Unsteady Simple Waves in Gases

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Introduction.

Flows for which the velocity vector $\mathbf{q}$ reduces to a function of just one independent variable are commonly referred to as "simple waves". In this paper simple waves in unsteady potential flow of a compressible fluid are considered. If $\lambda(x, y)$ is used to denote the fundamental independent variable then simple waves in steady flow have the analytic expression $\mathbf{q}(\lambda)$. The theory of simple waves in steady motion of an inviscid compressible fluid has been studied by several authors [2, 3, 4, 6] and the effect of vorticity in plane flow has also received attention. Analytically the theory of such flows is straightforward, the final equations determining the flow field being ordinary differential equations in terms of the parameter λ as fundamental independent variable. In all cases the curves $\lambda(x, y) = \text{constant}$, reduce to a system of straight lines. When steady rotational flow is considered these lines must be either parallel or concurrent, furthermore the form of the vorticity distribution is predetermined.

The purpose of this paper is to show how to obtain exact solutions of the quasi-linear partial differential equation for the velocity potential $\phi(x, y, t)$ in plane flow by focussing attention on flows of the type $\mathbf{q}(\lambda, t)$ where $\lambda = \lambda(x, y, t)$. In this way unsteady potential flows are obtained which may be referred to as "generalised simple waves" in unsteady flow. The curves $\lambda(x, y, t) = \text{constant}$ are again shown to reduce to a system of straight lines, but these lines are generated by an envelope which may not in general degenerate into a point. In this sense there exists no "simple centered wave" in unsteady potential flow of a compressible inviscid fluid. Two solutions which possess time varying envelopes are considered. In both cases the velocity is not uniquely defined and it is possible to obtain either of two possible flow fields.

1. General Method.

In effect the method used is a variation of the hodograph method in which the potential and fluid speed are taken as the independent variables. By applying the Legendre contact transformation it is then possible to separate these variables

for flows of the type considered. The application of these transformations yields three mixed equations of the Monge-Ampère type for the velocity vector and the Legendre transform of the potential. These equations are expressed in terms of (λ, t) as the independent variables whereas the original equation for the potential involved the three variables (x, y, t) . This simplification is due to the result that the Legendre transform of the potential becomes a function of (λ, t) only whereas the potential itself does not. Development of the general theory as a consequence of the fundamental assumption on the existence of solutions of the type $\mathbf{q}(\lambda, t)$ is then possible.

Two of the Monge-Ampère equations are parabolic, λ being the characteristic variable for one of these equations. The function $\lambda(x, y, t)$ is originally left unspecified and by suitable choice of λ it is possible to express the general solution of the system in terms of four functions of λ which satisfy a system of ordinary differential equations. This representation is sufficient to show that the λ -lines may never be concurrent in the physical plane if the flow is to be essentially unsteady and non-trivial.

2. General Equations of Unsteady Flow.

The equations of flow of an inviscid gas are

$$(1) \quad \rho_t + \operatorname{div}(\rho \mathbf{q}) = 0$$

$$(2) \quad \mathbf{q}_t + (\mathbf{q} \cdot \operatorname{grad}) \mathbf{q} + \rho^{-1} \operatorname{grad} p = 0$$

p denoting the pressure, ρ the density and $\mathbf{q}$ the vector velocity. In addition isentropic flow is assumed so that

$$p \rho^{-\gamma} = \text{constant},$$

throughout the field of flow. In terms of the acoustic speed $a = (\gamma p / \rho)^{\frac{1}{2}}$ the isentropic condition may be written

$$\rho = k a^{2/(\gamma-1)}$$

where k is a constant. (1) may now be written,

$$(3) \quad \frac{\partial}{\partial t} (a^{2/(\gamma-1)}) + \operatorname{div} (\mathbf{q} a^{2/(\gamma-1)}) = 0$$

Now

$$\rho^{-1} \operatorname{grad} p = \frac{2}{\gamma - 1} a \operatorname{grad} a$$

so that (2) may be integrated in the form

$$(4) \quad \frac{q^2}{2} + \frac{a^2}{\gamma - 1} + \phi_t = \frac{c^2}{2}$$

where ϕ is a potential so that $\mathbf{q} = \text{grad } \phi$, and c is a constant. It is possible to obtain from (4) the derivatives of a^2 and substitute in the expanded form of (3) to give a general equation for the velocity

$$(5) \quad a^2 \text{div } \mathbf{q} = q\mathbf{q} \cdot \text{grad } q + 2qq_t + \phi_{tt}$$

where a^2 is given in terms of q and ϕ by equation (4).

3. Simple Waves.

To investigate flows of the simple wave pattern for which the velocity vector assumes the form $\mathbf{q}(\lambda, t)$, where $\lambda = \lambda(\mathbf{r}, t)$, follow references [1, 2] and use the Legendre contact transformation to define the function Φ

$$(6) \quad \Phi + \mathbf{r} \cdot \mathbf{q} = \phi$$

where

$$\mathbf{q} = \text{grad } \phi$$

It follows that

$$d\Phi + \mathbf{r} \cdot \mathbf{q}_\lambda d\lambda = 0$$

so that Φ is a function of (λ, t) only and

$$(7) \quad \Phi_\lambda + \mathbf{r} \cdot \mathbf{q}_\lambda = 0$$

By successive differentiation of (6), (7) there results the following system of equations:

$$(8) \quad \phi_t = \Phi_t + \mathbf{r} \cdot \mathbf{q}_t$$

$$(9) \quad \phi_{tt} = \Phi_{tt} + \mathbf{r} \cdot \mathbf{q}_{tt} + (\Phi_{\lambda t} + \mathbf{r} \cdot \mathbf{q}_{\lambda t})\lambda_t$$

$$(10) \quad (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda}) \text{grad } \lambda + \mathbf{q}_\lambda = 0$$

$$(\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda})\lambda_t + (\Phi_{\lambda t} + \mathbf{r} \cdot \mathbf{q}_{\lambda t}) = 0$$

The relation (5) is written

$$(11) \quad (a^2 \mathbf{q}_\lambda - q\mathbf{q}_\lambda) \cdot \text{grad } \lambda = 2q(q_\lambda \lambda_t + q_t) + \phi_{tt}$$

The relation (4) becomes using (8)

$$(12) \quad \frac{q^2}{2} + \frac{a^2}{\gamma - 1} + \Phi_t + \mathbf{r} \cdot \mathbf{q}_t = \frac{c^2}{2}$$

which must now be used to give a^2 . When ϕ_{tt} , $\text{grad } \lambda$ and λ_t are substituted, equation (11) becomes,

$$(13) \quad a^2 q_\lambda^2 = (\Phi_{\lambda t} + q\mathbf{q}_\lambda + \mathbf{r} \cdot \mathbf{q}_{\lambda t})^2 - (\Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda})(\Phi_{tt} + 2qq_t + \mathbf{r} \cdot \mathbf{q}_{tt})$$

In these expressions the subscript t , when used with a function of the type $f(\lambda, t)$, denotes partial differentiation with respect to t with λ constant and not total differentiation with respect to time for which $\mathbf{r}$ is constant. In the method used the current independent variables (x, y, t) are replaced by (λ, ϕ, t) whilst the dependent function ϕ is itself replaced by its Legendre transform, but is retained as a fundamental independent variable where necessary. This procedure replaces the original partial differential equation for the velocity regarded as a function of three variables (x, y, t) by three partial differential equations for q, θ, Φ regarded as functions of the two variables (λ, t) . This reduction is achieved because the Legendre transform of the potential reduces to a function of (λ, t) only and is independent of ϕ . Although the velocity is a function of two independent variables, one of these is the time and the hodograph is instantaneously a curve; therefore the title "Simple Wave" is retained for the flows here discussed.

Equations (8), (9), (12) show that ϕ_t, ϕ_{tt} and a^2 must continue to be regarded as general functions of position and it is necessary to use their equivalent representations as functions of (λ, ϕ, t) .

