The Elementary Basic Principles of the Unified Theory of Relativity. C₄ General Case

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CONTENTS

Introduction	420
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CHAPTER I: Prerequisites

1. Previous results	422
2. The tensors $*g^{\lambda\nu}$, S_λ and $m_{\lambda\mu}$	427
3. The tensor $K_{\omega\mu\nu}$	431
4. The tensors $S_{\omega\mu\nu}$ and $U_{\nu\omega\mu}$	432

CHAPTER II: The third class

5. The tensors $K_{\omega\mu\nu}$, $U_{\nu\omega\mu}$ and $A_{\omega\mu}$	437
6. Maxwell's equations	439
7. Continuation	440
8. Gravitational field	443
9. Continuation and recapitulation	445
10. Mass density. Motion	446
11. Discussion of (10.5)	447
12. The law of inertia	448
13. Discussion of (12.3)	450
14. Auxiliary results	450
15. Newtonian laws	453

CHAPTER III: The first and second class

16. The tensors $K_{\omega\mu\nu}$, $U_{\nu\omega\mu}$ and $A_{\omega\mu}$	457
17. Maxwell's equations	458
18. Auxiliary lemmas	460
19. Gravitational field	461
20. The law of inertia	463
21. Newtonian laws	464

APPENDIX

22. Paths of photons	467
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INTRODUCTION

This is the last of a series of papers[†] dealing with “Problem C” of the unified field theory (see [9], [10], and [14]).*) It is divided into three chapters and a short appendix.

The first chapter recapitulates briefly the results derived in previous papers and needed later on in this paper (Section 1). Section 2 deals with the significance of the conditions (1.25a) of “Problem C” for the tensor $m_{\lambda\mu}$ which later will be identified with the electro-magnetic field. In Section 3 we express one of the basic tensors, namely $K_{\omega\mu\nu}$, by means of $m_{\lambda\mu}$, and a vector which will lead to the current vector. Section 4 presents necessary conditions for “Problem C” to be considered simultaneously for all three possible classes of the basic tensor $g_{\lambda\mu}$. These conditions are not satisfied for all fields $g_{\lambda\mu}$. Therefore, it is necessary to deal with “Problem C” for the third class in a separate (second) chapter, while the third chapter is devoted to “Problem C” for the first two classes, which can be handled simultaneously.

In the second chapter we consider first the necessary consequences of the results of the first chapter for the third class (Section 5). Sections 6 and 7 develop physical interpretations of the conditions (1.25a, b). These conditions lead to the Maxwell equations for the electromagnetic field $m_{\lambda\mu}$ and to the “electromagnetic field equations” whose solutions satisfy the Maxwell equations. Section 8 concerns the “gravitational field equations” derived from the condition (1.25c). In Section 9 the main results are recapitulated. Thus the electromagnetic and gravitational fields are interrelated. In the unified theory the electromagnetic field is always present; hence, we might look upon it as a primary field which according to the previous results creates the gravitational field. However, there is at least one known electromagnetic field which does not create a gravitational field (*i.e.*, the field of the plane wave in the electromagnetic theory of light). Section 10 is devoted to the appropriate definition of mass by means of electric charge. Moreover, the four identities derived from the gravitational field equations are interpreted as equations of motion. However, there are strong objections to these equations (Section 11). In order to dispose of these objections, in Sections 12 and 13 we replace conditions (1.25b) by another set of conditions and in this way obtain the equations of motion in the form of a “unified law of inertia.” A particle moving freely in the unified field $g_{\lambda\mu}$ describes an auto-parallel line of the unified connection $\Gamma_{\lambda\mu}^{\nu}$, defined by $g_{\lambda\mu}^{**}$). In Section 14 we present results necessary for a detailed discussion of the unified law of inertia. It turns out (Section 15) that there are coordinate

[†]The author's researches on the Unified Field Theory will be presented shortly in a book, *Geometry of Einstein's Unified Field Theory*, to be published by Noordhoff, Groningen.

*)The numbers in brackets refer to the bibliography at the end.

**)The law of inertia in the so called general relativity may be obtained from this law by replacing $g_{\lambda\mu}$ and $\Gamma_{\lambda\mu}^{\nu}$ by $h_{\lambda\mu}$ and $\{\lambda_{\mu}^{\nu}\}$.

systems for which the unified law of inertia reduces “in the first approximation” to the classical Newtonian law of gravitation while “the second approximation” reduces the unified law of inertia to the relativistic^{*)} second Newtonian law for the electromagnetic force vector. Hence we may say that the unified law of inertia replaces the gravitational force vector as well as the relativistic^{*)} electromagnetic force vector by the unified field $g_{\lambda\mu}$.

According to the results of Section 4, the first two classes can not be handled simultaneously with the third class. Therefore, Chapter 3 is devoted to “Problem C” for the first and second classes. Since the general working scheme of this chapter is the same as in the previous chapter, no detailed description is needed.

Section 22 is a short appendix devoted to the elementary geometry of the Michelson-Morley experiment as suggested by the unified field theory and by the assumption that the velocity of light is constant. The latter does not exclude—at least theoretically—Miller’s result when it is properly interpreted. This is an astonishing conclusion, since in classical physics Miller’s result has been looked upon as being at variance with Michelson & Morley’s result. The explanation is a simple one: in the unified field theory, even in the absence of a gravitational field, the first curvature of a path of a photon depends on its initial direction.**)

^{*)}We refer here to the so called “general” relativity *i.e.*, the gravitational field theory in the presence of an electromagnetic field.

^{**)}Unless otherwise stated, all Greek (Latin) indices in this paper run from I to IV (1 to 4). Both kinds of indices follow the usual rule of summation with the exception of x, y, z (see *e.g.* equation (1.15)).

CHAPTER I: *Prerequisites*

1. Previous results. In this section we recapitulate briefly and without proof results needed later in this paper.

Problem A.

The basic real tensor $g_{\lambda\mu}$, never symmetric, may be split into its symmetrical part $h_{\lambda\mu} \stackrel{\text{def}}{=} g_{[\lambda\mu]}$ and its skew symmetric part $k_{\lambda\mu} \stackrel{\text{def}}{=} g_{[\lambda\mu]}$. Denoting by $\mathfrak{h}$ the determinant of $h_{\lambda\mu}$, we assume that $h_{\lambda\mu}$ (which plays the role of metric tensor) is of signature $---+$, so that $\mathfrak{h} < 0$. Moreover, denoting by $\mathfrak{g}$, $\mathfrak{k}$ the determinants of $g_{\lambda\mu}$, $k_{\lambda\mu}$, we assume $\mathfrak{g} < 0$ and put

$$(1.1a) \quad g \stackrel{\text{def}}{=} \mathfrak{g}/\mathfrak{h} > 0,$$

$$(1.1b) \quad k \stackrel{\text{def}}{=} \mathfrak{k}/\mathfrak{h} \leq 0,$$

$$(1.2) \quad 2K \stackrel{\text{def}}{=} \frac{1}{2}k_{\alpha\beta}k_{\gamma\delta}h^{\alpha\gamma}h^{\beta\delta}, \quad D \stackrel{\text{def}}{=} K^2 - k.$$

The tensor $g_{\lambda\mu}$ belongs

to the first class, if $kD \neq 0$,

to the second class, if $k = 0$, $D \neq 0$,

to the third class, if $k = 0$, $D = 0$.*

For the first class and for the second class there are exactly four linearly independent null-vectors ${}_xe^{\nu}$ satisfying (cf. [15])

$$(1.3) \quad \lambda_x {}_xe^{\nu} = k_{\lambda}{}^{\nu} {}_xe^{\nu}$$

and

$$(1.4) \quad h_{ij} = h_{ij} \stackrel{\text{def}}{=} {}_ie^{\lambda} {}_je^{\mu} h_{\lambda\mu} = \begin{cases} 1 & \text{for } i = 1, \quad j = 2, \\ 1 & \text{for } i = 3, \quad j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

The system of inverse vectors ${}^ie_{\lambda}$ is defined by

$$(1.5) \quad {}_ie^{\lambda} {}^ie_{\lambda} = \delta_i^i, \quad {}_ie^{\nu} {}^ie_{\lambda} = \delta_{\lambda}^{\nu}.$$

The λ_x are solutions of

$$(1.6a) \quad \lambda^4 + 2\lambda^2 K + k = 0,$$

*There are no other possibilities. Cf. [6], [20].

i.e.,

$$(1.6b) \quad \lambda_1 = -\lambda_2 = i\sqrt{\sqrt{D} + K},$$

$$(1.6c) \quad \lambda_3 = -\lambda_4 = \sqrt{\sqrt{D} - K}.$$

The vectors ${}_xe''$ are called the eigenvectors of k_λ'' ; the scalars λ_x are the corresponding eigenvalues. It is not difficult to express ${}_xe''$, ${}_xe_\lambda$ as well as ${}_1e^{[\lambda}{}_2e^{\nu]}$, ${}_3e^{[\lambda}{}_4e^{\nu]}$ and ${}_1e_{[\lambda}{}_2e_{\mu]}$, ${}_3e_{[\lambda}{}_4e_{\mu]}$ in terms of $g_{\lambda\mu}$ by means of transvections and multiplications (cf. [15], [16]). The frame ${}_xe''$, ${}_xe_\lambda$ is used as a nonholonomic frame:

$$T_i^{\cdot\cdot\cdot} = {}_xe''_\nu e^\lambda \cdots T_\lambda^{\cdot\cdot\cdot}.$$

In particular, the nonholonomic components g_{ii} are given by

$$(1.7a) \quad ((g_{ii})) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 + \lambda_1 & 0 & 0 \\ 1 + \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + \lambda_3 \\ 0 & 0 & 1 + \lambda_4 & 0 \end{bmatrix}.$$

As far as the third class is concerned, there is a two parametric set of linearly independent vectors ${}_xe''$ (the ${}_4e''$ is the only eigenvector belonging to the single eigenvalue $\lambda = 0$) which gives rise to the nonholonomic components (cf. [15])

$$(1.7b) \quad ((g_{ii})) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

These vectors are given up to transformations

$$(1.8) \quad {}_xe'' = A_i^j {}_xe'',$$

where

$$(1.9) \quad ((A_i^j)) \stackrel{\text{def}}{=} \begin{bmatrix} \epsilon & u \sin v & u \cos v & \frac{1}{2}\epsilon u^2 \\ 0 & \epsilon & 0 & u \sin v \\ 0 & 0 & \epsilon' & \epsilon\epsilon' u \cos v \\ 0 & 0 & 0 & \epsilon \end{bmatrix}.$$

Here $\epsilon, \epsilon' = \pm 1$ and u, v are arbitrary parameters. Equation (1.7b) is invariant with respect to (1.8).

Problem B.*)

The basic connection $\Gamma_{\lambda\mu}^\nu$ is given by (cf. [4])

$$(1.10a) \quad \partial_\omega g_{\lambda\mu} = \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha} .$$

Put

$$(1.11a) \quad \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} \stackrel{\text{def}}{=} \frac{1}{2} h^{\nu\alpha} (\partial_\lambda h_{\alpha\mu} + \partial_\mu h_{\alpha\lambda} - \partial_\alpha h_{\lambda\mu}),$$

$$(1.11b) \quad S_{\lambda\mu}{}^\nu \stackrel{\text{def}}{=} \Gamma_{[\lambda\mu]}^\nu ,$$

$$(1.11c) \quad U^\nu{}_{\lambda\mu} \stackrel{\text{def}}{=} 2h^{\nu\alpha} S_{\alpha(\lambda} k_{\mu)\beta} .$$

Then the solution of (1.10a) is

$$(1.10b) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu}$$

where of course S and U are still unknown. In the case of the third class we have

$$(1.12a) \quad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^\alpha K_{\omega]\alpha\beta} k_\nu{}^\beta ,$$

where

$$(1.13) \quad \begin{aligned} K_{\omega\mu\nu} &\stackrel{\text{def}}{=} \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} + \nabla_\nu k_{\omega\mu} \\ &= 3\nabla_{[\omega} k_{\nu\mu]} + 2\nabla_\nu k_{\omega\mu} . \end{aligned}$$

In dealing with the first two classes, we easily describe $S_{\omega\mu\nu}$ by means of the nonholonomic frame of ${}_i e^\nu$, ${}^i e_\lambda$. Put

$$(1.14) \quad \lambda_{ijk} = 1 + \lambda_i \lambda_j + \lambda_j \lambda_k + \lambda_k \lambda_i ,$$

and denote by $L_{\mu\lambda\nu}$ the tensor whose nonholonomic components are given by

$$(1.15) \quad L_{xyz} \stackrel{\text{def}}{=} \frac{1}{\lambda_{xyz}} K_{xyz} .$$

Then

$$(1.12b) \quad 2S_{\omega\mu\nu} = L_{\omega\mu\nu} + 3L_{[\alpha\beta\nu]} k_{[\omega}{}^\alpha k_{\mu]}{}^\beta .$$

The construction (1.15) is possible if and only if

$$(1.16a) \quad g \neq 0 \quad \text{for the } \left\{ \begin{smallmatrix} \text{first} \\ \text{second} \end{smallmatrix} \right\} \text{ class.}$$

$$(1.16b) \quad g(g-2) \neq 0$$

The tensor $U_{\lambda\mu}^\nu$ follows easily from (1.12b) and (1.11c). Hence (1.16a, b) are necessary and sufficient conditions for the existence and uniqueness of the solution (1.10b) of (1.10a) for the first and second classes. As far as the third

*) Cf. [7], [8], [15], [16].

class is concerned, (1.10a) always admits a unique solution if the first of the conditions (1.1) is satisfied. If D_μ denotes the symbol of the covariant derivative with respect to the connection $\Gamma_{\lambda\mu}^\nu$, then (1.10a) is equivalent to

$$(1.10c) \quad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}.$$

Note. Put

$${}_xE_\lambda{}^\nu \stackrel{\text{def}}{=} {}_xe^\nu{}_ze_\lambda.$$

It has been shown^{*)} that the construction of the tensors ${}_xE_\lambda{}^\nu$ does not require a knowledge of the corresponding eigenvectors. On the other hand we have from (1.15)

$$L_{\omega\mu\nu} = K_{\alpha\beta\gamma} \sum_{x,y,z} \frac{1}{\lambda_{xyz}} {}_xE_\omega{}^\alpha {}_yE_\mu{}^\beta {}_zE_\nu{}^\gamma.$$

Hence the construction of $L_{\omega\mu\nu}$ from (1.15) does not require any knowledge of the eigenvectors.

Problem C.**)

Put

$$(1.17) \quad S_\lambda \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^\alpha$$

and

$$(1.18a) \quad R_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} \partial_\mu \Gamma_{\lambda\omega}{}^\nu - \partial_\omega \Gamma_{\lambda\mu}{}^\nu + \Gamma_{\alpha\mu}{}^\nu \Gamma_{\lambda\omega}{}^\alpha - \Gamma_{\alpha\omega}{}^\nu \Gamma_{\lambda\mu}{}^\alpha,$$

$$(1.18b) \quad R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha.$$

Then the conditions imposed by EINSTEIN [4] on the connection $\Gamma_{\lambda\mu}^\nu$ (*i.e.*, on $g_{\lambda\mu}$) amount to

$$(1.19a) \quad S_\lambda = 0,$$

$$(1.19b) \quad R_{\mu\lambda} = \partial_{[\mu} X_{\lambda]} ,$$

where X_λ is an arbitrary vector. The existence of the solution $g_{\lambda\mu}$, X_λ of (1.19) is assured by the results of LICHNEROWICZ [18], [19]. If we confine ourselves to the conditions (1.19), then there is only one skew symmetric tensor $m_{\lambda\mu}$ expressible in terms of $g_{\lambda\mu}$ and not containing the derivatives of $g_{\lambda\mu}$ such that (1.19a) implies

$$(1.20) \quad \partial[\omega m_{\mu\lambda}] = 0.$$

^{*)}Cf. (1.15) and (1.16) in [16].

^{**)}Cf. [9], [10], [11], [12], [13], [14].

This tensor is*)

$$(1.21) \quad m_{\lambda\nu} \stackrel{\text{def}}{=} \frac{1}{\sqrt{g}} [\kappa \sqrt{|k|} k_{\lambda\nu} - \tfrac{1}{2} \epsilon_{\lambda\nu\alpha\beta} \sqrt{|\mathfrak{h}|} k^{\alpha\beta}].$$

One can identify $m_{\lambda\nu}$ with the electromagnetic field and $h_{\lambda\mu}$ with the gravitational field. The main purpose of “Problem C” is to transcribe (1.19) in terms of the electromagnetic field, the gravitational field, and the electric charge, obtaining the electromagnetic field equations (MAXWELL’S equations) and, in a slightly changed theory, the “unified law of inertia.” In earlier papers ([10], [14]) we were concerned with Problem C under certain restrictive conditions, namely

$$(1.22) \quad U'_{\lambda\mu} = 0 \quad \text{in} \quad [10],$$

$$(1.23) \quad U'_{\lambda\mu} = 2p''h_{\lambda\mu} - \delta_\lambda'' p_\mu - \delta_\mu'' p_\lambda \quad \text{in} \quad [14].$$

In this paper we shall deal with “Problem C” without imposing any restrictive condition on $g_{\lambda\mu}$. However, we shall consider a more general condition than (1.19b), namely,

$$(1.24) \quad R_{\mu\lambda} = \partial_{[\mu} X_{\lambda]} - \varphi h_{\lambda\mu},$$

where φ is an unspecified scalar. For $\varphi = 0$ we again obtain (1.19b). Hence the basic conditions of the problem are

$$(1.25a) \quad S_\lambda = 0,$$

$$(1.25b) \quad R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]},$$

$$(1.25c) \quad R_{(\mu\lambda)} = -\varphi h_{\mu\lambda}.$$

Denote by $H_{\omega\mu\lambda}{}^\nu$ the curvature tensor of $h_{\lambda\mu}$

$$(1.26a) \quad H_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} \partial_\mu \left\{ \begin{matrix} \nu \\ \lambda\omega \end{matrix} \right\} - \partial_\omega \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \left\{ \begin{matrix} \nu \\ \alpha\mu \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \lambda\omega \end{matrix} \right\} - \left\{ \begin{matrix} \nu \\ \alpha\omega \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \lambda\mu \end{matrix} \right\}$$

and by $H_{\mu\lambda}$ its contraction

$$(1.26b) \quad H_{\mu\lambda} \stackrel{\text{def}}{=} H_{\omega\mu\lambda}{}^\omega.$$

*) $\epsilon_{\lambda\nu\alpha\beta}$ is the covariant indicator defined in [9], where also the contravariant indicator $\mathfrak{E}^{\lambda\nu\alpha\beta}$ was introduced. Moreover, $\kappa = \text{sgn } \mathfrak{E}^{\omega\mu\lambda\nu} k_{\omega\mu} l_{\lambda\nu}$.

