Asymptotic Solutions of the First Boundary Value Problem for a Fourth-Order Elliptic Partial Differential Equation

ROBERT B. DAVIS

Communicated by C. C. LIN

There is a rapidly growing literature of "reduction-of-order" problems in differential equations, engendered in part by asymptotic phenomena in fluid dynamics [cf. 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14]. In this literature, it is usual to find that two problems are compared:

Problem I: $L_\epsilon u_\epsilon = f$ in a region R
 u_ϵ satisfies prescribed boundary conditions (B.C. #1) on the boundary B of R .

Problem II: $L_0 u_0 = f$ in R
 u_0 satisfies boundary conditions (B.C. #2) on B .

Here L_ϵ is a differential operator of some sort (for example, in the case treated by LEVINSON [9],

$$L_\epsilon(u) \equiv \epsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u,$$

and $L_0(u)$ is the same operator with ϵ set equal to zero. Since, in Problem I, ϵ is assumed small, we can argue that the two solutions, u_ϵ and u_0 , should be quite similar, at least in some sense. Since, however, setting ϵ equal to zero eliminates at least some of the derivatives of highest order the possible boundary value problems are changed rather drastically, and certain marked differences between u_ϵ and u_0 must evidently occur, at least near portions of the boundary. In most cases of interest, the dilemma is resolved by proving that the difference between u_ϵ and u_0 consists of two terms, the first of which is small (say $O(\epsilon^n)$) uniformly in ϵ for all $R + B$, and the second of which is appreciable only near

the boundary, but is (say) $O(e^{-\delta/\epsilon})$ away from the boundary. This last term, often called the boundary-layer term, serves to adjust the discrepancies between the quite different boundary value problems appropriate to $\epsilon > 0$ and $\epsilon = 0$, while preserving our notion that u_ϵ and u_0 should not be too dissimilar. One case in point is the boundary layer due to viscosity in fluid dynamics.

The present paper applies the technique of LEVINSON [9] to a fourth-order elliptic partial differential equation which reduces, when ϵ is set equal to zero, to a second order equation, also elliptic.

Theorem 1. *Let R be a region, and let B denote its boundary. Let a and b be negative constants, and let $f(s)$ and $g(s)$ be functions defined on B , where s is arc-length along B . Suppose that R is bounded, that the parametric representing functions for $B \in C^7$, and that B , c , f , and g are sufficiently smooth that*

$$(1) \quad \left. \begin{aligned} L_\epsilon u_\epsilon &\equiv (\epsilon^2 \Delta + a)(\Delta + b)u_\epsilon = c && \text{in } R \\ u_\epsilon &= f && \text{on } B \\ \frac{\partial u_\epsilon}{\partial n} &= g && \text{on } B \end{aligned} \right\} \text{Problem I}$$

and

$$(2) \quad \left. \begin{aligned} L_0 u_0 &\equiv a(\Delta + b)u_0 = c && \text{in } R \\ u_0 &= f && \text{on } B \end{aligned} \right\} \text{Problem II}$$

determine unique solutions u_ϵ and u_0 , both of which $\in C^4$ on $R + B$. [The existence of solutions under suitable hypotheses on B , f , and g is well-known (cf., for example, BERG & LAX [1] regarding I, and COURANT & HILBERT [3, Chapter IV] regarding Problem II); the continuity of the derivatives up to the boundary follows, under suitable hypotheses on B , f , and g , from a recent theorem of LOUIS NIRENBERG¹.]

Conclusion. *Then*

$$u_\epsilon = u_0 + \bar{\Omega},$$

where

$$\begin{aligned} (\bar{\Omega}, \bar{\Omega}) &\equiv \iint_R \bar{\Omega}^2 dx dy = O(\epsilon^2) \\ (\bar{\Omega}_x, \bar{\Omega}_x) &= O(\epsilon^2) \\ (\bar{\Omega}_y, \bar{\Omega}_y) &= O(\epsilon^2). \end{aligned}$$

¹See the forthcoming proceedings of the Conference on Partial Differential Equations, Berkeley, summer of 1955.

Remark. Actually, we shall prove the theorem in the following apparently stronger form, which is in fact identical with the result as stated above. Moreover, as we shall see later, the present method cannot be used to strengthen the theorem by exhibiting the explicit form of the boundary layer terms.