The following expressions are needed:

$$\begin{aligned}
 (14) \quad & \mathbf{q}_{\lambda\lambda} + A_1 \mathbf{q} + B_1 \mathbf{q}_\lambda = 0 \\
 & \mathbf{q}_{\lambda t} + A_2 \mathbf{q} + B_2 \mathbf{q}_\lambda = 0 \\
 & \mathbf{q}_{tt} + A_3 \mathbf{q} + B_3 \mathbf{q}_\lambda = 0 \\
 & \mathbf{q}_t + A_4 \mathbf{q} + B_4 \mathbf{q}_\lambda = 0
 \end{aligned}$$

where the coefficients A_i, B_i are functions of (λ, t) only and are given by relations of the type

$$A_1 \mathbf{q} \times \mathbf{q}_\lambda = \mathbf{q}_\lambda \times \mathbf{q}_{\lambda\lambda}, \quad B_1 \mathbf{q} \times \mathbf{q}_\lambda = -\mathbf{q} \times \mathbf{q}_{\lambda\lambda}$$

therefore, explicitly,

$$\begin{aligned}
 (15) \quad & A_1 = \frac{1}{q^2 \theta_\lambda} \mathbf{q}_\lambda^2 \frac{\partial}{\partial \lambda} \left\{ \theta + \tan^{-1} \frac{q \theta_\lambda}{q_\lambda} \right\} & B_1 &= -\frac{\partial}{\partial \lambda} \log (q^2 \theta_\lambda) \\
 & A_2 = \frac{1}{q^2 \theta_\lambda} \mathbf{q}_\lambda^2 \frac{\partial}{\partial t} \left\{ \theta + \tan^{-1} \frac{q \theta_\lambda}{q_\lambda} \right\} & B_2 &= -\frac{1}{q \theta_\lambda} [q_\lambda \theta_t + (q \theta_\lambda)_t] \\
 & A_3 = \frac{q_\lambda}{q^3 \theta_\lambda} \frac{\partial}{\partial t} (q^2 \theta_t) + \theta_t^2 - \frac{q_{tt}}{q} & B_3 &= -\frac{1}{q^2 \theta_\lambda} \frac{\partial}{\partial t} (q^2 \theta_t) \\
 & A_4 = \frac{q_\lambda \theta_t - q_t \theta_\lambda}{q \theta_\lambda} & B_4 &= -\frac{\theta_t}{\theta_\lambda}
 \end{aligned}$$

Scalar multiplication of equations (14) by $\mathbf{r}$ and then using (6), (7) to eliminate the products $\mathbf{r} \cdot \mathbf{q}, \mathbf{r} \cdot \mathbf{q}_\lambda$ gives

$$\begin{aligned}
 \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda} &= (\Phi - \phi)A_1 + \Phi_\lambda B_1 \\
 \mathbf{r} \cdot \mathbf{q}_{\lambda t} &= (\Phi - \phi)A_2 + \Phi_\lambda B_2 \\
 \mathbf{r} \cdot \mathbf{q}_{t t} &= (\Phi - \phi)A_3 + \Phi_\lambda B_3 \\
 \mathbf{r} \cdot \mathbf{q}_t &= (\Phi - \phi)A_4 + \Phi_\lambda B_4
 \end{aligned}
 \tag{16}$$

and direct substitution into (13) gives a relation of the type

$$X(\lambda, t) + (\Phi - \phi)Y(\lambda, t) + (\Phi - \phi)^2 Z(\lambda, t) = 0$$

in which the functions X , Y , Z are obtained as functions of $\Phi(\lambda, t)$, $\mathbf{q}(\lambda, t)$ and their derivatives only and are therefore independent of ϕ . Since ϕ and λ are independent variables it follows that

$$X = Y = Z = 0$$

Explicitly,

$$Z(\lambda, t) = A_2^2 - A_1 A_3 \tag{17}$$

$$\begin{aligned}
 X(\lambda, t) &= (\Phi_{\lambda t} + B_2 \Phi_\lambda + q q_\lambda)^2 - (\Phi_{\lambda\lambda} + B_1 \Phi_\lambda)(\Phi_{t t} + B_3 \Phi_\lambda + 2q q_t) \\
 &\quad - (\gamma - 1)[\tfrac{1}{2}(c^2 - q^2) - \Phi_t - B_4 \Phi_\lambda] \mathbf{q}_\lambda^2
 \end{aligned}
 \tag{18}$$

$$\begin{aligned}
 Y(\lambda, t) &= 2A_2(\Phi_{\lambda t} + B_2 \Phi_\lambda + q q_\lambda) - A_1(\Phi_{t t} + B_3 \Phi_\lambda + 2q q_t) \\
 &\quad - A_3(\Phi_{\lambda\lambda} + B_1 \Phi_\lambda) + (\gamma - 1)A_4 \mathbf{q}_\lambda^2
 \end{aligned}
 \tag{19}$$

4. Equation for the Velocity.

Since only the equations (18), (19) of the simple wave equations involve Φ , it is possible to consider $\theta(\lambda, t)$ by concentrating on equation (17). For this purpose let the parameter $\lambda(x, y, t)$ equal the fluid speed $q(x, y, t)$. The original relation $q = q(\lambda, t)$ now reduces to simply $q = \lambda$ so that $q_t = 0$. The equations (15) give

$$\begin{aligned}
 q^2 \theta_q A_1 &= \mathbf{q}_q^2 \frac{\partial}{\partial q} (\theta + \tan^{-1} q \theta_q) & B_1 &= q(\theta_q^2 - A_1) \\
 q^2 \theta_q A_2 &= \mathbf{q}_q^2 \frac{\partial}{\partial t} (\theta + \tan^{-1} q \theta_q) & B_2 &= q(\theta_q \theta_t - A_2) \\
 q \theta_q A_3 &= \theta_{t t} + q \theta_q \theta_t^2 & B_3 &= q(\theta_t^2 - A_3) \\
 q \theta_q A_4 &= \theta_t & B_4 &= -q A_4
 \end{aligned}
 \tag{20}$$

If the variable $z = \log q$ is used and the notation

$$u = \theta_z, \quad v = \theta_t$$

is adopted then

$$uA_1e^{2z} = u(1 + u^2) + \theta_{zz}$$

$$uA_2e^z = v(1 + u^2) + \theta_{zt}$$

$$uA_3 = w^2 + \theta_{tt}$$

The equation (17) for the velocity becomes

$$(21) \quad \theta_{zz}\theta_{tt} - \theta_{zt}^2 + w^2\theta_{zz} + u(1 + u^2)\theta_{tt} - 2v(1 + u^2)\theta_{zt} - v^2(1 + u^2) = 0$$

This is an equation of the Monge-Ampère type in which the coefficients are functions only of the derivatives $\theta_z = u$, $\theta_t = v$. The characteristics are given by

$$du + u(1 + u^2) dz + m dt = 0$$

$$dv + m dz + w^2 dt = 0$$

m being given by the quadratic

$$m^2 - 2mv(1 + u^2) + v^2(1 + u^2)^2 = 0$$

Since $m = v(1 + u^2)$, (21) is parabolic, the coincident characteristics being given by

$$du + u(1 + u^2) dz + v(1 + u^2) dt = 0$$

$$dv + v(1 + u^2) dz + w^2 dt = 0$$

If a, b are arbitrary parameters of integration, the above system possesses the two integrals

$$t + \frac{u}{v} = a, \quad \frac{v^2 e^{2z}}{1 + u^2} = b$$

so that for the intermediate integral,

$$v^2 e^{2z} = (1 + u^2)f\left(t + \frac{u}{v}\right)$$

where f is arbitrary. The integral of this last relation may be written

$$\theta - \lambda = \cos^{-1} [(a - t)b^{\frac{1}{2}}e^{-z}]$$

where λ is a constant of integration. In terms of the fluid speed q ,

$$q \cos (\theta - \lambda) = (a - t)b^{\frac{1}{2}}$$

To obtain a general integral a, b are regarded as functions of λ to be eliminated between

