Electrohydrodynamics I.
The Equilibrium of a Charged Gas
in a Container

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Summary. The equilibrium of a uniformly charged gas in a perfectly conducting container is studied. Equilibrium occurs when the electric forces in the gas just balance the pressure forces. It is found that for any container and for each total mass of gas there is exactly one equilibrium distribution of that mass of gas in the container. In equilibrium the density and pressure attain their maxima at the container surface, and they are both constant on this surface. Furthermore, the density and pressure increase at each point as the total mass of gas increases. However, inside the container the density and pressure do not increase indefinitely as the total mass of gas does. Instead at each inner point the density and pressure both approach finite upper limits as the total mass of gas becomes infinite. Most of the gas accumulates in a thin layer near the surface when the total mass is large. The fact that the density and pressure cannot be made arbitrarily large at inner points by putting more gas into the container is considered to be the main physically interesting result.

1. **Introduction.** By 'electrohydrodynamics' we mean the science of the motions of fluids under the influence of electric fields. Such motions occur in vacuum tubes, where the fluid is an electron gas, in plasmas where the fluid is a neutral mixture of ions, in the ionosphere, *etc.* Whereas most previous studies of these motions have been based upon kinetic theory, the present investigation is completely macroscopic. The fluid is treated as a continuum, and this makes it possible to deal with more general fluids and to obtain more complete results.

In this article we consider the equilibrium of a mass M of uniformly charged gas (*e.g.*, an electron gas) within a container. If the gas had no pressure, we should conclude that all the charge would come to rest on the container wall, due to the mutual repulsion of the charges constituting the gas. However when

we take account of the pressure we find that the gas is distributed throughout the container, with its greatest density at the wall. Furthermore, the density has the same value at all points of the wall. As M increases, the density at every point within the container also increases.

Although the preceding results are physically obvious, our main result is not obvious. It is that *the density does not increase indefinitely with M at interior points*. Instead, at each inner point the density has an upper bound which depends upon the shape of the container and the location of the point. As the point approaches the boundary, the bound increases indefinitely. Thus as M increases, most of the gas accumulates in a thin layer near the surface. The fact that the density cannot be made arbitrarily large at points inside a container by putting more gas into the container may be of some practical importance.

We also find that there exists a solution in which M is infinite but the density at each interior point is finite. This generalizes previous results of L. BIEBERBACH [2] and H. RADEMACHER [3], concerning a charged ideal gas. Finally we present some additional results for an ideal gas and discuss the work of M. VON LAUE [1].

2. Formulation of the problem. We consider the equilibrium of a charged fluid within a container. Such a fluid achieves equilibrium when the pressure forces and electrostatic forces in it balance each other. In terms of the pressure p , the mass density ρ , the charge density $a\rho$, and the electric field vector E , this equilibrium condition is

$$(1) \quad \nabla p = a\rho E.$$

The constant a in (1) is the ratio of electric charge density to mass density in the fluid. It is constant because we assume that the fluid is uniformly charged. If each molecule of the fluid has mass m and charge e then $a = em^{-1}$, which is the case when the fluid is an electron gas.

The fact that the charge is a source of the field is expressed by the equation

$$(2) \quad \nabla \cdot E = 4\pi a\rho.$$

If the container surface S is a perfect conductor then

$$(3) \quad E_{\text{tangential}} = 0 \quad \text{on } S.$$

The fluid is characterized by an equation of state which expresses p as a function of ρ and the temperature. When the temperature is constant, as we assume it to be, the equation of state takes the form

$$(4) \quad p = p(\rho).$$

The function $p(\rho)$ is a non-negative increasing function defined for all non-negative ρ , and becomes infinite as ρ does. The total mass of fluid within the container D is given by

$$(5) \quad M = \int_D \rho \, dV.$$

Physically it seems that the preceding conditions are the only ones which must be satisfied by a charged gas in a perfectly conducting container. If this is so, then equations (1)–(5) should have one and only one solution p , ρ , and E for each value of the total mass M . In the next section we shall show that this is the case, and we shall also determine various properties of the solution.

