(\theta, \gamma)$ such that (2.25) is met throughout D . This confirms SLOBODYANSKY's [10] conjecture²² regarding the exceptional role of integral values of 4ν . We consider next a second reduction theorem associated with the Boussinesq-Papkovich solution (1.2), (1.3).

Let D be an open region of space which permits a choice of Cartesian coordinates (x, y, z) such that any straight-line segment parallel to the z -axis which connects two points of D , lies wholly in D .²³ Moreover, assuming such a choice of coordinates, let $x^2 + y^2$ be bounded for all points P in D . Let $\mathbf{u}(P)$ be defined in D and admit the representation

$$\mathbf{u} = \nabla(\mathbf{r} \cdot \psi) - 4(1 - \nu)\psi, \quad \nabla^2 \psi = 0, \quad (2.27)$$

where $\psi(P)$ is of Class 2 in D . Then $\mathbf{u}(P)$ also admits the representation

$$\mathbf{u} = \nabla(\varphi' + \mathbf{r} \cdot \psi') - 4(1 - \nu)\psi', \quad \nabla^2 \varphi' = 0, \quad \nabla^2 \psi' = 0, \quad (2.28)$$

where $\varphi'(P)$ and $\psi'(P)$ are analytic in D and

$$\mathbf{k} \cdot \psi' = 0, \quad (2.29)$$

if $\mathbf{k}$ is a vector parallel to the z -axis.

We aim first at conditions necessary for the existence of the alternative representation (2.28), (2.29). The function $\psi(P)$, by virtue of its harmonicity, is analytic in D . Equating the right members of the first of (2.27), (2.28), and taking the curl of the resulting equation, we conclude the existence of a function $H(P)$, analytic in D , such that

$$\psi' = \psi - \nabla H. \quad (2.30)$$

Subtracting (2.28) from (2.27), and using (2.30), we reach

$$\varphi' = \mathbf{r} \cdot \nabla H - \alpha H + c_1, \quad (2.31)$$

in which c_1 is an arbitrary constant and α is again given by (2.4). Equation (2.30), the second of (2.27), and the third of (2.28), yield

$$\nabla^2 H = c_2, \quad (2.32)$$

c_2 being another arbitrary constant. Finally, (2.29) and (2.30) imply

$$\mathbf{k} \cdot (\psi - \nabla H) = 0 \quad \text{or} \quad \frac{\partial H}{\partial z} = \psi_z, \quad (2.33)$$

if ψ_z is the scalar z -component of ψ .

²²See also SOKOLNIKOFF [4], p. 333.

²³Note that this is true for all choices of the coordinate system if D is convex. Cf. Footnote No. 7.

Conversely, the existence of a simultaneous, analytic solution $H(P)$ to

$$\frac{\partial H}{\partial z} = \psi_z, \quad \nabla^2 H = 0, \quad (2.34)$$

is also *sufficient* for the truth of the theorem.²⁴ For, we may then *define* $\varphi'(P)$ and $\psi'(P)$ by means of (2.30) and (2.31) with $c_1 = 0$. Substitution for ψ from (2.30) into the first of (2.27) and use of (2.31) lead to the first of (2.28); the remaining (2.28), as well as (2.29), hold by virtue of (2.34).

We have yet to exhibit a solution, analytic in D , to the system (2.34). To this end consider the function

$$H_1(x, y, z) = \int_{z_0(x, y)}^z \psi_z(x, y, \zeta) d\zeta, \quad (2.35)$$

where $Q_0(x, y, z_0)$ is one of the points of intersection with B of the corresponding straight line parallel to the z -axis.²⁵ By (2.35) and the second of (2.27),

$$\frac{\partial H_1}{\partial z} = \psi_z, \quad \nabla^2 \frac{\partial}{\partial z} H_1 = \frac{\partial}{\partial z} \nabla^2 H_1 = 0, \quad (2.36)$$

whence

$$\nabla^2 H_1 = h(x, y). \quad (2.37)$$

Define another function H_2 through the logarithmic potential,

$$\left. \begin{aligned} H_2(x, y) &= \frac{-1}{2\pi} \iint_{D_*} h(\xi, \eta) \log R d\xi d\eta, \\ R^2 &= (x - \xi)^2 + (y - \eta)^2, \end{aligned} \right\} \quad (2.38)$$

