Absolute Convergence and Integrability of Trigonometric Series

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1. Introduction. Suppose that $f(x)$ is defined on $(0, \pi)$ by the absolutely convergent sine series $\sum b_n \sin nx$. (By this we mean that the series converges absolutely for all x , so that $\sum |b_n|$ converges; similarly for other trigonometric series.) If we expand $f(x)$ in a cosine series $\sum a_n \cos nx$, this cosine series will not necessarily be absolutely convergent (Theorem 10, below). It is therefore of interest to find sufficient conditions for the absolute convergence of the cosine series. A rather general one that we shall obtain is

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\frac{1}{2}n} \frac{|b_{n+k} - b_{n-k}|}{k} < \infty.$$

This implies some more special criteria, for example, that the cosine series converges absolutely if b_n is ultimately decreasing, or if $\sum |b_n| \log n$ converges, or if $\sum n |\Delta b_n|$ converges.

When $f(x)$ is defined by an absolutely convergent sine series, the improper integral $\int_{-\pi}^{\pi} x^{-1} f(x) dx$ always converges [1]. We shall show that when the cosine series also converges absolutely, $|x^{-1} f(x)|$ is necessarily integrable. Thus any sufficient condition for the absolute convergence of the cosine series of a function $f(x)$ defined by an absolutely convergent sine series is in particular a sufficient condition for the integrability of $|x^{-1} f(x)|$. A number of known theorems, and some new ones, thus follow from our criteria. (However, these theorems can be proved much more easily directly.) By partial summation, a condition on b_n that makes $|x^{-1} f(x)|$ integrable is equivalent to a condition on Δa_n that makes $\sum a_n \cos nx$ a Fourier series (*i.e.*, the Fourier series of a Lebesgue integrable function). Thus we also obtain a number of theorems on the Fourier character of a cosine series. The most general of these is that $\sum a_n \cos nx$ is a Fourier

series if $a_n \rightarrow 0$ and both the sequences $\{a_n\}$ and $\{a_n^*\}$ are of bounded variation, with

$$a_n^* = \sum_{m \neq n} a_m \frac{(-1)^{m-n} - 1}{m - n}.$$

Although the integrability of $|x^{-1}f(x)|$ is presumably not sufficient for $f(x)$ to have an absolutely convergent cosine series when it has an absolutely convergent sine series, it is curious that all the simple sufficient conditions for $|x^{-1}f(x)|$ to be integrable in fact imply the stronger result.

There are similar, but somewhat more complicated, results connected with expanding an absolutely convergent cosine series $f(x) = \sum a_n \cos nx$ in a sine series. Here it is evidently necessary to assume that $f(0) = f(\pi) = 0$ if the sine series is to have any chance of converging absolutely. That this necessary condition is not sufficient is known: a function behaving like $1/\log x$ at 0 can have an absolutely convergent cosine series without having an absolutely convergent sine series (LÉVY [8]). A further necessary condition is the integrability of $|x^{-1}f(x)|$, and a sufficient condition is the convergence of $\sum |a_n| \log n$. We can deduce theorems on the integrability of $|x^{-1}f(x)|$, and on the Fourier character of a sine series, as special cases.

A somewhat more general way of looking at these problems is to consider a general absolutely convergent Fourier series $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, put $f_+(x) = f(x)$ for $0 < x < \pi$, $f_+(x) = 0$ for $-\pi < x < 0$, and ask when $f_+(x)$ has an absolutely convergent Fourier series; in fact, this is the formulation which will be used in the body of the paper. (For a sine or cosine series, $2f_+ - f$ is the function with the corresponding cosine or sine series, so that for even or odd functions our problem has not been changed.) We are, then, asking for conditions for the absolute convergence of the Fourier series of the product $f(x) \cdot \frac{1}{2}(1 + \operatorname{sgn} x)$. This suggests the general question of which multipliers $g(x)$ preserve the absolute convergence of Fourier series. In full generality this question has a trivial answer: the product fg has an absolutely convergent Fourier series whenever f has an absolutely convergent Fourier series if and only if g has an absolutely convergent Fourier series (sufficiency by the convolution formula, necessity by taking $f(x) \equiv 1$). However, if we require more of f , namely that both f and f_+ have absolutely convergent Fourier series, we can relax the requirement on g : it is enough if $g(t) + \lambda t$, with some λ , has an absolutely convergent Fourier series. This in effect admits multipliers g which fail to have absolutely convergent Fourier series only for the trivial reason that they happen not to have the same values at $\pm\pi$. In particular, any $g(t)$ which has a derivative of bounded variation on the closed interval $[-\pi, \pi]$ becomes (when restricted to $[-\pi, \pi]$ and then continued periodically) a multiplier for functions with absolutely convergent Fourier series, a fact which is of interest in connection with some problems on entire functions.