Using (1.10b) we may write (cf. [9] pp. 120 and 121)

$$(1.27a) \quad R_{[\omega\mu]} = \nabla_\alpha S_{\omega\mu}{}^\alpha + 2U^\beta{}_{\alpha[\omega} S_{\mu]\beta}{}^\alpha + (U_\beta + S_\beta) S_{\omega\mu}{}^\beta + \partial_{[\omega} S_{\mu]}$$

$$(1.27b) \quad R_{(\mu\lambda)} = H_{\mu\lambda} + S_{\alpha\mu}{}^\beta S_{\lambda\beta}{}^\alpha - (S_\beta + U_\beta) U^\beta{}_{\lambda\mu} \\ - \nabla^\alpha U_{\alpha\lambda\mu} + U^\beta{}_{\alpha\mu} U^\alpha{}_{\beta\lambda} + \nabla_{(\mu} (U_{\lambda)} + S_{\lambda)}) \\ U_\mu \stackrel{\text{def}}{=} U^\alpha{}_{\alpha\mu} .$$

These formulæ may be transcribed in terms of $U^\nu{}_{\lambda\mu}$ as follows:

$$(1.28a) \quad R_{[\omega\mu]} = \tfrac{1}{2} \nabla^\nu K_{\omega\mu\nu} + A_{\omega\mu} + \partial_{[\omega} S_{\mu]} + S^\beta (\tfrac{1}{2} K_{\omega\mu\beta} - 2U_{\alpha\beta[\mu} k_{\omega]}{}^\alpha)$$

$$(1.28b) \quad R_{(\mu\lambda)} = H_{\mu\lambda} - \tfrac{1}{4} K_{\alpha\mu}{}^\beta K_{\beta\lambda}{}^\alpha - B_{\mu\lambda} + \nabla_{(\mu} S_{\lambda)} - S_\beta U^\beta{}_{\lambda\mu} ,$$

where

$$(1.29a) \quad A_{[\omega\mu]} = A_{\omega\mu} \stackrel{\text{def}}{=} 2\nabla^\alpha U_{\beta\alpha[\omega} k_{\mu]}{}^\beta \\ + 2U^\beta{}_{\alpha[\omega} (\tfrac{1}{2} K_{\mu]\beta}{}^\alpha - k_{\mu]}{}^\gamma U_\gamma{}^\alpha{}_\beta + U^\gamma{}_\mu k_{\beta\gamma}{}^\alpha) \\ + U^\beta (\tfrac{1}{2} K_{\omega\mu\beta} - 2U_{\alpha\beta[\mu} k_{\omega]}{}^\alpha)$$

$$(1.29b) \quad -B_{(\mu\lambda)} = -B_{\mu\lambda} \stackrel{\text{def}}{=} -\nabla^\alpha U_{\alpha\lambda\mu} + U^\beta{}_{\alpha\mu} U^\alpha{}_{\beta\lambda} + \nabla_\mu U_\lambda - U_\beta U^\beta{}_{\lambda\mu} \\ - (K_{\lambda\beta}{}^\alpha U_\gamma{}^\beta{}_{[\mu} + K_{\mu\beta}{}^\alpha U_\gamma{}^\beta{}_{\lambda]}) k_{\alpha]}{}^\gamma + 4U_\gamma{}^\beta{}_{[\mu} k_{\alpha]}{}^\gamma U_\delta{}^\alpha{}_{\beta\lambda]}{}^\delta .$$

Later we shall often use equations (1.28).

2. The tensors $*g^{\lambda\nu}$, S_λ , and $m_{\lambda\mu}$. According to (1.1a), there is only one tensor $*g^{\lambda\nu}$ satisfying the condition

$$(2.1) \quad g_{\lambda\mu} *g^{\lambda\nu} = g_{\mu\lambda} *g^{\nu\lambda} = \delta_\mu{}^\nu .$$

We denote by $*h^{\lambda\nu}$, $*k^{\lambda\nu}$ its symmetric and skew symmetric part.

Theorem (2.2). *We have*

$$(2.2a) \quad *h^{\lambda\nu} = \frac{1}{g} [h^{\lambda\nu}(1 + 2K) + k^{\nu\alpha} k_\alpha{}^\lambda],$$

$$(2.2b) \quad *k^{\lambda\nu} = \frac{1}{g} \left[\mathfrak{h} k^{\lambda\nu} + \frac{\kappa}{2} \sqrt{\mathfrak{f}} \mathfrak{E}^{\lambda\nu\alpha\beta} k_{\alpha\beta} \right].$$

Proof. First, we have obviously

$$(2.3a) \quad \mathfrak{g} = \frac{1}{4!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} g_{\lambda\nu} g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} g_{\alpha_3\beta_3} ,$$

so that

$$(2.4) \quad \mathfrak{g} * g^{\lambda\nu} = \frac{1}{3!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} g_{\alpha_3\beta_3}.$$

The right hand term is the sum of the following four products:

$$(2.5a) \quad \frac{1}{3!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} h_{\alpha_1\beta_1} h_{\alpha_2\beta_2} h_{\alpha_3\beta_3} = \mathfrak{h} h^{\lambda\nu},$$

$$(2.5b) \quad \frac{3}{3!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} h_{\alpha_1\beta_1} h_{\alpha_2\beta_2} k_{\alpha_3\beta_3} = \mathfrak{h} k^{\lambda\nu},$$

$$(2.5c) \quad \frac{3}{3!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} k_{\alpha_1\beta_1} k_{\alpha_2\beta_2} h_{\alpha_3\beta_3} = \mathfrak{h}(k^\nu{}_\alpha k^{\alpha\lambda} + 2h^{\lambda\nu} K),$$

$$(2.5d) \quad \frac{1}{3!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} k_{\alpha_1\beta_1} k_{\alpha_2\beta_2} k_{\alpha_3\beta_3} = \frac{1}{2} \kappa \sqrt{|\mathfrak{f}|} \mathfrak{E}^{\lambda\nu\alpha\beta} k_{\alpha\beta}^{*}).$$

The equations (2.2) follow immediately from (2.4), (2.5), and

$$(2.6) \quad *g^{\lambda\nu} = *h^{\lambda\nu} + *k^{\lambda\nu}.$$

Note I. Formula (2.2b) has been already derived in [9] from the equation equivalent to (2.3a),

$$(2.3b) \quad g = 1 + 2K + k.$$

In subsequent considerations we shall need the following

Lemma (2.1). *The tensor $*h^{\lambda\nu}$ is of rank 4.*

Proof. Consider first the first two classes. Then we have, by virtue of (1.3) and (1.5),

$$(2.7a) \quad k_\lambda{}^\nu = \Sigma \lambda_x{}^x e_\lambda{}^x e^\nu{}_x,$$

$$(2.7b) \quad k^{\nu\alpha} k_{\alpha\lambda} = \Sigma \lambda_x{}^2 e_\lambda{}^x e^\nu{}_x,$$

^{*)}In order to compute these formulæ we have to use the equations I(2.2) in [9], I(2.4) in [9] and I(3.2) in [9]. We state here one application of the formulæ:

$$\begin{aligned} & \frac{3}{3!} \mathfrak{E}^{\lambda\alpha_1\alpha_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} h_{\alpha_1\beta_1} h_{\alpha_2\beta_2} k_{\alpha_3\beta_3} \\ &= \frac{1}{2} h^{\lambda\omega} \mathfrak{E}_{\omega\beta_1\beta_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} k^{\alpha_3}{}_{\beta_3} \\ &= \frac{1}{2} h^{\lambda\omega} \mathfrak{E}_{\omega\beta_1\beta_2\alpha_3} \mathfrak{E}^{\nu\beta_1\beta_2\beta_3} h k^{\alpha_3}{}_{\beta_3} = \frac{4}{2} \mathfrak{h} h^{\lambda\omega} \delta_{[\omega\alpha_3]}^{\nu\beta_3} k^{\alpha_3}{}_{\beta_3} \\ &= \mathfrak{h} h^{\lambda\omega} (\delta_{\omega\alpha_3}^{\nu\beta_3} - \delta_{\alpha_3\omega}^{\nu\beta_3}) k^{\alpha_3}{}_{\beta_3} \\ &= -\mathfrak{h} h^{\lambda\beta_3} k^{\nu}{}_{\beta_3} = \mathfrak{h} k^{\lambda\nu}. \end{aligned}$$

so that

$$(2.8a) \quad {}^*h_{\lambda}^{\nu} \stackrel{\text{def}}{=} {}^*h^{\mu\nu} h_{\mu\lambda} = \frac{1}{g} \Sigma_x e^{\nu}{}_x e_{\lambda}(1 + 2K + \lambda_x^2),$$

or, in the nonholonomic frame,

$$(2.8b) \quad {}^*h_x{}^y = \frac{1}{g} \delta_x^y (1 + 2K + \lambda_x^2).$$

The ranks of ${}^*h^{\lambda\nu}$, ${}^*h_{\lambda}{}^{\nu}$ and ${}^*h_x{}^y$ are equal. According to (1.6) the rank of ${}^*h_x{}^y$ is 4.*)

In the case of the third class we have, by virtue of (1.7b),

$$k^{ip} k_p{}^i = k^{i1} k_1{}^i + k^{i2} k_2{}^i = \delta_{i4}^{ij} k^{42} k_2{}^4 = \delta_{i4}^{ii}$$

so that according to (2.2a)

$$(2.9) \quad (({}^*h^{ii})) = \begin{bmatrix} 0 & 0 & 0 & 1/g \\ 0 & -1/g & 0 & 0 \\ 0 & 0 & -1/g & 0 \\ 1/g & 0 & 0 & 1/g \end{bmatrix};$$

the determinant of this matrix, and hence also the determinant of the matrix ${}^*h^{\lambda\nu}$, is different from zero.

Using this lemma we are able to prove the following

Theorem (2.2). *The condition (1.25a) is equivalent to either one of the following equivalent conditions. The density g is covariantly constant:*

$$(2.10a) \quad D_{\mu} g = 0.$$

The tensor $m_{\lambda\mu}$ defined by (1.21) satisfies the condition

$$(2.11a) \quad \partial_{[\omega} m_{\mu\lambda]} = 0 \quad **),$$

equivalent to

$$(2.11b) \quad m_{\mu\lambda} = \partial_{[\mu} m_{\lambda]}.$$

Proof. The equivalence of (1.25a) and (2.10a) follows at once from the equation

$$(2.12) \quad D_{\mu} g = 2S_{\mu} g,$$

*)The rank of ${}^*h_x{}^y$ would be less than four if for some x

$$(*) \quad \lambda_x^2 = -(1 + 2K).$$

This condition is equivalent to $g = 0$ which contradicts (11a).

**)In the first section we mentioned only the fact that (2.11a) is a consequence of (1.25a).

derived in a slightly different notation in [7] p. 28. On the other hand if we put

$$(2.13a) \quad \mathfrak{F}^{\lambda\nu} = -\sqrt{|\mathfrak{g}|} \, {}^*k^{\lambda\nu},$$

we have, according to I (3.6b) in [9],

$$(2.13b) \quad m_{\lambda\nu} = \tfrac{1}{2} e_{\lambda\nu\alpha\beta} \mathfrak{F}^{\alpha\beta}, \quad \mathfrak{F}^{\lambda\nu} = \tfrac{1}{2} \mathfrak{E}^{\lambda\nu\alpha\beta} m_{\alpha\beta},$$

and by virtue of Theorem (7.3) in [7]

$$(2.14) \quad \partial_\lambda \mathfrak{F}^{\lambda\nu} = S_\lambda \sqrt{|\mathfrak{g}|} \, {}^*h^{\lambda\nu}.$$

This equation may be rewritten according to (2.13b) as

$$(2.15) \quad 3\partial_{[\omega} m_{\mu\lambda]} = S_\alpha \sqrt{|\mathfrak{g}|} \, {}^*h^{\alpha\nu} e_{\omega\mu\lambda\nu}.$$

The equivalence of (1.25a) and (2.11a) follows at once from (2.15) and from Lemma (2.1). The remaining part of the theorem, concerning (2.11b), is obvious.

Note I. Unless the contrary is stated explicitly, we always assume the validity of (1.25a).

Note II. LICHNEROWICZ [18], [19], suggests that we identify the gravitational field with ${}^*h^{\lambda\nu}$. This also would have the advantage that in ${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}$ the tensor ${}^*k^{\lambda\nu}$ could be identified with the “dual” electromagnetic field, as follows from (2.14) and (1.25a).

In subsequent considerations we denote by $\mathfrak{m}$ the determinant of $m_{\lambda\nu}$ and put

$$(2.16) \quad m = \mathfrak{m}/\mathfrak{h}, \quad N = 1 + \tfrac{1}{2} m_{\alpha\beta} m^{\alpha\beta} + m.$$

We know that $\mathfrak{m} = \mathfrak{f}$ (I (4.2) in [9]).

Agreement (2.1). *Throughout this paper we assume*

$$(2.17) \quad N(m+1) \neq 0.$$

According to this agreement we have, by virtue of Theorem I (4.3) in [9],

$$(2.18a) \quad g = \frac{(1-m)^2}{N},$$

$$(2.18b) \quad \sqrt{G} \stackrel{\text{def}}{=} \frac{\sqrt{g}}{1-m} = \frac{1}{\sqrt{N}},$$

so that (1.21) admits unique solution

$$(2.19a) \quad k_{\lambda\nu} = \frac{1}{\sqrt{N}} \left[\tfrac{1}{2} \sqrt{|\mathfrak{h}|} e_{\lambda\nu\alpha\beta} m^{\alpha\beta} - \mu \sqrt{|m|} m_{\lambda\nu} \right]^*,$$

$$\mu \stackrel{\text{def}}{=} \text{sgn } \mathfrak{E}^{\omega\mu\lambda\nu} m_{\omega\mu} m_{\lambda\nu},$$

^{*}Cf. Theorem I(4.2) in [9].

expressed in terms of $m_{\lambda\nu}$ and $h_{\lambda\nu}$. Later we shall use the tensors

$$(2.20a) \quad e_{\omega\mu\lambda\nu} = \sqrt{|\mathfrak{h}|} e_{\omega\mu\lambda\nu}, \quad E^{\omega\mu\lambda\nu} = \frac{1}{\sqrt{|\mathfrak{h}|}} \mathfrak{E}^{\omega\mu\lambda\nu}$$

so that

$$(2.20b) \quad E^{\omega\mu\lambda\nu} = -e_{\omega\mu\lambda\nu}, \quad e^{\omega\mu\lambda\nu} = -E^{\omega\mu\lambda\nu}.$$

In particular, if we put

$$(2.21) \quad \begin{aligned} M^{\lambda\nu} &\stackrel{\text{def}}{=} \sqrt{G} m^{\lambda\nu}, \\ L_{\lambda\nu} &\stackrel{\text{def}}{=} -\mu \sqrt{|m|} M_{\lambda\nu} = -\mu \sqrt{|m|} G m_{\lambda\nu}, \end{aligned}$$

we may rewrite (2.19) as follows:

$$(2.19b) \quad k_{\lambda\nu} = \frac{1}{2} e_{\lambda\nu\alpha\beta} M^{\alpha\beta} + L_{\lambda\nu}.$$

Later we shall use this formula very often.

Note. The equations (1.21) and (2.19a) show that the following two equations are equivalent:

$$(2.22) \quad k_{\lambda\mu} = 0, \quad m_{\lambda\mu} = 0.$$

However, we assume throughout this paper that $k_{\lambda\mu} \neq 0$, so that $m_{\lambda\mu} \neq 0$.

3. The tensor $K_{\omega\mu\nu}$. In the following theorems we shall use the abbreviation

$$(3.1a) \quad W^\nu \stackrel{\text{def}}{=} \nabla_\lambda M^{\lambda\nu},$$

where ∇_λ as usual denotes the symbol of covariant differentiation with respect to Christoffel symbols (1.11a). From (3.1a) we obtain

$$(3.1b) \quad \sqrt{|\mathfrak{h}|} W^\nu = \nabla_\lambda \sqrt{|\mathfrak{h}|} M^{\lambda\nu} = \partial_\lambda \sqrt{|\mathfrak{h}|} M^{\lambda\nu},$$

so that

$$(3.1c) \quad \partial_\nu \sqrt{|\mathfrak{h}|} W^\nu = \sqrt{|\mathfrak{h}|} \nabla_\nu W^\nu = 0.$$

Another abbreviation used later is

$$(3.2a) \quad A_{\{\omega B_{\mu\lambda}\}} \stackrel{\text{def}}{=} A_\omega B_{\lambda\mu} + A_\mu B_{\omega\lambda} + A_\lambda B_{\omega\mu},$$

which in the case of $B_{\mu\lambda} = B_{[\mu\lambda]}$ may be written as

$$(3.2b) \quad A_{\{\omega B_{\mu\lambda}\}} = 3A_{\{\omega B_{\lambda\mu}\}} + 2A_\lambda B_{\omega\mu}.$$

Theorem (3.1). *Put*

$$(3.3) \quad {}^0K_{\omega\mu\nu} \stackrel{\text{def}}{=} e_{\omega\mu\alpha\beta} \nabla_\nu M^{\alpha\beta} + \nabla_{[\omega} L_{\mu\nu]}.$$

Then we have for the tensor $K_{\omega\mu\nu}$ defined by (1.13)

$$(3.4) \quad K_{\omega\mu\nu} = W^\alpha e_{\omega\nu\mu\alpha} + {}^0K_{\omega\mu\nu},$$

and ${}^0K_{\omega\mu\nu}$ does not contain W^ν .

Proof. First of all we have from (2.19b) and (1.13)

$$(3.5) \quad K_{\omega\mu\nu} = \nabla_{[\omega} k_{\mu\nu]} = \frac{1}{2} \nabla_{[\omega} e_{\mu\nu]\alpha\beta} M^{\alpha\beta} + \nabla_{[\omega} L_{\mu\nu]}.$$

On the other hand

$$(3.6) \quad \frac{1}{2} \nabla_{[\omega} e_{\mu\nu]\alpha\beta} M^{\alpha\beta} = \frac{3}{2} e_{\alpha\beta[\nu\mu} \nabla_{\omega]} M^{\alpha\beta} + (\nabla_\nu M^{\alpha\beta}) e_{\omega\mu\alpha\beta}.$$

Put

$$(3.7a) \quad M_{\nu\mu\omega} \stackrel{\text{def}}{=} \frac{3}{2} e_{\alpha\beta(\nu\mu} \nabla_{\omega]} M^{\alpha\beta};$$

then

$$(3.7b) \quad M_{\nu\mu\omega} E^{\nu\mu\omega\lambda} = \frac{3}{2} e_{\alpha\beta[\nu\mu} \nabla_{\omega]} M^{\alpha\beta} E^{\nu\mu\omega\lambda} = 6 \nabla_\alpha M^{\alpha\lambda} = 6W^\lambda,$$

so that

$$(3.7c) \quad M_{\nu\mu\omega} E^{\nu\mu\omega\lambda} e_{\xi\eta\zeta\lambda} = 6M_{\xi\eta\zeta} = 6W^\alpha e_{\xi\eta\zeta\alpha}.$$

The equation (3.4) follows from (3.5)–(3.7). The remaining part of the theorem is obvious.

4. The tensors $S_{\omega\mu\nu}$ and $U_{\nu\omega\mu}$. A simple inspection of (1.14) shows that $\lambda_{ijk} = 1$ if and only if $g_{\lambda\mu}$ is of the third class (i.e., if all eigenvalues are equal to zero). In such a case the formula (1.12b) derived for the first two classes reduces to

$$(4.1a) \quad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{[\alpha\beta\nu]} k_{[\omega}{}^\alpha k_{\mu]}{}^\beta,$$

while (1.12a) derived for the third class is unchanged:

$$(4.1b) \quad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^\alpha K_{\omega]\alpha\beta} k_{\nu]}{}^\beta.$$

Let us investigate the circumstances under which formulæ (4.1a) and (4.1b) are identical. To this effect we shall first prove the following

Lemma (4.1). *The equations*

$$(4.2) \quad (a+2)e^\lambda{}_\alpha e^\mu{}_\nu \nabla_\mu{}^\alpha e_\lambda{}^\nu = 0 \quad (\text{unsummed})$$

$$a = 1, 2, \quad a = 2, 4$$

are invariant with respect to (1.8).