We shall prove that

$$u_\epsilon = u_0 + \Psi + \epsilon \Psi^{(1)} + \Omega,$$

where Ψ , $\Psi^{(1)}$, and Ω have properties which we now develop. For Ψ we introduce a co-ordinate t along B , and a co-ordinate n along the inner normals to B , so that the equation of B becomes $n = 0$. There exists a neighborhood T_1^* of B (with respect to $\bar{R}$) and a neighborhood T_2^* of T_1^* with respect to $\bar{R}$ such that, if we define

$$T_1 \equiv T_1^* - B,$$

$$T_2 \equiv T_2^* - T_1^*,$$

we have in T_1

$$\Psi_n = e^{-g/\epsilon} h, \quad \Psi = \int_0^n e^{-g/\epsilon} h \, dn,$$

where g and h are independent of ϵ , h is bounded, and $g > 0$ in T_1 , $g = 0$ on B . In $R - T_2^*$, $\Psi \equiv 0$. The region T_2 contains interpolation between the values in T_1 and those in $R - T_2^*$; in T_2 , Ψ and its first derivatives (we can in fact obtain as many further derivatives here as we desire) are $O(\epsilon)$.

For $\Psi^{(1)}$ we have the corresponding properties.

Finally, Ω satisfies

$$(\Omega, \Omega) = O(\epsilon^2)$$

$$(\Omega_x, \Omega_x) = O(\epsilon^2)$$

$$(\Omega_y, \Omega_y) = O(\epsilon^2)$$

(That this fails to exhibit the boundary layer terms follows from the fact that Ψ is of the same order as Ω . Why this cannot be improved will be discussed presently.)

Proof of Theorem 1. To begin with, we define Ψ and $\Psi^{(1)}$ over T_1 . (The definition of T_1 will be given presently.) We wish to substitute $\partial\Psi/\partial n = e^{-g/\epsilon} h$ into the left-hand side of equation (1) and determine g and h so that

$$L_\epsilon(\Psi) = O(1), \tag{1'}$$

that is, so that

$$L_\epsilon(\Psi) = 0$$

insofar as terms of the order of ϵ^{-2} and ϵ^{-1} are concerned. We utilize the fact that equation (1) has constant coefficients; hence $L_\epsilon(v) = 0$ implies

$$L_\epsilon\left(\frac{\partial v}{\partial n}\right) = 0, \quad (1'')$$

and, in fact,

$$\int_0^n L_\epsilon\left(\frac{\partial v}{\partial n}\right) dn = 0 = L_\epsilon\left(\int_0^n \frac{\partial v}{\partial n} dn\right),$$

so that for Ψ to satisfy (1') it is necessary and sufficient that

$$\Psi = \int_0^n e^{-\sigma'/\epsilon} h \, dn, \quad (2)$$

where g and h are determined by substitution into (1''). We make this substitution; from the terms in ϵ^{-2} and ϵ^{-1} we get

$$g_x^2 + g_y^2 = -a \quad (\text{in } T_1) \quad (3)$$

$$2ag_x h_x + 2ag_y h_y + Ah = 0 \quad (\text{in } T_1), \quad (4)$$

where

$$A = -4(g_{xx}g_x^2 + g_{yy}g_y^2) - 8g_x g_{xy} g_y + a\Delta g.$$

To equations (3) and (4) we must add appropriate initial conditions, namely

$$g = 0 \quad \text{on } B \quad (3')$$

$$h = \frac{\partial u_\epsilon}{\partial n} - \frac{\partial u_0}{\partial n} \quad \text{on } B. \quad (4')$$

1.1 We now consider the initial value problems posed by equations (3), (4), (3'), and (4'). In this section we shall define T_1 , which will be a portion of R near the boundary B , and we shall show that g and h are well-defined in T_1 by means of equations (3), (3'), (4), and (4').

For equations (3) and (3'), we must show first that the characteristic base

curves of (3) are nowhere tangent to B . But this is easily done, since the characteristics of (3) are given by [3, p. 63 ff., especially p. 65].

$$\frac{dx}{ds} = 2g_x, \quad \frac{dy}{ds} = 2g_y, \quad \frac{dg}{ds} = -2a, \quad \frac{dp}{ds} = 0, \quad \frac{dq}{ds} = 0,$$

where s is a parameter along the characteristics. Letting t be arc-length along B , we have

$$g = 0 \quad \text{on } B$$

$$g_x \frac{dx}{dt} + g_y \frac{dy}{dt} = 0 \quad \text{on } B. \quad (5)$$

To study the angle between B and the characteristic base curves, we consider the dot product between, say,

$$\left(\frac{dx}{ds}, \frac{dy}{ds} \right)$$

and the normal to B ,

$$\left(\frac{dy}{dt}, -\frac{dx}{dt} \right),$$

which would be zero in case of tangency. However, on B ,

$$\left(\frac{dx}{ds}, \frac{dy}{ds} \right) \cdot \left(\frac{dy}{dt}, -\frac{dx}{dt} \right) = 2g_x \frac{dy}{dt} - 2g_y \frac{dx}{dt} = \pm 2\sqrt{-a} \neq 0 \quad (6)$$

where the evaluation is obtained by comparison with equations (3) and (5).