The l^p theory ($p > 1$) corresponding to the l^1 theory discussed here has been

investigated by STECHKIN [9]. It is a good deal simpler, since any $g(x)$ of bounded variation is an l^p multiplier, and so, in particular, f_+ has its Fourier coefficients in l^p ($p > 1$) whenever f does.

2. Necessary conditions. Let $f_+(x)$ be defined as in §1.

Theorem 1. *If $f(x)$ and $f_+(x)$ both have absolutely convergent Fourier series on $(-\pi, \pi)$, then $|x^{-1}f(x)|$ is integrable.*

Corollary. *If, for some number μ , $f(x)$ and $(f(x) + \mu|x|)_+$ both have absolutely convergent Fourier series, $|x^{-1}f(x)|$ is integrable.*

The corollary follows because $|x|$ has an absolutely convergent Fourier series.

In Theorem 1, only the neighborhood of $x = 0$ is significant. Since $f_+(x)$ must be continuous (when continued with period 2π), the hypotheses imply $f(0) = f(\pi) = 0$.

We introduce the entire functions

$$(2.1) \quad \begin{aligned} F(z) &= \int_{-\pi}^{\pi} f(t)e^{izt} dt, \\ G(z) &= \int_0^{\pi} f(t)e^{izt} dt = \int_{-\pi}^{\pi} f_+(t)e^{izt} dt; \end{aligned}$$

then our hypotheses are

$$\sum_{-\infty}^{\infty} |F(n)| < \infty, \quad \sum_{-\infty}^{\infty} |G(n)| < \infty.$$

We first prove that

$$(2.2) \quad \int_{-\infty}^{\infty} |F(x)| dx < \infty.$$

To do this it is enough to prove that

$$(2.3) \quad \int_{-\infty}^{\infty} |G(x)| dx < \infty, \quad \int_{-\infty}^{\infty} |H(x)| dx \equiv \int_{-\infty}^{\infty} |F(x) - G(x)| dx < \infty,$$

where $H(x)$ is related to $f(x) - f_+(x)$ as $G(x)$ is related to $f_+(x)$. The first of the integrals in (2.3) is typical of both. If $\phi(z) = G(z)e^{-i\pi z/2}$, then $|\phi(x)| = |G(x)|$ for real x and so we may consider $\phi(x)$ instead of $G(x)$. Now

$$\begin{aligned} \phi(z) &= \int_0^{\pi} f_+(t)e^{-\frac{1}{2}\pi iz}e^{izt} dt \\ &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f_+(t + \tfrac{1}{2}\pi)e^{izt} dt \end{aligned}$$

has $\sum |\phi(n)| < \infty$ and its Fourier transform vanishes outside $(-\pi/2, \pi/2)$. By a theorem of WIENER [9, p. 80], these two facts imply $\int_{-\infty}^{\infty} |\phi(x)| dx < \infty$, and (2.2) follows.

Since, on $(-\pi, \pi)$,

$$f(t) \operatorname{sgn} t = 2f_+(t) - f(t),$$

it follows that

$$\int_{-\pi}^{\pi} f(t) (\operatorname{sgn} t) e^{izt} dt$$

also belongs to $L(-\infty, \infty)$. This, by a theorem of HILLE & TAMARKIN [6], implies that $|x^{-1}f(x)|$ is integrable on $(-\pi, \pi)$.

By using (2.2) we can establish another necessary condition, which is the theorem on multipliers stated in the introduction.

Theorem 2. *If $g(x) + \lambda x$ has, for some λ , an absolutely convergent Fourier series on $(-\pi, \pi)$, then, for every $f(x)$ such that f and f_+ both have absolutely convergent Fourier series, $f(x)g(x)$ has an absolutely convergent Fourier series.*

Since the Fourier series of $f(x)[g(x) + \lambda x]$ is trivially absolutely convergent, it is necessary only to prove that the Fourier series of $xf(x)$ is absolutely convergent.