Proof. Put $a = 1$, $\alpha = 2$. Then we have by virtue of (1.8) and (1.9)

$$\begin{aligned} {}_3e^\lambda {}_2e^\mu \nabla_\mu {}^1e_\lambda &= {}_3e^\lambda {}_2e^\mu \nabla_\mu {}_4e_\lambda \\ &= \epsilon'({}_3e^\lambda + \epsilon u \cos v {}_4e^\lambda)(\epsilon {}_2e^\mu + u \sin v {}_4e^\mu) \nabla_\mu \epsilon {}_4e_\lambda \\ &= \epsilon \epsilon'({}_3e^\lambda + u \cos v {}_4e^\lambda)(\epsilon {}_2e^\mu + u \sin v {}_4e^\mu) \nabla_\mu {}^1e_\lambda^{*)} \\ &= \epsilon'({}_3e^\lambda {}_2e^\mu \nabla_\mu {}^1e_\lambda + \epsilon \epsilon' u \sin v {}_3e^\lambda {}_4e^\mu \nabla_\mu {}^1e_\lambda). \end{aligned}$$

Hence if (4.2) holds for $a = 1$, $\alpha = 2$, so does

$${}_3e^\lambda {}_2e^\mu \nabla_\mu {}^1e_\lambda = 0.$$

The invariance of remaining equations may be proven in the same way.

Theorem (4.1). *The equations (4.2) are necessary and sufficient conditions for the equivalence of (4.1a) and (4.1b).*

In order to prove this theorem we shall make use of the first three auxiliary equations (10.3) in [15], namely,

$$(4.3a) \quad K_{4q4} = 0, \quad q = 2, 3,$$

$$(4.3b) \quad K_{422} = 0,$$

$$(4.3c) \quad K_{i[24]} = 0 \dagger).$$

Comparing (4.1a) and (4.1b) we see that a necessary and sufficient condition for their equivalence is

$$2k_\nu{}^\beta k_{[\omega}{}^\alpha K_{\mu]\alpha\beta} = 3K_{[\alpha\beta\nu]} k_\omega{}^\alpha k_\mu{}^\beta,$$

or, in the nonholonomic frame (1.7b),

$$(4.4) \quad k_h{}^p k_i{}^r K_{irp} - k_h{}^p k_j{}^r K_{irp} = 3K_{(prh)} k_i{}^p k_j{}^r.$$

According to (1.7b) we have

$$(4.5) \quad k_1{}^2 = k_2{}^4 = 1, \text{ the remaining } k_i{}^j = 0.$$

Hence, (4.4) splits into three groups of conditions**)

*) We have ${}_4e^\lambda {}_1e_\lambda = {}_1e^\lambda {}_1e_\lambda = 0$ and consequently

$$(\nabla_\mu {}^1e_\lambda) {}_4e^\lambda = (\nabla_\mu {}^1e_\lambda) {}_1e_\mu h^{\lambda\mu} = 0.$$

†) As a matter of fact, we proved (4.3c) only for $i = 3, 4$. However, for the third class we have for any i

$$K_{ikh} k^{ik} = -k^{ik} \nabla_i k_{jh} = 0,$$

so that, according to (1.7b), the equation (4.3c) holds for any i .

**) Here and in subsequent considerations the Latin indices a, b, c (f, g) run from 1 to 2 (from 3 to 4). Moreover, we shall use also the starred indices a^*, b^* , defined as follows: when $a = 1$ ($a = 2$), then $a^* = 2$ ($a^* = 4$), and similarly for b^* .

(1) For $h = f$, we obtain

$$(4.6a) \quad K_{[p r f]} k_i^p k_i^r = 0.$$

When i or j is equal to 3 or 4, the equation (4.6a) reduces to an identity. The same holds true for $i = a \neq b = j$, $f = 4$. Hence the only remaining equation (4.6a) is

$$(4.7a) \quad K_{[243]} = 0.$$

(2) For $h = c$, $i = 1$, $j = 2^{*)}$, the equation (4.4) yields by virtue of (4.3a, b)

$$(4.6b) \quad -K_{14c*} = 3K_{[24c]},$$

and this equation splits into two:

$$(4.7b) \quad K_{144} = 0 \quad \text{for } c = 2,$$

$$(4.7c) \quad K_{2[41]} = 0 \quad \text{for } c = 1.$$

(3) Finally for $h = c$, $i = a$, $j = f$, we obtain from (4.4)

$$(4.6c) \quad K_{fa^*c*} = 0,$$

and the same result holds for $h = c$, $i = f$, $j = a$. Hence for $a = 1$ we obtain two equations which, by virtue of (4.3a, b), reduce to

$$(4.7d) \quad K_{322} = 0,$$

$$(4.7e) \quad K_{324} = 0.$$

Moreover, for $a = 2$ we obtain two equations from (4.6c). The first one is

$$(4.7f) \quad K_{342} = 0,$$

while the second one is identical with (4.3a) for $q = 3$.

In order to derive some consequences from (4.7) we introduce the nonholonomic components of the Christoffel symbols:

$$(4.8a) \quad \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} = -{}_i e^\lambda {}_k e^\mu \nabla_\mu {}^i e_\lambda,$$

so that obviously

$$(4.8b) \quad \left\{ \begin{matrix} 1 \\ 4 \quad k \end{matrix} \right\} = 0.$$

Hence we have from (4.7f), (1.7b), and (4.8b)

$$(4.9a) \quad \begin{aligned} K_{342} &= \nabla_3 k_{24} + \nabla_4 k_{32} + \nabla_2 k_{34} \\ &= - \left[\left\{ \begin{matrix} 1 \\ 4 \quad 3 \end{matrix} \right\} k_{21} + \left\{ \begin{matrix} 1 \\ 3 \quad 4 \end{matrix} \right\} k_{12} \right] = \left\{ \begin{matrix} 1 \\ 3 \quad 4 \end{matrix} \right\} = 0. \end{aligned}$$

***)**The equation (4.4) is skew symmetric in i, j . Hence the same result may be obtained for $i = 2, j = 1$.

Therefore,

$$\begin{aligned}
 K_{243} &= \nabla_2 k_{34} + \nabla_4 k_{23} + \nabla_3 k_{24} \\
 &= -\left[\left\{ \begin{smallmatrix} 1 \\ 3 & 4 \end{smallmatrix} \right\} k_{21} + \left\{ \begin{smallmatrix} 1 \\ 4 & 3 \end{smallmatrix} \right\} k_{21} \right] = -\left\{ \begin{smallmatrix} 1 \\ 3 & 4 \end{smallmatrix} \right\} = 0, \\
 K_{324} &= \nabla_3 k_{42} + \nabla_2 k_{34} + \nabla_4 k_{32} \\
 &= -\left[\left\{ \begin{smallmatrix} 1 \\ 4 & 3 \end{smallmatrix} \right\} k_{12} + \left\{ \begin{smallmatrix} 1 \\ 3 & 4 \end{smallmatrix} \right\} k_{12} \right] = \left\{ \begin{smallmatrix} 1 \\ 3 & 4 \end{smallmatrix} \right\} = 0.
 \end{aligned}$$

Therefore, (4.7a) and (4.7e) are identically satisfied because of (4.7f). Hence the only remaining equations to be considered are (4.7b, c, d). These equations are equivalent to

$$(4.9b) \quad K_{144} = 2\nabla_4 k_{14} = -2\left\{ \begin{smallmatrix} 2 \\ 4 & 4 \end{smallmatrix} \right\} k_{12} = 2\left\{ \begin{smallmatrix} 2 \\ 4 & 4 \end{smallmatrix} \right\} = 0,$$

$$(4.9c) \quad K_{2[41]} = \nabla_2 k_{14} = -\left\{ \begin{smallmatrix} 2 \\ 4 & 2 \end{smallmatrix} \right\} k_{12} = \left\{ \begin{smallmatrix} 2 \\ 4 & 2 \end{smallmatrix} \right\} = 0,$$

$$(4.9d) \quad K_{322} = 2\nabla_2 k_{32} = -2\left\{ \begin{smallmatrix} 1 \\ 3 & 2 \end{smallmatrix} \right\} k_{12} = 2\left\{ \begin{smallmatrix} 1 \\ 3 & 2 \end{smallmatrix} \right\} = 0.$$

The equations (4.9a, b, c, d) together with (4.8a) yield (4.2).

The question arises whether or not the conditions (4.2) are inherent to the third class. This question is answered in the following

Theorem (4.2). *The set of all tensors $g_{\lambda\mu}$ of the third class for which the conditions (4.2) $\left\{ \begin{smallmatrix} \text{are} \\ \text{are not} \end{smallmatrix} \right\}$ satisfied is not an empty set.*

The proof will be achieved by exhibiting such tensors. Denote by γ a real scalar function of coordinates and consider the tensor

$$(4.10) \quad ((g_{\lambda\mu})) = \begin{pmatrix} 0 & -\gamma^{-1} & 0 & 1 \\ \gamma^{-1} & -\gamma^{-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

which is obviously of the third class. The vector frame which leads to (1.7b) and satisfies (1.5) is

$$\begin{aligned}
 (4.11) \quad {}_1e^\nu(1 \ 0 \ 0 \ 0) & \quad {}^1e_\lambda(1 \ 0 \ 0 \ 0) \\
 {}_2e^\nu(0 \ \gamma \ 0 \ 0) & \quad {}^2e_\lambda(0 \ \gamma^{-1} \ 0 \ 0) \\
 {}_3e^\nu(0 \ 0 \ 1 \ 0) & \quad {}^3e_\lambda(0 \ 0 \ 1 \ 0) \\
 {}_4e^\nu(0 \ 0 \ 0 \ 1) & \quad {}^4e_\lambda(0 \ 0 \ 0 \ 1).
 \end{aligned}$$

Hence we have according to (1.11a)

$${}_4e^\lambda {}_2e^\mu \nabla_\mu {}^2e_\lambda = - \left\{ \begin{matrix} \nu \\ \lambda \mu \end{matrix} \right\} {}_4e^\lambda {}_2e^\mu {}^2e_\nu = - \left\{ \begin{matrix} \text{II} \\ \text{IV II} \end{matrix} \right\} = \partial_{\text{IV}} \ln \gamma,$$

and in a similar manner

$${}_4e^\lambda {}_4e^\mu \nabla_\mu {}^2e_\lambda = {}_3e^\lambda {}_4e^\mu \nabla_\mu {}^1e_\lambda = {}_3e^\lambda {}_2e^\mu \nabla_\mu {}^1e_\lambda = 0.$$

Hence if $\gamma \begin{cases} \text{is not} \\ \text{is} \end{cases}$ a function of x^{IV} , then the conditions (4.2) $\begin{cases} \text{are} \\ \text{are not} \end{cases}$ satisfied. According to Lemma (4.1), this property is independent of the transformation (1.8) of the vectors (4.11).

Theorem (4.3). *As for as "Problem C" is concerned there are two unified theories, one for the first two classes, and one for the third class. These two theories coincide in the special case when for the third class conditions (4.2) are satisfied.*

Proof. One of the cornerstones of Problem C is the substitution into (1.25b, c) from (1.10b), where $U''_{\lambda\mu}$ is given by (1.11c) and $S_{\lambda\mu}''$ is given either by (1.12a) in the case of the third class, or by (1.12b) in the case of the first two classes. In the case of the third class this latter equation reduces to (1.12a) if and only if (4.2) is satisfied. Thus in this case "Problem C" could be dealt with for the third class by means of (1.12b).

Note I. This theorem justifies our subsequent method in dealing with the third class separately. (Cf. the next chapter.)

Note II. There is a question concerning $U''_{\lambda\mu}$ which is somewhat similar to the problem handled in Theorem (4.1). Using (1.12a) and (1.11c) we obtain for the third class

$$(4.12) \quad U_{\nu\omega\mu} = K_{\nu(\omega} {}^\alpha k_{\mu)\alpha} + k_{\gamma}{}^\beta [k_{\nu}{}^\alpha k_{(\mu}{}^\gamma K_{\omega)\alpha\beta} - k_{(\omega}{}^\alpha k_{\mu)}{}^\gamma K_{\nu\alpha\beta}].$$

Yet if one compares the equation

$$(4.13) \quad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 4U_{\alpha\nu[\mu} k_{\omega]}{}^\alpha$$

(cf. equation (2.14) in [7]), which holds for all three classes, with the equation (1.12a), then one sees that for the third class the tensor $U''_{\lambda\mu}$, which is defined by (4.12), must satisfy

$$(4.14) \quad 2U_{\alpha\nu[\mu} k_{\omega]}{}^\alpha = k_{[\mu}{}^\alpha K_{\omega]\alpha\beta} k_{\nu]}{}^\beta.$$

That this relation is indeed satisfied one may verify by using the nonholonomic frame, in which (1.7b) holds.

CHAPTER II: *The third class*

5. The tensors $K_{\omega\mu\nu}$, $U_{\nu\omega\mu}$, and $A_{\omega\mu}$. Throughout this chapter we shall deal with the case of the third class

$$(5.1a) \quad k = 0,$$

$$(5.1b) \quad D = K^2 = 0,$$

so that

$$(5.2a) \quad g = 1$$

$$(5.2b) \quad U_\mu \stackrel{\text{def}}{=} U^\alpha_{\alpha\mu} = 0^*).$$

The results will be somewhat similar to the results of [10]. Therefore, we shall not need to discuss all the details and shall instead refer to [10].

For the scalars and tensors involved in (2.21) we have, according to (5.1), (5.2) and $m = k$,

$$(5.3a) \quad G = 1/N = 1,$$

$$(5.3b) \quad L_{\lambda\nu} = 0,$$

$$(5.3c) \quad M_{\lambda\mu} = m_{\lambda\mu} ,$$

so that the equations (1.21) and (2.19b) reduce to

$$(5.4a) \quad m_{\lambda\nu} = -\frac{1}{2}e_{\lambda\nu\alpha\beta}k^{\alpha\beta},$$

$$(5.4b) \quad k_{\lambda\nu} = \frac{1}{2}e_{\lambda\nu\alpha\beta}m^{\alpha\beta},$$

or

$$(5.4c) \quad m^{\lambda\nu} = \frac{1}{2}E^{\lambda\nu\alpha\beta}k_{\alpha\beta} ,$$

$$(5.4d) \quad k^{\lambda\nu} = -\frac{1}{2}E^{\lambda\nu\alpha\beta}m_{\alpha\beta} .$$

Hence in accordance with (5.3a) we have

$$(5.5) \quad \begin{aligned} m_{\alpha\beta}m^{\alpha\beta} &= -\frac{1}{4}e_{\lambda\nu\alpha\beta}k^{\lambda\nu}E^{\omega\mu\alpha\beta}k_{\omega\mu} \\ &= -\delta_{[\lambda\nu]}k^{\lambda\nu}k_{\omega\mu} = -k_{\alpha\beta}k^{\alpha\beta} = -4K = 0. \end{aligned}$$

Agreement (5.1). Throughout this chapter we assume

$$(5.6) \quad S_{\omega\mu\lambda}S^{\omega\mu\lambda} \leq 0,$$

and define the vector v^ν by

$$(5.7) \quad v^\nu \stackrel{\text{def}}{=} E^{\nu\omega\mu\lambda}S_{\omega\mu\lambda} .$$

* Cf. the equation (8.7) in [7].

Theorem (5.1). *The vector v^r satisfies the identities*

$$(5.8a) \quad v^r \equiv \nabla_\lambda m^{\lambda r},$$

$$(5.8b) \quad \nabla_\lambda v^\lambda \equiv 0,$$

and is identical with the vector W^r defined by (3.1a). Moreover,

$$(5.8c) \quad v^\lambda v_\lambda \geq 0.$$

Proof. The equations (3.1a) and (5.3c) lead to $W^r = v^r$. The proof of (5.8b) then follows easily from (3.1c). The proof of (5.8a, c) follows the pattern of the proof of Theorem II (1.3) in [10].

Theorem (5.2). *Put*

$$(5.9a) \quad {}^0K_{\omega\mu\nu} = e_{\omega\mu\alpha\beta} \nabla_\nu m^{\alpha\beta}.$$

Then

$$(5.9b) \quad K_{\omega\mu\nu} = v^\alpha e_{\alpha\omega\mu\nu} + {}^0K_{\omega\mu\nu}.$$

The proof follows at once from Theorem (3.1) and from equations (5.3).

In the next theorem we use the following abbreviation:

$$(5.10) \quad 2M_{\nu\omega\mu} \stackrel{\text{def}}{=} m^{\alpha\beta} \{ e_{\alpha\beta(\mu} \delta_{\omega)\nu}^{[\eta\xi]} + [e_{\alpha\beta\nu} \delta_{(\omega}^{[\eta} \delta_{\xi]}^{\lambda]} + \delta_\nu^{[\eta} e_{\alpha\beta(\omega}^{\xi]}] m_{\mu)}{}^\gamma m_\gamma{}^\zeta \}.$$

Theorem (5.3). *Put*

$$(5.11a) \quad {}^0U_{\nu\omega\mu} \stackrel{\text{def}}{=} {}^0K_{\alpha\beta\gamma} M_{\nu\omega\mu}^{\alpha\beta\gamma}.$$

Then

$$(5.11b) \quad U_{\nu\omega\mu} = v^\lambda e_{\lambda\alpha\beta\gamma} M_{\nu\omega\mu}^{\alpha\beta\gamma} + {}^0U_{\nu\omega\mu}.$$

None of the tensors

$$(5.12) \quad e_{\lambda\alpha\beta\gamma} M_{\nu\omega\mu}^{\alpha\beta\gamma}, \quad {}^0U_{\nu\omega\mu},$$

contains v^r .

Proof. The equation (4.12) may be rewritten as follows:

$$U_{\nu\omega\mu} = K_{\alpha\beta\gamma} M_{\nu\omega\mu}^{\alpha\beta\gamma}.$$

Substituting from (5.9b) into this equation, one obtains (5.11b). The remaining part of the theorem follows by simple inspection of (5.12), (5.9a), and (5.10).

Theorem (5.4). *The tensor $A_{\omega\mu}$ defined by (1.29)*¹ may be expressed as follows:*

$$(5.13a) \quad A_{\omega\mu} = \frac{1}{2} a_{\lambda\nu\omega\mu} \nabla^\lambda v^\nu + {}^0A_{\omega\mu} + v^\lambda A_{\lambda\omega\mu} + v^\lambda v^\nu A_{\lambda\nu\omega\mu},$$

where the tensors $a_{\lambda\nu\omega\mu}$, ${}^0A_{\omega\mu}$, $A_{\lambda\omega\mu}$, $A_{\lambda\nu\omega\mu}$ have the following properties:

*¹Of course, we have $S_\lambda = 0$ (and $U_\lambda = 0$).