Since $dg/ds = -2a > 0$, we need only make a suitable choice of the sign of s in order to assure that there exist a neighborhood T_1^* of B with respect to $R + B$ such that g exists in T_1^* , $g \in C^7$, and $g > 0$ in the region $T_1 \equiv T_1^* - B$.

Finally, for h we consider the characteristic base curves of equation (4) as given by

$$\frac{dx}{2ag_x} = \frac{dy}{2ag_y}$$

and find them to be conveniently identical with those of (3). Hence h exists and belongs to C^6 in T_1^* .

1.2 For T_1^* the definition of Ψ is now complete:

$$\Psi = \int_0^n e^{-\sigma/\epsilon} h \, dn \quad \text{in } T_1^* .$$

1.3 We now define $h^{(1)}$ and $\Psi^{(1)}$, in T_1^* . $h^{(1)}$ is the first order correction obtained by requiring

$$L_\epsilon(\Psi + \epsilon\Psi^{(1)}) = O(\epsilon) \quad \text{in } T_1^* ,$$

where

$$\Psi^{(1)} = \int_0^n e^{-\sigma/\epsilon} h^{(1)} \, dn, \quad \text{in } T_1^* .$$

By substituting we find

$$2ag_x h_x^{(1)} + 2ag_y h_y^{(1)} + Ah^{(1)} = B \quad (\text{in } T_1) \quad (7)$$

where B is defined by

$$\begin{aligned} B \equiv & h_x(8g_{xy}g_y + 4g_xg_{yy} + 12g_{xx}g_x) + h_y(8g_xg_{xy} + 4g_{xx}g_y + 12g_{yy}g_y) \\ & + h(4g_{xx}g_y + 2g_{xx}g_{yy} + 4g_{xy}^2 + 4g_xg_{xyy} + 4g_{yyy}g_y + 3g_{yy}^2 + 4g_{xxx}g_x + 3g_{xx}^2) \\ & + 2g_y^2h_{xx} + 2g_x^2h_{yy} + 8g_xg_yh_{xy} + 6g_y^2h_{yy} + 6g_x^2h_{xx} + a\Delta h. \end{aligned}$$

To equation (7) we add the evident initial value

$$h^{(1)} = 0 \quad \text{on } B;$$

noting that (7) has the same characteristic base curves as the equations for g and h , we conclude that $h^{(1)}$ is well-defined in T_1^* and belongs to C^4 .

We now turn to the problem of extending these definitions to the remainder of the region R .

2. In this section we define Ψ and $\Psi^{(1)}$ for the remainder of the region R . First, we define T_2^* as a neighborhood (with respect to $R + B$) of T_1^* . T_2 is now defined by

$$T_2 \equiv T_2^* - T_1^* .$$

In $R - T_2^*$, we define $\Psi \equiv \Psi^{(1)} \equiv 0$.

In what follows, we shall modify T_1 and T_2 by making them smaller, if necessary, so as to secure smooth boundaries. We define B_1^* as the boundary of T_1^* , and define

$$B_1 \equiv B_1^* - B .$$

Similarly we define B_2^* as the boundary of T_2 , and define

$$B_2 \equiv B_2^* - B_1 .$$

We now suppose that the parametric representing functions for B_1 and B_2 belong to C^4 . Following LEVINSON [9, p. 436], we readily interpolate Ψ and $\Psi^{(1)}$ so that each belongs to C^4 on $R + B$, and each is zero in $R - T_2^*$, while their definitions in T_1 remain unchanged.

3. Thus far we have

$$u_\epsilon = u_0 + \Psi\epsilon + \Psi^{(1)} + \dots ,$$

the three dots indicating that not all of the discrepancy between u_ϵ and u_0 has yet been accounted for. We now define Ω by

$$u_\epsilon = u_0 + \Psi + \epsilon\Psi^{(1)} + \Omega .$$

The main burden of the proof is now contained in our estimate of Ω . First, let $\mathfrak{I}$ denote a neighborhood of B (with respect to $R + B$). Then, as we shall show by a lemma in Section 4, we have the following results:

Ψ and all of its partial derivatives up to order 4 are $O(\epsilon)$ in $R - \mathfrak{I}$,
 $\epsilon\Psi^{(1)}$ and all of its partial derivatives up to order 4 are $O(\epsilon)$ in $R - \mathfrak{I}$.