Introduce again the entire function $F(z)$ of (2.1), which we showed satisfies (2.2). It is known [2, p. 211] that, since $F(z)$ is of exponential type, this implies

$$\int_{-\infty}^{\infty} |F'(x)| dx < \infty,$$

which in turn implies [2, p. 101]

$$\sum_{-\infty}^{\infty} |F'(n)| < \infty,$$

and this is the absolute convergence of the Fourier series of $xf(x)$.

For other conditions which are necessary under auxiliary hypotheses, see the discussion after Theorem 7, and Theorem 10.

3. General formulas. In order to find sufficient conditions for the absolute convergence of the Fourier series of $f_+(x)$ when the Fourier series of $f(x)$ converges absolutely, we need formulas connecting the Fourier coefficients of the two series, in which the functions themselves do not appear explicitly. Let $f(x)$ have the absolutely convergent Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$. Then the Fourier coefficients of $f_+(x)$ are given by

$$\begin{aligned} (3.1) \quad c_n^* &= \frac{1}{2\pi} \int_0^{\pi} e^{-inx} \sum_{m=-\infty}^{\infty} c_m e^{imx} dx \\ &= \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} \frac{c_m \lambda_{m-n}}{m-n} + c_n, \end{aligned}$$

where the prime indicates omission of the term with $m = n$, and

$$(3.2) \quad \lambda_k = (-1)^k - 1$$

(an abbreviation which we shall use consistently). For $n > 0$, we may write

$$\begin{aligned} 2\pi i(c_n^* - c_n) &= -\frac{c_0\lambda_n}{n} + \sum_{m=1}^{\infty} \frac{c_m\lambda_{m-n}}{m-n} - \sum_{m=1}^{\infty} \frac{c_{-m}\lambda_{m-n}}{m+n} \\ &= \sum_{m=1}^{\infty} c_m\lambda_{m-n} \left(\frac{1}{m-n} + \frac{1}{m+n} \right) - \sum_{m=1}^{\infty} \frac{c_m + c_{-m}}{m+n} \lambda_{m-n} - \frac{c_0\lambda_n}{n}, \\ (3.3) \quad 2\pi i(c_n^* - c_n) &= 2 \sum_{m=1}^{\infty} c_m\lambda_{m-n} \frac{m}{m^2 - n^2} - \sum_{m=0}^{\infty} \frac{c_m + c_{-m}}{m+n} \lambda_{m+n} \\ &= c_n^{\Delta} + c_n^{\square}, \end{aligned}$$

say, where in $\sum^*$ the term with $m = 0$ is to be multiplied by $\frac{1}{2}$, and λ_k is defined in (3.2). For $n < 0$, similarly,

$$\begin{aligned} (3.4) \quad 2\pi i(c_n^* - c_n) &= -2 \sum_{m=1}^{\infty} c_{-m}\lambda_{m-n} \frac{m}{m^2 - n^2} + \sum_{m=0}^{\infty} \frac{c_m + c_{-m}}{m-n} \lambda_{m-n} \\ &= c_n^{\Delta} + c_n^{\square}. \end{aligned}$$

Since c_n^* is defined by (3.1) or by (3.3) and (3.4) whenever $\sum |c_m/m|$ converges, we can consider $\{c_n^*\}$ as a transform of $\{c_n\}$ without reference to $f_+(x)$, even when $f(x)$ is not defined by an absolutely convergent Fourier series. We can reformulate Theorem 1 as follows.

Theorem 3. *If $\sum |c_n|$ converges, $f(x) = \sum c_n e^{inx}$, c_n^* is defined by (3.1), and $\sum |c_n^*|$ converges, then $|x^{-1}f(x)|$ is integrable.*

Corollary. *In Theorem 3 the convergence of $\sum |c_n^*|$ can be replaced by the convergence of $\sum |c_n^* + i\mu/n|$ for some real μ .*

If $\sum |c_n|$ and $\sum |c_n^*|$ both converge, the definition of c_n^* in terms of $f_+(x)$ shows that $f(0) = \sum_{n=-\infty}^{\infty} c_n = 0$.

If $\sum |c_n|$ does not necessarily converge, but $c_n \rightarrow 0$ and $\{c_n\}$ is of bounded variation, i.e. if $\sum |\Delta c_n|$ converges, the trigonometric series $\sum c_n e^{inx}$ converges except perhaps at the points $2k\pi$. It is not, in general, the Fourier series of an integrable function [11, p. 111]. The following theorem is equivalent to Theorem 3.