where D_* is the projection of D on to the plane $z = 0$ and, by hypothesis is, bounded. Recalling²⁶ that

$$\nabla^2 H_2 = -h(x, y) \text{ in } D_*, \quad (2.39)$$

we note from (2.35) to (2.39) that

$$H(x, y, z) \equiv H_1(x, y, z) + H_2(x, y), \quad (2.40)$$

in view of the analyticity of ψ , and because of the assumed convexity property of D , is analytic throughout D and satisfies both of (2.34). The proof of the theorem is now complete. It follows at once that *for a region D and a choice of the coordinates system, meeting the hypothesis of this theorem, ψ_z in (1.2), (1.3) may be taken as zero without loss in completeness.* Clearly, the same is true of ψ_x or ψ_y under analogous circumstances.

²⁴NEUBER [8], without restricting D , takes for granted the existence of such an $H(P)$.

²⁵If the straight lines parallel to the z -axis do not intersect B (e.g., if B is a cylinder whose generators are parallel to the z -axis), we may take $z_0 = 0$.

²⁶See KELLOGG [13], p. 174.

3. Axisymmetric displacement fields; reduction to two stress functions.

Let (ρ, γ, z) be cylindrical coordinates defined by the transformation

$$\left. \begin{aligned} x &= \rho \cos \gamma, & y &= \rho \sin \gamma, & z &= z; \\ 0 &\leq \rho < \infty, & 0 &\leq \gamma < 2\pi. \end{aligned} \right\} \quad (3.1)$$

A displacement field $\mathbf{u}$ is characterized by *torsionless axisymmetry* with respect to the z -axis if

$$u_\rho = u_\rho(\rho, z), \quad u_\gamma = 0, \quad u_z = u_z(\rho, z), \quad (3.2)$$

where (u_ρ, u_γ, u_z) are the cylindrical scalar components of the displacement vector. In this section we state and prove a theorem according to which both ψ_x and ψ_y in the Boussinesq-Papkovich solution (1.2) may be set zero without loss of completeness, if $\mathbf{u}$ has the properties (3.2).

Let D be an open region of space whose boundary B is a surface of revolution. Let $\mathbf{u}(P)$ in D admit the representation (1.2), (1.3), where $\varphi(P)$ and $\psi(P)$ are of Class 2 in D . Moreover, let $\mathbf{u}$ have the properties (3.2) where the z -axis coincides with the axis of revolution of B . Then $\mathbf{u}(P)$ also admits the representation

$$\mathbf{u} = \nabla(\varphi' + \mathbf{r} \cdot \psi') - 4(1 - \nu)\psi', \quad \nabla^2 \varphi' = 0, \quad \nabla^2 \psi' = 0, \quad (3.3)$$

in which

$$\varphi' = \varphi'(\rho, z), \quad \psi' = \mathbf{k}\psi'(\rho, z), \quad (3.4)$$

and $\mathbf{k}$ is a unit-vector parallel to the z -axis. The functions $\varphi'(\rho, z)$, $\psi'(\rho, z)$ are single-valued and analytic for (ρ, z) in any simply connected subregion of D_* , where D_* is the plane region which D has in common with the meridional half-plane $\rho > 0, \gamma = \text{constant}$.

Conditions necessary for the existence of the alternative representation (3.3), (3.4) are readily obtained. To this end, equate the right members in the first of (1.2), (3.3), operate on the resulting equation with the curl and the divergence, respectively, and take notice of the harmonicity of φ , ψ , φ' , and ψ' . Thus,

$$\nabla \times \psi' = \nabla \times \psi, \quad \nabla \cdot \psi' = \nabla \cdot \psi. \quad (3.5)$$

In view of (3.4), there results in cylindrical coordinates,²⁷

$$\left. \begin{aligned} \psi'_{,\rho} &= \psi_{z,\rho} - \psi_{\rho,z}, \\ \psi'_{,z} &= \frac{1}{\rho}(\rho\psi_{\rho})_{,\rho} + \frac{1}{\rho}\psi_{\gamma,\gamma} + \psi_{z,z}, \end{aligned} \right\} \quad (3.6)$$

if $(\psi_\rho, \psi_\gamma, \psi_z)$ are the cylindrical scalar components of ψ and subscripts preceded by a comma denote partial differentiation with respect to the argument indicated. Equations (1.2), (3.3), (3.4) at the same time imply directly,