- a) They are well-defined functions of $h_{\lambda\mu}$, $m_{\lambda\mu}$ and their first two derivatives, and are skew symmetric in $\omega\mu$.
- b) They contain neither v^ν nor $\nabla^\lambda v^\nu$.
- c) In particular

$$(5.13b) \quad \frac{1}{2}a_{\lambda\nu\omega\mu} \stackrel{\text{def}}{=} e_{\nu\alpha\beta\gamma} M_{\delta\lambda[\omega} e_{\mu]}^{\alpha\beta\gamma} \delta_{\xi\eta} m^{\xi\eta}.$$

- d) The tensor ${}^0A_{\omega\mu}$ is obtained from (1.29a)*¹ where instead of $K_{\omega\mu\nu}$ and $U_{\omega\mu\nu}$ we have to write ${}^0K_{\omega\mu\nu}$, ${}^0U_{\omega\mu\nu}$ and substitute for $k_{\mu\lambda}$ from (5.4).
- e) None of these tensors contain

$$(5.14) \quad \Delta m_{\lambda\nu} \stackrel{\text{def}}{=} h^{\alpha\beta} \nabla_\alpha \nabla_\beta m_{\lambda\nu}.$$

The proof follows by substitution into (1.29a)*¹ from (5.4b, d), (5.9b) and (5.11b) and by a simple inspection of the formula so obtained.

6. Maxwell's equations.

Theorem (6.1). The condition (1.25b) is equivalent to

$$(6.1a) \quad \partial_{[\omega} X_{\mu]} = \frac{1}{2}(\nabla^\lambda v^\nu)(e_{\nu\lambda\omega\mu} + a_{\lambda\nu\omega\mu}) + {}^0A_{\omega\mu} + \frac{1}{2}e_{\omega\mu\lambda\nu} \Delta m^{\lambda\nu} \\ + v^\lambda A_{\lambda\omega\mu} + v^\lambda v^\nu A_{\lambda\nu\omega\mu}.$$

Proof. We have from (5.9)

$$(6.2) \quad \frac{1}{2}\nabla^\lambda K_{\omega\mu\lambda} = \frac{1}{2}(\nabla^\lambda v^\nu)e_{\nu\omega\mu\lambda} + \frac{1}{2}e_{\omega\mu\alpha\beta}\Delta m^{\alpha\beta}.$$

The proof then follows easily from (1.25a, b), (1.28a), (5.13), and (6.2).

The system (6.1a) represents six differential equations for twenty-two unknowns,

$$X_\lambda, m_\lambda^{*\nu}, v^\nu, h_{\lambda\mu}.$$

Denote by

$$(6.3) \quad m_\lambda = m_\lambda(X_\cdot, v^\cdot, h_{\cdot\cdot}),$$

the solution of (6.1a) expressed in terms of $X_\lambda, v^\nu, h_{\lambda\mu}$.

Theorem (6.2). Upon substituting from (6.3) into the identity (5.8a) we obtain a system of four equations

$$(6.4) \quad v^\nu = \nabla_\lambda m^{\lambda\nu},$$

for the unknowns $X_\lambda, v^\nu, h_{\lambda\mu}$.

Proof. According to (6.3) m_λ is a function of X_λ , while the left hand term of the identity (5.8a) does not contain this argument. Hence, the substitution

*¹Cf. (2.11b).

of (6.3) into (5.8a) leads to a system of four equations—namely, (6.4)—for the unknowns X_λ , v^ν , $h_{\lambda\mu}$.

The equations (2.11a) and (6.4) *i.e.*,

$$(6.5a) \quad \partial_{[\omega} m_{\mu\lambda]} = 0,$$

$$(6.5b) \quad v^\nu = \nabla_\lambda m^{\lambda\nu},$$

have the form of MAXWELL's equations in proper units. In particular, if we put

$$(6.6) \quad \mathfrak{v}^\nu \stackrel{\text{def}}{=} v^\nu \sqrt{|\mathfrak{h}|}, \quad m_{\lambda\nu} = m_{\lambda\nu} \sqrt{|\mathfrak{h}|},$$

equation (6.5b) reduces to

$$(6.7) \quad \mathfrak{v}^\nu = \partial_\lambda m^{\lambda\nu},$$

while (5.8b) leads at once to

$$(6.8) \quad \partial_\lambda \mathfrak{v}^\lambda = 0.$$

Also, as we did in [10], we shall again identify $m_{\lambda\mu}$ with the *electromagnetic field*, $\mathfrak{v}^\nu$ with the *world current density vector*. Then (6.8) is to be regarded as the *conservation law*. Whenever $\mathfrak{v}^2 > 0^{*)}$, we shall identify $\mathfrak{v}$ with the electric charge density. Moreover, we shall introduce the four dimensional velocity unit vector

$$(6.9) \quad u^\nu \stackrel{\text{def}}{=} \mathfrak{v}^\nu / \sqrt{\mathfrak{v}^2} = v^\nu / \sqrt{v^2} \quad (v^2 = v^\lambda v_\lambda).$$

Theorem (6.4). *Put*

$$(6.10) \quad G_{\omega\mu\lambda\nu} \stackrel{\text{def}}{=} H_{\omega\mu\lambda\nu} - 2H_{\nu[\omega} h_{\mu]\lambda}.$$

Then every solution of Maxwell's equations (6.5) is also a solution of

$$(6.11) \quad \Delta m_{\omega\mu} = 2\nabla_{[\omega} v_{\mu]} + G_{\omega\mu\lambda\nu} m^{\nu\lambda}.$$

The proof follows the pattern of the proof of Theorem III (1.2) in [14].

Note. Sometimes the equations (6.11) will be referred to as the “general” Maxwell's equation.

Theorem (6.5). *The equations (6.1a) are equivalent to*

$$(6.1b) \quad \partial_{[\omega} X_{\mu]} = \frac{1}{2}(\nabla^\lambda v^\nu)(e_{\lambda\nu\omega\mu} + a_{\lambda\nu\omega\mu}) + {}^0A_{\omega\mu} + \frac{1}{2}G^{\alpha\beta\gamma\delta}e_{\omega\mu\alpha\beta}m_{\delta\gamma} \\ + v^\lambda A_{\lambda\omega\mu} + v^\nu A_{\lambda\nu\omega\mu}.$$

Proof. One obtains (6.1b) by eliminating $\Delta m_{\lambda\nu}$ from (6.1a) and (6.11).

7. Continuation.

Theorem (7.1). *A necessary and sufficient condition for*

$$(7.1) \quad v^\nu = 0,$$

*) According to (5.8c) we have $\mathfrak{v}^2 \stackrel{\text{def}}{=} \mathfrak{v}^\nu \mathfrak{v}_\nu \geq 0$.

is

$$(7.2a) \quad S_{[\omega\mu\lambda]} = 0.$$

The proof follows at once from (5.7).

Note. The condition (7.2a) is equivalent to

$$(7.2b) \quad \partial_{[\omega} k_{\mu\lambda]} = 0^{*)}$$

and leads to pure radiation,

$$(7.3a) \quad \partial_{[\omega} m_{\mu\lambda]} = 0,$$

$$(7.3b) \quad \partial_{\lambda} m^{\lambda\nu} = 0.$$

Theorem (7.2). *The equations (6.1a) are equivalent to Maxwell's equations (6.5) in the sense that every solution of (6.1a) is also a solution of (6.5).*

The proof will be accomplished in three steps.

1) Denote by $Q_{\omega\mu}$ the right hand term of (6.1a). Then we see from (1.28a), (1.25b), and (6.1a) that if we disregard (1.25a) we can obtain a more general system instead of (6.1a), namely,

$$(7.4) \quad \partial_{[\omega} X_{\mu]} = Q_{\omega\mu} + S_{\omega\mu},$$

where

$$(7.5) \quad S_{\omega\mu} = \partial_{[\omega} S_{\mu]} + S^{\beta}(\frac{1}{2}K_{\omega\mu\beta} - 2U_{\alpha\beta[\mu}k_{\omega]}^{\alpha}).$$

Using (2.15), (5.9), and (5.11), one could express $S_{\omega\mu}$ in terms of $h_{\lambda\mu}$, $m_{\lambda\mu}$, and v^{ν} . In dealing with the third class we are imposing the conditions

$$(7.6) \quad (5.1a, b), \quad (1.25a, b, c).$$

on $g_{\lambda\mu}$

The condition (1.25a) reduces $S_{\omega\mu}$ to zero,

$$(7.7) \quad S_{\omega\mu} = 0,$$

so that (7.4) reduces to

$$(7.8) \quad \partial_{[\omega} X_{\mu]} = Q_{\omega\mu},$$

with a solution satisfying (6.5a). However, we cannot exclude *a priori* a solution which satisfies (7.7) and $\partial_{[\omega} m_{\mu\lambda]} \neq 0$. Hence, we must investigate this possibility from the point of view of the conditions (7.6) which we are imposing on $g_{\lambda\mu}$. The tensor $S_{\omega\mu}$ is a function of $k_{\lambda\mu}$ and its first two derivatives. On the other hand, the first two derivatives of (5.1a, b) do not yield $\nabla_{\omega} k_{\mu\lambda}$ with two free indices which appear in $S_{\omega\mu}$. Hence, (5.1a, b) can not yield (7.7), and obviously the same holds for (1.25b). Yet (1.27b) shows that (1.25c) cannot yield (7.7). Thus the only condition left is (1.25a) which is equivalent to (6.5a) and leads

*)Cf. the equation (7.25) in [7]. There is an obvious misprint in this equation.

to (7.7). Hence, under conditions (7.6), every solution of (7.8) must satisfy (6.5a).

2) Assume that we are dealing with a charged particle, so that (7.2a) is not satisfied. Then, in the same manner that we obtained (6.5b) from the identity (5.8a) by means of (6.1a), it is clear that every solution of (6.1a) satisfies (6.5b).

3) Let us put

$$(7.9) \quad Z_{\omega\mu} = \frac{1}{2}(\nabla^\lambda v^\nu)(e_{\nu\lambda\omega\mu} + \alpha_{\lambda\mu\omega\mu}) + v^\lambda A_{\lambda\omega\mu} + v^\lambda v^\nu A_{\lambda\nu\omega\mu},$$

so that (6.1a) may be written

$$(7.10) \quad \partial_{[\omega} X_{\mu]} = Z_{\omega\mu} + {}^0A_{\omega\mu} + \frac{1}{2}e_{\omega\mu\lambda\nu}\Delta m^{\lambda\nu}.$$

If we are dealing with pure radiation, then we are restricting our problem by a new condition, (7.2a), in addition to (7.6). Then, of course,

$$(7.11) \quad Z_{\omega\mu} = 0,$$

is an identity and (7.10) reduces to

$$(7.12) \quad \partial_{[\omega} X_{\mu]} = {}^0A_{\omega\mu} + \frac{1}{2}e_{\omega\mu\lambda\nu}\Delta m^{\lambda\nu},$$

with a solution satisfying (7.3b). On the other hand, we can not exclude *a priori* a solution for which (7.11) and $\partial_\lambda m^{\lambda\nu} \neq 0$ are satisfied. Hence we have to investigate this possibility from the point of view of conditions (7.6) which we are imposing on $g_{\lambda\mu}$. It is obvious that (1.25) cannot yield (7.11). Yet we see from (5.7) that the same holds for the conditions (5.1). Hence (7.11) cannot be derived from (7.6). In other words, as far as the conditions (7.6) are concerned (7.11) is not an identity and it therefore *represents a new condition* in addition to (7.6). The only admissible new condition besides (7.6) is (7.2a), *which reduces (7.11) to an identity*, so that every solution of (7.12) must satisfy the condition (7.3b).

Corollary (7.1). *The system (6.1a) and the system of general Maxwell's equations (6.11) are equivalent in the sense that every solution of (6.1a) is a solution of (6.11).*

The proof follows at once from Theorems (6.4) and (7.2).

Note I. According to the previous results the equations (6.1), closely connected with the general Maxwell's equations, will be referred to as the *electromagnetic field equations*.

Note II. According to Theorem (2.2) we may replace the condition (1.25a) by the equivalent condition (2.11b). We shall then have ten equations (6.5b) and (6.1a) for twenty-two unknowns X_λ , v^ν , m_λ , and $h_{\lambda\mu}$. However, X_λ could be eliminated if, instead of (6.1a), we considered its integrability conditions:

$$(7.13) \quad \delta_{[\omega\mu}^{\alpha\beta} \nabla_{\xi]} \{ \frac{1}{2}(\nabla^\lambda v^\nu)(e_{\nu\lambda\alpha\beta} + \alpha_{\lambda\nu\alpha\beta}) + {}^0A_{\alpha\beta} + \frac{1}{2}e_{\alpha\beta\lambda\nu}\Delta m^{\lambda\nu} + v^\lambda A_{\lambda\alpha\beta} + v^\lambda v^\nu A_{\lambda\nu\alpha\beta} \} = 0.$$

Then our system of equations reduces to eight equations (6.5b) and (7.13) for eighteen unknowns m_λ , v^ν , and $h_{\lambda\mu}$. However, this reduction is only a formal one.

8. Gravitational fields. In this section we deal with the condition (1.25c). First we prove a lemma about the tensor $B_{\mu\lambda}$ defined by (1.29b).

Lemma (8.1). *The tensor $B_{\mu\lambda}$ may be expressed in the following way:*

$$(8.1) \quad B_{\mu\lambda} = b_{\alpha\beta\lambda\mu} \nabla^\alpha v^\beta + {}^0B_{\mu\lambda} + v^\alpha B_{\alpha\lambda\mu} + v^\alpha v^\beta B_{\alpha\beta\lambda\mu}$$

and the tensors $b_{\alpha\beta\lambda\mu}$, ${}^0B_{\mu\lambda}$, $B_{\alpha\lambda\mu}$, $B_{\alpha\beta\lambda\mu}$, symmetric in $\lambda\mu$, have the following properties:

- a) They are well defined functions of $h_{\lambda\mu}$, $m_{\lambda\mu}$ and their first two derivatives.
- b) They contain neither v^ν nor $\nabla^\lambda v^\nu$.
- c) In particular,

$$(8.2) \quad b_{\alpha\beta\lambda\mu} \stackrel{\text{def}}{=} M_{\alpha\lambda\mu}^{\xi\eta\zeta} e_{\beta\xi} e_{\eta\zeta}.$$

- d) The tensor ${}^0B_{\mu\lambda}$ is obtained from (1.29b)*, where for $K_{\omega\mu\nu}$ and $U_{\omega\mu\nu}$ we have to write ${}^0K_{\omega\mu\nu}$ and ${}^0U_{\omega\mu\nu}$ and substitute for $k_{\lambda\mu}$ from (5.4).

The proof follows by substitution into (1.29b) from (5.4b), (5.9b) and (5.11b) and by simple inspection of the formula so obtained.

In the next agreement we shall use the following abbreviation:

$$(8.3) \quad B \stackrel{\text{def}}{=} B_{\lambda\mu} h^{\lambda\mu}.$$

Agreement (8.1). *The scalar φ involved in (1.25c) is defined as follows:*

$$(8.4) \quad 4\varphi \stackrel{\text{def}}{=} m^{\alpha\beta} (\nabla_\alpha v_\beta + \tfrac{1}{2} m^{\delta\gamma} G_{\alpha\beta\gamma\delta}) + B.$$

In order to deal with the condition (1.25c) let us first prove

Lemma (8.2). *We have*

$$(8.5) \quad \tfrac{1}{4} K_{\alpha\mu}{}^\beta K_{\beta\lambda}{}^\alpha = \tfrac{1}{2} [v_\lambda v_\mu - h_{\mu\lambda} v^2 + 2v^\alpha \nabla_{(\mu} m_{\lambda)\alpha} + \tfrac{1}{2} e_{\mu\alpha\beta\gamma} e_{\lambda\xi\eta\zeta} (\nabla^\xi m^{\beta\gamma}) (\nabla^\alpha m^{\eta\zeta})].$$

The proof follows from (5.9b).

Theorem (8.1). *Put*

$$(8.6) \quad {}^*T_{\mu\lambda} \stackrel{\text{def}}{=} \tfrac{1}{4} K_{\alpha\mu}{}^\beta K_{\beta\lambda}{}^\alpha + B_{\mu\lambda},$$

where the first (second) term on the right hand side is defined by (8.5) ((8.1)). Then (1.25c) is equivalent to

$$(8.7) \quad H_{\lambda\mu} - {}^*T_{\lambda\mu} = -\varphi h_{\lambda\mu}.$$

The proof follows at once from Lemmas (8.1), (8.2) and from the equations (1.25a, c) and (1.28b).

The next two theorems will be proven simultaneously.

*Of course, we have $S_\lambda = 0$ (and $U_\lambda = 0$).

Theorem (8.2). *We have according to Agreement (8.1)*

$$(8.8) \quad H \stackrel{\text{def}}{=} H_{\lambda\mu} h^{\lambda\mu} = \frac{1}{2} v^2.$$

Theorem (8.3). *The equation (1.25c) is equivalent to*

$$(8.9) \quad H_{\lambda\mu} - \frac{1}{2} H h_{\lambda\mu} = T_{\lambda\mu},$$

where

$$(8.10) \quad T_{\lambda\mu} \stackrel{\text{def}}{=} \frac{1}{2} (v_{\lambda} v_{\mu} + P_{\mu\lambda}) = T_{\mu\lambda}$$

and

$$(8.11a) \quad \begin{aligned} \frac{1}{2} P_{\lambda\mu} = \frac{1}{2} P_{\mu\lambda} \stackrel{\text{def}}{=} & b_{\alpha\beta\lambda\mu} \nabla^{\alpha} v^{\beta} + v^{\lambda} (\nabla_{(\mu} m_{\lambda)\alpha} + B_{\alpha\lambda\mu}) \\ & + v^{\alpha} v^{\beta} (B_{\alpha\beta\lambda\mu} - \frac{3}{4} h_{\mu\lambda} h_{\alpha\beta}) \\ & + {}^0 B_{\mu\lambda} + \frac{1}{4} e_{\mu\alpha\beta\gamma} e_{\lambda\xi\eta\zeta} (\nabla^{\xi} m^{\beta\gamma}) \nabla^{\alpha} m^{\eta\zeta} - \varphi h_{\lambda\mu} \end{aligned}$$

Proof. First we obtain from (6.11), (8.4) and (8.7)

$$(8.12) \quad \begin{aligned} H &= {}^* T_{\lambda\mu} h^{\lambda\mu} - 4\varphi \\ &= \frac{1}{2} (v^2 + m_{\alpha\beta} \Delta m^{\alpha\beta}) - 4\varphi + B^* \\ &= \frac{1}{2} v^2 + m^{\alpha\beta} (\nabla_{\alpha} v_{\beta} + \frac{1}{2} G_{\alpha\beta\gamma\delta} m^{\delta\gamma}) + B - 4\varphi \\ &= \frac{1}{2} v^2. \end{aligned}$$

Theorem (8.3) follows then by simple inspection from (8.7) and (8.8).