Theorem 4. *If $c_n \rightarrow 0$ and $\{c_n\}$ and $\{c_n^*\}$ are both of bounded variation, $\sum c_n e^{inx}$ is the Fourier series of an integrable function.*

To show that Theorem 3 implies Theorem 4, suppose that $\{c_n\}$ and $\{c_n^*\}$ are of bounded variation and $c_n \rightarrow 0$. Then $g(x) = \sum (\Delta c_n) e^{inx}$ is an absolutely convergent Fourier series and, since $\Delta c_n^* = (\Delta c_n)^*$, $g_+(x)$ also has an absolutely

convergent Fourier series. Hence $|x^{-1}g(x)|$ is integrable. By partial summation we have, since $c_n \rightarrow 0$,

$$\begin{aligned} g(x) &= \sum c_n(e^{inx} - e^{i(n-1)x}) \\ &= \sum c_n e^{inx}(e^{ix} - 1). \end{aligned}$$

Since $|g(x)/(e^{ix} - 1)|$ is integrable with $|x^{-1}g(x)|$, $\sum c_n e^{inx}$ (which converges on $(-\pi, \pi)$ except perhaps at 0) is a Fourier series.

That Theorem 4 implies Theorem 3 follows by reversing the argument. If $\sum |c_n|$ and $\sum |c_n^*|$ converge and $f(x) = \sum c_n e^{inx}$, put $C_n = \sum_{k=n}^{\infty} c_k$ so that $\Delta C_n = C_n - C_{n+1} = c_n$. Partial summation gives

$$f(x) = \sum C_n e^{inx}(e^{ix} - 1).$$

Since $f(0) = 0$, $C_n \rightarrow 0$ as $|n| \rightarrow \infty$ and so Theorem 4 shows that $\sum C_n e^{inx}$ is a Fourier series. Thus $|f(x)/(e^{ix} - 1)|$ is integrable, and so is $|x^{-1}f(x)|$.

4. Transformation of c_n^* . We turn now to conditions, expressed in terms of c_n alone, for $\sum |c_n^*|$ to converge if $\sum |c_n|$ converges. Each such condition will yield, as a corollary, a theorem on the integrability of $|x^{-1}f(x)|$ (or of $f(x)$), but since these theorems are mostly known and can in any case be proved more directly, the corollaries are chiefly interesting as indications of the scope of the main results.

We begin by transforming c_n^Δ , the first sum on the right of (3.3) or (3.4), into a more convenient form. We consider only the case $n > 0$, since for $n < 0$ we have the same transform of c_{-n} . We have, for $n > 0$,

$$\begin{aligned} \frac{1}{2}c_n^\Delta &= \sum_{m=1}^{\infty}{}' c_m \lambda_{m-n} \frac{m}{m^2 - n^2} \\ &= \sum_{m>2n} + \sum_{1 \leq m \leq 2n}{}' \\ &= S_1(n) + S_2(n). \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=1}^{\infty} |S_1(n)| &\leq 2 \sum_{n=1}^{\infty} \sum_{m>2n} \frac{m |c_m|}{m^2 - n^2} \\ &= \sum_{m=1}^{\infty} |c_m| \sum_{n<\frac{1}{2}m} \frac{m}{m^2 - n^2}, \end{aligned}$$

and since the inner sum has at most $m/2$ terms of magnitude at most $1/(3m)$, $\sum |c_n^\Delta|$ converges if and only if $\sum |S_2(n)|$ converges.

Next,

$$\begin{aligned}
 S_2(n) &= -\sum_{m=1}^{n-1} c_m \lambda_{m-n} \frac{m}{n^2 - m^2} + \sum_{m=n+1}^{2n} c_m \lambda_{m-n} \frac{m}{m^2 - n^2} \\
 &= -\sum_{k=1}^{n-1} \lambda_k c_{n-k} \frac{n-k}{n^2 - (n-k)^2} + \sum_{k=1}^n \lambda_k c_{n+k} \frac{n+k}{(n+k)^2 - n^2} \\
 &= \sum_{k=1}^n \lambda_k \left\{ \frac{(n+k)c_{n+k}}{k(2n+k)} - \frac{(n-k)c_{n-k}}{k(2n-k)} \right\} \\
 &= \sum_{k=1}^n \lambda_k \left\{ \frac{(n+k)c_{n+k} - (n-k)c_{n-k}}{k(2n-k)} + \frac{(n+k)c_{n+k}}{k} \left(\frac{1}{2n+k} - \frac{1}{2n-k} \right) \right\} \\
 &= S_3(n) + S_4(n).
 \end{aligned}$$