Now, in T_1 , we have

$$\begin{aligned} L_\epsilon(\Psi + \epsilon\Psi^{(1)}) &= O(\epsilon), & L_\epsilon(u_\epsilon) &= c, \\ L_0(u_0) &= c, & L_\epsilon(u_0) &= \epsilon^2\Delta(\Delta + b)u_0 . \end{aligned}$$

Consequently, in T_1 ,

$$L_\epsilon(\Omega) = -\epsilon^2\Delta(\Delta + b)u_0 + O(\epsilon) .$$

Further, in $R - T_1$, we have

$$\begin{aligned} L_\epsilon(\Psi + \epsilon\Psi^{(1)}) &= O(\epsilon), & L_\epsilon(u_\epsilon) &= c, \\ L_\epsilon(u_0) &= \epsilon^2\Delta(\Delta + b)u_0 + c, & L_\epsilon(\Omega) &= O(\epsilon) . \end{aligned}$$

Combining, we see that in all of R

$$L_\epsilon(\Omega) = f' , \tag{8}$$

where $f' = O(\epsilon)$ in R .

For boundary values, we evidently have

$$\Omega = \Omega_n = 0 \quad \text{on } B . \tag{9}$$

It now remains to use equations (8) and (9) to obtain estimates on Ω . We do this by following a suggestion of PETER LAX and PAUL BERG. With the

usual notation $(f, g) \equiv \int_R fg \, dA$, we have

$$\epsilon^2 \Delta \Delta \Omega + (a + \epsilon^2 b) \Delta \Omega + ab \Omega = f'$$

$$\epsilon^2 (\Delta \Omega, \Omega) + (a + \epsilon^2 b) (\Delta \Omega, \Omega) + ab (\Omega, \Omega) = (f', \Omega),$$

and, integrating by parts,

$$\epsilon^2 (\Delta \Omega, \Delta \Omega) - (a + \epsilon^2 b) D[\Omega] + ab (\Omega, \Omega) = (f', \Omega), \quad (10)$$

where

$$D[\Omega] \equiv \int_R (\Omega_x^2 + \Omega_y^2) \, dA.$$

Now,

$$\epsilon^2 (\Delta \Omega, \Delta \Omega) \geq 0, \quad -(a + \epsilon^2 b) D[\Omega] \geq 0,$$

so that

$$ab (\Omega, \Omega) \leq (f', \Omega) \leq \sqrt{(f', f')} \sqrt{(\Omega, \Omega)},$$

or

$$\sqrt{(\Omega, \Omega)} \leq \frac{1}{ab} \sqrt{(f', f')} = O(\epsilon)$$

and $(\Omega, \Omega) = O(\epsilon^2)$.

We now use this last estimate in equation (10), recalling once more that $\epsilon^2 (\Delta \Omega, \Delta \Omega)$ is positive:

$$\begin{aligned} D[\Omega] &\leq \frac{1}{-(a + \epsilon^2 b)} (f', \Omega) - ab (\Omega, \Omega) \\ &= O(\epsilon^2). \end{aligned}$$

This completes the proof of Theorem 1. (There is no difficulty in establishing the properties of $\bar{\Omega}$ from the lemma of Section 4.)

4. Lemma. *With*

$$\Psi(n, t) \equiv \int_0^n e^{-s/\epsilon} h \, dn,$$

and

$$\Psi^{(1)}(n, t) \equiv \int_0^n e^{-s/\epsilon} h^{(1)} \, dn,$$

we wish to show that Ψ , $\epsilon \Psi^{(1)}$, and all of their partial derivatives of order four or less, are $O(\epsilon)$ in $T_1 - \mathfrak{I}$, where $\mathfrak{I}$ is any neighborhood of B . (That these estimates

can then be maintained in the interpolation strip follows as in LEVINSON [9, pp. 436, 437]. That they hold where Ψ and $\Psi^{(1)}$ are identically zero is evident.) We give the details only for Ψ . That

$$\frac{\partial \Psi}{\partial n} = O(\epsilon) \quad \text{in } R - \mathfrak{J}$$

is immediate. The same applies to all derivatives involving at least one differentiation with respect to n .