3. Existence, uniqueness and monotonicity of the solution. By eliminating E from (1) and (2) we obtain

$$(6) \quad \nabla \cdot (\rho^{-1} \nabla p) = 4\pi a^2 \rho.$$

Since (4) expresses p as a function of ρ , (6) is an equation involving ρ alone. Alternatively, since $p(\rho)$ is monotonic, ρ may be expressed in terms of p by means of (4) and then (6) becomes an equation for p . It is more convenient, however, to introduce instead a new function v defined by

$$(7) \quad v = \int_{p_0}^p \frac{dp}{\rho(p)}.$$

Here p_0 is an arbitrary constant greater than or equal to $p(0)$. In terms of v , (6) becomes

$$(8) \quad \nabla^2 v = f(v).$$

The function $f(v)$ is defined in terms of the inverse functions $\rho(p)$ and $p(v)$ by

$$(9) \quad f(v) = 4\pi a^2 \rho[p(v)].$$

The function $f(v)$ is a non-negative increasing function of v .

From (3) and (1) we see that ∇p is normal to the container surface S , which is therefore a level surface of p . Then from (4) and (7) we conclude that p , ρ and v are constant on S if S is connected, or on each connected component of S if S is not connected. Assuming for simplicity that S is connected we find that v is a solution of (8) which is constant on S . Now let α denote the value of v on S . From the existence, uniqueness, and monotonicity theorems for solutions of (8) when f is increasing we have the following result: There is one and only one solution v of (8) having the value α on the surface S of the container. This solution increases at every point as α increases. Furthermore, from the fact that f is non-negative it follows that v is a subharmonic function. Therefore $v \leq \alpha$ throughout the container, by the maximum principle for subharmonic functions. Corresponding results also apply to p and ρ since they are increasing functions of v .

In the Appendix it is shown that α can be uniquely chosen so that (5) is satisfied for any given non-negative number M and that α is an increasing function of M . It then follows that equations (1)–(5) have one and only one solution for any $M \geq 0$. In this solution $\rho > 0$ and $p > p(0)$ if $M > 0$. Both ρ and p are constant on S , attain their maximum values on S and are increasing

functions of M at each point of D . Corresponding results can be obtained for two-dimensional containers by slightly modifying the foregoing proofs. For a one-dimensional container (*i.e.*, the region between two parallel planes) the boundary is not connected, but the same results are obtained if it is assumed that ρ has the same value on the two planes.

4. Bounds on the solution. We have just seen that ρ and p increase with M . We shall now show that if

$$(10) \quad \int_{p(1)}^{\infty} \frac{dp}{\rho(p)(p - p_0)^{\frac{1}{2}}} < \infty$$

then there exists a function $g(R)$ determined solely by $\rho(p)$ such that

$$(11) \quad v(P) \leq g[R(P)].$$

The function $g(R)$ is decreasing and $g(R) \rightarrow \infty$ as $R \rightarrow 0$. In (11) P denotes a point in D and $R(P)$ denotes the distance from P to S . Equation (11) shows that $v(P)$, and therefore also ρ and p , are bounded above independently of M for any gas satisfying (10). Thus v , ρ and p tend to finite upper limits as M becomes infinite, for all points P in D . We note that (10) depends only upon the behavior of $\rho(p)$ as $p \rightarrow \infty$ and it is satisfied if ρ increases faster than $p^{\frac{1}{2}}$. For a polytropic gas with adiabatic exponent γ , ρ is proportional to $p^{1/\gamma}$ so (10) holds for $\gamma < 2$.

Before proving (10) we observe that the upper bound on $\rho(p)$ (and on v and p) tends to infinity as P tends to S . Thus as M increases ρ tends to a finite upper limit at each inner point of D and ρ becomes infinite on S . Therefore, most of the gas accumulates in a thin layer near S as M increases. There is also a solution of (1) to (5) with M infinite, provided (10) holds. In this solution v , ρ and p are finite at each inner point of D but infinite on S . This follows from a result in [4].

Equation (11) also follows from a theorem in [4] on solutions of (8) when $f(v)$ is positive and increasing, and for which

$$(12) \quad \int_1^{\infty} \left[\int_0^x f(z) dz \right]^{-\frac{1}{2}} dx < \infty.$$

Now $f(v) = \rho$ is positive when $M > 0$, and f is increasing. Therefore the theorem applies when (12) holds, and (12) becomes (10) if the function $\rho(p)$ inverse to (4) is introduced. A method of determining the bound $g(R)$ is given in [4] and will not be described here.