Note. According to (8.9) and (8.10) and (5.7) the tensor $P_{\lambda\mu}$ may be expressed also in the form

$$(8.11b) \quad \frac{1}{2} P_{\lambda\mu} = H_{\lambda\mu} - \frac{1}{2} H h_{\lambda\mu} - \frac{1}{2} e_{\lambda\alpha\beta\gamma} e_{\mu\xi\eta\zeta} S^{\alpha\beta\gamma} S^{\xi\eta\zeta}.$$

Theorem (8.4). *The tensor $T_{\lambda\mu}$ satisfies the following conditions:*

$$(8.13) \quad \nabla_{\lambda} T^{\lambda\nu} = 0^{**}.$$

The proof follows from (8.9) and from the known consequences of the Bianchi identity

$$(8.14) \quad \nabla_{\lambda} (H^{\lambda\nu} - \frac{1}{2} H h^{\lambda\nu}) = 0.$$

Agreement (8.2). *Since the equation (8.9) has the form of the gravitational field equation of "general relativity" we identify*

- a) $h_{\lambda\mu}$ with the gravitational field,
- b) $T_{\lambda\mu}$ with the momentum energy tensor, and
- c) the equations (8.13) with the law of conservation of momentum and energy.

^{*}In this case one must use equation (5.5) and its consequence

$$(\nabla_{\gamma} m_{\alpha\beta}) \nabla^{\gamma} m^{\alpha\beta} + m^{\alpha\beta} \Delta m_{\alpha\beta} = 0.$$

^{**}These four conditions are substantially the same as the "four identities" of EINSTEIN [5], BOSE [3], and LICHNEROWICZ [18] and [19], restricted by field equations.

Note. In Section II 6 of [10] we specified the properties which characterize a metric tensor. It turned out that $h_{\lambda\mu}$ satisfies these requirements.

9. Continuation and recapitulation.

Theorem (9.1). *Both fields, the electromagnetic and the gravitational, are interdependent. The $\left\{\begin{smallmatrix} \text{electromagnetic} \\ \text{gravitational} \end{smallmatrix}\right\}$ field may be expressed in terms of the $\left\{\begin{smallmatrix} \text{gravitational} \\ \text{electromagnetic} \end{smallmatrix}\right\}$ field (and the vector X_λ).*

Proof. If we substitute from (6.4) into $\left\{\begin{smallmatrix} (6.1a) \\ (8.9) \end{smallmatrix}\right\}$, we obtain a set of $\left\{\begin{smallmatrix} \text{six} \\ \text{ten} \end{smallmatrix}\right\}$ equations for $\left\{\begin{smallmatrix} m_{\lambda\nu} \\ h_{\lambda\nu} \end{smallmatrix}\right\}$ expressed in terms of $\left\{\begin{smallmatrix} h_{\lambda\nu} \\ m_{\lambda\nu} \end{smallmatrix}\right\}$ and X_λ .

Note. According to our assumption $g_{\lambda\mu}$ is never symmetric, $k_{\lambda\mu} \neq 0$. Hence $m_{\lambda\nu} \neq 0$. Therefore, we may think of $m_{\lambda\nu}$ as a primary field. Then of course we could reword Theorem (9.1) by stating that the electromagnetic field creates a gravitational field. That this, however, is not always the case is shown in the following

Theorem (9.2). *There are electromagnetic fields which do not create gravitation. In particular, this is the case of the electromagnetic field of the plane wave in the electromagnetic theory of light.*

Proof. Consider the field $g_{\lambda\mu}$ used in the proof of Theorem II (7.2) in [9]. We have

$$(9.1a) \quad U_{\nu\lambda\mu} = 0,$$

$$(9.1b) \quad S_{[\lambda\mu\nu]} = 0,$$

so that in particular

$$(9.2a) \quad v^\nu = 0,$$

$$(9.2b) \quad B_{\lambda\mu} = 0.$$

Since $h_{\lambda\mu}$ is defined here by

$$(9.3) \quad h_{\lambda\mu} = \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right],$$

we have according to (8.4) and (9.2)

$$(9.4) \quad \varphi = 0.$$

Hence if we take for X_λ a gradient, by virtue of (9.1) and (9.4) our Theorem is identical with the statement (2) of Theorem I (5.2) in [10].

Note. The electromagnetic and gravitational field equations (6.1) and (8.9) have been derived under the assumption $m_{\lambda\nu} \neq 0$. If we admit for a moment the possibility $m_{\lambda\nu} = 0$, then these equations reduce to

$$(9.5a) \quad \partial_{[\mu} X_{\lambda]} = 0,$$

$$(9.5b) \quad H_{\lambda\mu} = 0.$$

The first equation shows that $X_\lambda = \partial_\lambda X$ is a gradient. The second equation is identical with the gravitational field equation for empty space in the pure gravitational theory. Hence, one could say that in the absence of electromagnetism the unified theory reduces to the pure gravitational theory for empty space (outside of the "central body"). This suggests the idea that mass is somehow connected with the electromagnetic field. We shall discuss this problem later on.

Recapitulating our results, we have the following statements.

Theorem (9.3). *The unified theory is governed by twenty-two unknowns*

$$(9.6) \quad m_\lambda, X_\lambda, v_\lambda, h_{\lambda\mu}.$$

The world vector m_λ is connected with the electromagnetic field $m_{\lambda\mu}$ by the equation

$$(9.7) \quad m_{\lambda\mu} = \partial_{[\lambda} m_{\mu]},$$

*equivalent to (1.25a). Altogether we have twenty equations for twenty-two unknowns (9.6) namely, six electromagnetic field equations (6.1a) equivalent to (1.25b), which lead to four Maxwell's equations (6.7), and ten gravitational field equations (8.9) equivalent to (1.25c).**

10. Mass density. Motion. In the Section II 7 of [10] we defined (rest) mass density $\mathfrak{M} = \mathfrak{M}(H)$ as a function of H such that $\mathfrak{M}(0) = 0$. Moreover, we required for $\mathfrak{M} \neq 0$ the validity of the continuity equation

$$(10.1) \quad \partial_\mu \mathfrak{M} u^\mu = 0.$$

These requirements lead to the formula for mass M

$$(10.2) \quad M \stackrel{\text{def}}{=} \mathfrak{M} / \sqrt{|\mathfrak{h}|} = |v/\mu|^{**} \quad (v = \sqrt{v^2}),$$

where μ is a function of position, *constant along the path of the particle*, $v^\lambda \partial_{\lambda\mu} = 0$. Hence we have by virtue of (8.8)

$$(10.3) \quad M = \frac{\sqrt{2|H|}}{|\mu|},$$

*The equations (8.13) are identically satisfied and do not represent any new conditions.

**If we admit this equation also for $M = 0$, then we may say that a particle moving with the velocity of light (photon) has no mass.

so that $\mathfrak{M} = \mathfrak{M}(H)$ and $\mathfrak{M}(0) = 0$ as required. Therefore, we have according to (6.9) and by virtue of (10.2)

$$(10.4) \quad v'' = |\mu| Mu''.$$

If we substitute this value into (8.10), then, except for the function of position μ constant along the path of the particle, the momentum energy tensor has the same arguments as in the *gravitational theory* (“general” theory) in the presence of the electromagnetic field. We may even use the momentum energy tensor for deriving the equations of motion as in the “general” theory.

Theorem (10.1). *The equations of motion are*

$$(10.5) \quad v^\lambda \nabla_\lambda v'' + \nabla_\lambda P^{\lambda\prime} = 0.$$

The proof follows at once from (8.13), (8.10), and (5.8b).

11. Discussion of (10.5). There are several reasons for disappointment as far as the result of Theorem (10.1) is concerned.

First, the equation (10.5) does not show the role of the unified connection $\Gamma_{\lambda\mu}^{\nu\prime}$ (cf. (1.10)) with which we began. The equations of motion are expressed in terms of Christoffel symbols $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ only.

Second, the gravitational theory replaced the gravitational force vector by the gravitational field: the equations of a trajectory of a particle moving freely in the gravitational field may be brought into the form

$$(11.1) \quad \frac{d^2 x''}{dp} + \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0,$$

which not only expresses the “gravitational law of inertia”, but also yields, in the first approximation, the gravitational law of Newton. In the unified theory, where instead of the simple gravitational field $h_{\lambda\mu}$ we have the unified field

$$(11.2) \quad g_{\lambda\mu} = h_{\lambda\mu} + \frac{1}{2} e_{\lambda\mu\alpha\beta} m^{\alpha\beta},$$

the equation (10.5) replaces the gravitational force vector by the gravitational field $h_{\lambda\mu}$ but introduces *besides the electromagnetic field* also a kind of “electromagnetic force vector” $\nabla_\lambda P^{\lambda\prime}$, so that from this point of view (10.5) does not represent the “unified law of inertia” but rather a generalization of the second law of Newton.

The third objection concerns the application to celestial mechanics. Assuming that the electromagnetic field (of the sun) is so weak that quadratic (and higher) products of its components may be neglected, we obtain from (8.9) the equations

$$(11.3) \quad H_{\lambda\mu} = 0,$$

which represent the gravitational field equations of the "general" theory for empty space, whose Schwarzschild solution $h_{\lambda\mu}$ is known. Substituting it into (10.5)*) we obtain a system different from (11.1). In other words, the predictions (based on (10.5)) concerning the three famous phenomena of celestial mechanics will be at variance with the predictions obtained in the "general" theory (although both theories use the same Schwarzschild field $h_{\lambda\mu}$). This discrepancy of these two predictions constitutes a serious objection concerning Theorem (10.1).

In order to dispose of these objections and to find a solution for these difficulties we must change some of the basic conditions in (1.25).

12. The law of inertia. As a leading principle for this change we assume that the predictions of the above-mentioned three phenomena by means of "general" theory have been confirmed within reasonable limits by experimental tests. Therefore, rather than to *impose* some conditions on $g_{\lambda\mu}$ we shall try to formulate some general conditions in such a way that, in the special case of celestial mechanics, they will yield predictions confirming experimental tests which have been regarded as "proof" of the gravitational theory.

From this point of view, it seems reasonable to maintain the conditions (1.25a) and (1.25c). The latter leads to the gravitational field equations as in the gravitational theory; the former is equivalent to the first set of Maxwell's equations. Therefore, our desired change may affect only (1.25b). It is irrelevant whether we consider (1.25b) or its integrability conditions,

$$(12.1) \quad \partial_{[\omega} R_{\mu\lambda]} = 0.$$

In this form we have sixteen unknowns instead of twenty, and we have four equations (12.1) instead of six equations (1.25b).

Agreement (12.1). *Henceforth in this chapter, whenever $S_{[\omega\mu\lambda]} \neq 0$ **) we replace the conditions (1.25b) (i.e., four conditions (12.1)) by another set of four conditions*

$$(12.2) \quad D_{\lambda} P^{\lambda\nu} = 2U'_{\lambda\mu} H^{\lambda\mu} \dagger).$$

Note I. Using (8.11a) and (5.11b) we may transcribe (12.2) in a form containing $m_{\lambda\mu}$, $h_{\lambda\mu}$ (and their first two derivatives) and v^{ν} and its first derivatives.

Note II. As in [10], we may also prove here that (12.2) leads to the second set of Maxwell's equations (6.5b). Hence, we have eighteen equations (6.5b), (12.2) and (8.9) for eighteen unknowns m_{λ} , v^{ν} , $h_{\lambda\mu}$. With the equation (8.9) not being changed, we still have the law of conservation of momentum and

*)And neglecting only the cubic (and higher) products of $m_{\lambda\mu}$.

**)If we are dealing with radiation, $S^{[\omega\mu\lambda]} = 0$, so that $v^{\nu} = 0$ and we do not have any equations of motion (although $\nabla_{\lambda} T^{\lambda\nu} = 0$); hence in this case it makes no sense to change the condition (1.25b).

†) D_{λ} as usual is the symbol of a covariant derivative with respect to the unified connection and $P^{\lambda\nu}$ is defined by (8.11).

energy in the form (8.13). However, the consequences here are different from those (namely (10.5)) based on (1.25b). This situation is described in the following two theorems.

Theorem (12.1). *The equations of motion are*

$$(12.3a) \quad v^\lambda D_\lambda v^\nu = 0.$$

They are equivalent to

$$(12.3b) \quad \frac{d^2 x^\nu}{dp} + \Gamma_{\lambda\mu}^\nu \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0,$$

where

$$(12.3c) \quad \frac{dx^\nu}{dp} \stackrel{\text{def}}{=} v^\nu.$$

Theorem (12.2). (*The “unified” law of inertia*). *A particle moving freely in the unified field (11.2) describes an autoparallel line of the unified connection.*

Proof. Theorem (12.2) is the description of (12.3b), which is obviously equivalent to (12.3a). Thus it suffices to prove (12.3a). We know* that

$$(12.4a) \quad U_{\nu\{\lambda\mu\}} = 0,$$

$$(12.4b) \quad U_{(\nu\lambda\mu)} = 0.$$

Hence we have by virtue of (5.2b)

$$(12.5) \quad 6U_{(\nu\lambda\mu)} h^{\lambda\mu} = 2U_{\nu\lambda\mu} h^{\lambda\mu} = 0;$$

consequently according to (8.9)

$$(12.6) \quad \begin{aligned} 2U_{\lambda\mu}^\nu T^{\lambda\mu} &= 2U_{\lambda\mu}^\nu (H^{\lambda\mu} - \tfrac{1}{2} H h^{\lambda\mu}) \\ &= 2U_{\lambda\mu}^\nu H^{\lambda\mu}. \end{aligned}$$

Therefore, the condition (12.2) is equivalent to

$$(12.7) \quad D_\lambda P^{\lambda\nu} = 2U_{\lambda\mu}^\nu T^{\lambda\mu}.$$

On the other hand, we have

$$\begin{aligned} D_\lambda P^{\lambda\nu} &= \nabla_\lambda P^{\lambda\nu} + U_{\alpha\lambda}^\nu P^{\alpha\lambda}, \\ v^\lambda D_\lambda v^\nu &= v^\lambda \nabla_\lambda v^\nu + U_{\alpha\lambda}^\nu v^\alpha v^\lambda, \end{aligned}$$

so that according to (8.10) and (8.13)

$$v^\lambda D_\lambda v^\nu + D_\lambda P^{\lambda\nu} = 2\nabla_\lambda T^{\lambda\nu} + 2U_{\alpha\lambda}^\nu T^{\alpha\lambda} = 2U_{\alpha\lambda}^\nu T^{\alpha\lambda}.$$

This equation reduces by virtue of (12.7) to (12.3a).

* Cf. Lemma (8.1) in [7].

13. Discussion of (12.3). Theorems (12.1) and (12.2) dispose of the first two objections discussed in Section 11. Therefore, we have to deal only with the third objection of that section. Neglecting the quadratic and higher products of $m_{\lambda\mu}$ we obtain, as before, the equation (11.3) with the Schwarzschild solution $h_{\lambda\mu}$ of the gravitational theory. Now we have to substitute this solution into (12.3a) or into its equivalent form (12.3b), which may be written

$$(13.1) \quad \frac{d^2 x^\nu}{dp^2} + \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} + U'_{\lambda\mu} v^\lambda v^\mu = 0.$$

In this equation we have to neglect only cubic and higher products of $m_{\lambda\mu}$. Now $U'_{\lambda\mu}$ is according to (5.11b) at least of the same order of magnitude as a quadratic product of $m_{\lambda\mu}$. Consequently $U'_{\lambda\mu} v^\lambda v^\mu$ is of the order of magnitude of biquadratic products of $m_{\lambda\mu}$ and therefore (13.1) reduces in our approximation to (11.1) *which is the equation of motion in the gravitational theory. Hence our theory must yield the same predictions of the three aforementioned phenomena as in the gravitational theory. This result disposes of the third objection.*

By means of the unified law of inertia we replaced gravitational force vectors as well as electromagnetic force vectors by the unified field $g_{\lambda\mu}$. Now we have to show that this unified law of inertia may be brought into a form which expresses—at least at a point—the corresponding Newtonian laws. Only then could this theory be looked upon as a generalization of classical results. We shall deal with this problem in the next two sections.

14. Auxiliary results.* In this section we deal with the tensor $g_{\lambda\mu}$ and the vector v^ν which are involved in (12.3). Let us denote by O a general point of our space-time whose coordinates we assume to be $x^\nu = 0$. Moreover, let us assume that $g_{\lambda\mu}$ may be expanded in a Taylor series in the neighborhood of O :

$$(14.1a) \quad g_{\lambda\mu} = (g_{\lambda\mu})_O + x^\alpha (\partial_\alpha g_{\lambda\mu})_O + \frac{1}{2} x^\alpha x^\beta (\partial_\alpha \partial_\beta g_{\lambda\mu})_O + \dots$$

for sufficiently small x 's. In subsequent considerations we shall use the equality sign $\stackrel{0}{=}$ ($\stackrel{1}{=}$) whenever we wish to stress that an equation holds only if we neglect linear (quadratic) and higher products of x 's.** In particular, we assume

$$(14.1b) \quad k_{\lambda\mu} = g_{[\lambda\mu]} \stackrel{0}{=} 0,$$

so that

$$(14.1c) \quad {}^0 h_{\lambda\mu} \stackrel{\text{def}}{=} (h_{\lambda\mu})_O \stackrel{0}{=} g_{\lambda\mu}.$$

* In Sections 14 and 15 the Greek (Latin) indices run from 1 to 4 (from 1 to 3).

** The symbol $\stackrel{0}{=}$ is evidently equivalent to the evaluation at the point O .

These relations hold for any coordinate system. As far as the vector v^r is concerned it is easily seen from

$$(14.2a) \quad D_\omega h_{\lambda\mu} = 2S_{\omega(\mu}{}^\alpha g_{\lambda)\alpha}{}^*,$$

that (12.3a) admits a first integral

$$(14.2b) \quad h_{\lambda\mu} v^\lambda v^\mu = \gamma^2 (= \text{const.}).$$

We shall confine ourselves to the case $\gamma \neq 0$. Thus, according to the results of Section 6, we have $\gamma^2 > 0$.

Lemma (14.1). *There is always a coordinate system for which*

$$(14.3a) \quad v^r \stackrel{0}{=} \gamma \delta_4^r,$$

$$(14.3b) \quad ((h_{\lambda\mu})) \stackrel{0}{=} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof. The tangential space of our space-time at O has as its ideal space I_3 a projective three dimensional space. The components ${}^0h_{\lambda\mu}$ may be regarded as projective homogenous coordinates of a quadric Q in I_3 . This quadric is a non-ruled one for ${}^0h_{\lambda\mu}$ is of signature $(---+)$. Moreover, the components $(v^r)_0$ may be looked upon as projective homogeneous coordinates of a point in I_3 . This point is inside of O , since $\gamma^2 > 0$. If we refer Q to a coordinate system of a self polar tetrahedron, then we can easily obtain (14.3b). If we take a self polar tetrahedron whose vertex inside of Q is the point v^r , then we obtain (14.3b) and

$$v^r \stackrel{0}{=} c \delta_4^r.$$

Comparing this equation with (14.2b) we obtain $\gamma^2 = c^2$. Hence a suitable orientation of the x^4 -axis then yields (14.3a).

Definition (14.1). *Every coordinate system x^r for which*

a) *the point O has the coordinates $x^r = 0$,*

b) *the equations (14.3) hold*

will be referred to as an admissible coordinate system.

Note. If x^r and x'^r are two admissible coordinate systems, we assume that the transformation $x'^r \leftrightarrow x^r$ may be expanded in a Taylor series

$$(14.4a) \quad x'^r = C_\lambda^r x^\lambda + \frac{1}{2} C_{\lambda\mu}^r x^\lambda x^\mu + \dots,$$

$$(14.4b) \quad x^r = C_{\lambda'}^r x^{\lambda'} + \frac{1}{2} C_{\lambda'\mu'}^r x^{\lambda'} x^{\mu'} + \dots$$

*This equation is a consequence of (1.10c).

