Simple Extension of a Hypo-elastic Body of Grade Zero

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1. Introduction. The constitutive equations of an isotropic hypo-elastic body, as defined by TRUESDELL (1955a), are

$$\begin{aligned} \bar{s}^{ik} g_{ki} = & h_0 d_k^k \delta_j^i + h_1 d_j^i + h_2 d_k^k s_j^i + h_3 M \delta_j^i \\ & + \frac{1}{2} h_4 (d_k^i s_j^k + s_k^i d_j^k) + h_5 d_k^k s_r^i s_j^r \end{aligned} \quad (1.1)$$

$$\begin{aligned} & + h_6 M s_j^i + h_7 N \delta_j^i + \frac{1}{2} h_8 (d_r^i s_m^r s_j^m + s_r^i s_m^r d_j^m) \\ & + h_9 M s_r^i s_j^r + h_{10} N s_j^i + h_{11} N s_k^i s_j^k, \end{aligned}$$

$$d_{i,j} = \frac{1}{2} (v_{i,i} + v_{j,i}), \quad M = s_j^i d_i^j, \quad N = s_j^i s_k^j d_i^k, \quad (1.2)$$

$$\bar{s}^{ik} = \frac{\partial s^{ik}}{\partial t} + v^m s_{,m}^{ik} - s^{im} v_{,m}^k - s^{mk} v_{,m}^i + s^{ik} d_m^m, \quad (1.3)$$

where g_{ik} is the covariant metric tensor of a general fixed system of coordinates, v_i is the covariant velocity vector, $2\mu s_j^i$ is the stress tensor, μ being the rigidity modulus of classical elasticity, and where a comma denotes covariant differentiation. Also $h_1, h_2, \dots, h_{11}$ are dimensionless analytic functions of the principal invariants of s_j^i . To complete the system of differential equations we add the equations of motion and continuity in the form

$$2\mu s_{,i}^{ij} = \rho \left(\frac{\partial v^i}{\partial t} + v^i_{,j} v^j \right), \quad (1.4)$$

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0,$$

where ρ is the density and body forces are zero.

TRUESDELL defines a hypo-elastic body of grade zero to be such that (1.1) has the special form

$$\mathfrak{g}^{ik} g_{ki} = \frac{\nu}{1 - 2\nu} d_k^k \delta_i^i + d_i^i, \quad (1.5)$$

where ν is the classical Poisson ratio. Using equations (1.4) and (1.5), TRUESDELL (1955b) has studied the problem of simple extension. The solution is obtained, without approximations, for an accelerationless motion in which inertia terms are identically zero and, as soon as the extension is large, the results are very different from those obtained by neglecting inertia effects entirely.

GREEN (1956) has studied the theory of hypo-elasticity for incompressible bodies, with reference to the theory of plasticity. In particular, using a special form of the general equations, the problem of simple extension was solved completely, including inertia terms, with no restriction on the speed or acceleration of the motion. From the solution it could be seen that, in certain circumstances, it was legitimate to neglect inertia effects, the results being identical with those usually found by neglecting inertia terms in the equations of motion.

The purpose of this note is to examine further the simple extension problem for a compressible hypo-elastic body of grade zero whose constitutive equations are given by (1.5). An exact solution of equations (1.4) and (1.5) is obtained which is completely different in character from TRUESDELL's accelerationless motion. Moreover, the inertia terms in the solution can be made small in a precise way, the resulting stresses being similar in form to those which would be obtained by neglecting inertia in the differential equations. In the solution given here, however, the way in which the motion takes place is restricted. The results of the present discussion do not necessarily apply to other problems and other hypo-elastic bodies.

2. Extension. We take a fixed system of Cartesian axes (x, y, z) and denote velocity components by (v_x, v_y, v_z) and stress components by $2\mu s_{xx}, 2\mu s_{xy}, \dots$. For simple extension we take the velocity field

$$v_x = \dot{\epsilon}x, \quad v_y = \dot{\eta}y, \quad v_z = \dot{\eta}z, \quad (2.1)$$

where ϵ, η are functions of t . The second equation in (1.4) then becomes

$$\frac{\partial \rho}{\partial t} + \rho(\dot{\epsilon} + 2\dot{\eta}) + \dot{\epsilon}x \frac{\partial \rho}{\partial x} + \dot{\eta} \left(y \frac{\partial \rho}{\partial y} + z \frac{\partial \rho}{\partial z} \right) = 0, \quad (2.2)$$

and the solution of this equation under the condition that $\rho = \rho_0$, a constant,

when $t = 0$, is

$$\rho = \rho_0 e^{-\epsilon - 2\eta}, \quad (2.3)$$

provided ϵ and η vanish at $t = 0$.