Other bounds, both upper and lower, can be obtained from the integral equation (A4) which is satisfied by v . These bounds will involve α , the value of v on the boundary. First we introduce the notation $F[v]$ for the right side of (A4):

$$(13) \quad F[v] \equiv \alpha - \int_D f[v(r')] G(r, r') dr'.$$

Since $f(v)$ is an increasing function, we see at once that $F[v]$ is a decreasing functional, *i.e.*, if $v_1 > v_2$ then $F[v_1] < F[v_2]$. Now (A4) may be written as $v = F[v]$, and $v \leq \alpha$ since $f \geq 0$. We now define the sequence of functions v_n by

$$(14) \quad v_0 = \alpha, \quad v_n = F[v_{n-1}], \quad n = 1, 2, \dots$$

From the decreasing property of F and the fact that $v \leq v_0$ we find $v_1 \leq v$. By induction we conclude that the v_{2n} form a decreasing sequence of upper bounds on v and the v_{2n+1} form an increasing sequence of lower bounds. These bounds may be shown to converge to v , and in fact the existence of v can be proved in this way.

5. Example: The ideal gas. To exemplify the preceding considerations, let us consider an ideal gas, for which the equation of state (4) is

$$(15) \quad p = \frac{RT}{m} \rho.$$

In (15) T is the constant temperature, R is the gas constant, and m is the average mass of the molecules in the gas. Then v becomes, if $p_0 = 1$,

$$(16) \quad v = \frac{RT}{m} \log p.$$

From (9), (15) and (16), we see that $f(v)$ is now

$$(17) \quad f(v) = \frac{4\pi a^2 m}{RT} \exp \left[\frac{m}{RT} v \right].$$

Then (8) becomes

$$(18) \quad \nabla^2 v = \frac{4\pi a^2 m}{RT} \exp \left[\frac{m}{RT} v \right].$$

If we define a new variable u by

$$(19) \quad u = \frac{m}{RT} v + \log 4\pi \left[\frac{am}{RT} \right]^2$$

then (18) becomes

$$(20) \quad \nabla^2 u = e^u.$$

If the container is a sphere, a cylinder, or a pair of parallel planes, we may assume that the solution u of (20) is a function of one variable only. This variable, which we will denote by r , is the distance from the center of the sphere, from the axis of the cylinder, or from the median plane in the three-, two-, and one-dimensional cases respectively. If $u = u(r)$ and n denotes the dimension, equa-

tion (20) becomes

$$(21) \quad u_{rr} + \frac{n-1}{r} u_r = e^u.$$

The regularity of u at the center of the sphere or axis of the cylinder requires that

$$(22) \quad u_r(0) = 0.$$

If we also assume (22) for $n = 1$, the solution in that case will be symmetric in the median plane.

Although we have proved the existence of a solution of the equilibrium problem for each of the three containers under consideration here, we have not shown that the solutions for the sphere and cylinder are functions of r only. However, this can be shown by directly proving the existence of such solutions, making use of the ordinary differential equation (21) or the corresponding equation for any other gas. Then by the uniqueness theorem, the symmetric solution is the only solution in each of these cases.

In the one-dimensional case (21) can be solved explicitly and we find

$$(23) \quad \rho(r) = \rho_0 \sec^2 \left[r \sqrt{\frac{2\pi a^2 m \rho_0}{RT}} \right].$$

Here ρ_0 is the value of ρ at $r = 0$. The total mass per unit area between the median plane and the plane $r = \text{constant}$ is

$$(24) \quad M(r) = \int_0^r \rho(x) dx = \sqrt{\frac{\rho_0 RT}{2\pi a^2 m}} \tan \left[r \sqrt{\frac{2\pi a^2 m \rho_0}{RT}} \right].$$

If the distance between the planes is $2L$, and if $M(L)$ is given, then ρ_0 is determined by setting $r = L$ in (24) and solving for ρ_0 . We see from (23) and (24) that ρ and M become infinite at

$$r = \sqrt{\frac{RT}{2\pi a^2 m \rho_0}} \cdot \frac{\pi}{2}.$$

But any finite M will be attained for some value of r less than this. Thus L must be less than this value, and we conclude that no matter how large $M(L)$ is, we have

$$(25) \quad \rho_0 \leq \frac{\pi RT}{8a^2 m L^2}.$$

This bound on ρ_0 is a special instance of the type of bound on ρ found in Section 4.