For Ψ ,

$$|\Psi| \leq \int_0^n e^{-\sigma/\epsilon} |h| \, dn.$$

In $T_1 - \mathfrak{J}$, $g \geq n\delta$ for some $\delta > 0$; moreover, we are assuming $\epsilon > 0$, so that

$$|\Psi| \leq \int_0^n e^{-n\delta/\epsilon} |h| \, dn \leq m \int_0^n e^{-n\delta/\epsilon} \, dn,$$

where

$$|h| \leq m \quad \text{in } T_1.$$

Consequently,

$$|\Psi| \leq m \left(-\frac{\epsilon}{\delta} \right) (e^{-n\delta/\epsilon} - 1) < \frac{m}{\delta} \epsilon.$$

For Ψ_t , we use an analytical translation of an argument long familiar to aeronautical engineers [5, p. 51]. We have

$$\Psi_t = \int_0^n \left(-\frac{1}{\epsilon} \right) g_t e^{-\sigma/\epsilon} h \, dn + \int_0^n e^{-\sigma/\epsilon} h_t \, dn.$$

Now, h_t being bounded, the second integral above is of a type already studied, and is $O(\epsilon)$. We consider the first integral

$$\begin{aligned} \int_0^n \left(-\frac{1}{\epsilon} \right) g_t e^{-\sigma/\epsilon} h \, dn &= \int_0^n \frac{g_t}{g_n} h e^{-\sigma/\epsilon} \left(-\frac{1}{\epsilon} g_n \right) \, dn \\ &= \left[g_t \frac{h}{g_n} e^{-\sigma/\epsilon} \right]_0^n - \int_0^n \left(g_t \frac{h}{g_n} \right)_n e^{-\sigma/\epsilon} \, dn, \end{aligned}$$

where we are of course relying on our knowledge that g_n is bounded away from zero near B . Continuing,

$$\begin{aligned} \int_0^n \left(-\frac{1}{\epsilon} \right) g_t e^{-\sigma/\epsilon} h \, dn &= g_t \frac{h}{g_n} e^{-\sigma/\epsilon} - \int_0^n \left(g_t \frac{h}{g_n} \right)_n e^{-\sigma/\epsilon} \, dn \\ &= O(\epsilon). \end{aligned}$$

Evidently this may be repeated indefinitely for higher derivatives; the integrated term gives rise to derivative terms that are exponentially small, while the integral term may be integrated by parts, in every case possessing an appropriate factor g_ϵ , with $g_\epsilon = 0$ on B .

This completes the proof of the lemma. That no stronger estimate than $O(\epsilon)$ is obtained may at first appear disappointing, in view of the apparent exponential smallness; however, one readily constructs examples (e.g., $g(n, t) = n$) which show that in general, no better result is possible.

5. *Remark.* We now discuss the limitations of the method which has been used to establish Theorem 1. One might hope, by taking enough $\Psi^{(n)}$, to obtain a residual term Ω of the order of

$$(\Omega, \Omega) = O(\epsilon^k)$$

for fixed k , and hence to exhibit the boundary layer terms $\Psi, \Psi^{(1)}$, in explicit form, not obscured by Ω . The present technique, which follows LEVINSON, cannot be used to do this. In fact, referring to pages 441 and 442, we see that there is no difficulty in T_1 , nor does the $\epsilon^2 \Delta(\Delta + b)u_0$ in $R - T_1$ cause trouble, for this may be dealt with by introducing $u_0^1, u_0^2, \dots$, analogous to LEVINSON's $U^{(n)}$, where, with $u_0 \equiv u_0^0$, $L_\epsilon \equiv L_0 + L_{\epsilon-0}$,

$$L_0[u_0^n] = -L_{\epsilon-0}[u_0^{n-1}],$$

$$u_0^n = 0 \quad \text{on } B \quad \text{for } n \geq 1.$$

The obstacle which cannot be overcome lies in the estimate of

$$L_\epsilon(\Psi + \epsilon\Psi^{(1)})$$

in $R - T_1$. Within $R - T_1$ the Ψ 's are solutions of no special differential equation, and this term is estimated only in terms of the general smallness of Ψ and its derivatives. Now, in the second-order case considered by LEVINSON, the correct estimate is $O(\epsilon^n)$ for all n (Ψ and its derivatives are exponentially small). Hence this term never becomes a problem at any approximation. In the present case, however, we have merely that Ψ and its derivatives are $O(\epsilon)$, and it is impossible to utilize refined approximations.

Nor is this difficulty merely one of our estimates. For a fourth-order equation of the present type, one does not get solely exponentially-decreasing boundary layer terms; cf. some forthcoming results of GORDON LATTA. In fact, LATTA suggests for the rectangle as an illustrative example

$$u_\epsilon = u_0 + \epsilon(e^{-x/\epsilon}h^{(0)} + e^{-(1-x)/\epsilon}k^{(0)} + m^{(0)}) + O(\epsilon^2),$$

and one encounters the boundary layer term $\epsilon m^{(0)}$, which is not exponentially small in the interior.