$$(22) \quad q \cos (\theta - \lambda) = \beta$$

$$(23) \quad q \sin (\theta - \lambda) = \beta_\lambda$$

where

$$\beta(\lambda, t) = [h(\lambda) - t]g(\lambda)$$

and $a = h(\lambda)$, $b^{\frac{1}{2}} = g(\lambda)$ are arbitrary. The parameter λ here invoked may be identified in the form

$$(24) \quad \lambda = \theta - \cot^{-1}(q\theta_{\lambda}/q_{\lambda})$$

To show this take the λ -derivative of (22) and use (23); then

$$q_{\lambda} \cos(\theta - \lambda) = q\theta_{\lambda} \sin(\theta - \lambda)$$

from which (24) follows at once. The λ -lines (7) intersect the streamlines at the angle $\Gamma(\lambda, t)$ given by

$$\cot \Gamma = q\theta_{\lambda}/q_{\lambda}$$

It follows that the λ -lines intersect the streamlines at the angle $(\theta - \lambda)$, the inclination of each λ -line to the direction $\theta = 0$ being λ . It follows also from (22), (23) that $\lambda(q, t)$ may be regarded as being given by

$$(25) \quad q^2 = \beta^2 + \beta_{\lambda}^2$$

5. Characteristic Variable.

The linear equation (19) for $\Phi(q, t)$ becomes, with $\lambda = q$,

$$(26) \quad A_3 \Phi_{qq} - 2A_2 \Phi_{qt} + A_1 \Phi_{tt} + (A_3 B_1 + A_1 B_3 - 2A_2 B_2) \Phi_q = 2qA_2 + (\gamma - 1)A_4 q^2$$

Since $A_1 A_3 = A_2^2$ equation (26) is parabolic, the characteristics being given by

$$A_3 dt^2 + 2A_2 dq dt + A_1 dq^2 = 0$$

that is

$$A_1 dq + A_2 dt = 0$$

But from (20), (24),

$$q^2 \theta_q A_1 = q^2 \lambda_q, \quad q^2 \theta_q A_2 = q^2 \lambda_t$$

therefore

$$\lambda_q dq + \lambda_t dt = 0$$

It follows that the characteristic curves of (26) are simply the curves $\lambda(q, t) = \text{constant}$. If therefore the parameter $\lambda(\mathbf{r}, t)$ of §3 is chosen to coincide with the variable λ defined by (24) then the equation (26) will be reduced to the standard form for parabolic equations in which (λ, t) are the fundamental variables. It is convenient to introduce the function β of §4 in terms of which q, θ are given by equations (22), (23). From these the derivatives of q, θ may be calculated

directly giving

$$\begin{aligned} q^2\theta_\lambda &= \beta(\beta + \beta_{\lambda\lambda}) & q^2\theta_t &= g^2h_\lambda \\ qq_\lambda &= \beta_\lambda(\beta + \beta_{\lambda\lambda}) & qq_t &= -(g\beta + g_\lambda\beta_\lambda) \end{aligned}$$

The equations (15) now give for the coefficients A_i , B_i

$$\begin{aligned} A_1 &= \frac{1}{\beta}(\beta + \beta_{\lambda\lambda}) & B_1 &= -\frac{\partial}{\partial\lambda} \log \beta(\beta + \beta_{\lambda\lambda}) \\ B_2 &= -\frac{\partial}{\partial t} \log(\beta + \beta_{\lambda\lambda}) & B_4 &= \frac{-gh_\lambda}{(h-t)(\beta + \beta_{\lambda\lambda})} \\ A_4 &= \frac{1}{h-t} & A_2 &= A_3 = B_3 = 0 \end{aligned}$$

Equation (19) now reduces to the required canonical form

$$(27) \quad \Phi_{tt} + 2qq_t = (\gamma - 1)g(\beta + \beta_{\lambda\lambda})$$

With the aid of (25) it is possible to integrate (27) directly to give

$$(28) \quad \Phi(\lambda, t) = k(\lambda) + f(\lambda)(h-t) + E(\lambda)(h-t)^2 + G(\lambda)(h-t)^3$$

where $f(\lambda)$, $k(\lambda)$ are functions of integration and

$$\begin{aligned} (29) \quad 2E(\lambda) &= g[2\gamma g_\lambda h_\lambda + (\gamma - 1)gh_{\lambda\lambda}] \\ 6G(\lambda) &= (\gamma + 1)g^2 + 2g_\lambda^2 + (\gamma - 1)gg_{\lambda\lambda} \end{aligned}$$

The four functions $f(\lambda)$, $g(\lambda)$, $h(\lambda)$, $k(\lambda)$ are not independent, they must be chosen so that together they yield a solution $\Phi(\lambda, t)$ satisfying the additional Monge-Ampère relation $X = 0$ (equation (18)). With the present choice of λ this equation becomes

$$\begin{aligned} (\Phi_{\lambda t} + B_2\Phi_\lambda + qq_\lambda)^2 &= (\gamma - 1)q(\beta + \beta_{\lambda\lambda})(\Phi_{\lambda\lambda} + B_1\Phi_\lambda) \\ &\quad + (\gamma - 1)[\tfrac{1}{2}c^2 - q^2] - \Phi_t - B_4\Phi_\lambda]q_\lambda^2 \end{aligned}$$

When the functions B_1 , B_2 , B_4 are inserted this becomes

$$\begin{aligned} (30) \quad (H\Phi_{\lambda t} - H_t\Phi_\lambda + LH^2)^2 \\ = (\gamma - 1)H^2[gH\Phi_{\lambda\lambda} - (gH)_\lambda\Phi_\lambda + \tfrac{1}{2}H^2(c^2 - q^2 - 2\Phi_t)] \end{aligned}$$

where

$$\begin{aligned} (31) \quad L(\lambda, t) &= \beta_\lambda \\ H(\lambda, t) &= \beta + \beta_{\lambda\lambda} \\ q^2 &= \beta^2 + \beta_\lambda^2 \\ \beta &= g(h-t) \end{aligned}$$

6. Envelope of the λ -lines.

The relations obtained by eliminating $\Phi(\lambda, t)$ between (28), (30) are so complex that it is desirable first to investigate the possible existence of certain types of flow field. If the lines (7) are instantaneously concurrent, then the flow will be a 'simple centered' wave. It will be shown that, in this case, a fixed center is not possible if the flow is to be essentially unsteady. More generally there is the possibility of a non-degenerate envelope (λ -lines neither parallel nor concurrent) with shape independent of the time but which may move as a whole with non-uniform velocity.

The envelope of the λ -lines is given by the equations

$$(32) \quad \Phi_\lambda + \mathbf{r} \cdot \mathbf{q}_\lambda = 0$$

$$(33) \quad \Phi_{\lambda\lambda} + \mathbf{r} \cdot \mathbf{q}_{\lambda\lambda} = 0$$

These equations have the solution

$$(34) \quad \mathbf{r}(\lambda, t) = (\Phi_\lambda \mathbf{Q}_{\lambda\lambda} - \Phi_{\lambda\lambda} \mathbf{Q}_\lambda) |\mathbf{q}_\lambda \times \mathbf{q}_{\lambda\lambda}|^{-1}$$

where $\mathbf{Q}$ denotes the vector $(-q \sin \theta, q \cos \theta)$. Introducing the unit vector $\mathbf{b}(\lambda)$,

$$\mathbf{b} = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta$$

then the vector expression for the velocity may be written, from (22), (23)

$$\mathbf{q} = \beta \mathbf{b} + \beta_\lambda \mathbf{b}_\lambda$$

so that, since $\mathbf{b}_{\lambda\lambda} + \mathbf{b} = 0$,

$$\mathbf{q}_\lambda = H \mathbf{b}_\lambda$$

$$\mathbf{q}_{\lambda\lambda} = H_\lambda \mathbf{b}_\lambda - H \mathbf{b}$$

H being defined by (31). Equation (34) now gives, for any point on the envelope,

$$(35) \quad \mathbf{r}_0(\lambda, t) = \Psi_\lambda \mathbf{b} - \Psi \mathbf{b}_\lambda$$