²⁷See, for example, WEATHERBURN [15], p. 18, for the cylindrical transforms of the gradient, curl, divergence, and Laplacian operators.

$$\left. \begin{aligned} (\varphi' + z\psi')_{,\rho} &= \varphi_{,\rho} + (\rho\psi_{\rho})_{,\rho} + z\psi_{z,\rho} - \alpha\psi_{\rho}, \\ (\varphi' + z\psi')_{,z} &= \alpha\psi' + \varphi_{,z} + \rho\psi_{\rho,z} + (z\psi_z)_{,z} - \alpha\psi_z, \end{aligned} \right\} \quad (3.7)$$

in which $\alpha = 4(1 - \nu)$.

We now turn to the proof of the theorem. Guided by (3.6), (3.7), we *define* functions ψ' and φ' as follows:

$$\psi' = \int_{(\rho_0, z_0)}^{(\rho, z)} \left\{ [\psi_{z,\rho} - \psi_{\rho,z}] d\rho + \left[\frac{1}{\rho} (\rho\psi_{\rho})_{,\rho} + \frac{1}{\rho} \psi_{\gamma,\gamma} + \psi_{z,z} \right] dz \right\}, \quad (3.8)$$

$$\begin{aligned} \varphi' &= -z\psi' + \int_{(\rho_0, z_0)}^{(\rho, z)} \{ [\varphi_{,\rho} + (\rho\psi_{\rho})_{,\rho} + z\psi_{z,\rho} - \alpha\psi_{\rho}] d\rho \\ &\quad + [\alpha\psi' + \varphi_{,z} + \rho\psi_{\rho,z} + (z\psi_z)_{,z} - \alpha\psi_z] dz \}, \end{aligned} \quad (3.9)$$

where (ρ_0, z_0) are the coordinates of a fixed point in D_* . An elementary computation, based on (1.2), (1.3), and (3.2), confirms that the line integrals in (3.8), (3.9) are independent of the path in any simply connected subregion of D_* ; furthermore, the functions φ' and ψ' so defined are found to be independent of γ and harmonic in D_* .

Finally, we *define* a vector field $\mathbf{u}'$ through

$$\mathbf{u}' = \nabla(\varphi' + \mathbf{r} \cdot \boldsymbol{\psi}') - \alpha\boldsymbol{\psi}', \quad \boldsymbol{\psi}' = \mathbf{k}\psi' \quad (3.10)$$

and verify with the aid of (3.8), (3.9) that $\mathbf{u} \equiv \mathbf{u}'$ in D , where $\mathbf{u}$ is the displacement field given by (1.2), (1.3), and (3.2). The details of these trivial computations may again be omitted safely. This completes the proof. We emphasize that the present theorem does not involve any convexity assumptions regarding D .

Equations (3.3), (3.4) in cylindrical coordinates appear as

$$\left. \begin{aligned} u_{\rho} &= \varphi'_{,\rho} + z\psi'_{,\rho} - 4(1 - \nu)\psi', \quad u_{\gamma} = 0, \quad u_z = \varphi'_{,z} + z\psi'_{,z} - (3 - 4\nu)\psi'; \\ \nabla^2 \varphi' &= 0, \quad \nabla^2 \psi' = 0. \end{aligned} \right\} \quad (3.11)$$

This solution to the displacement equations of equilibrium in the case of rotational symmetry has been associated with the name of BOUSSINESQ.²⁸ As pointed out in the Introduction, the completeness of BOUSSINESQ's solution may also be inferred from an application of the Stokes-Helmholtz resolution to an axisymmetric displacement field satisfying (1.1), the remaining argument being analogous to that employed by MINDLIN [3] in the general three-dimensional case.²⁹

²⁸Cf. Footnote No. 4.

²⁹ See MARGUERRE [9]. It should be noted, however, that any restrictions placed upon the Stokes-Helmholtz scalar and vector potentials, by virtue of the assumed rotational symmetry of the displacement field, are in need of justification. Cf. Footnote No. 8.

In connection with various alternative forms of the general solution to the equilibrium equations in the presence of axisymmetry, we cite the valuable bibliography given by MARGUERRE [9]. To the references listed there we add a paper by MICHELL [16], who developed what is probably the earliest systematic approach to the rotationally symmetric problem in the classical theory of elasticity.

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