6. We indicate briefly one further extension of this technique. By a *composed problem* we shall mean a boundary value problem of the following type:

$$\begin{aligned} L_1(u) &= \rho_1 \quad \text{in } R, & u &= f_1 \quad \text{on } B, \\ L_2(\rho_1) &= \rho_2 \quad \text{in } R, & \rho_1 &= f_2 \quad \text{on } B, \\ &\cdot \\ &\cdot \\ &\cdot \\ L_n(\rho_{n-1}) &= \rho_n \quad \text{in } R, & \rho_{n-1} &= f_n \quad \text{on } B, \end{aligned}$$

where ρ_n and the f_k 's are given functions, and $u, \rho_1, \dots, \rho_{n-1}$ are to be determined.

One can establish the following theorem for a simple composed problem:

Theorem 2. *Let R be a bounded, simply-connected region, with a boundary B . We require that the parametric representing functions for B belong to C^4 . Let a, b , and f be functions of x and y , with*

$$(1) \quad a, b, f \in C^6 \quad \text{on } R + B$$

$$(2) \quad a < -c_1 < 0 \quad \text{on } R + B$$

$$(3) \quad b < -c_2 < 0 \quad \text{on } R + B$$

$$(4) \quad |f| < M \quad \text{on } R + B$$

$$(5) \quad \left| \Delta \frac{f}{a} \right| < m \quad \text{on } R + B, \quad \text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Define u by

$$L_1 L_2(u) \equiv (\epsilon^2 \Delta + a)(\Delta + b)u = f \quad \text{in } R$$

$$u = 0 \quad \text{on } B$$

$$L_2(u) \equiv \Delta u + bu = 0 \quad \text{on } B.$$

Define U by

$$\Delta U + bU = \frac{f}{a} \quad \text{in } R$$

$$U = 0 \quad \text{on } B.$$

Conclusion. *Then*

$$u = U + \phi + \zeta + w,$$

where

$$\begin{aligned} &\phi = 0 \text{ on } B; \\ &L_2(\phi) = e^{-g/\epsilon} h \text{ in a certain neighborhood } T_1 \text{ of } B, \text{ with } g > 0 \text{ in } R \cap T_1, \\ &\quad \text{and with } h \text{ bounded and (in fact) independent of } \epsilon; \\ &L_2(\phi) = 0 \text{ at points of } R \text{ distant from } B; \\ &\phi = O(\epsilon) \text{ uniformly in any compact subset of the open region } R; \\ &\phi \text{ is bounded in } R, \text{ independently of } \epsilon; \\ &\quad \zeta = 0 \text{ on } B; \\ &L_2(\zeta) = O(\epsilon) \text{ in } R \\ &\quad \zeta = O(\epsilon) \text{ in } R \end{aligned} \left. \vphantom{\begin{aligned} &L_2(\phi) = e^{-g/\epsilon} h \text{ in a certain neighborhood } T_1 \text{ of } B, \text{ with } g > 0 \text{ in } R \cap T_1, \\ &\quad \text{and with } h \text{ bounded and (in fact) independent of } \epsilon; \\ &L_2(\phi) = 0 \text{ at points of } R \text{ distant from } B; \\ &\phi = O(\epsilon) \text{ uniformly in any compact subset of the open region } R; \\ &\phi \text{ is bounded in } R, \text{ independently of } \epsilon; \\ &\quad \zeta = 0 \text{ on } B; \\ &L_2(\zeta) = O(\epsilon) \text{ in } R \\ &\quad \zeta = O(\epsilon) \text{ in } R \end{aligned}} \right\} \text{uniformly in } \epsilon;$$

and

$$|w| \leq \frac{m\epsilon^2}{c_1 c_2} + O(e^{-\delta/\epsilon}) \quad \text{on } R + B.$$

The proof of this theorem follows that of Theorem 1, except for the following lemma.

Lemma. *Consider ϕ defined by*

*$L_2(\phi) = e^{-g/\epsilon} h \equiv z$ in some neighborhood T_1 of B , $g = 0$ on B , $g > 0$ in $T_1 - B$,
The functions g and h being obtained as in Theorem 1;*

$L_2(\phi) = z$ in $R - T_1$, where z is extrapolated to zero outside of some larger neighborhood of B ;

$\phi = 0$ on B .

Then we have $\phi = O(\epsilon)$ uniformly in any compact subset of the open region R .

Proof. Let R_0 be a compact subset of R . For some $\gamma > 0$, we define T_3 as the set of all points of R whose distance from B does not exceed γ . Evidently, for a suitable choice of γ , we shall have

$$d(R_0, T_3) = \mu > 0,$$

for some positive μ , where $d(R_0, T_3)$ denotes the distance from R_0 to T_3 .