An explicit solution can also be obtained in the two-dimensional case, by following the method of [1], but no explicit solution has been found for the spherical case. However, in this case, as well as in the others, upper and lower bounds on the solution can be obtained from equation (14). To construct these bounds we first introduce the Green's function for equation (21) which satisfies

(22) and which also vanishes on the container $r = L$. The Green's functions for $n = 1, 2, 3$ are

$$(26) \quad G(r, r') = \begin{cases} L - r' & r' \log L/r' & (r')^2 \left(\frac{1}{r'} - \frac{1}{L} \right), & r \leq r' \\ L - r & r' \log L/r & (r')^2 \left(\frac{1}{r} - \frac{1}{L} \right), & r' \leq r. \end{cases}$$

From (14), with $v_0 = \alpha$, we find for $n = 3$

$$(27) \quad v_1 = \alpha - \frac{4\pi a^2 m}{RT} \exp \left[\frac{m}{RT} \alpha \right] r^2 \log L/r \leq v \leq \alpha = v_0.$$

The analogues of (27) for $n = 1$ and $n = 2$ are obtained by replacing $r^2 \log L/r$ by $\frac{1}{2}(L^2 - r^2)$ and $rL - r^2$ respectively. This yields the bounds on ρ for $n = 3$

$$(28) \quad \rho_1 = \bar{\rho} \left(\frac{r}{L} \right)^{4\pi a^2 \bar{\rho} r^2} \leq \rho \leq \bar{\rho}.$$

Corresponding bounds can be found for $n = 1$ and $n = 2$, although exact solutions are available in these cases. Here $\bar{\rho} = m/RT \exp [m/RT \alpha]$ is the value of ρ on the boundary $r = L$ where $v = \alpha$. Improved bounds can be obtained by continued use of (14).

Equation (20) was studied by M. VON LAUE [1] in connection with the equilibrium of an electron gas. He deduced (20) by using the statistical mechanical result that in equilibrium the density at any point is proportional to an exponential function in which the exponent is the negative of the potential energy function at the point divided by KT . The potential was then assumed to be just the electrostatic potential, which limited the considerations to an ideal gas.

LAUE considered the problem of solving (20) for the potential u in a half-space, where u is a given constant on the boundary plane and a linear function of distance from the plane at infinity. He also treated the corresponding problem for the exterior of a circle, with appropriate conditions on u at infinity. In both these problems (20) becomes an ordinary differential equation which can be solved explicitly. In addition, he observed that by means of conformal mapping, the corresponding problem for the exterior of any cylinder could be obtained from that for a circular cylinder if $\rho = \infty$ (*i.e.*, $u = \infty$) on the cylinder. From our results we see that this solution provides an upper bound for any solution with ρ finite on the cylinder.

L. BIEBERBACH [2] proved that in the two-dimensional case within a smooth closed curve, (20) has a unique solution which becomes infinite to the same order as s^{-2} as the boundary curve is approached, where s denotes the distance of a point from the boundary curve. H. RADEMACHER [3] extended BIEBERBACH'S

result to three dimensions. Both of these results—excepting the uniqueness—are contained in the preceding section, which applies to any gas for which (10) is satisfied, and not merely to an ideal gas.

Appendix

It remains to be shown that α can be uniquely chosen so that (5) is satisfied for any given non-negative number M . To this end we note that the right side of (5) is an increasing function of α , so that (5) has at most one solution α for each M . Thus we need only show that the right side of (5) is zero for some choice of α , and that it becomes arbitrarily large as α increases. Then it will follow that there is one and only one α for each M and thus one and only one solution to equations (1)–(5). From the preceding remarks it is already clear that α is an increasing function of M , and thus v , p and ρ , which are increasing functions of α , will also be increasing functions of M .

To prove that the right side of (5) is zero for some choice of α , we first let v_0 be the value of v corresponding to $\rho = 0$. This value of v is determined by setting $\rho = 0$ in (4), which yields $p(0)$, and then inserting this value of p into (7). Then we have $f(v_0) = 0$, as we see from (9). Thus $v = v_0$ is a solution of (8) with the constant boundary value $\alpha = v_0$, and for this solution $\rho = 0$. But the right side of (5) is zero if $\rho = 0$. Thus we have shown that for $\alpha = v_0$ the right side of (5) is zero.