In this case the C 's are constants and symmetric in their lower indices, and in particular,

$$\begin{aligned}
 {}^0h_{\lambda\mu} &= C_{\lambda}^{\alpha'} C_{\mu}^{\beta'} \delta_{\alpha'}^{\omega} \delta_{\beta'}^{\nu} {}^0h_{\omega\nu},^{*)} \\
 {}^0h_{\alpha\beta} \delta_{\omega}^{\alpha} \delta_{\nu}^{\beta} &= C_{\omega}^{\alpha} C_{\nu}^{\beta} {}^0h_{\alpha\beta}, \\
 \delta_{4'}^{\nu'} &= C_{4'}^{\nu'}, \quad \delta_{4'}^{\mu} = C_{4'}^{\mu}, \\
 C_{\lambda'\mu'}^{\nu} &= -C_{\lambda}^{\alpha} C_{\mu}^{\beta} C_{\gamma}^{\nu} C_{\alpha\beta}^{\gamma}.
 \end{aligned}
 \tag{14.4c}$$

Lemma (14.2). *If x' and x'' are two admissible coordinate systems, then we have*

$$\partial_{\omega} k_{\mu\lambda} \stackrel{0}{=} C_{\omega}^{\alpha} C_{\mu}^{\beta} C_{\lambda}^{\gamma} \partial_{\alpha} k_{\beta\gamma}
 \tag{14.5a}$$

$$(\partial_{\xi} k^{\beta'\alpha'}) (\partial_{\beta} k_{\alpha}{}^{\nu'}) \stackrel{0}{=} C_{\xi}^{\gamma} C_{\delta}^{\nu'} (\partial_{\gamma} k^{\beta\alpha}) (\partial_{\beta} k_{\alpha}{}^{\delta}).
 \tag{14.5b}$$

Proof. One obtains first

$$\partial_{\omega} k_{\mu\lambda} = \frac{\partial x^{\alpha}}{\partial x^{\omega'}} \left[\left(\partial_{\alpha} \frac{\partial x^{\beta}}{\partial x^{\mu'}} \frac{\partial x^{\gamma}}{\partial x^{\lambda'}} \right) k_{\beta\gamma} + \frac{\partial x^{\beta}}{\partial x^{\mu'}} \frac{\partial x^{\gamma}}{\partial x^{\lambda'}} \partial_{\alpha} k_{\beta\gamma} \right],$$

for any coordinate systems x' and x'' , so that by virtue of (14.1b) and (14.4b), we obtain (14.5a). The equation (14.5b) may be proven in the same way.

In the next Lemma we use the following abbreviations:

$$\mathcal{K}_{\omega\mu\nu} \stackrel{\text{def}}{=} [\partial_{\omega} k_{\nu\mu} + \partial_{\mu} k_{\omega\nu} + \partial_{\nu} k_{\omega\mu}]_0,
 \tag{14.6a}$$

$$\mathcal{U}_{\nu\omega\mu\xi} \stackrel{\text{def}}{=} -(\partial_{\xi} k_{(\mu}{}^{\alpha)} \mathcal{K}_{\omega) \nu \alpha}.
 \tag{14.6b}$$

Lemma (14.3). *The tensor $U_{\nu\omega\mu}$ defined by (4.12) satisfies the following conditions*

$$U_{\nu\omega\mu} \stackrel{0}{=} 0,
 \tag{14.7a}$$

$$U_{\nu\omega\mu} \stackrel{1}{=} x^{\xi} \mathcal{U}_{\nu\omega\mu\xi}.
 \tag{14.7b}$$

Moreover, if x' and x'' are two admissible coordinate systems then

$$\mathcal{U}_{\nu'\omega'\mu'\xi'} = C_{\nu'}^{\alpha} C_{\omega'}^{\beta} C_{\mu'}^{\gamma} C_{\xi'}^{\delta} \mathcal{U}_{\alpha\beta\gamma\delta}.
 \tag{14.8}$$

Proof. The equations (14.7) follow immediately from (4.12) and (14.1a, b). The equation (14.8) is an immediate consequence of (14.5a).

The last of the lemmas we shall need in the next section is

Lemma (14.4). *We have*

$${}^{\lambda}D_{\lambda} v^{\nu} \stackrel{0}{=} v^{\lambda} \nabla_{\lambda} v^{\nu},
 \tag{14.9a}$$

$${}^{\lambda}D_{\lambda} v^{\nu} \stackrel{1}{=} (v^{\lambda} \nabla_{\lambda} v^{\nu})_0 + x^{\xi} [(\partial_{\xi} v^{\lambda} \nabla_{\lambda} v^{\nu})_0 + \mathcal{U}_{\omega\mu\xi}^{\nu} (v^{\omega} v^{\mu})_0].
 \tag{14.9b}$$

^{*)}Here we put

$$\delta_{\lambda'}^{\nu'} = \delta_{\lambda}^{\nu} = \delta_{\lambda'}^{\nu'} \stackrel{\text{def}}{=} \delta_{\lambda}^{\nu}.$$

Proof. First we have

$$(14.10) \quad v^\lambda D_\lambda v^\nu = \frac{d^2 x^\nu}{dp^2} + \left\{ \begin{matrix} \nu \\ \lambda \mu \end{matrix} \right\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} + U^\nu{}_{\lambda\mu} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp},$$

and

$$(14.11a) \quad v^\nu = (v^\nu)_o + x^\lambda (\partial_\lambda v^\nu)_o + \frac{1}{2} x^\lambda x^\mu (\partial_\lambda \partial_\mu v^\nu)_o + \dots$$

$$(14.11b) \quad \left\{ \begin{matrix} \nu \\ \lambda \mu \end{matrix} \right\} = \left\{ \begin{matrix} \nu \\ \lambda \mu \end{matrix} \right\}_o + x^\alpha \left(\partial_\alpha \left\{ \begin{matrix} \nu \\ \lambda \mu \end{matrix} \right\} \right)_o + \dots$$

Substituting from (14.11) into (14.10) and remembering (14.7), we obtain at once (14.9).

15. Newtonian laws.

Theorem (15.1). *There is always an admissible coordinate system x^ν for which*

$$(15.1a) \quad \partial_4 h_{\lambda 4} \stackrel{0}{=} 0,$$

$$(15.1b) \quad \partial_\xi (v^\lambda \nabla_\lambda v^\nu) + \mathfrak{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu + (\partial_\xi k^{\beta\alpha}) \partial_\beta k_\alpha{}^\nu \stackrel{0}{=} 0.$$

Proof. Let $x^{\nu'}$, x^ν be two admissible coordinate systems. Then a simple inspection based on (14.1), (14.3), and (14.4) leads to

$$(15.2) \quad \partial_{4'} h_{\lambda' 4'} \stackrel{0}{=} C_{4'\lambda'}^4 + C_{\lambda'}^\alpha C_{4'4'}^\beta {}^0 h_{\alpha\beta} + C_{\lambda'}^\alpha (\partial_4 h_{\alpha 4})_o,$$

and in particular

$$(15.3a) \quad \partial_{4'} h_{4' 4'} \stackrel{0}{=} 2C_{4'4'}^4 + (\partial_4 h_{44})_o,$$

$$(15.3b) \quad \partial_{4'} h_{i' 4'} \stackrel{0}{=} C_{4'i'}^4 + C_{i'}^a (C_{4'4'}^b {}^0 h_{ab} + (\partial_4 h_{a4})_o).$$

Hence if we put

$$(15.4) \quad \begin{aligned} C_{4'4'}^4 &\stackrel{\text{def}}{=} -\frac{1}{2}(\partial_4 h_{44})_o \\ C_{4'i'}^4 &\stackrel{\text{def}}{=} -C_{i'}^a (C_{4'4'}^b {}^0 h_{ab} + (\partial_4 h_{a4})_o), \end{aligned}$$

the left hand term of (15.2) is equal to zero at O . Suppressing the indices we obtain (15.1a). In order to get (15.1b) let us introduce the abbreviations

$$\begin{aligned} P^\nu &\stackrel{\text{def}}{=} v^\lambda \nabla_\lambda v^\nu, \\ X_\xi^\nu &\stackrel{\text{def}}{=} \mathfrak{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu + (\partial_\xi k^{\beta\alpha}) (\partial_\beta k_\alpha{}^\nu). \end{aligned}$$

The left hand term in (15.1b) is the value of

$$\partial_\xi P^\nu + X_\xi^\nu \quad \text{at } O.$$

Then we obtain by virtue of Lemmas (14.2) and (14.3)

$$(15.5) \quad \partial_\xi P^{\nu'} + X_{\xi'}^{\nu'} \stackrel{0}{=} C_{\xi'}^\alpha \{ P^\beta C_{\alpha\beta}^{\nu'} + C_{\beta'}^{\nu'} [\partial_\alpha P^\beta + X_\alpha^\beta] \}_o.$$

Let us consider the conditions

$$(P^\beta)_o C_{\alpha\beta}^{\nu'} = -C_{\beta}^{\nu'} (\partial_\alpha P^\beta + X_\alpha^\beta)_o$$

equivalent to (cf. the last of (14.4c))

$$(15.6) \quad (P^\beta)_o C_{\beta}^{\beta'} C_{\mu'\beta'}^\nu = C_{\mu'}^\alpha (\partial_\alpha P^\nu + X_\alpha^\nu)_o ,$$

for which the left hand term in (15.5) vanishes at O . The equation (15.6) yields for $\nu = 4, \mu' = 4'$

$$(15.7a) \quad (P^4)_o C_{4'4'}^4 + (P^b)_o C_b^{b'} C_{4'b'}^4 = (\partial_4 P^4 + X_4^4)_o ,$$

or according to (15.4)

$$(15.7b) \quad -\frac{1}{2}(P^4 \partial_4 h_{44})_o - (P^a)_o [C_{4'4'}^{b'} {}^0 h_{ab} + (\partial_4 h_{a4})_o] = (\partial_4 P^4 + X_4^4)_o .$$

This is a condition for the coefficient $C_{4'4'}^{b'}$, which was not defined by (15.4). Hence if we choose $C_{4'4'}^{b'}$ according to (15.7b), then (15.7a) is satisfied for $C_{4'4'}^{b'}$ defined by (15.4). The remaining $C_{\mu'\lambda'}^\nu$ then may be obtained from (15.6) (which does not define them in a unique way). Hence with this choice the left hand term in (15.5) vanishes at O . Suppressing the indices, we obtain (15.1b).

Definition (15.1). Every admissible coordinate system for which (15.1) holds will be defined as an allowable system for the third class.

In order to formulate the main theorem of this section we have to remind the reader that in the so called “general” theory (*i.e.*, in the theory based on $h_{\lambda\mu}$) the electromagnetic force vector F^ν is defined by

$$(15.8) \quad F^\nu \stackrel{\text{def}}{=} \nabla_\lambda E^{\lambda\nu} ,$$

where the symmetric tensor $E^{\lambda\nu}$ is the corresponding momentum energy tensor of electromagnetism

$$(15.9) \quad E_\lambda^\nu \stackrel{\text{def}}{=} m^{\nu\alpha} m_{\alpha\lambda} + \frac{1}{4} \delta_\lambda^\nu m^{\alpha\beta} m_{\alpha\beta} .$$

We shall call F^ν the “classical relativistic electromagnetic force vector.” By the words “classical relativistic”, we stress the fact that we are dealing with the so called “general” relativity, based on $h_{\lambda\mu}$. Moreover, the equality sign $\stackrel{0}{=}$ ($\stackrel{1}{=}$) will be described as “equal at O ” (“equal in the linear neighborhood of O ”).

Theorem (15.2). Every allowable coordinate system for the third class x^ν has the following properties:

a) The unified law of inertia (12.3a) expressed in x^ν reduces at O to the classical Newtonian law of gravitation.

b) The unified law of the inertia (12.3a) expressed in x^ν reduces in the linear neighborhood of O to the classical relativistic second law of Newton for the classical relativistic electromagnetic force vector F^ν (defined by (15.8)).

Proof. Let us first prove the assertion (a). We obtain from (12.3a) (equivalent to (12.3b)) and (14.9a)

$$(15.10) \quad (v^\lambda \nabla_\lambda v^\nu)_o = \left(\frac{d^2 x^\nu}{dp^2} + \left\{ \begin{matrix} \nu \\ \lambda \mu \end{matrix} \right\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} \right)_o = 0.$$

Let us now identify t with x^4 so that we have according to (14.3a)

$$(15.11) \quad \left(\frac{dx^\nu}{dt} \right)_o = \delta_4^\nu.$$

Moreover, we have from (14.2b)

$$\frac{dt}{dp} = \gamma \left(h_{\lambda\mu} \frac{dx^\lambda}{dt} \frac{dx^\mu}{dt} \right)^{-\frac{1}{2}},$$

and therefore according to (15.11) and (15.1a)

$$(15.12) \quad \left(\frac{d^2 t}{dp^2} \right)_o = 0.$$

Therefore the equations (15.10) may be rewritten as follows:

$$(15.13a) \quad \left(\frac{d^2 x^a}{dt^2} + \left\{ \begin{matrix} a \\ 44 \end{matrix} \right\} \right)_o = 0,$$

$$(15.13b) \quad \left(\left\{ \begin{matrix} 4 \\ 44 \end{matrix} \right\} \right)_o = 0.$$

Now we have according to (14.3b) and (15.1a)

$$(15.14a) \quad \left\{ \begin{matrix} a \\ 44 \end{matrix} \right\}_o = \frac{1}{2} h^{ab} (2\partial_4 h_{b4} - \partial_b h_{44})_o = -\frac{1}{2} h^{ab} (\partial_b h_{44})_o,$$

$$(15.14b) \quad \left\{ \begin{matrix} 4 \\ 44 \end{matrix} \right\}_o = \frac{1}{2} h^{44} (\partial_4 h_{44})_o = 0.$$

The equation (15.13b) reduces to an identity by virtue of (15.14b). On the other hand, if we put

$$\Phi \stackrel{\text{def}}{=} \frac{1}{2} h_{44},$$

we obtain from (15.13a) and (15.14a)

$$(15.15) \quad \frac{d^2 x^a}{dt^2} - h^{ab} \partial_b \Phi \stackrel{o}{=} 0.$$

If we replace the symbol $\stackrel{o}{=}$ by the symbol “=” we obtain the classical Newtonian law of gravitation (the force vector $\partial_b \Phi$ acting on a particle of unit mass). This proves the assertion (a).

In order to prove assertion (b), we observe first that by virtue of (5.4) and (5.5) we obtain from (15.9)

$$E_\lambda^\nu = k^{\nu\alpha} k_{\alpha\lambda},$$

so that

$$(15.16a) \quad F^\nu = \nabla_\lambda k_\alpha{}^\lambda k^\nu{}^\alpha.$$

This equation could be rewritten. As a matter of fact the equation (6.5) is equivalent by virtue of (5.4) to

$$(15.17) \quad \nabla_\lambda k_\alpha{}^\lambda = 0.$$

Hence (15.16a) could be written

$$(15.16b) \quad F^\nu = k^{\beta\alpha} \nabla_\beta k_\alpha{}^\nu,$$

and yields

$$(15.18) \quad F^\nu \stackrel{1}{=} x^\xi (\partial_\xi k^{\beta\alpha}) \partial_\beta k_\alpha{}^\nu.$$

On the other hand, we obtain from (12.3a) and (14.9b)

$$(15.19) \quad (v^\lambda \nabla_\lambda v^\nu)_o + x^\xi [\partial_\xi v^\lambda \nabla_\lambda v^\nu + \mathfrak{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu] \stackrel{1}{=} 0.$$

If we denote by s the arc length of our trajectory, then we have from (14.2b)

$$\frac{ds}{dp} = \gamma,$$

so that

$$\frac{dx^\nu}{dp} = \frac{dx^\nu}{ds} \gamma, \quad \frac{d^2 x^\nu}{dp^2} = \frac{d^2 x^\nu}{ds^2} \gamma^2.$$

Therefore,

$$(15.20) \quad v^\lambda \nabla_\lambda v^\nu = \gamma^2 \frac{dx^\lambda}{ds} \nabla_\lambda \frac{dx^\nu}{ds}.$$

On the other hand, we have from (15.18) and (15.1b)

$$(15.21) \quad x^\xi (\partial_\xi v^\lambda \nabla_\lambda v^\nu + \mathfrak{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu) \stackrel{1}{=} -F^\nu,$$

so that (15.19) could be rewritten

$$(15.22) \quad \gamma^2 \frac{dx^\lambda}{ds} \nabla_\lambda \frac{dx^\nu}{ds} \stackrel{1}{=} F^\nu.$$

If we replace the symbol " $\stackrel{1}{=}$ " in (15.22) by the symbol "=", we obtain the classical relativistic second law of Newton for the electromagnetic force vector F^ν . This proves the assertion (b).^{*}

Note. Theorem (15.2) requires a privileged coordinate system in the neighborhood of O . A similar theorem could be proven without involving such a privileged coordinate system. Then one has to impose some conditions on $g_{\lambda\mu}$ itself. This point of view has been adopted in [13].

^{*}A similar result has been obtained by BONNOR [2] who also started by changing the original set of Einstein's conditions.

CHAPTER III: *First and second class.*

16. The tensors $K_{\omega\mu\nu}$, $U_{\nu\omega\mu}$, and $A_{\omega\mu}$. In this chapter we shall deal exclusively with the first two classes, so that we have either

$$(16.1) \quad kD \neq 0 \quad \text{for the first class}$$

or

$$(16.2) \quad k = 0, \quad D = K^2 \neq 0 \quad \text{for the second class.}$$

In both cases we shall use the formulæ (1.21) and (2.19) *i.e.*,

$$(16.3a) \quad m_{\lambda\nu} \stackrel{\text{def}}{=} \frac{1}{\sqrt{g}} (\kappa \sqrt{|k|} k_{\lambda\nu} - \tfrac{1}{2} e_{\lambda\nu\alpha\beta} k^{\alpha\beta}),$$

$$(16.3b) \quad k_{\lambda\nu} = \frac{1}{\sqrt{N}} (\tfrac{1}{2} e_{\lambda\nu\alpha\beta} m^{\alpha\beta} - \mu \sqrt{|m|} m_{\lambda\nu}) = \tfrac{1}{2} e_{\lambda\nu\alpha\beta} M^{\alpha\beta} + L_{\lambda\nu}$$

remembering that

$$(16.4) \quad k = m.$$

In subsequent considerations we shall use one or another of the following vectors:

$$(16.5a) \quad W^\nu \stackrel{\text{def}}{=} \nabla_\lambda M^{\lambda\nu},$$

$$(16.5b) \quad v^\nu \stackrel{\text{def}}{=} \nabla_\lambda m^{\lambda\nu},$$

$$(16.5c) \quad V^\nu \stackrel{\text{def}}{=} \sqrt{G} v^\nu = \frac{1}{\sqrt{N}} v^\nu.$$

If we put

$$(16.6) \quad u_\lambda = \partial_\lambda \sqrt{G},$$

we have

$$(16.7) \quad W^\nu = V^\nu + u_\lambda m^{\lambda\nu}.$$

Moreover, if we put (*cf.* Theorem (3.1))

$$(16.8a) \quad \begin{aligned} K^0_{\omega\mu\nu} &\stackrel{\text{def}}{=} {}^0K_{\omega\mu\nu} + u_\alpha m^{\alpha\xi} e_{\omega\nu\mu\xi} \\ &= e_{\omega\mu\alpha\beta} \nabla_\nu M^{\alpha\beta} + \nabla_{\{\omega} L_{\mu\nu\}} + u_\alpha m^{\alpha\xi} e_{\xi\omega\mu\nu}, \end{aligned}$$

we have

$$(16.8b) \quad K_{\omega\mu\nu} = V^\xi e_{\xi\omega\mu\nu} + K^0_{\omega\mu\nu},$$

and $K^0_{\omega\mu\nu}$ does not contain V^ν .