An elementary solution for $\Delta u - k^2 u = 0$ can be found in the form

$$I_0(kr) \log r + V,$$

where $I_0(kr)$ denotes the modified Bessel function of the first kind, of order zero and where V is regular in R [7, p. 141 ff.]. From this fundamental solution

we may construct a Green's function, by solving the appropriate boundary value problem. Denote this Green's function by $G(P, Q)$. Then, since the non-negative-minimum argument is applicable to the equation $\Delta u - k^2 u = 0$, we deduce that $G \geq 0$ in R . The integral of G over all of R is bounded, $\int G(P, Q) dA_Q < N$.

Finally, the problem

$$\begin{aligned}\Delta\phi + b\phi &= z \quad \text{in } R \quad (b < 0) \\ \phi &= 0 \quad \text{on } B\end{aligned}$$

determines ϕ as

$$\phi(P) = - \int_R G(P, Q) z(Q) dA_Q ,$$

We note that, for $Q \in R - T_3$, we have

$$z(Q) = O(e^{-\delta/\epsilon})$$

for some $\delta > 0$.

The verification that $\phi(P) = O(\epsilon)$ uniformly for $P \in R_0$ can now proceed without difficulty.

$$\begin{aligned}|\phi(P)| &\leq \left| \int_{R-T_3} G(P, Q) z(Q) dA_Q \right| + \left| \int_{T_3} G(P, Q) z(Q) dA_Q \right| \\ &\leq O(e^{-\delta/\epsilon}) \int_{R-T_3} G(P, Q) dA_Q + L \cdot \int_{T_3} z(Q) dA_Q \\ &= O(e^{-\delta/\epsilon}) + L \cdot \int_{T_3} z(Q) dA_Q ,\end{aligned}$$

where $G(P, Q) < L$ for $P \in R_0$, $Q \in T_3$, by virtue of the fact that $d(R_0, T_3) = \mu > 0$.

It remains to estimate $\int_{T_3} z(Q) dA_Q$. To do this, we introduce the arc-length co-ordinate t along B , and the co-ordinate n along the inner normals to B . As in LEVINSON [9], if T_3 is sufficiently small we are guaranteed that

$$0 < \frac{\partial(x, y)}{\partial(n, t)} < \infty$$

in T_3 . Let $n = 0$ be the equation of B . Suppose, also, that $T_3 \subset T_1$. Then

$$\int_{T_3} z(Q) dA_Q = \iint_T z(Q) \frac{\partial(x, y)}{\partial(n, t)} dn dt ,$$

and the first integration, on n , is readily verified to yield a result that is $O(\epsilon)$:

$$\begin{aligned}
 \left| \int_0^\gamma e^{-\sigma/\epsilon} h \frac{\partial(x, y)}{\partial(n, t)} dn \right| &= \left| \int_0^\gamma e^{-\sigma/\epsilon} g_n \frac{h}{g_n} \frac{\partial(x, y)}{\partial(n, t)} dn \right| \\
 &\leq \left| \int_0^\gamma e^{-\sigma/\epsilon} g_n P dn \right| \\
 &= \left| (P)(-\epsilon) \int_0^\gamma e^{-\sigma/\epsilon} \left(\frac{-g_n}{\epsilon} \right) dn \right| \\
 &= \epsilon P \left| 1 - e^{-\sigma(\gamma)/\epsilon} \right| \\
 &< \epsilon P,
 \end{aligned}$$

where

$$\left| \frac{h}{g_n} \frac{\partial(x, y)}{\partial(n, t)} \right| < P,$$

which must hold for sufficiently small T_3 . The integration on t is over a bounded interval, and does not change this estimate.

This completes the proof that $\phi(P) = O(\epsilon)$ for $P \in R_0$.

Remark. In connection with Theorem 1, one might inquire how to determine the appropriate boundary value problem for u_0 . We show a systematic approach to this question by employing a device of OLEJNIK [14]. It is necessary to assume *a priori* that one is interested in boundary layer terms of the form

$$\Psi = \epsilon e^{-\sigma/\epsilon} h + \epsilon R,$$

where $u_\epsilon = u_0 + \Psi$, at least near the boundary.

We consider the equation

$$\epsilon^2 \Delta \Delta u_\epsilon + A \Delta u_\epsilon + B(u_\epsilon)_x + C(u_\epsilon)_y + D u_\epsilon = f$$

with boundary conditions

$$\left. \begin{aligned} B_1(u_\epsilon) &\equiv a u_\epsilon + b \frac{\partial}{\partial n} (u_\epsilon) = g \\ B_2(u_\epsilon) &\equiv c u_\epsilon + e \frac{\partial}{\partial n} (u_\epsilon) = k \end{aligned} \right\} \quad \text{on } B$$

We now suppose that $B_1(u_0) = g$ on B , and we seek to determine the range of permissible choices of a , b , c , and e . On B , we have

$$B_1(\Psi) = 0, \quad B_2(\Psi) = k - B_2(u_0) \equiv H.$$

These equations become

$$a\epsilon h + a\epsilon R + b(-g_n)h + b\epsilon h_n + b\epsilon R_n = 0,$$

$$c\epsilon h + c\epsilon R + e(-g_n)h + e\epsilon h_n + e\epsilon R_n = H.$$

Evidently we must have, from terms in ϵ^0 ,

$$b = 0,$$

But then there is no loss of generality if we assume $a = 1$, $c = 0$, $e = 1$.