where

$$(36) \quad \Psi(\lambda, t) = \frac{1}{H} \Phi_\lambda$$

The λ -lines (32) now become

$$\Psi + \mathbf{r} \cdot \mathbf{b}_\lambda = 0$$

By differentiation (35) gives

$$(37) \quad \mathbf{r}_{0\lambda} = (\Psi_{\lambda\lambda} + \Psi) \mathbf{b}$$

If s denotes the arc length along the envelope then

$$\mathbf{r}_{0\lambda} = \mathbf{r}_{0s} s_\lambda$$

therefore, from (37), since $\mathbf{r}_{0s}$, $\mathbf{b}$ are unit vectors,

$$s_\lambda = \Psi_{\lambda\lambda} + \Psi$$

Since λ is the local inclination of the tangent to the envelope, this expression for s_λ gives the radius of curvature. The envelope will move without change of shape if $(\Psi_{\lambda\lambda} + \Psi)$ reduces to a function of λ only. If $\mathbf{r}_0 = \mathbf{r}_0(t)$ the envelope reduces to a point; from (37) it is necessary to have

$$\Psi_{\lambda\lambda} + \Psi = 0$$

so that

$$\Psi(\lambda, t) = a(t) \cos \lambda + b(t) \sin \lambda$$

where $a(t)$, $b(t)$ are arbitrary functions of integration. (35) then gives the point of concurrency to be $[b(t), -a(t)]$.

To find the equations satisfied by $\Psi(\lambda, t)$ differentiate (27), (30) with respect to λ and substitute for Φ_λ from equation (36); then

$$(38) \quad \frac{\partial^2}{\partial t^2} (\beta + \beta_{\lambda\lambda}) \Psi = (\gamma + 1)g_\lambda(\beta + \beta_{\lambda\lambda}) + 2\beta_\lambda(g + g_{\lambda\lambda}) + (\gamma - 1)g(\beta + \beta_{\lambda\lambda})_\lambda$$

$$(39) \quad (\gamma - 1)g(\Psi_{\lambda\lambda} + \Psi) = (\Psi_t + \beta_\lambda)[2\Psi_{\lambda t} + (\gamma + 1)\beta_{\lambda\lambda} + (\gamma - 1)\beta]$$

Simple Centered Wave. It is possible to obtain from (38), (39) solutions for which

$$(40) \quad \Psi_{\lambda\lambda} + \Psi = 0$$

and therefore, from (37), for which the λ -lines are concurrent. From the equations of §5 it is necessary to find a solution for which $\beta + \beta_{\lambda\lambda} \neq 0$; otherwise a degenerate flow results in which $\mathbf{q}$ is a function of time only. If (40) is satisfied at least one of the factors on the right hand side of (39) must vanish identically. If

$$\Psi_t + \beta_\lambda = 0$$

then from (40),

$$(\beta_{\lambda\lambda} + \beta)_\lambda = 0$$

therefore, since $\beta(\lambda, t)$ is linear in t ,

$$\beta_{\lambda\lambda} + \beta = g_0(h_0 - t)$$

g_0 , h_0 being constants. Equation (38) now reduces to

$$\Psi_{tt} = (\gamma + 1)g_\lambda = -\beta_{\lambda t} = g_\lambda$$

since $\Psi_t = -\beta_\lambda$ and $\beta_t = -g$. It follows that $\Psi_{tt} = g_\lambda = 0$; therefore the λ -lines are concurrent and move with constant velocity in a fixed direction. Since $g = g_0$

the solution for β , Ψ may be exhibited

$$\begin{aligned}\beta &= g_0(h_0 - t) + a_1 \cos \lambda + b_1 \sin \lambda \\ \Psi &= (a_0 + a_1 t) \sin \lambda - (b_0 + b_1 t) \cos \lambda \\ \mathbf{q} &= \mathbf{q}_0 + g_0(h_0 - t)\mathbf{b}(\lambda)\end{aligned}$$

where $\mathbf{q}_0$ denotes the constant velocity of the center with components (a_1, b_1) . At any time t the fluid is moving instantaneously along the radius from the center with relative speed $g_0(h_0 - t)$.

When the second possibility is considered

$$2\Psi_{\lambda t} + (\gamma + 1)\beta_{\lambda\lambda} + (\gamma - 1)\beta = 0$$

and the equation for β is obtained using (40) then, if $n^2 = (\gamma - 1)/(\gamma + 1)$,

$$\left(\frac{\partial^2}{\partial \lambda^2} + 1\right)(\beta_{\lambda\lambda} + n^2\beta) = 0$$

therefore

$$\beta = a_1 \cos n\lambda + a_2 \sin n\lambda + a_3 \cos \lambda + a_4 \sin \lambda$$

in which the coefficients a_i may be linear functions of t . The associated value of Ψ_t is

$$\Psi_t = a_3 \sin \lambda - a_4 \cos \lambda$$

When these forms are inserted in (38) it is found that the only solution satisfying $\beta_{\lambda\lambda} + \beta \neq 0$ occurs when the coefficients a_i reduce to constants. In this case β , Ψ_t are independent of the time and the λ -lines again are translated with constant speed in a fixed direction. Since $\beta = \beta(\lambda)$ therefore $\mathbf{q} = \mathbf{q}(\lambda)$ and, referred relative to the moving center, the flow is essentially steady. In particular no unsteady simple centered wave (non-radial flow) exists having a fixed center. These results show that it is necessary to consider non-degenerate envelopes.

7. The Flow in the Physical Plane.

To construct the flow it is necessary to draw the envelope according to the relation

$$(41) \quad \mathbf{r}_0 = \Psi_\lambda \mathbf{b} - \Psi \mathbf{b}_\lambda$$

From each point of the envelope the tangent line is drawn. Along any such line the velocity is instantaneously constant and given by the expression

$$(42) \quad \mathbf{q} = \beta \mathbf{b} + \beta_\lambda \mathbf{b}_\lambda$$

The field equations

$$\Phi + \mathbf{r} \cdot \mathbf{q} = \phi \quad \Phi_\lambda + \mathbf{r} \cdot \mathbf{q}_\lambda = 0$$

may be solved to give a parametrisation of the plane in the form

$$(43) \quad \beta \mathbf{r}(\lambda, \phi, t) = (\phi - \Phi + \beta_\lambda \Psi) \mathbf{b} - \beta \Psi \mathbf{b}_\lambda$$

after using (42), (36). If (41) is used then (43) may be expressed

$$(44) \quad \beta(\mathbf{r} - \mathbf{r}_0) = (\beta_\lambda \Psi - \beta \Psi_\lambda - \Phi + \phi) \mathbf{b}$$

which expresses the coordinates of any point in the plane in terms of the origin $\mathbf{r}_0$ of the relevant λ -line and the distance μ along this line given by

$$(45) \quad \beta \mu = \beta_\lambda \Psi - \beta \Psi_\lambda - \Phi + \phi$$

Along an instantaneous streamline a relation between λ and ϕ exists. From (45) is obtained by differentiation

$$(46) \quad \beta^2 \mathbf{r}_\lambda = \beta \mathbf{b} \phi_\lambda + (\beta_\lambda \Psi - \beta \Psi_\lambda - \Phi + \phi)(\beta \mathbf{b}_\lambda - \beta_\lambda \mathbf{b})$$

This will be tangent to the streamline vector (42) if

$$\beta \beta_\lambda \phi_\lambda = (\beta^2 + \beta_\lambda^2)(\beta_\lambda \Psi - \beta \Psi_\lambda - \Phi + \phi)$$

that is, when expressed in terms of the tangential distance μ from the envelope,

$$\beta_\lambda \phi_\lambda = \mu(\beta^2 + \beta_\lambda^2)$$

therefore (46) becomes

$$(47) \quad \beta_\lambda \mathbf{r}_\lambda = \mu(\beta \mathbf{b} + \beta_\lambda \mathbf{b}_\lambda)$$

The unit tangent vector is

$$(48) \quad \mathbf{r}_s = (\beta \mathbf{b} + \beta_\lambda \mathbf{b}_\lambda)(\beta^2 + \beta_\lambda^2)^{-\frac{1}{2}}$$