The preceding argument fails when $v_0 = -\infty$, *i.e.*, when the integral in (7) diverges for $p = p(0)$. In this case we may solve (1)–(5) directly without introducing v , and the solution is $\rho = E = 0$, $p = p(0)$. For this solution the right side of (5) is zero, as was to be shown. Furthermore, as α tends to $-\infty$, the right side of (5) tends to zero since ρ tends to zero uniformly (this is true because the maximum of ρ occurs on S and tends to zero). Therefore we have shown that in this case, too, the right side of (5) tends to zero as α tends to $-\infty$, and that it can be made equal to zero by choosing an appropriate solution which, we may say, corresponds to $\alpha = -\infty$.

The above results, together with the fact that v is an increasing function of α , show that for $\alpha > v_0$ we have $v > v_0$. From this it follows that $\rho > 0$ and $p > p(0)$ when $\alpha > v_0$. Thus, as was to be expected, the density is non-negative.

It now remains to be shown that the right side of (5) increases indefinitely with increasing α . In order to show this we convert (8) into an integral equation by using the Green's function $G(r, r')$ for Laplace's equation. This function is defined by

$$(A1) \quad \nabla^2 G(r, r') = 0, \quad r' \neq r,$$

$$(A2) \quad G(r, r') = 0, \quad r' \text{ on } S,$$

$$(A3) \quad G(r, r') = \frac{1}{|r - r'|} + R(r, r'), \quad |R| \leq B.$$

In (A1)–(A3) r and r' denote two points in the container and R denotes a bounded function, whose bound is B . It is known that such a function G exists and is unique, that $G(r, r') > 0$ if neither r nor r' is on S , and that $G(r, r') = G(r', r)$. By applying Green's theorem to v and G and using (8) we obtain

$$(A4) \quad v(r) = \alpha - \int_D f[v(r')]G(r, r') dr'.$$

From the definition of $f(v)$ this can be rewritten as

$$(A5) \quad v(r) = \alpha - 4\pi a^2 \int_D \rho(r')G(r, r') dr'.$$

Now, using (A5), we shall show that the right side of (5) becomes arbitrarily large as α increases. Suppose, on the contrary, that there is a constant C which bounds the right side of (5), *i.e.*,

$$(A6) \quad \int_D \rho(r') dr' \leq C \quad \text{for all } \alpha.$$

If $v(r) \rightarrow \infty$ as $\alpha \rightarrow \infty$, then $\rho(r) \rightarrow \infty$, since ρ is an increasing function of v which becomes infinite as v does. By (A6), ρ is bounded for almost all r , so v must be similarly bounded. But from (A5) this implies that

$$(A7) \quad \int_D \rho(r')G(r, r') dr' \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

Making use of (A3) we have

$$(A8) \quad \int_D \rho(r')G(r, r') dr' \leq \int_D \frac{\rho(r')}{|r - r'|} dr' + BC.$$

From (A7) and (A8) we conclude

$$(A9) \quad \int_D \frac{\rho(r')}{|r - r'|} dr' \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

To prove that (A9) contradicts (A6) we introduce a sphere σ_ϵ of radius $\epsilon > 0$ and center r , and denote by $D = \sigma_\epsilon$ the part of D which is outside σ_ϵ and by $D \cap \sigma_\epsilon$ the part of D inside σ_ϵ . Then we have, recalling that $\rho \geq 0$ and using (A6),

$$(A10) \quad \begin{aligned} \int_D \frac{\rho(r')}{|r - r'|} dr' &= \int_{D - \sigma_\epsilon} \frac{\rho(r')}{|r - r'|} dr' + \int_{D \cap \sigma_\epsilon} \frac{\rho(r')}{|r - r'|} dr' \\ &\leq \frac{C}{\epsilon} + \int_{D \cap \sigma_\epsilon} \rho(r') |r - r'| d|r - r'| d\omega'. \end{aligned}$$

In the last integral $d\omega'$ indicates the angular part of the integration. Now from (A10) and (A9) we conclude that for almost all r in D , for all $\epsilon > 0$,

$$(A11) \quad \int_{D \cap \sigma_\epsilon} \rho(r') |r - r'| d|r - r'| d\omega' \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

Thus for almost all r in D , $\rho(r) \rightarrow \infty$ as $\alpha \rightarrow \infty$. But then

$$(A12) \quad \int_D \rho(r') dr' \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

Since (A12) contradicts (A6) we conclude that (A6) is false. But the alternative is (A12), which must therefore apply. Thus we have proved that (5) has one and only one solution α for any $M \geq 0$ since we have shown that the right side of (5) is zero for some value of α and that it increases monotonically to ∞ as α does.

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