Here $\sum |S_4(n)|$ converges, since

$$\begin{aligned}
 \sum_{n=1}^{\infty} |S_4(n)| &\leq 4 \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{n+k}{4n^2 - k^2} |c_{n+k}| \\
 &\leq 4 \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{2n}{3n^2} |c_{n+k}| \\
 &\leq \frac{8}{3} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=n}^{2n} |c_i| \\
 &= \frac{8}{3} \sum_{i=1}^{\infty} |c_i| \sum_{\frac{1}{2}i \leq n \leq i} \frac{1}{n},
 \end{aligned}$$

which is at most a constant multiple of $\sum |c_i|$. Thus $\sum |c_n^\Delta|$ converges if and only if $\sum |S_3(n)|$ converges. We can write $S_3(n)$ in the form

$$\sum_{k=1}^n \lambda_k \left\{ \frac{n(c_{n+k} - c_{n-k})}{k(2n-k)} + \frac{c_{n+k} + c_{n-k}}{2n-k} \right\}.$$

By arguments similar to those used for $S_1(n)$ and $S_4(n)$, we find that each of the following sums contributes at most a constant multiple of $\sum |c_n|$ to $\sum |S_3(n)|$:

$$\begin{aligned}
 &\sum_{k=1}^n \frac{|c_{n+k}|}{2n-k}, \quad \sum_{k=1}^{\frac{1}{2}n} \frac{|c_{n-k}|}{2n-k}, \\
 &\sum_{k > \frac{1}{2}n} \frac{n |c_{n+k}|}{k(2n-k)}, \quad \sum_{k > \frac{1}{2}n} |c_{n-k}| \left(\frac{n}{k(2n-k)} - \frac{1}{2n-k} \right).
 \end{aligned}$$

Hence $\sum |S_3(n)|$, and therefore $\sum |c_n^\Delta|$, converges or diverges with

$$\sum_{n=1}^{\infty} n \left| \sum_{\frac{1}{2}n \leq k \leq n} \frac{\lambda_k (c_{n+k} - c_{n-k})}{k(2n-k)} \right|.$$

Summing up, we have, for $n > 0$,

$$(4.1) \quad \begin{aligned} 2\pi i(c_n^* - c_n) &= 2n \sum_{1 \leq k \leq \frac{1}{2}n} \lambda_k \frac{c_{n+k} - c_{n-k}}{k(2n-k)} + c_n^\square + \alpha_n \\ &= 2c_n^\circ + c_n^\square + \alpha_n, \end{aligned}$$

where $c_n^\square$ is defined in (3.3) and $\sum |\alpha_n|$ converges; there is a similar formula for $n < 0$.

5. Special sufficient conditions. The sums c_n° and $c_n^\square$ behave rather differently, the second being less likely to yield an absolutely convergent series. The following theorem enables us to deal with $c_n^\square$.

Theorem 5. *If $c_n^\square$ is defined by (3.3) and (3.4), if $\sum_{-\infty}^{\infty} c_n = \sum_{-\infty}^{\infty} (-1)^n c_n = 0$, and if $\sum |c_k| \log |k|$ ($k \neq 0$) converges, or even if $\sum |c_k|$ and $\sum_1^{\infty} |c_k + c_{-k}| \log k$ converge, then*

$$(5.1) \quad \sum |c_n^\square| = \sum_{n=-\infty}^{\infty} \left| \sum_{m=0}^{\infty} \frac{c_m + c_{-m}}{m + |n|} \lambda_{m-n} \right|$$

converges.

Under the hypotheses of Theorem 4, we change (5.1) by an absolutely convergent series if we replace $(m + |n|)^{-1}$ by $(m + |n|)^{-1} - |n|^{-1}$ for $n \neq 0$. In fact, for $n > 0$

$$\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{|c_m|}{n} = \sum_{m=1}^{\infty} |c_m| \sum_{n=1}^m \frac{1}{n}$$

converges, and hence

$$(5.2) \quad \sum_{n=1}^{\infty} \left| \sum_{m=n}^{\infty} \frac{c_m \lambda_{m-n}}{n} \right|$$

converges. Similarly,

$$(5.3) \quad \sum_{n=1}^{\infty} \left| \sum_{m=-\infty}^{-n} \frac{c_m \lambda_{m-n}}{n} \right|$$