From (47), (48) it follows that

$$(49) \quad \beta_\lambda s_\lambda = \mu(\beta^2 + \beta_\lambda^2)^{\frac{1}{2}}$$

Differentiation of (48) and the use of (49) produces

$$\mu \mathbf{r}_{ss} = \beta \beta_\lambda (\beta + \beta_{\lambda\lambda})(\beta \mathbf{b}_\lambda - \beta_\lambda \mathbf{b})(\beta^2 + \beta_\lambda^2)^{-2}$$

and therefore, for the streamline curvature,

$$(50) \quad k = \beta \beta_\lambda (\beta + \beta_{\lambda\lambda})(\beta^2 + \beta_\lambda^2)^{-\frac{3}{2}} \mu^{-1}$$

The paths of the fluid particles are given by the relation

$$\mathbf{r}_i = \mathbf{q}$$

Since $\mathbf{q}$ is known as a function of (λ, t) and not (x, y, t) the simplest procedure is to obtain the relation between λ and t valid along a particle path. The above equation then gives $\mathbf{q}$ parametrically in terms of either λ or t . To find the relation between λ and t , differentiate the field equation

$$(51) \quad \Psi + \mathbf{r} \cdot \mathbf{b}_\lambda = 0$$

then

$$(\Psi_\lambda + \mathbf{r} \cdot \mathbf{b}_{\lambda\lambda})\lambda_t + (\Psi_t + \mathbf{q} \cdot \mathbf{b}_\lambda) = 0$$

With the aid of (42) and since $\mathbf{b}_{\lambda\lambda} + \mathbf{b} = 0$, this becomes

$$(52) \quad (\Psi_\lambda - \mathbf{r} \cdot \mathbf{b})\lambda_t + (\Psi_t + \beta_\lambda) = 0$$

Further differentiation produces

$$(\Psi_\lambda - \mathbf{r} \cdot \mathbf{b})\lambda_{tt} + (2\Psi_{\lambda t} + \beta_{\lambda\lambda} - \beta)\lambda_t + (\Psi_{\lambda\lambda} - \mathbf{r} \cdot \mathbf{b}_\lambda)\lambda_t^2 + (\Psi_{tt} + \beta_{\lambda t}) = 0$$

Use of (51), (52) to eliminate the terms $\mathbf{r} \cdot \mathbf{b}_\lambda$, $\mathbf{r} \cdot \mathbf{b}$ yields

$$(53) \quad (\Psi_t + \beta_\lambda)\lambda_{tt} = (2\Psi_{\lambda t} + \beta_{\lambda\lambda} - \beta)\lambda_t^2 + (\Psi_{\lambda\lambda} + \Psi)\lambda_t^3 + (\Psi_{tt} + \beta_{\lambda t})\lambda_t$$

When known solutions $\Psi(\lambda, t)$, $\beta(\lambda, t)$ are inserted, equation (53) gives an ordinary differential equation for $\lambda(t)$. Equations (51), (52) may be solved for $\mathbf{r}$ to give

$$(54) \quad \mathbf{r} = [\Psi_\lambda + (\Psi_t + \beta_\lambda)t_\lambda]\mathbf{b} - \Psi\mathbf{b}_\lambda$$

When the solution obtained from (53) is used to substitute for λ or t , (54) will give the position of the fluid elements also in terms of λ or t . This is illustrated in §8.

8. A solution of the equation (30) may be obtained on the assumption $h(\lambda) = 0$. In this case

$$\begin{aligned} \beta &= -tg(\lambda) \\ H &= -(g + g_{\lambda\lambda})t \\ \Phi &= k(\lambda) - f(\lambda)t - G(\lambda)t^3 \\ (55) \quad \Psi &= \psi_0(\lambda) + \psi(\lambda)t^2 - \frac{k_\lambda}{H} \end{aligned}$$

where

$$(56) \quad \psi_0(\lambda) = \frac{f'(\lambda)}{g + g_{\lambda\lambda}}$$

$$(57) \quad \psi(\lambda) = \frac{G'(\lambda)}{g + g_{\lambda\lambda}}$$

When these results are substituted in equation (30) a quadratic expression in t is obtained and the conditions that this should be satisfied identically produces the following system:

$$(58) \quad k'(\lambda) = 0$$

$$(59) \quad gaf_{\lambda\lambda} - (ga)_\lambda f_\lambda + \frac{1}{2}a^2(c^2 + 2f) = 0$$

$$(60) \quad (2G_\lambda - ag_\lambda)^2 = (\gamma - 1)[gaG_{\lambda\lambda} - (ga)_\lambda G_\lambda + \frac{1}{2}a^2(6G - g^2 - g_\lambda^2)]$$

where

$$a(\lambda) = g + g_{\lambda\lambda}$$

From (59), by differentiation to remove the constant c ,

$$(61) \quad \psi_{0\lambda\lambda} + \psi_0 = 0$$

after substituting for f_λ from equation (56). If A, B are constants the solution of (61) is

$$(62) \quad \psi_0 = A \sin \lambda - B \cos \lambda$$

therefore

$$f_\lambda = (g + g_{\lambda\lambda})(A \sin \lambda - B \cos \lambda)$$

The parameter lines (51),

$$\Psi - x \sin \lambda + y \cos \lambda = 0$$

now become, using (55), (62) and (58),

$$(63) \quad (y - B) \cos \lambda - (x - A) \sin \lambda + t^2 \psi(\lambda) = 0$$

It is clear from this relation that the constants A, B correspond only to change of origin and may be taken to be zero; therefore from (62) no generality is lost by taking $\psi_0 = 0$. The substitution of G_λ from (57) in (60) gives the equation for $\psi(\lambda)$.

$$(64) \quad (2\psi - g_\lambda)^2 = \frac{2}{5}[g\psi_\lambda - g_\lambda\psi + 3G - \frac{1}{2}(g^2 + g_\lambda^2)]$$

where the value $7/5$ for γ has been used. To reduce (64) the function $v(\lambda)$ is used where

$$(65) \quad v = g^{-7/3} a^{-2/3} \quad a = g + g_{\lambda\lambda}$$

From (57) and the equation defining G (equation (29)),

$$\psi = \frac{1}{15} \left(11g_\lambda + \frac{ga_\lambda}{a} \right)$$

therefore

$$(66) \quad \frac{v_\lambda}{v} = \frac{5}{g} (g_\lambda - 2\psi)$$

When (66) is inserted into (64) a linear equation for $v(\lambda)$ is obtained

$$(67) \quad gv_{\lambda\lambda} - 7av = 0$$

in which $a(\lambda)$, $v(\lambda)$ are related to $g(\lambda)$ by (65). A solution of the system is given by

$$(68) \quad \begin{aligned} g &= be^{n\lambda} \\ v &= (1 + n^2)^{-\frac{3}{2}} g^{-3} \\ \psi &= \frac{4}{5} bne^{n\lambda} \end{aligned}$$

provided $n^2 = 7/2$. The λ -lines become

$$(69) \quad y \cos \lambda - x \sin \lambda + \frac{4}{5} nbt^2 e^{n\lambda} = 0$$

The expression (42) for the velocity becomes

$$(70) \quad \mathbf{q} = bte^{n\lambda} \operatorname{cosec} \lambda_0 [\mathbf{i} \sin (\lambda - \lambda_0) - \mathbf{j} \cos (\lambda - \lambda_0)]$$

where $n = \cot \lambda_0$.