This is precisely the problem considered in Theorem 1. This result also confirms our expectations. If one compares with equation (3.7) of LIN [11], one finds that, in a generalized sense, u_ϵ corresponds to a viscous fluid-flow problem, while u_0 solves the corresponding inviscid problem. We should expect to obtain the correct boundary values for u_0 by allowing slip, *i.e.*, flow tangential to the boundary, while restricting flow normal to the boundary just as in the viscous case. Now u corresponds to the fluid dynamical stream function Ψ . Therefore u_ϵ corresponds to a normal component of velocity, while u_n corresponds to a tangential component. By relaxing the prescribed value of u_n , we are, in fact, allowing slip. This is precisely what one would expect.

Presented to the American Mathematical Society in Maryland, October, 1955.

The preparation of this paper has been supported, in part, by the Office of Ordnance Research, U. S. Army, under Contract DA-19-020-ORD-2245.

University of New Hampshire

BIBLIOGRAPHY

1. PAUL W. BERG & PETER D. LAX, *Fourth order operators*, Rendiconti del Sem. Mat. di Torino **11**, 1951-2, pp. 343-358.
2. G. F. CARRIER, *Boundary layer problems in applied mechanics*, Advances in Applied Mechanics **3**, 1953, pp. 1-19.
3. R. COURANT & D. HILBERT, *Methoden der mathematischen Physik* **2**, Berlin, 1937.
4. K. O. FRIEDRICHS, *Asymptotic phenomena in mathematical physics*, Gibbs Lecture, Pittsburgh meeting of the American Mathematical Society, December 27, 1954.
5. S. GOLDSTEIN, *Modern Developments in Fluid Dynamics* **1**, London, 1938.
6. I. S. GRADŠTEIN, *Linear equations with variable coefficients and small parameters in the highest derivatives*, Matematičeskii Sbornik (N. S.) **27** (69), 1950, pp. 47-68 (American Mathematical Society Translation No. 82, 1953).
7. FRITZ JOHN, *Special topics in partial differential equations*, Institute for Mathematics and Mechanics, New York University, New York (mimeographed notes, 1953).
8. J. J. LEVIN, *First order partial differential equations containing a small parameter*, Journal of Rational Mechanics and Analysis **4**, 1955, pp. 481-501.
9. N. LEVINSON, *The first boundary value problem for $\epsilon \Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ for small ϵ* , Annals of Mathematics **51**, 1950, pp. 428-445.
10. C. C. LIN, *Hydrodynamic stability*, Proceedings of the Fifth Symposium in Applied Mathematics of the American Mathematical Society, 1954, pp. 1-18.
11. ———, *On the stability of two-dimensional parallel flows*, Part I, Quarterly of Applied Mathematics **3**, 1945, pp. 117-142.
12. CARLO MIRANDA, *Equazioni alle Derivate Parziali di Tipo Ellittico*, Berlin, 1955 (especially p. 174).
13. O. A. OLEJNIK [Олейник], *On the second boundary value problem for elliptic equations with a small parameter in the highest derivatives*, Doklady A. N. U.S.S.R. **79**, 1951, pp. 735-737.
14. ———, *On equations of elliptic type with a small parameter in the highest derivatives*, Math. Sbornik **31**, 1952, pp. 104-117.
15. ERICH ROTHE, *Über asymptotische Entwicklungen bei Randwertaufgaben elliptischer partieller Differentialgleichungen*, Mathematische Annalen **108**, 1933, pp. 578-594.
16. ———, *Über asymptotische Entwicklungen bei Randwertaufgaben der Gleichung $\Delta \Delta u + \lambda u = \lambda^k \Psi$* , Mathematische Annalen **109**, 1934, pp. 267-272.
17. WOLFGANG WASOW, *Asymptotic solution of boundary value problems for the differential equation $\Delta U + \lambda(\partial/\partial x)U = \lambda f(x, y)$* , Duke Journal **11**, 1944, pp. 405-415.
18. ———, *Asymptotic solution of the differential equation of hydrodynamic stability in a domain containing a transition point*, Annals of Mathematics **58**, 1953, pp. 222-253.