converges. Also, if $n > 0$,

$$\sum_{m=-n+1}^{n-1} \frac{c_m \lambda_{m-n}}{n} = - \sum_{|m| > n} \frac{c_m \lambda_{m-n}}{n}$$

if $\sum_{m=-\infty}^{\infty} c_m \lambda_{m-n} = 0$, and hence (since $\lambda_k = (-1)^k - 1$) if $\sum c_m = \sum (-1)^m c_m = 0$. Hence the convergence of (5.2) and (5.3) implies the convergence of

$$\sum_{n=1}^{\infty} \left| \sum_{m=-n+1}^{n-1} \frac{c_m \lambda_{m-n}}{n} \right|.$$

Similar arguments apply when $n < 0$. Consequently (5.1) converges with

$$\sum_{n \neq 0} \sum_{m=0}^{\infty} {}^* |c_m + c_{-m}| \left(\frac{1}{|n|} - \frac{1}{m + |n|} \right) = \sum_{m=0}^{\infty} {}^* |c_m + c_{-m}| \sum_{n \neq 0} \left(\frac{1}{|n|} - \frac{1}{m + |n|} \right)$$

and this converges since the inner sum is $O(\log m)$.

Corollary. If $\sum |c_n|$ converges and the even component $g(x)$ of $f(x)$ vanishes at 0 and π , and $\sum |c_m + c_{-m}| \log m$ converges, then $\sum |c_n^*|$ converges or diverges with $\sum |c_n^\circ|$.

Here c_n° is defined in (4.1). In particular, $\sum |c_n^*|$ converges with $\sum |c_n^\circ|$ if $f(x)$ is odd.

We now consider c_n° . The most general of our criteria is as follows.

Theorem 6. $\sum |c_n^\circ|$ converges if $\sum |c_n|$ converges and

$$(5.4) \quad \sum_{|n|=1}^{\infty} \sum_{k=1}^{\frac{1}{2}|n|} \frac{|c_{n+k} - c_{n-k}|}{k}$$

converges.

Consider positive values of n . Then, referring to (4.1), we see that in c_n° , $n/(2n - k) \leq \frac{2}{3}$; hence $\sum_{n>0} |c_n^\circ|$ converges with the part of (5.4) in which $n > 0$. A similar argument applies to negative values of n .

Corollary 1. If $f(x)$ is odd, $\sum |c_n^*|$ converges if $\sum |c_n|$ and (5.4) converge.

Corollary 2. If $f(x)$ is odd and $\sum |c_n|$ and (5.4) converge, $|x^{-1}f(x)|$ is integrable.

Corollary 3. The trigonometric series $\sum a_n \cos nx$ with $a_n \rightarrow 0$ and $\sum |\Delta a_n| < \infty$ is a Fourier series if

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\Delta a_{n+k} - \Delta a_{n-k}|}{k} < \infty.$$

We can deduce several simpler special criteria from Theorem 6.

Theorem 7. If $\sum' |c_n| \log |n|$ converges, $\sum |c_n^\circ|$ converges.

Here the prime denotes the omission of the term with $n = 0$.

By Theorem 6 it is certainly enough to show that

$$(5.5) \quad \sum_{n=1}^{\infty} \sum_{k=1}^{\frac{1}{2}n} \frac{|c_{n+k}| + |c_{n-k}|}{k},$$

and the corresponding series for $n < 0$, converge. The series (5.5) is

$$\sum_{n=1}^{\infty} \sum_{j=n+1}^{\frac{3}{2}n} \frac{|c_j|}{j - n} + \sum_{n=1}^{\infty} \sum_{j=\frac{1}{2}n}^{n-1} \frac{|c_j|}{n - j} = \sum_{n=1}^{\infty} \sum_{j=\frac{3}{2}n}^{\frac{3}{2}n} \frac{|c_j|}{|n - j|} = \sum_{j=1}^{\infty} |c_j| \sum_{n=\frac{3}{2}j}^{2j} \frac{1}{|n - j|},$$

and converges because the inner sum is $O(\log j)$. The discussion for $n < 0$ is similar.