The envelope of the system (69) is given by the equations

$$(71) \quad \begin{cases} x = \frac{4}{5} bnt^2 e^{n\lambda} \operatorname{cosec} \lambda_0 \cos (\lambda - \lambda_0) \\ y = \frac{4}{5} nbt^2 e^{n\lambda} \operatorname{cosec} \lambda_0 \sin (\lambda - \lambda_0) \end{cases}$$

Since

$$\Psi = \frac{4}{5} nbt^2 e^{n\lambda}, \quad \beta = -bte^{n\lambda}$$

the equation (53) for the paths of the fluid elements becomes

$$nt\lambda_{,t} = 6nt^2\lambda_{,t}^3 + \frac{2}{2}t\lambda_{,t}^2 + n\lambda_{,t}$$

The integral of this relation may be expressed

$$t^2 = (a_1 e^{-2n\lambda} + a_2 e^{-\lambda/(2n)}) e^{-n\lambda}$$

where a_1 , a_2 are constants serving to identify the fluid elements. The equation (54) now gives for the position of the element (a_1, a_2) ,

$$(72) \quad \frac{5}{b} \mathbf{r} = -4n(a_1 e^{-2n\lambda} + a_2 e^{-\lambda/(2n)}) \mathbf{b}_\lambda + \frac{1}{4}(32a_2 e^{-\lambda/(2n)} - 7a_1 e^{-2n\lambda}) \mathbf{b}$$

Along a streamline the local slope is, by (70),

$$(73) \quad \frac{dy}{dx} = -\cot (\lambda - \lambda_0)$$

If x and y are eliminated in turn between (69) and (73), linear equations to determine these quantities in terms of λ are obtained. These equations have the solutions,

$$(74) \quad \begin{aligned} x &= \frac{4}{5} nbt^2 \sec 2\lambda_0 e^{n\lambda} \sin (\lambda - 2\lambda_0) + Ce^{\lambda/n} \cos \lambda \\ y &= -\frac{4}{5} nbt^2 \sec 2\lambda_0 e^{n\lambda} \cos (\lambda - 2\lambda_0) + Ce^{\lambda/n} \sin \lambda \end{aligned}$$

where C is a parameter of integration which determines the streamlines.

According to equations (71) the envelope is a logarithmic spiral. From each point of the envelope the tangent line is drawn, the conditions of velocity being instantaneously determined by the value of λ associated with the line. Because the envelope is a spiral curve, each tangent line will intersect the envelope again an infinite number of times. Each such line may not however be continued beyond its first intersection with the envelope since the tangent lines will then intersect and the velocity will no longer be uniquely defined. The difficulty arises from equation (69) which fails to define λ uniquely, there being an infinite number of solutions corresponding to any set (x, y, t) . Therefore it is not possible to consider the section of the same λ -line beyond the first intersection with the envelope as giving rise to a possible flow field. It is necessary to use a configuration of λ -lines compatible with (69) so that these lines never intersect, except possibly on the envelope. Such a configuration is shown in Fig. 1. In this system each λ -line is interrupted at its first intersection with the

FIG. 1.

envelope to be continued, with a different value of λ , as the tangent to the envelope at the intersection. If this scheme is adopted then the entire flow field is mapped out by a series of straight segments of λ -lines intersecting only on the envelope. Along each segment the velocity is constant, but the value of λ and therefore the velocity, is discontinuous each time the polygon crosses the envelope. The velocity is finite everywhere but the acceleration becomes infinite on the envelope where the flow is supersonic. The solution is one-valued everywhere except on the envelope where the discontinuity persists.

For negative values of C the instantaneous streamlines according to (74) are spiral curves similar to the envelope and never intersecting it. At the origin these curves converge, elsewhere no singular points exist in the field of flow. The paths of the fluid elements given by (72) when the parameters a_1 , a_2 are positive are also spiral curves extending to infinity without the formation of discontinuities. For negative values of the time these elements converge onto the origin and then diverge away from it. At $t = 0$ the envelope degenerates and the λ -lines are concurrent through the origin, the fluid being everywhere at rest.

The instantaneous streamlines given by equations (74) for positive values of C also represent spiral curves extending to infinity but, for such values of C , cusps are formed and the envelope becomes the locus of these cusps (Fig. 2).

FIG. 2.

The two branches of each streamline which meet to form the cusp must be interpreted separately, since the streamlines of different branches intersect. On the side of the envelope near the origin the adjacent instantaneous streamlines have the form of continuous spiral curves. On the side of the envelope remote from the origin each of the possible branches is abruptly reflected on the envelope and is not tangent to it generally. The solution is many valued and not physically possible, even if attention is confined to branches of the same family. The envelope becomes a curve of sources or sinks where fluid is ejected or absorbed. Although the breakdown of the theory occurs only on the envelope, the breakdown is not a simple discontinuity in the fluid speed such as previously existed on crossing the stream paths. Further because the envelope is a spiral curve extending to infinity, it is not possible to exclude the envelope in the consideration of the flow field in general.

9. Accelerated Motion.

A solution having the property that the envelope accelerates uniformly without change of shape may be obtained by selecting

$$g(\lambda) = b \cos \lambda$$

where b denotes a constant. The function $h(\lambda)$ may then be expressed in terms of elliptic functions.

The following reductions follow at once from (31), (29) when the above form of $g(\lambda)$ is substituted.

$$(75a) \quad \beta = b(h - t) \cos \lambda$$

$$(75b) \quad H = b(h_{\lambda\lambda} \cos \lambda - 2h_{\lambda} \sin \lambda)$$

$$(75c) \quad 5E = b^2(h_{\lambda\lambda} \cos^2 \lambda - 7h_{\lambda} \sin \lambda \cos \lambda)$$

$$3G = b^2$$

When these relations are used the equation obtained from (28), (30) reduces to a quadratic expression in $(h - t)$ the coefficients of which, when equated to zero, give the following system:

$$(76) \quad \frac{5}{2}[2(E_{\lambda} + b^2 h_{\lambda}) - H g_{\lambda}]^2 = gH(E_{\lambda\lambda} + b^2 h_{\lambda\lambda}) - (gH)_{\lambda}(E_{\lambda} + b^2 h_{\lambda}) + \frac{1}{2}b^2 H^2 \\ 5[2(E_{\lambda} + b^2 h_{\lambda}) - H g_{\lambda}][f_{\lambda} + 2E h_{\lambda} - g h_{\lambda} H]$$

$$(77) \quad = gH(f_{\lambda\lambda} + 4E_{\lambda} h_{\lambda} + 2E h_{\lambda\lambda} + 2b^2 h_{\lambda}^2) \\ - (gH)_{\lambda}(f_{\lambda} + 2E h_{\lambda}) + (2E - g g_{\lambda} h_{\lambda}) H^2$$

$$(78) \quad \frac{5}{2}(f_{\lambda} + 2E h_{\lambda} - g h_{\lambda} H)^2 = gH(k_{\lambda\lambda} + 2f_{\lambda} h_{\lambda} + f h_{\lambda\lambda} + 2E h_{\lambda}^2) \\ - (gH)_{\lambda}(k_{\lambda} + f h_{\lambda}) + \frac{1}{2}H^2(c^2 - g^2 h_{\lambda}^2 + 2f)$$

To solve (76) write

$$gHu(\lambda) = (E_{\lambda} + b^2 h_{\lambda})$$

then (76) becomes

$$5(2u + \tan \lambda)^2 = 2u_{\lambda} + \sec^2 \lambda$$

of which a particular solution is $u = -\frac{1}{2} \tan \lambda$; therefore

$$2(E_{\lambda} + b^2 h_{\lambda}) = -b \sin \lambda H$$

With the aid of (75b) it is found that the following integral exists:

$$(79) \quad 2E = -b^2(h + h_{\lambda} \sin \lambda \cos \lambda)$$

When the expression (75c) for E is substituted in (79) the equation for $h(\lambda)$ is obtained,

$$(80) \quad 2h_{\lambda\lambda} - 9h_{\lambda} \tan \lambda + 5h \sec^2 \lambda = 0$$

The substitution

$$(81) \quad z = -\tan^2 \lambda$$

converts (80) into a hypergeometric equation,

$$(82) \quad 8z(1-z)h_{zz} + 2(2+3z)h_z - 5h = 0$$

With the usual notation for the hypergeometric functions, the solution of (82) may be expressed

$$h = h_1 F\left(-\frac{5}{4}, -\frac{1}{2}, \frac{1}{2}, z\right) + h_2 F\left(-\frac{3}{4}, 0, \frac{3}{2}, z\right) z^{\frac{1}{2}}$$