Theorem (16.1). Denote by $2U_{\lambda\nu\omega\mu}$ and ${}^0U_{\nu\omega\mu}$ the tensors whose nonholonomic components are

$$(16.9a) \quad 2U_{i\alpha\beta\gamma} \stackrel{\text{def}}{=} \frac{1}{\lambda_{\alpha\beta\gamma}} [-3M_{[i\alpha}h_{\beta]\gamma} - 3M_{[i\alpha}h_{\beta]\gamma} + 2L_{(\alpha}{}^i e_{\beta)\gamma}],$$

$$(16.9b) \quad 2{}^0U_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \frac{2}{\lambda_{\alpha\beta\gamma}} K_{\alpha(\beta}{}^i [\frac{1}{2}M^{\alpha\gamma}e_{\gamma)i} + L_{\alpha})_i].$$

Then

$$(16.9c) \quad U_{\nu\omega\mu} = V^\xi U_{\xi\nu\omega\mu} + {}^0U_{\nu\omega\mu},$$

and the tensors ${}^0U_{\nu\omega\mu}$ and $U_{\xi\nu\omega\mu}$ do not contain V_ξ .

Proof. We have according to (5.5) in [15]

$$(16.10) \quad 2U_{\nu\omega\mu} = k_\mu{}^\alpha L_{\nu\omega\alpha} + k_\omega{}^\alpha L_{\nu\mu\alpha},$$

where the nonholonomic components of L are defined by (1.15). The final formula (16.9c) follows easily from (1.15), (16.8b), and (16.10). The remaining part of our theorem follows from inspection of (16.9b).

Theorem (16.2). The tensor $A_{\omega\mu}$ defined by (1.29a)* may be written as follows:

$$(16.11) \quad A_{\omega\mu} = \frac{1}{2}a_{\lambda\nu\omega\mu}\nabla^\lambda V^\nu + {}^0A_{\omega\mu} + V^\lambda A_{\mu\omega\lambda} + V^\lambda V^\nu A_{\lambda\nu\omega\mu},$$

where the tensors a and A have properties similar to properties a)–d) introduced in Theorem (5.4), but for

$$(16.12) \quad \frac{1}{2}a_{\lambda\nu\omega\mu} \stackrel{\text{def}}{=} U_{\nu\beta\lambda[\omega}(e_{\mu]\delta\xi\eta}h^{\delta\beta}M^{\xi\eta} + 2L_\mu{}^\beta).$$

The proof, based on the previous theorem, follows the pattern of the proof of Theorem (5.4).

17. Maxwell's equations. In the next theorem we shall use the following abbreviations:

$$(17.1a) \quad N_\lambda \stackrel{\text{def}}{=} -\mu\partial_\lambda\sqrt{|m|}G,$$

$$(17.1b) \quad A_{\omega\mu}^0 \stackrel{\text{def}}{=} {}^0A_{\omega\mu} + \frac{1}{2}[(\nabla^\lambda N_{[\omega}m_{\mu\lambda]} + 3N^\lambda\nabla_\lambda m_{\omega\mu} \\ + \nabla^\lambda u_\alpha m^{\alpha\xi}e_{\omega\lambda\mu\xi} + e_{\omega\mu\alpha\beta}\Delta M^{\alpha\beta} - 2\mu\sqrt{|m|}G\Delta m_{\omega\mu}],$$

$$(17.1c) \quad A_{\lambda\omega\mu}^1 \stackrel{\text{def}}{=} A_{\lambda\omega\mu} + \sqrt{N}h_{\lambda[\mu}N_{\omega]}.$$

Theorem (17.1). The condition (1.25b) is equivalent to

$$(17.2) \quad \partial_{[\omega}X_{\mu]} = \frac{1}{2}(\nabla^\lambda V^\nu)(a_{\lambda\nu\omega\mu} - e_{\lambda\nu\omega\mu}) + A_{\omega\mu}^0 + V^\lambda A_{\lambda\omega\mu}^1 + V^\lambda V^\nu A_{\lambda\nu\omega\mu}.$$

The tensors $a_{\lambda\nu\omega\mu} - e_{\lambda\nu\omega\mu}$, $A_{\omega\mu}^0$, $A_{\lambda\omega\mu}^1$, $A_{\lambda\nu\omega\mu}$ do not contain V^λ or ${}^\lambda V^\nu$.

*)Of course, we have $S_\lambda = 0$.

Proof. We have from (16.8)

$$(17.3a) \quad \frac{1}{2} \nabla^\lambda K_{\omega\mu\lambda} = \frac{1}{2} (\nabla^\lambda V^\nu) e_{\nu\omega\mu\lambda} + \frac{1}{2} e_{\omega\mu\alpha\beta} \Delta M^{\alpha\beta} + \frac{1}{2} \nabla^\lambda \nabla_{[\omega} L_{\mu\lambda]} + \frac{1}{2} \nabla^\lambda u_\alpha m^{\alpha\epsilon} e_{\epsilon\omega\mu\lambda},$$

and moreover, according to (2.21),

$$(17.3b) \quad \frac{1}{2} \nabla^\lambda \nabla_{[\omega} L_{\mu\lambda]} = \sqrt{N} N_{[\omega} V_{\mu]} + \frac{1}{2} [(\nabla^\lambda N_{[\omega} m_{\mu\lambda]} + 3N^\lambda \nabla_\lambda m_{\omega\mu} - 2\mu \sqrt{|m|} G \Delta m_{\omega\mu}],$$

so that

$$\frac{1}{2} \nabla^\lambda K_{\omega\mu\lambda} = -\frac{1}{2} (\nabla^\lambda V^\nu) e_{\lambda\nu\omega\mu} + V^\lambda \sqrt{N} N_{[\omega} h_{\mu]\lambda} + A_{\omega\mu}^0 - {}^0 A_{\omega\mu}.$$

The proof of (17.2) follows easily from (1.25a, b), and (1.28a) and (16.11). The remaining part of the theorem is obvious.

Note. From (17.2) and the identity (5.8a) we obtain as before (*cf.* Theorem (6.2)) the second set of Maxwell's equations

$$(17.4) \quad v^\nu = \nabla_\lambda m^{\lambda\nu},$$

so that we apply here the same identification of the electromagnetic field and of the current density vector v^ν as before. (*Cf.* equations (6.5)–(6.9).) Of course, Theorem (6.4) holds for each of the three classes.

Theorem (17.2). *A necessary and sufficient condition for pure radiation,*

$$(17.5) \quad v^\nu = 0,$$

may be expressed in terms of $k_{\lambda\mu}$ as follows

$$(17.6) \quad U_\omega (\kappa \sqrt{|k|} k_{\mu\lambda} - \frac{1}{2} e_{\mu\lambda\alpha\beta} k^{\alpha\beta}) \left(h^{\omega\mu} h^{\lambda\nu} + \frac{\kappa \sqrt{|k|}}{2} E^{\omega\mu\lambda\nu} \right) = \kappa (\nabla_\lambda \sqrt{|k|}) k^{\lambda\nu} + \frac{1}{2} E^{\omega\mu\lambda\nu} [(1 - k) \nabla_\omega k_{\mu\lambda} + \sqrt{|k|} (\nabla_\omega \sqrt{|k|}) k_{\mu\lambda}].$$

Proof. From the equation (4.1c) in [12] we see that (1.25a) is equivalent to

$$(17.7a) \quad 2\nabla_\alpha k^{\alpha\nu} = E^{\omega\mu\lambda\nu} (\nabla_\omega \sqrt{|k|} k_{\mu\lambda} - \sqrt{g} U_\omega m_{\mu\lambda}).$$

On the other hand, we obtain from (16.3), (17.4) a necessary and sufficient condition for (17.5) in the form

$$(17.7b) \quad -U_\lambda m^{\lambda\nu} + \frac{1}{\sqrt{g}} [\kappa \nabla_\lambda \sqrt{|k|} k^{\lambda\nu} + \frac{1}{2} E^{\omega\mu\lambda\nu} \nabla_\omega k_{\mu\lambda}] = 0.$$

The equations (17.7a, b) lead directly to (17.6).

Note. In the case of pure radiation the equation (17.2) reduces to

$$(17.8) \quad \partial_{[\omega} X_{\mu]} = A_{\omega\mu}^0.$$

Theorem (17.3). *The equations (17.2) are equivalent to Maxwell's equations (6.5) in the sense that any solution of (17.2) is also a solution of (6.5).*

The proof follows the pattern of the proof of Theorem (7.2).

A corollary similar to (7.1) also holds.

Corollary (17.1). *The system (17.2) and the system of general Maxwell's equations (6.11) are equivalent in the sense that every solution of (17.2) is also a solution of (6.11).*

18. Auxiliary lemmas.

Lemma (18.1). *The tensor $B_{\mu\lambda}$ defined by (1.29b) may be expressed as follows*

$$(18.1) \quad B_{\lambda\mu} = U_{\beta\alpha\lambda\mu} \nabla^\alpha V^\beta + {}^0B_{\mu\lambda} + V^\alpha B_{\alpha\lambda\mu} + V^\alpha V^\beta B_{\alpha\beta\lambda\mu},$$

where the tensors ${}^0B_{\mu\lambda}$, $B_{\alpha\lambda\mu}$, $B_{\alpha\beta\lambda\mu}$ have properties similar to properties of the corresponding tensors involved in Lemma (8.1).

The proof based on (16.9c) follows the pattern of the proof of Lemma (8.1).

In the next lemma we shall use the following abbreviations:

$$(18.2a) \quad \begin{aligned} K'_{\omega\mu\nu} &\stackrel{\text{def}}{=} K^0_{\omega\mu\nu} - e_{\omega\mu\alpha\beta} \nabla_\nu M^{\alpha\beta} \\ &= \nabla_{[\omega} L_{\mu\nu]} + u_\alpha m^{\alpha\beta} e_{\omega\nu\mu\beta} \end{aligned}$$

$$(18.2b) \quad \begin{aligned} Q_{\nu\mu\lambda} &\stackrel{\text{def}}{=} u^\alpha [m_{\alpha\nu} h_{\mu\lambda} - m_{\alpha(\lambda} h_{\mu)\nu} - h_{\alpha(\mu} m_{\lambda)\nu}] \\ &\quad + \frac{1}{2} h_{\gamma(\lambda} K'_{\mu)\alpha\beta} E^{\alpha\beta\gamma\delta} h_{\delta\nu} = Q_{\nu\lambda\mu}. \end{aligned}$$

Lemma (18.2). *We have*

$$(18.3) \quad \begin{aligned} \frac{1}{4} K_{\alpha\mu}{}^\beta K_{\beta\lambda}{}^\alpha &= \frac{1}{4} K^0_{\alpha\mu}{}^\beta K^0_{\beta\lambda}{}^\alpha + V^\nu (\sqrt{G} \nabla_{(\mu} m_{\lambda)\nu} - Q_{\nu\mu\lambda}) \\ &\quad + \frac{1}{2} V_\alpha V_\beta (\delta_{\lambda\mu}^{\alpha\beta} - h^{\alpha\beta} h_{\lambda\mu}), \end{aligned}$$

and the tensors

$$(18.4) \quad K^0_{\alpha\mu}{}^\beta K^0_{\beta\lambda}{}^\alpha, \quad \sqrt{G} \nabla_{(\mu} m_{\lambda)\nu} - Q_{\nu\mu\lambda},$$

do not contain V^ν or $\nabla^\alpha V^\beta$.

Proof. According to (18.2a) we may rewrite (16.8b) in the following way:

$$(18.5) \quad \begin{aligned} K_{\omega\mu\nu} &= V^\xi e_{\xi\omega\mu\nu} + K^0_{\omega\mu\nu} \\ &= V^\xi e_{\xi\omega\mu\nu} + K'_{\omega\mu\nu} + e_{\omega\mu\alpha\beta} \nabla_\nu M^{\alpha\beta}. \end{aligned}$$

The proof of (18.3) follows from (18.5) and (18.2). The statement about the tensors (18.4) follows by simple inspection.

Lemma (18.3). *Put*

$$(18.6) \quad {}^*T_{\lambda\mu} = \frac{G}{2} (v_\lambda v_\mu - v^2 h_{\lambda\mu} + 2v^\alpha \nabla_{(\mu} m_{\lambda)\alpha}) + (\nabla^\alpha V^\beta) U_{\beta\alpha\lambda\mu} \\ + {}^0B_{\lambda\mu} + \frac{1}{4} K_{\alpha\mu}^0 {}^\beta K_{\beta\lambda}^0 {}^\alpha + \sqrt{G} v^\alpha (B_{\alpha\mu\lambda} - Q_{\alpha\mu\lambda}) + G v^\alpha v^\beta B_{\alpha\beta\lambda\mu}.$$

Then (1.25c) is equivalent to

$$(18.7a) \quad -\varphi h_{\lambda\mu} = H_{\lambda\mu} - {}^*T_{\lambda\mu}.$$

Moreover,

$$(18.7b) \quad {}^*T_{\lambda\mu} h^{\lambda\mu} = \frac{G}{2} v^2 + {}^0TG,$$

where the scalar 0T does not contain v^2 .

The proof of (18.7a) follows by substitution from (16.5c), (18.1), and (18.3) into (1.28b) as a consequence of (1.25a) and (1.25c). In order to prove (18.7b) we observe first that

$$(18.8a) \quad \frac{G}{2} (v_\lambda v_\mu - v^2 h_{\lambda\mu} + 2v^\alpha \nabla_{(\mu} m_{\lambda)\alpha}) h^{\lambda\mu} = \frac{G}{2} (v^2 - 4v^2 + 2v^2) \\ = -\frac{G}{2} v^2.$$

On the other hand, a simple inspection shows that the product of $h^{\lambda\mu}$ and any one of the tensors

$$(18.9) \quad U_{\beta\alpha\lambda\mu}, \quad {}^0B_{\lambda\mu}, \quad B_{\alpha\mu\lambda} - Q_{\alpha\mu\lambda}, \quad B_{\alpha\beta\lambda\mu},$$

never leads to v^2 . Therefore, besides (18.8a) the only term in (18.6) whose contraction could lead to v^2 is $K_{\alpha\mu}^0 {}^\beta K_{\beta\lambda}^0 {}^\alpha$. We have from (16.8a)

$$(18.8b) \quad \frac{1}{4} K_{\alpha\mu}^0 {}^\beta K_{\beta\lambda}^0 {}^\alpha h^{\mu\lambda} = Gv^2 + \text{terms not containing } v^2.$$

The equation (18.7b) follows from (18.8a, b).

19. Gravitational field. In the following agreement we shall use the result (18.7).

Agreement (19.1). *The function φ , up to now arbitrary, will be defined as follows:*

$$(19.1) \quad 4\varphi = {}^0TG.$$

Theorem (19.1). *We have*

$$(19.2) \quad H \stackrel{\text{def}}{=} H_{\lambda\mu} h^{\lambda\mu} = \frac{G}{2} v^2.$$

The proof follows at once from (18.7) and (19.1).

Theorem (19.2). *Put*

$$(19.3a) \quad \begin{aligned} \frac{G}{2} P_{\lambda\mu} \stackrel{\text{def}}{=} & G(B_{\alpha\beta\lambda\mu} - \tfrac{3}{4}h_{\alpha\beta}h_{\lambda\mu})v^\alpha v^\beta \\ & + v^\alpha [G\nabla_{(\mu}m_{\lambda)\alpha} + \sqrt{G}(B_{\alpha\mu\lambda} - Q_{\alpha\mu\lambda}) + u^\beta U_{\alpha\beta\lambda\mu}] \\ & + \sqrt{G}U_{\beta\alpha\lambda\mu}\nabla^\alpha v^\beta + {}^0B_{\lambda\mu} + \tfrac{1}{4}{}^0K_{\alpha\mu}{}^\beta K_{\beta\lambda}{}^\alpha - \tfrac{1}{4}h_{\lambda\mu}{}^0TG, \end{aligned}$$

$$(19.3b) \quad T_{\lambda\mu} = v_\lambda v_\mu + P_{\lambda\mu}.$$

Then (1.25c) is equivalent to

$$(19.4) \quad H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \frac{G}{2}T_{\lambda\mu}.$$

Proof. One obtains (19.4) from (16.5c), (18.7a), and (19.2).

Theorem (19.3). *The tensor $T_{\lambda\mu}$ satisfies the identity*

$$(19.5) \quad \nabla_\lambda GT^{\lambda\nu} = 0.$$

The proof follows from (19.4) and (8.14).

Note. The four conditions (19.5) are substantially the same as the “four identities” mentioned in the footnote following Theorem (8.13).

Agreement (19.2). *Since the equations (19.4) have the form of gravitational field equations in “general relativity” we identify*

- a) $h_{\lambda\mu}$ with the gravitational field,
- b) $T_{\lambda\mu}$ with the momentum energy tensor,
- c) G with the gravitational function, and
- d) the equation (19.5) with the law of conservation of momentum and energy.

Note I. In the case of the third class we had $G = 1$. On the other hand, the idea of *gravitational function* is not a new one (cf. [17]). In our case, according to (2.16), the gravitational function $G = 1/N$ depends on the *electromagnetic field*.

Note II. The considerations of §9 could be repeated here almost without change. Hence in particular, we have again the interdependence of both fields $h_{\lambda\mu}$ and $m_{\lambda\mu}^{*})$.

Note III. There are cases of pure radiation satisfying our field equations. We shall exhibit at least one particular case for $\varphi = 0$. Put

$$(19.6) \quad r \stackrel{\text{def}}{=} x^I, \quad \Theta \stackrel{\text{def}}{=} x^{II}, \quad \Phi \stackrel{\text{def}}{=} x^{III}, \quad t \stackrel{\text{def}}{=} x^{IV},$$

*)In the case of an electromagnetic field not creating gravitation we must have $v^2 = 0$ according to (19.2). Hence, v^ν is either a null vector or $v^\nu = 0$.

and

$$(19.7a) \quad m^{\lambda\nu} \stackrel{\text{def}}{=} \mathfrak{G}^{1\lambda\nu 4} \frac{\epsilon \sqrt{|A|}}{r^4 \sin \Theta},$$

$$(19.7b) \quad ((h_{\lambda\mu})) \stackrel{\text{def}}{=} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2 \Theta & 0 \\ 0 & 0 & 0 & B(1 - Ar^{-4}) \end{array} \right]$$

where $A < 0$, $B > 0$ are constants and $\epsilon = \pm 1$. This system together with $\varphi = 0$, $X_\lambda = \partial_\lambda X$ satisfies (18.7a), (17.2) as well as Maxwell's equations,

$$\partial_{[\alpha} m_{\mu\lambda]} = 0 \quad \partial_\lambda m^{\lambda\nu} = 0^*).$$

Note IV. It is not difficult to express $h_{\lambda\mu}(m^{\lambda\nu})$ according to (19.7) in terms of $m^{\lambda\nu}(h_{\lambda\mu})$, and the coordinates r , Θ .