Corollary 1. *If the even component of $f(x)$ vanishes at 0 and π , and in particular if $f(x)$ is odd, $\sum |c_n^*|$ converges if $\sum' |c_n| \log |n|$ converges.*

Corollary 2. *If $\sum' |c_n| \log |n|$ converges and $\sum c_n = 0$, then $|x^{-1}f(x)|$ is integrable.*

Corollary 2 can be proved much more directly. Indeed, for a sine series we have

$$\int_0^\pi x^{-1} |f(x)| dx \leq \sum_1^\infty |c_n| \int_0^\pi x^{-1} |\sin nx| dx = \sum_1^\infty |c_n| \int_0^{\pi n} u^{-1} |\sin u| du,$$

and the integral is $O(\log n)$. For a cosine series see [5]. Similar remarks apply to other corollaries.

Corollary 3. [11, p. 112] *The trigonometric series $\sum c_n e^{inx}$ is a Fourier series if $c_n \rightarrow 0$ and $\sum' |\Delta c_n| \log |n|$ converges.*

Corollary 1 is a best possible result as far as concerns the even component of $f(x)$, since if c_n are even and ultimately positive, $\sum c_n = 0$, and $|x^{-1}f(x)|$ is integrable, $\sum' c_n \log |n|$ converges [5]. However, Theorem 6 is a better result for the odd component, since (5.4) does not imply that $\sum' |c_n| \log |n| < \infty$. For an example illustrating this, take c_m odd and $c_m = m^{-1}(\log m)^{-2}$ for $m \geq 3$. The derivative of $x^{-1}(\log x)^{-2}$ does not exceed in absolute value, for $x \geq 3$, a constant multiple of $x^{-2}(\log x)^{-2}$. Hence $|c_i - c_{i+2k}| \leq Akj^{-2}(\log j)^{-2}$ with some constant A , and

$$\begin{aligned} \sum_{n=1}^\infty \sum_{k=1}^{\frac{1}{2}n} \frac{|c_{n+k} - c_{n-k}|}{k} &= \sum_{k=1}^\infty \frac{1}{k} \sum_{j=k}^\infty |c_j - c_{i+2k}| \\ &\leq A \sum_{k=1}^\infty \sum_{j=k}^\infty j^{-2}(\log j)^{-2} \\ &= A \sum_{j=1}^\infty j^{-2}(\log j)^{-2} \sum_{k=1}^j 1 \\ &= A \sum_{j=1}^\infty j^{-1}(\log j)^{-2} < \infty. \end{aligned}$$

Theorem 8. $\sum |c_n^\circ|$ converges if c_n decreases as $|n|$ increases and $\sum c_n$ converges.

Consider (5.4) for positive n (similar considerations apply to negative n). We have $c_{n+k} - c_{n-k} \geq 0$, and so the part of (5.4) for positive n converges with

$$\begin{aligned} \sum_{n=1}^\infty \sum_{1 \leq k \leq \frac{1}{2}n} \frac{c_{n+k} - c_{n-k}}{k} &= \sum_{k=1}^\infty \frac{1}{k} \sum_{n \geq 2k} (c_{n+k} - c_{n-k}) \\ &= \sum_{k=1}^\infty \frac{1}{k} \sum_{k \leq j < 3k} c_j \\ &= \sum_{j=1}^\infty c_j \sum_{\frac{1}{3}j < k \leq j} \frac{1}{k}, \end{aligned}$$

and the inner sum is bounded.

Corollary 1. *If $f(x)$ is odd and has ultimately decreasing sine coefficients b_n , with $\sum b_n$ convergent, then $\sum |b_n^*|$ converges. In particular, $|x^{-1}f(x)|$ is integrable.*

The second clause also follows directly from the fact that $\int_{-\pi}^{\pi} x^{-1} f(x) dx$ exists [1], while $f(x) \geq 0$ near 0 [4], [7].

Corollary 2. [11, p. 109] *A trigonometric cosine series whose coefficients tend to 0 and are ultimately convex is a Fourier series.*

We can deduce from Theorem 6 any number of criteria involving differences, or other operators similar to those considered by CESARI [3]. If D is a linear operator such that