The second hypergeometric function in this expression has the value unity, the first is expressible in integral form and it is possible to write

$$h = h_1 z^{1/2} \int z^{-3/2} (1-z)^{5/4} dz + h_2 z^{1/2}$$

In terms of λ defined by equation (81) this becomes,

$$(83) \quad h = h_1 \tan \lambda \int (\sec \lambda)^{5/2} \operatorname{cosec}^2 \lambda d\lambda + h_2 \tan \lambda$$

apart from constant factors which may be absorbed in h_1, h_2 . To express the velocity in suitable form introduce the function

$$(84) \quad X(\lambda) = bh_\lambda \cos^2 \lambda$$

then

$$(85) \quad X_\lambda = b(h_{\lambda\lambda} - 2h_\lambda \tan \lambda) \cos^2 \lambda$$

If $h_\lambda, h_{\lambda\lambda}$ are eliminated between (80), (84) and (85) the following expression for h is obtained:

$$(86) \quad bh = X \tan \lambda - \frac{2}{5} X_\lambda$$

From equation (75a) the function $\beta(\lambda, t)$ is now given by

$$(87) \quad \beta = X \sin \lambda - \frac{2}{5} X_\lambda \cos \lambda - bt \cos \lambda$$

and the expression (42) for the velocity becomes

$$(88) \quad \mathbf{q} = -(bt + \frac{2}{5} X_\lambda) \mathbf{i} + X \mathbf{j}$$

It is clear from (84) and (88) that the effect of the term involving h_2 in (83) is simply to add a constant term to X and $\mathbf{q}$. This term therefore will henceforth be omitted. It then follows from (83), (84) that

$$(89) \quad X(\lambda) = \frac{5}{2} bh_1 \int (\sec \lambda)^{5/2} d\lambda$$

This may be written explicitly in the form

$$(90) \quad X(\lambda) = \frac{5bh_1}{6} \left[2 \tan \lambda (\sec \lambda)^{\frac{1}{2}} + \sqrt{2} \operatorname{sd}^{-1} \left(\sqrt{2} \tan \frac{\lambda}{2} \right) \right]$$

the modulus of the elliptic function being $2^{-\frac{1}{2}}$.

The expression (83) for $h(\lambda)$ may also be written,

$$(91) \quad \frac{h}{h_1} = \frac{1}{3}(5 \sin^2 \lambda - 3)(\sec \lambda)^{5/2} + \frac{5}{6}\sqrt{2} \tan \lambda \cdot \text{sd}^{-1}\left(\sqrt{2} \tan \frac{\lambda}{2}\right)$$

In virtue of (87), (89) the value of $H(\lambda)$ according to (31) is given by

$$(92) \quad H = X_\lambda \sec \lambda$$

When (79) is inserted in (77) an equation for $f(\lambda)$ is obtained,

$$gH(f_\lambda + 2Eh_\lambda)_\lambda - (gH)_\lambda(f_\lambda + 2Eh_\lambda) + (gg_\lambda h_\lambda - b^2 h)H^2 = 0$$

After division by $g^2 H^2$ this may be integrated to give

$$f_\lambda + 2Eh_\lambda + g_\lambda h H = c_1 g H$$

On substituting the value of E from (79) it is possible to obtain $f(\lambda)$ explicitly,

$$(93) \quad f = -gg_\lambda h h_\lambda + c_1 g^2 h_\lambda + c_2$$

For simplicity the constants of integration c_1, c_2 will be taken to be zero.

Equation (78) which is used to determine $k(\lambda)$ may be expressed

$$(94) \quad \frac{\partial}{\partial \lambda} [(k_\lambda + fh_\lambda)(gH)^{-1}] = \psi_\lambda$$

where

$$(95) \quad \psi(\lambda) = \frac{1}{2} \int \left(6h_\lambda^2 - 14hh_\lambda \tan \lambda + 5h^2 \tan^2 \lambda - \frac{c^2}{b^2} \sec^2 \lambda \right) d\lambda$$

The value of Φ_λ obtained from (28), *viz.*,

$$\Phi_\lambda = (k_\lambda + fh_\lambda) + (f_\lambda + 2Eh_\lambda)(h - t) + (E_\lambda + 3Gh_\lambda)(h - t)^2$$

now reduces to, after using (79), (93), (94), (86),

$$(96) \quad \Phi_\lambda = bX_\lambda[\psi + \frac{1}{2}(h^2 - t^2) \tan \lambda]$$

From (36), (92) the function $\Psi(\lambda, t)$ corresponding to (96) is

$$(97) \quad \Psi = b[\psi \cos \lambda + \frac{1}{2}(h^2 - t^2) \sin \lambda]$$

It follows that $(\Psi_{\lambda\lambda} + \Psi)$ reduces to a function of λ only and therefore by §6 (equation (37)) that the envelope moves without change of shape. From (88), (89),

$$\mathbf{q}_\lambda = (-\mathbf{i} \tan \lambda + \mathbf{j})X_\lambda$$

and when this is substituted together with (96) in (32) the λ -lines become

$$(98) \quad y - x \tan \lambda + b[\psi + \frac{1}{2}(h^2 - t^2) \tan \lambda] = 0$$

in which the functions $h(\lambda)$, $\Psi(\lambda)$ are given by (91), (95). The envelope of the system (98) is given by equation (98) used in conjunction with its λ -derivative

$$x \sec^2 \lambda - b[\psi_\lambda + \frac{1}{2}(h^2 - t^2) \sec^2 \lambda + hh_\lambda \tan \lambda] = 0$$

When the value of ψ given by (95) is substituted,

$$b^{-2}(2bx + c^2 + b^2t^2) \sec^2 \lambda = h^2 + 6(h_\lambda - h \tan \lambda)^2$$

finally, by substituting h , h_λ using (84), (85),

$$(99) \quad (2bx + c^2 + b^2t^2) \sec^2 \lambda = 6(X + \frac{2}{3}X_\lambda \tan \lambda)^2 + (X \tan \lambda - \frac{2}{3}X_\lambda)^2$$

The function $X(\lambda)$ may be obtained from (89) or (90). Equation (99) provides the x -coordinate of the current point on the envelope in terms of λ , the local inclination at the point. This representation is sufficient to obtain the general form of the envelope.

The general form of the envelope for $0 \leq \lambda \leq \pi/2$ is shown in Fig. 3. For

FIG. 3.

FIG. 4.

$-\pi/2 \leq \lambda \leq 0$ a second branch exists which is a reflection in the x -axis of the first branch (Fig. 5). A number of different solutions may be obtained depending on the branches selected and on which tangents are used. Each tangent to the envelope may be drawn in one of two directions and the flow field depends on which system of lines is chosen. For negative values of the time the envelope is moving to the right without change of shape and with uniform retardation b . At $t = 0$ the envelope comes instantaneously to rest and begins to accelerate uniformly to the left. The velocity of the envelope is $-b\alpha$; therefore by (88) the fluid velocity relative to the envelope is a function of λ only. Conditions of velocity relative to the envelope are independent of the time, but it is not possible to obtain this solution from any essentially steady motion since the envelope accelerates. The equations (1), (2) will not reduce to the equations of steady motion when referred relative to accelerated axes.

FIG. 5.

FIG. 6.

When only the upper branch of the envelope is considered, which corresponds to the range $0 \leq \lambda \leq \pi/2$, there are two possible flow fields. In the first scheme the instantaneous streamlines sweep round from the x -axis for negative values of the time to meet the envelope. The envelope moves to the right and overtakes the fluid in front of it. For $t > 0$ the local fluid overtakes the envelope and

passes through it. To obtain the second scheme the tangent lines to the envelope are extended downwards from the point of tangency and map out the entire lower half plane as well as the region to the right of the envelope. The streamlines of this solution meet those of the first on the envelope in a cusp. To obtain the flow field to the left of the envelope it is necessary to draw the tangent lines of the second scheme from each of the two branches. These tangent lines will cross the opposite branch and map out the region to the left of the envelope (Figs. 7, 8).

FIG. 7.

FIG. 8.