Theorem (19.4). *The equations of motion are*

$$(19.8) \quad v^\lambda \nabla_\lambda G v^\nu + \nabla_\lambda G P^{\lambda\nu} = 0.$$

The proof follows at once from (19.5), (19.4) and (8.14).

Note. As far as the concept of mass density is concerned, we have only to repeat the corresponding considerations of §10. This also holds for the validity of the objections expressed in §11, these now being directed against (19.8).

20. The law of inertia. In order to dispose of these objections we shall make an agreement similar to that of §12.

Agreement (20.1). *Whenever we are dealing with charged particles, so that (17.6) does not hold,**) we replace in subsequent considerations the conditions (1.25b) (i.e., four conditions (12.1)) by another set of four conditions, namely,*

$$(20.1) \quad D_\lambda G P^{\lambda\nu} = 2U^\nu{}_{\lambda\mu} H^{\lambda\mu} + U_\alpha (2h^{\nu\alpha} H + P^{\alpha\nu} G).$$

The following two theorems will be proven simultaneously.

Theorem (20.1a). *The equations of motion are*

$$(20.2) \quad v^\lambda D_\lambda G v^\nu = 0.$$

Note. Denote by

$$(20.3) \quad v^\nu = \frac{dx^\nu}{dp}$$

*) (Cf. Theorem I(9.7) in [9].

**) In the case of pure radiation when (17.6) holds, we do not have any equations of motion, although the law of conservation (19.5) still holds.

the vector involved in (20.2) and by $x^\nu = x^\nu(p)$ a solution of this latter equation. Then G may be considered as a function of p along our trajectory.

Theorem (20.1b). *Put*

$$(20.4) \quad q = c_1 \int \frac{dp}{G} + c_2 \quad (c_1 \neq 0, c_2, \text{const.}).$$

Then (20.2) reduces to

$$(20.5) \quad \frac{d^2 x^\nu}{dq^2} + \Gamma_{\lambda\mu}^\nu \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0.$$

This equation admits a first integral

$$(20.6) \quad h_{\lambda\mu} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = c \quad (c = \text{const.}).$$

Proof. First of all we obtain according to (19.8) and (19.3b)

$$(20.7) \quad v^\lambda D_\lambda G v^\nu + D_\lambda G P^{\lambda\nu} = G(U_{\lambda\mu}^\nu T^{\lambda\mu} + U_\alpha P^{\alpha\nu}),$$

or by virtue of (19.4) and (12.4b)

$$(20.8) \quad v^\lambda D_\lambda G v^\nu + D_\lambda G P^{\lambda\nu} = 2U_{\lambda\mu}^\nu H^{\lambda\mu} + U_\alpha (2h^{\nu\alpha} H + P^{\alpha\nu} G).$$

According to (20.1) this equation reduces to (20.2), and this latter equation is equivalent to

$$(20.9) \quad \frac{d}{dp} G \frac{dx^\nu}{dp} + \Gamma_{\lambda\mu}^\nu \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} G = 0.$$

A simple inspection shows that the transformation (20.4) reduces this equation to the form (20.5). The proof of (20.6) follows from (14.2a).

Theorem (20.2). *The “unified” law of inertia as described in Theorem (12.2) holds also in our case of the first two classes.*

The proof, based on (20.5), is obvious.

Note. A discussion of (20.2) (of (20.5)) similar to the one given in §13 could be applied also to this case. In particular we have to show that (20.5) yields Newton’s laws in an appropriate approximation.

21. Newtonian laws. This section is based on the results of §14 and on (20.5). In order to be able to use the results of §14 without changing notation, we denote in this section by v^ν the vector $\frac{dx^\nu}{dq}$:

$$(21.1) \quad v^\nu \stackrel{\text{def}}{=} \frac{dx^\nu}{dq}^*,$$

*1) No confusion with (20.3) will arise, for in this section we shall not use the equations (20.2) and (20.3).

so that (20.5) may be rewritten in the form (12.3a), *i.e.*,

$$(21.2) \quad v^\lambda D_\lambda v^\nu = \frac{d^2 x^\nu}{dq^2} + \Gamma_{\lambda\mu}^\nu \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0.$$

Then we may use here all definitions and results of §14 except Lemmas (14.3), and (14.4). In regard to the tensor $U_{\nu\omega\mu}$ involved in these lemmas, let us first observe that we have according to (14.1) and (1.6)

$$\lambda_i \stackrel{0}{=} 0, \quad \lambda_i \stackrel{1}{=} x^\xi (\partial_\xi \lambda_i)_0,$$

so that we obtain from (1.14), (1.15), and (14.6a)

$$\lambda_{ijk} \stackrel{1}{=} 1$$

$$L_{\omega\mu\nu} \stackrel{0}{=} \mathcal{K}_{\omega\mu\nu}.$$

Hence, if we use (14.6b) we see from (16.10) that even Lemma (14.3) holds, and therefore the same is true for Lemma (14.4). Hence all results and definitions of §14 may also be used here.

Theorem (21.1). *There is always an admissible coordinate system x^ν for which*

$$(21.3a) \quad \partial_{IV} h_{IV} \stackrel{0}{=} 0$$

$$(21.3b) \quad \partial_i v^\lambda \nabla_\lambda v^\nu + \mathfrak{U}_{\omega\mu\xi}^\nu v^\omega v^\mu + (\partial_\xi k^{\beta\alpha}) \partial_\beta k_\alpha^\nu - \frac{1}{2} (\partial^\nu k_{\alpha\beta}) \partial_\xi k^{\alpha\beta} \stackrel{0}{=} 0.$$

The proof follows the pattern of the proof of Theorem (15.1).

Definition (21.1). *Every admissible coordinate system for which (21.3) holds will be termed an allowable system for the first two classes.*

Theorem (21.2). *Every allowable coordinate system for the first two classes x^ν has the following properties:*

a) *The unified law of inertia (21.2) expressed in x^ν reduces at O to the classical Newtonian law of gravitation.*

b) *The unified law of inertia (21.2) expressed in x^ν reduces in the linear neighborhood of O to the classical relativistic second law of Newton for the classical relativistic electromagnetic force vector F^ν (defined by (15.8)).*

Proof. First of all, we have according to (14.1)

$$(21.4a) \quad \nabla_\lambda k^{\lambda\nu} \stackrel{0}{=} \partial_\lambda k^{\lambda\nu},$$

$$(21.4b) \quad \nabla_\beta k^{\beta\alpha} k_\alpha^\nu \stackrel{1}{=} x^\xi [(\partial_\beta k^{\beta\alpha})(\partial_\xi k_\alpha^\nu) + (\partial_\beta k_\alpha^\nu)(\partial_\xi k^{\beta\alpha})],$$

$$(21.4c) \quad \frac{1}{4} \nabla^\nu k^{\beta\alpha} k_{\beta\alpha} \stackrel{1}{=} \frac{1}{2} x^\xi (\partial^\nu k_{\alpha\beta})(\partial_\xi k^{\alpha\beta}).$$

Moreover, by virtue of (16.3) and (14.1)

$$(21.5) \quad m_{\lambda^r} \stackrel{0}{=} 0, \quad m_{\lambda^r} \stackrel{1}{=} -\frac{1}{2} {}^0 e_{\lambda^r \alpha \beta} x^\xi \partial_\xi k^{\alpha \beta},$$

where

$${}^0 e_{\lambda^r \alpha \beta} = \sqrt{|\mathfrak{h}|_0} \, e_{\lambda^r \alpha \beta}.$$

Hence, the first set of Maxwell's equations yields

$$(21.6) \quad \partial_\lambda k^{\lambda^r} \stackrel{0}{=} 0.$$

On the other hand, we obtain from (16.3), (15.8), and (15.9)

$$(21.7a) \quad \frac{g}{1-k} E_\lambda^r = k^{\nu \alpha} k_{\alpha \lambda} + \frac{1}{4} \delta_\lambda^\nu k^{\alpha \beta} k_{\alpha \beta},$$

$$(21.7) \quad F^r = (\nabla_\beta k^{\beta \alpha} k_{\alpha}{}^r + \frac{1}{4} \nabla^r k^{\alpha \beta} k_{\alpha \beta}) \frac{1-k}{g} + E^{\lambda^r} \partial_\lambda \ln \left(\frac{1-k}{g} \right)$$

so that according to (21.4) and (21.6)

$$(21.8) \quad F^r \stackrel{1}{=} x^\xi [(\partial_\xi k^{\beta \alpha}) \partial_\beta k_{\alpha}{}^r + \frac{1}{2} (\partial^r k_{\alpha \beta}) \partial_\xi k^{\alpha \beta}].$$

From this point on the proof follows the pattern of the proof of Theorem (15.2).

APPENDIX

22. Paths of photons.

In the book of P. G. BERGMANN [1] we find the following lines:

“The experiment of MICHELSON and MORLEY has been repeated many times under varying conditions. All these new experiments have confirmed the original results, with the exception of those which were carried out by D. C. MILLER. It is very difficult to decide why MILLER’s experiments show an effect which appears to indicate an “ether drift” of about 10 km/sec. However, since all the evidence of other experiments point to the accuracy of the Lorentz transformation equations, it is reasonable to assume that MILLER’s results were caused by a systematic experimental error which has not yet been discovered.”

Far from being able to discover this error,^{*)} we wish to investigate here the most elementary geometric idea which from the point of view of the unified field theory furnishes the basis of the experiment.

Denote by $t_{(a)}$ ($a = 1, 2$) the time needed by the light signal sent from a source O to reach a point $P_{(a)}$. We shall investigate the difference $t_{(1)} - t_{(2)}$ under the following assumptions:

- 1) The points $O, P_{(1)}, P_{(2)}$ are linearly independent and

$$(22.1) \quad \overline{OP_{(1)}} = \overline{OP_{(2)}}$$

- 2) *The velocity of light is constant.* We shall denote it by 1.

- 3) The gravitational part of the unified field is so weak that it may be neglected.**)

^{*)}In 1955 R. S. SHANKLAND, S. W. MCCUSKEY, F. C. LEONE, & G. KUERTI subjected MILLER’s observations to a new analysis [21]. They summarized their results as follows: “The small periodic fringe displacements found by MILLER are due in part to statistical fluctuations in the readings of the fringe positions in a very difficult experiment. The remaining systematic effects are ascribed to local temperature conditions. . . . MILLER’s extensive Mount Wilson data contain no effect of the kind predicted by the æther theory and, within the limitations imposed by local disturbances, are entirely consistent with a null result at all epochs during a year.”

^{***)}Unified fields exist without gravitation. (Cf. for instance, Theorem (9.2)). According to the footnote on page 462 in our case, where $v'' \neq 0$, v'' must be a null vector. In neglecting the gravitational field, we come as close as possible to the situation of the Michelson-Morley experiment.

Using the assumption 3) we may put

$$(22.2) \quad ((h_{\lambda\mu})) = \begin{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \end{bmatrix}^{*})$$

In the subsequent considerations quite often we shall use the tensor

$$(22.3) \quad a_{\lambda\mu} \stackrel{\text{def}}{=} -h_{\lambda\mu} ,$$

so that

$$(22.4) \quad (dx)^2 + (dy)^2 + (dz)^2 - (dt)^2 = a_{\lambda\mu} dx^\lambda dx^\mu .$$

According to Assumption 2), we have for the tangential vector $\frac{dx^\nu}{dq}$ of a path of a photon

$$(22.5a) \quad a_{\lambda\mu} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0 ,$$

$$(22.5b) \quad \left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2 - \left(\frac{dt}{dq}\right)^2 = 0 .$$

(22.5) is a first integral of the law of inertia (20.5) for a photon. The latter equation could be rewritten according to (22.2)

$$(22.6) \quad \frac{d^2 x^\nu}{dq^2} + U_{\lambda\mu}^\nu \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0 .$$

According to (22.5b), the solution of (22.6) may be looked upon as a non-minimal curve C in our three-dimensional Euclidean space E_3 with the arc length element dt . In order to derive the differential equations of C in E_3 , we have to eliminate the parameter q from (22.6) by means of the last of these equations

$$(22.7) \quad \frac{d^2 t}{dq^2} + U_{\lambda\mu}^4 \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0 .$$

*)In this section the Greek (Latin) indices run from 1 to 4 (from 1 to 3). Moreover, very often we shall write the letters x, y, z, t for x^1, x^2, x^3, x^4 and consider x, y, z as orthogonal coordinates in our three-dimensional Euclidean space E_3 .

To this effect we introduce first the vector

$$(22.8) \quad v^r \stackrel{\text{def}}{=} \frac{dx^r}{dt}.$$

Then the result of elimination is

$$(22.9) \quad \dot{v}^a - \Omega^a_{\lambda\mu} v^\lambda v^\mu = 0,$$

where $\dot{v}^a = \frac{d}{dt} v^a$ and

$$(22.10) \quad \Omega^r_{\lambda\mu} \stackrel{\text{def}}{=} 2v^{[r} U^{4]}_{\lambda\mu}.$$

The corresponding first integral of (22.9) may be written according to (22.5b)

$$(22.11) \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1 = a_{b,c} v^b v^c.$$

Hence v^a is the tangent unit vector of C . From (22.10) we have $\Omega^4_{\lambda\mu} = 0$, so that in particular by virtue of (22.9)

$$(22.12) \quad \begin{aligned} a_{b,c} \dot{v}^b v^c &= a_{b,c} \Omega^b_{\lambda\mu} v^\lambda v^\mu v^c = a_{r,\omega} \Omega^r_{\lambda\mu} v^\lambda v^\mu v^\omega \\ &= a_{r,\omega} v^r v^\omega U^4_{\lambda\mu} v^\lambda v^\mu - a_{r,\omega} U^r_{\lambda\mu} v^\lambda v^\mu v^\omega \\ &= 0 \text{ *)}. \end{aligned}$$

Hence $\dot{v}^b$ is perpendicular to v^a , as was to be expected. Its length in E_3 is obviously the first curvature k of C :

$$(22.13) \quad \begin{aligned} k^2 &= a_{b,c} \dot{v}^b \dot{v}^c = a_{b,c} \Omega^b_{\lambda\mu} v^\lambda v^\mu \dot{v}^c \\ &= a_{r,\omega} \Omega^r_{\lambda\mu} v^\lambda v^\mu \dot{v}^\omega = -h_{r,\omega} \Omega^r_{\lambda\mu} v^\lambda v^\mu \dot{v}^\omega \\ &= -v^r \dot{v}^\omega h_{r,\omega} U^4_{\lambda\mu} v^\lambda v^\mu + U_{\omega\lambda\mu} \dot{v}^\omega v^\lambda v^\mu \\ &= U_{r\lambda\mu} v^\lambda v^\mu \dot{v}^r \text{ **)}. \end{aligned}$$

*) We have

$$\begin{aligned} a_{r,\omega} v^r v^\omega &= 0, \\ U_{r\lambda\mu} v^\lambda v^\mu v^r &= U_{(\nu\lambda\mu)} v^\lambda v^\mu v^\nu = 0. \end{aligned}$$

**) We have

$$-h_{r,\omega} v^r \dot{v}^\omega = -h_{b,c} v^b \dot{v}^c = a_{b,c} v^b \dot{v}^c = 0.$$

Hence a necessary and sufficient condition for C to be a straight line is

$$(22.14) \quad U_{\nu\lambda\mu} v^\lambda v^\mu v^\nu = 0.$$

This equation is satisfied if, for instance,

$$(22.15) \quad U^\nu_{\lambda\mu} = 2p^\nu h_{\lambda\mu} - \delta_\lambda^\nu p_\mu - \delta_\mu^\nu p_\lambda^{**}.$$

In any case, if we denote by $\delta_{(a)}$ the length of $\overline{OP_{(a)}}$ we have for a sufficiently small $\delta_{(a)}$ ^{**)}

$$(22.16a) \quad \delta_{(a)}^2 = t_{(a)}^2 - \frac{t_{(a)}^4}{4!} 2(k_{(a)}^2)_O + \dots$$

where $(k_{(a)}^2)_O$ is the value at O of the square of the first curvature

$$(22.17) \quad (k_{(a)}^2)_O = (U_{\nu\lambda\mu} v_{(a)}^\lambda v_{(a)}^\mu \dot{v}_{(a)}^\nu)_O,$$

and $v_{(a)}^\nu$ is the tangential vector of the path of photon joining O and $P_{(a)}$. If we disregard terms containing q^{th} powers of t , $q > 3$, then we may write instead of (22.16a)

$$(22.16b) \quad \delta_{(a)} = t_{(a)} \left(1 - \frac{t_{(a)}^2 k_{(a)}^2}{4!} \right),$$

so that according to (22.1)

$$(22.18) \quad 4!(t_{(1)} - t_{(2)}) = t_{(1)}^3 (k_{(1)}^2)_O - t_{(2)}^3 (k_{(2)}^2)_O.$$

When an electromagnetic field is not present or when (22.15) holds we obtain instead of (22.18)

^{*)}(22.15) is a necessary and sufficient condition for the following three requirements to be satisfied:

- a) $U^\nu_{\lambda\mu} v^\lambda v^\mu$ is for any arbitrary null vector v^ν in space-time proportional to v^ν .
- b) For any other vector $v^\nu \neq p^\nu$ which is not a null vector in space-time $U^\nu_{\lambda\mu} v^\lambda v^\mu$ is not proportional to v^ν .
- c) $U^\nu_{\lambda\mu}$ cannot be expressed in terms of v^ν without the restrictive condition (1.19a).

The statements a) and b) follow from Theorem (10.1) in [7]. We did not mention in this theorem the requirement c) because at that state of the investigation (*i.e.*, Problem B) we did not need to express U (and S) in terms of the current vector. However, this vector enters into the picture through the "back door" of Theorem (10.3) in [7], for $K_{\omega\mu\nu}$ might be expressed in terms of a current vector.

^{**) Cf.} HLAVATÝ, VÁCLAV, *Differentialgeometrie der Kurven und Flächen und Tensorrechnung*, Noordhoff (Groningen), 1945, page 47.

$$(22.19) \quad t_{(2)} = t_{(1)} .$$

The formula (22.18) may be of particular interest as shown by the following remarks:

(1) (22.18) may be considered as a theoretical basis for an experimental test of the unified field theory. The only theoretical complication is the computation of $U'_{\lambda\mu}$ from the given data.*)

(2) Even in the presence of some electromagnetic fields the formula (22.18) might reduce to (22.19). Then, of course, we would expect MICHELSON & MORLEY's result.

(3) For other fields the formula (22.18) does not reduce to (22.19), in spite of the fact that it has been derived under the assumption that the velocity of light propagation is constant. One cannot help thinking of MILLER's result.

Hence from the point of view of the unified field theory and under the assumption of the constant velocity of light, one has to expect either Michelson & Morley's result or Miller's result. In other words, from the point of view of the unified field theory Miller's result, properly interpreted, is not necessarily at variance with the assumption of the constant velocity of light propagation.

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*)Cf. (4.12b) and (16.10).

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