We now seek for a solution of equations (1.4), (1.5), (2.1) and (2.3) which is such that

$$s_{xx} = P, \quad s_{yy} = Q, \quad s_{zz} = R, \quad s_{xy} = s_{yz} = s_{zx} = 0. \quad (2.4)$$

In this case the first set of equations in (1.4) reduces to

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{\rho}{2\mu} (\ddot{\epsilon} + \dot{\epsilon}^2)x = \frac{\rho_0}{2\mu} e^{-\epsilon - 2\eta} (\ddot{\epsilon} + \dot{\epsilon}^2)x, \\ \frac{\partial Q}{\partial y} &= \frac{\rho}{2\mu} (\ddot{\eta} + \dot{\eta}^2)y = \frac{\rho_0}{2\mu} e^{-\epsilon - 2\eta} (\ddot{\eta} + \dot{\eta}^2)y, \\ \frac{\partial R}{\partial z} &= \frac{\rho}{2\mu} (\ddot{\eta} + \dot{\eta}^2)z = \frac{\rho_0}{2\mu} e^{-\epsilon - 2\eta} (\ddot{\eta} + \dot{\eta}^2)z, \end{aligned} \quad (2.5)$$

and equations (1.5) and (1.3) yield

$$\begin{aligned} \frac{\partial P}{\partial t} + \dot{\epsilon}x \frac{\partial P}{\partial x} + \dot{\eta} \left(y \frac{\partial P}{\partial y} + z \frac{\partial P}{\partial z} \right) - P(\dot{\epsilon} - 2\dot{\eta}) &= \frac{\nu(\dot{\epsilon} + 2\dot{\eta})}{1 - 2\nu} + \dot{\epsilon}, \\ \frac{\partial Q}{\partial t} + \dot{\epsilon}x \frac{\partial Q}{\partial x} + \dot{\eta} \left(y \frac{\partial Q}{\partial y} + z \frac{\partial Q}{\partial z} \right) + Q\dot{\epsilon} &= \frac{\nu(\dot{\epsilon} + 2\dot{\eta})}{1 - 2\nu} + \dot{\eta}, \\ \frac{\partial R}{\partial t} + \dot{\epsilon}x \frac{\partial R}{\partial x} + \dot{\eta} \left(y \frac{\partial R}{\partial y} + z \frac{\partial R}{\partial z} \right) + R\dot{\epsilon} &= \frac{\nu(\dot{\epsilon} + 2\dot{\eta})}{1 - 2\nu} + \dot{\eta}. \end{aligned} \quad (2.6)$$

For our purpose a suitable form of solution of equations (2.6) is

$$\begin{aligned} P &= P_0(t) + e^{\epsilon - 2\eta} f(xe^{-\epsilon}, ye^{-\eta}, ze^{-\eta}), \\ Q &= Q_0(t) + e^{-\epsilon} g(xe^{-\epsilon}, ye^{-\eta}, ze^{-\eta}), \\ R &= Q_0(t) + e^{-\epsilon} h(xe^{-\epsilon}, ye^{-\eta}, ze^{-\eta}), \end{aligned} \quad (2.7)$$

where f, g, h are arbitrary functions of their arguments, and where $P_0(t)$, $Q_0(t)$, are integrals of the equations

$$\frac{dP_0}{dt} - P_0(\dot{\epsilon} - 2\dot{\eta}) = \frac{\nu(\dot{\epsilon} + 2\dot{\eta})}{1 - 2\nu} + \dot{\epsilon}, \quad (2.8)$$

$$\frac{dQ_0}{dt} + Q_0\dot{\epsilon} = \frac{\nu(\dot{\epsilon} + 2\dot{\eta})}{1 - 2\nu} + \dot{\eta}.$$