If both branches of the envelope with outward drawn tangent lines are considered then the configurations of Figs. 5, 6 result, in which the streamlines are non-intersecting. The inward drawn tangent lines from both branches of the envelope cannot be used together because the corresponding stream paths intersect.

10. The Wave $q(\lambda)$.

A special theory of simple waves may be obtained by considering flows of the type $q(\lambda)$. The time enters into q only through the parameter λ . In this case the partial derivatives, q_t , $q_{\lambda t}$, q_{tt} are all zero and from (15) all the functions A_i , B_i vanish except A_1 , B_1 which are now functions of λ only.

The equations (17), (18), (19), reduce to

$$\begin{aligned} Z &= 0 \\ (100) \quad Y &= -A_1(\lambda)\Phi_{tt} \\ X &= (\Phi_{\lambda t} + q q_{\lambda})^2 - \Phi_{tt}(\Phi_{\lambda\lambda} + B_1\Phi_{\lambda}) - \frac{1}{2}(\gamma - 1)(c^2 - q^2 - 2\Phi_t)q_{\lambda}^2 \end{aligned}$$

The condition $Y = 0$ shows that at least one of the functions $A_1(\lambda)$, Φ_{tt} must vanish. If $A_1 = 0$ then by equation (15)

$$\theta + \tan^{-1}(q\theta_a) = \text{constant}$$

therefore conditions of velocity are constant along each of a series of parallel straight lines. Alternatively when $\Phi_{tt} = 0$ then Φ_t reduces to a function of λ only. Equation (35) may then be used to show that r_t is a function of λ only. Therefore each point of the envelope moves with constant speed in a fixed direction. In particular the 'center' of a simple centered wave of the type $q(\lambda)$ moves with constant velocity in a fixed direction, and the λ -lines move along with the center without rotation. It follows that the flow is essentially steady when axes moving with the point of concurrency are considered. If the expression for $\Phi(\lambda, t)$

$$(101) \quad \Phi(\lambda, t) = tg(\lambda) + f(\lambda)$$

is inserted in (100), an equation for $q(\lambda)$ is obtained,

$$2(g_{\lambda} + q q_{\lambda})^2 = (\gamma - 1)(c^2 - q^2 - 2g)q_{\lambda}^2$$

Regarding θ, g as functions of q this becomes,

$$(102) \quad 2(g_q + q)^2 = (\gamma - 1)(c^2 - q^2 - 2g)(1 + q^2\theta_q^2)$$

The functions $g(q)$, $f(q)$ may be chosen arbitrarily; then (102) will give on integration the function $\theta(q)$. The field equation (32) becomes, with the aid of (101),

$$(103) \quad tg_q + f_q + r \cdot q_a = 0$$

Take for example

$$g(q) = \frac{1}{2}q^2, \quad f(q) = 0$$

then, if $c^2 = 2b^2$,

$$a^2 = (\gamma - 1)(b^2 - q^2)$$

Let

$$q^2 = b^2(\sin^2 z + \rho^2 \cos^2 z)$$

where

$$\rho^2 = \frac{\gamma - 1}{\gamma + 3}$$

then

$$a = 2b\rho \cos z$$

so that from (102),

$$(104) \quad q\theta_a = \rho^{-1} \tan z$$

when the positive sign for the square root is selected. Expressed in terms of z ,

$$\rho\theta_z = (1 - \rho^2)(1 + \rho^2 \cot^2 z)^{-1}$$

This integrates to give, by suitable choice of the constant of integration,

$$(105) \quad \theta = \rho^{-1}z + \tan^{-1}(\rho \cot z) + \pi - \frac{\pi}{2\rho}$$

The angle Γ which the streamlines make with the lines of constant velocity (103) is given by,

$$(106) \quad \cot \Gamma = q\theta_a$$

therefore, using (104),

$$\tan \Gamma = \rho \cot z$$

so that (105) becomes

$$(107) \quad \theta = \rho^{-1}z + \Gamma + (2 - \rho^{-1})\frac{\pi}{2}$$

The relation (103) becomes

$$(108) \quad qt + x(q \cos \theta)_a + y(q \sin \theta)_a = 0$$

On performing the differentiation, using (106) and expressing x, y in terms of polar coordinates (r, ψ) , (108) gives

$$qt \sin \Gamma = r \sin (\theta - \Gamma - \psi)$$

Finally in terms of z , using (106), (107),

$$(109) \quad R \sin \left(\rho^{-1} z - \psi - \frac{\pi}{2\rho} \right) = \cos z$$

where $R = r(\rho b t)^{-1}$. The envelope of the system (109) is given additionally by

$$(110) \quad R \cos \left(\rho^{-1} z - \psi - \frac{\pi}{2\rho} \right) = -\rho \sin z$$

and therefore the parametric equations may be expressed in terms of z in the form

$$\psi = \rho^{-1} z - \tan^{-1}(\rho \tan z) - \frac{\pi}{2\rho} + \frac{3\pi}{2}$$

$$R^2 = \cos^2 z + \rho^2 \sin^2 z$$

The envelope is similar to an epicycloid which increases linearly in size since $r = R \rho b t$. The instantaneous streamlines (Fig. 9) are perpendicular to the tangent

FIG. 9.

generated from the point of the envelope given by $z = 0$, and approach asymptotically the direction parallel to the tangent at $z = \frac{1}{2}\pi$. The λ -line $z = 0$ along which the flow commences, moves normal to itself with velocity ρb , equal to that of the fluid along it.

11. Parallel λ -lines.

The solution of the Monge-Ampère equation (21) for the velocity was expressed by equations (22), (23) in terms of the parameter λ , satisfying (24). If however conditions are constant along each of a series of parallel straight

lines, the parameter λ which is the inclination of each such line will be independent of the coordinates x, y . In this case q, θ are no longer expressible as functions of λ and the representation (22), (23) as the solution of (21) is no longer applicable. Similarly when the simplified wave $\mathbf{q}(\lambda)$ of §10 is considered the possibility still occurs that the function $A_1(\lambda)$ vanishes identically, in which case conditions are constant along each of a series of straight lines parallel to a fixed direction. In this paper we omit the consideration of these flows, which are degenerate instances of the general theory and may be treated by direct methods.

12. Conclusion.

In all cases of unsteady potential flow of the simple wave pattern the lines of constant (q, θ) are straight. The lines may be parallel but in general are generated by tangents to an envelope which varies with the time. Apart from flows of a radial pattern it is shown that the lines of constant velocity may never be concurrent if the flow is to remain essentially unsteady. These deductions are in contrast to simple wave phenomena in steady flow. For steady potential flow of the simple wave class the λ -lines may be parallel, concurrent or possess a non-degenerate envelope. If the steady flows become rotational, but remain iso-energetic, then the lines never possess a non-degenerate envelope and must be either parallel or concurrent. The latter alternative is the very case which is excluded when the simple wave becomes unsteady. There exists no 'simple centered wave' in unsteady potential flow.

When local breakdown of the flow is considered it is found that the phenomena resembles the corresponding phenomena which occur when steady flow is considered. In general where the breakdown occurs the fluid paths form cusps, of which the envelope of the lines of constant speed is the locus. However in such circumstances it is no longer necessary for the local fluid speed to be supersonic. §9 provides an example in which conditions near the envelope, where local collapse occurs, may be either subsonic or supersonic (*cf.* [5]).

The Legendre contact transformation has been shown to be specially suited to the study of certain degenerate flows. The criterion depends on the existence of a suitable gradient field vector constant along each of a one-parameter family of curves $\lambda(x, y, t) = \text{constant}$. In the investigated cases of fluid flow these curves have reduced to straight lines. The contact transformation has been used by GHAFFARI [1] to study simple waves in steady potential flow in which the fluid speed q is used as the independent variable. GIESE [2] used the same transformation to discuss similar phenomena including, in addition to the simple wave $\mathbf{q}(\lambda)$, the "double wave" $\mathbf{q}(\lambda, \mu)$ in steady potential flow. In addition it is shown in [3] how the theory of the rotational wave $\mathbf{q}(\lambda)$, originally obtained in [4], may be derived by an analogous method.

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