$$c_n = \sum_{t=-\infty}^{\infty} \mu_t Dc_{n+t},$$

we can introduce this expression into (5.4) and express the resulting condition in terms of the μ 's and Dc_n . We carry this out only in the simple case when $Dc_n = \Delta c_n$. Here we can write, for $n > 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2k}^{\infty} |c_{n+k} - c_{n-k}| &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2k}^{\infty} \left| \sum_{j=n-k+1}^{n+k} \Delta c_j \right| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2k}^{\infty} \sum_{j=n-k+1}^{n+k} |\Delta c_j| \\ &= \sum_{k=1}^{\infty} \left\{ \sum_{j=3k}^{\infty} \sum_{n=j-k}^{j+k-1} 1 + \sum_{j=k}^{3k} \sum_{n=2k}^{j+k-1} 1 \right\} |\Delta c_j| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \sum_{j=3k}^{\infty} 2k |\Delta c_j| + \sum_{j=k}^{3k} (j-k) |\Delta c_j| \right\} \\ &= 2 \sum_{k=1}^{\infty} \sum_{j=3k}^{\infty} |\Delta c_j| + \sum_{k=1}^{\infty} \sum_{j=k}^{3k} \frac{j-k}{k} |\Delta c_j|, \end{aligned}$$

which converges if $\sum_{j=1}^{\infty} j |\Delta c_j|$ converges. Hence we have the following theorem.

Theorem 9. $\sum |c_n^{\circ}|$ converges if $\sum |c_n|$ and $\sum n |\Delta c_n|$ converge.

Corollary 1. If $f(x)$ is odd, $\sum |c_n^*|$ converges if $\sum |c_n|$ and $\sum n |\Delta c_n|$ converge.

Corollary 2. If $f(x)$ is odd, $|x^{-1} f(x)|$ is integrable if $\sum |c_n|$ and $\sum n |\Delta c_n|$ converge.

Corollary 3. [11, p. 109] *A trigonometric cosine series $\sum a_n \cos nx$ is a Fourier series if $a_n \rightarrow 0$ and $\sum |\Delta a_n|$ and $\sum n |\Delta^2 a_n|$ converge.*

In a similar way we can prove that $\sum |c_n^{\circ}|$ converges if $\sum |c_n| < \infty$ and either $\sum |\Delta^{\sigma} c_n| < \infty$, $-1 < \sigma < 0$, or $\sum n^{\sigma} |\Delta^{\sigma} c_n| < \infty$, $0 < \sigma < 1$. The equivalent theorems on the Fourier character of a trigonometric series were given by CESARI [3].

6. A gap theorem. Our final theorem is a gap theorem which furnishes a partial converse to Theorem 7.

Theorem 10. If $f(x)$ is odd, $c_n = 0$ for $n > 0$ except for $n = n_k$, where $n_{k+1}/n_k \geq \theta > 1$, and both $\sum |c_n|$ and $\sum |c_n^*|$ converge, then $\sum' |c_n| \log |n|$ converges.

This theorem provides us with a simple example of a function with an absolutely convergent sine series but not an absolutely convergent cosine series: e.g., $f(x) = \sum n^{-2} \sin 2^n x$.

In §4 we showed that for an odd $f(x)$, with $\sum |c_n| < \infty$, $\sum |c_n^*|$ converges if and only if

$$\sum_{n=1}^{\infty} \left| \sum_{1 \leq k \leq \frac{1}{2}n} \lambda_k \frac{c_{n+k} - c_{n-k}}{k(2n - k)} \right|$$

converges. Slight modifications in the reasoning show that in the inner sum $\frac{1}{2}n$ can be replaced by $n\mu$, where μ is any positive number less than 1. Now suppose that $c_n = 0$ except for $n = n_p$, where $n_{p+1}/n_p > \theta > 1$. If μ is small enough, for each n in an interval $n_p(1 - \mu) < n < n_p$ the difference $c_{n+k} - c_{n-k}$ is zero except for the one value of k which makes it equal to c_{n_p} , namely $k = n_p - n$. Consequently, if $\sum |c_n^*|$ converges, so does

$$\sum_{p=1}^{\infty} \sum_{n_p(1-\mu) < n < n_p} \frac{n\lambda_{n_p-n} |c_{n_p}|}{(n_p - n)(3n - n_p)} \geq \frac{1}{2} \sum_{p=1}^{\infty} |c_{n_p}| \sum_{n_p(1-\mu) < n < n_p} \frac{\lambda_{n_p-n}}{n_p - n},$$

and here the inner sum is of the order of $\log n_p$.

Added in proof: While this paper was in press, there appeared a paper by S. IZUMI & T. TSUCHIKURA, Absolute convergence of Fourier expansions, *Tôhoku Math. J.* (2) **7**, 243–251 (1955), dealing with the same subject by similar methods. These authors give an example of the phenomenon mentioned in the third sentence of §1 and establish all the sufficient conditions for absolute convergence stated in §1. They deal only with pure sine or cosine series.

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