Inspection of equations (2.5) and (2.7) leads us to specialize the values of P, Q, R to

$$\begin{aligned} P &= P_0(t) + \tfrac{1}{2}Ax^2e^{-2\epsilon}e^{\epsilon-2\eta} = P_0(t) + \tfrac{1}{2}Ax^2e^{-\epsilon-2\eta}, \\ Q &= Q_0(t) + \tfrac{1}{2}By^2e^{-2\eta}e^{-\epsilon} = Q_0(t) + \tfrac{1}{2}By^2e^{-\epsilon-2\eta}, \\ R &= Q_0(t) + \tfrac{1}{2}Bz^2e^{-2\eta}e^{-\epsilon} = Q_0(t) + \tfrac{1}{2}Bz^2e^{-\epsilon-2\eta}, \end{aligned} \quad (2.9)$$

where A, B are constants. These values will satisfy equations (2.5) provided

$$\ddot{\epsilon} + \dot{\epsilon}^2 = \frac{2\mu A}{\rho_0}, \quad \ddot{\eta} + \dot{\eta}^2 = \frac{2\mu B}{\rho_0}, \quad (2.10)$$

or

$$\frac{d^2}{dt^2}e^\epsilon = \frac{2\mu A}{\rho_0}e^\epsilon, \quad \frac{d^2}{dt^2}e^\eta = \frac{2\mu B}{\rho_0}e^\eta, \quad (2.11)$$

and these equations may be integrated completely. We now, however, observe, from (2.8), that $Q_0(t)$ may be taken to be identically zero provided

$$\eta = -\nu\epsilon. \quad (2.12)$$

In general this relation is only compatible with equations (2.11) if

$$\epsilon = kt, \quad \eta = -\nu kt, \quad (2.13)$$

where k is a constant, and

$$A = \frac{\rho_0 k^2}{2\mu}, \quad B = \frac{\rho_0 \nu^2 k^2}{2\mu}. \quad (2.14)$$

Also, the first equation in (2.8) then becomes

$$\frac{dP_0}{dt} - k(1 + 2\nu)P_0 = (1 + \nu)k, \quad (2.15)$$

and hence

$$P_0(t) = \frac{1 + \nu}{1 + 2\nu} [e^{(1+2\nu)\epsilon} - 1], \quad (2.16)$$

if, for simplicity, we restrict our attention to the case $P_0(0) = 0$. From (2.9), (2.14) and (2.16) we see that the final components of stress are

$$\begin{aligned} P &= \frac{1 + \nu}{1 + 2\nu} [e^{(1+2\nu)\epsilon} - 1] + \frac{\rho_0 k^2}{4\mu} e^{-(1-2\nu)\epsilon} x^2, \\ Q &= \frac{\rho_0 k^2 \nu^2}{4\mu} e^{-(1-2\nu)\epsilon} y^2, \quad R = \frac{\rho_0 k^2 \nu^2}{4\mu} e^{-(1-2\nu)\epsilon} z^2, \end{aligned} \quad (2.17)$$

where ϵ is given by (2.13).

If $\nu < \frac{1}{2}$ and we consider stresses at fixed finite values of x, y, z we see that the inertia terms in (2.17) rapidly decrease as t increases so that, approximately,

$$P = \frac{1 + \nu}{1 + 2\nu} [e^{(1+2\nu)\epsilon} - 1], \quad Q = R = 0. \quad (2.18)$$

In any case, provided $\rho_0 k^2 L^2 \ll 4\mu$, where L is some standard dimension of the body, the inertia terms can be ignored.

If inertia terms are ignored in the differential equations (1.4) and we seek for a solution in which the only non-zero stress is $s_{xx} = P$, we are led to equations (2.12) and (2.18). In this case, however, ϵ is an arbitrary increasing function of the time (in simple extension) instead of having the special value (2.13), but in both solutions the stress at any instant depends only on the amount of extension. The solution obtained by neglecting inertia terms in the differential equations would, therefore, appear to give an adequate description of one dynamically possible motion.

It should be observed that other types of solutions are possible in which the cross stresses Q, R are not necessarily small or zero. These are given by (2.8) and (2.9), where ϵ, η are determined from (2.11). These solutions include TRUESDELL'S

accelerationless motion as a special case when $A = B = 0$. In addition, other stress systems may be possible in which the shear stresses are non-zero. For a hypo-elastic body of grade zero we have an infinity of dynamically possible solutions corresponding to a velocity field (2.1).

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