Subharmonic and Ultraharmonic Solutions for Weakly Non-linear Systems

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CONTENTS

Introduction	354
§1. Description of the method and proof of convergence . . .	356
§2. Existence of periodic solutions	360
§3. Variations of the method	362
§4. Further considerations for real systems	365
§5. Cases when the Jacobian is identically zero	369
§6. The Duffing and Mathieu equations	376
§7. Liénard type equation with a forcing term	383
§8. The Reuter equation	387
§9. Applications to systems	389
§10. Remarks on autonomous systems	392
Bibliography	393

INTRODUCTION

Consider a system of differential equations of the form

$$(1) \quad \dot{y}_j = \rho_j y_j + \epsilon q_j(y_1, \dots, y_n, t, \epsilon), \quad j = 1, 2, \dots, n,$$

where ϵ is a small parameter, $\rho_1, \dots, \rho_n$ are constants, and each q_j is a complex valued function, periodic in the real variable t of period $2\pi/\omega$. Such systems have been studied by M. LINSTEDT [16], A. LIAPOUNOFF [15], J. H. POINCARÉ [23], G. DUFFING [7], N. KRYLOFF & N. BOGOLIUBOFF [13], I. G. MALKIN [18], M. L. CARTWRIGHT [2], S. LEFSCHETZ [14], E. A. CODDINGTON & N. LEVINSON [4], G. COLOMBO [5], D. GRAFFI [9], and many others. The problem of interest is to determine periodic solutions of (1) whose dominant term has period equal to $2\pi/\omega$ (harmonics), or a submultiple of $2\pi/\omega$ (subharmonics), or a multiple of $2\pi/\omega$ (ultra-subharmonics).

In case $n = 2$, general theorems for the existence of periodic solutions may be obtained by use of topological methods (*cf.* [2]), though these methods do not give the solution explicitly. By application of the topological method to systems of order higher than 2 some results have been obtained recently by D. GRAFFI [9].

In general, for systems (1) with $n > 2$, the existence of periodic solutions is obtained, for instance, by some method of successive approximations, following the lines of POINCARÉ and LIAPOUNOFF or by applying the periodicity conditions of POINCARÉ. This latter approach is the one used by E. A. CODDINGTON & N. LEVINSON in a recent paper [4]. The series solutions obtained there are proved to be convergent and applications are given.

The aim of the present paper is to give general sufficient conditions for the existence of harmonics, subharmonics, ultraharmonics and ultra-subharmonics of system (1) for ϵ small. At the same time, we determine their approximate expressions through a convergent process. Various general statements are given concerning systems of the type (1) from which the existence of periodic solutions can be inferred through a minimum of computations, while a number of terms of the solution can be obtained by further computations.

The method used is a method of successive approximations which is a variant of the Poincaré method of casting out the secular terms. It is similar to the method used by L. CESARI [3] for linear systems with periodic coefficients, and also similar to the method used by J. K. HALE [11] for the determination of cycles of autonomous systems [systems (1) where q_i do not depend on t]. It may be pointed out that the series obtained by this method are proved to be convergent, while the classical method (LINSTEDT), seem to give for the same problems semi-convergent series (see for instance, M. L. CARTWRIGHT [2], p. 198).

In §1, the above mentioned method of successive approximations is defined, and the convergence is proved. In §2, general sufficient conditions are given

for the periodic solutions of (1). In particular, the already known condition that a certain Jacobian (we shall refer to this as the *classical Jacobian*) be different from zero is obtained (Theorem 2.2.i) [4]. In §3, variations of the method of §1 are discussed. In particular, the case where (1) contains a "large" periodic forcing term is considered.

In §4, we give (Theorem 4.2.i) an existence theorem for real periodic solutions in the case where the classical Jacobian is not zero. In §5, we give sufficient conditions (Theorems 5.3.i, ii, iii, iv) for real periodic solutions of the system of second order equations

$$(2) \quad \ddot{y}_j + \sigma_j^2 y_j = \epsilon q_j(y_1, \dots, y_n, \dot{y}_1, \dots, \dot{y}_n, t, \epsilon), \quad j = 1, \dots, n,$$

for which the classical Jacobian may vanish.

In §6, §7, §8, several particular weakly non-linear systems of differential equations are discussed as applications of the methods of the preceding paragraphs, and therefore, with a uniform procedure. Harmonics, subharmonics, and ultraharmonics are determined, along with their approximate expressions. In §6, the equation $\ddot{y} + c\dot{y} + \sigma^2 y = B \cos 2\omega t + \epsilon(\alpha \cos 2\omega t)y + \epsilon by^3$ is discussed in the three cases $B \neq 0, c = 0$ (non-linear Mathieu equation with a large forcing term), $\alpha = 0, c = 0$ (Duffing equation with a large forcing term), and $\alpha = 0, c \neq 0$ (Duffing equation with damping). In §7, the equation $\ddot{y} + c\dot{y} + \sigma^2 y = \epsilon p_0(y) + \epsilon p_1(y) \dot{y} + \epsilon b \cos(\omega t + \alpha)$, where $p_0(y), p_1(y)$ are polynomials, is discussed and, in particular, the case where $p_0(y) \equiv 0, p_1(y) = 1 - y^{2n}, c = 0$ (generalized Van der Pol equation with a forcing term) is discussed. In §8, the equation $\ddot{y} + \sigma^2 y = \epsilon[3\nu \cos 2t - y^2] - \epsilon^2[\lambda \dot{y} + \mu y] - \epsilon^3 \mu y^2$ (Reuter equation) is discussed for the cases where $\lambda \neq 0$, and $\lambda = 0$.

In order to illustrate how the theory may be applied to systems of differential equations, we discuss two particular fourth order systems in §9.

Due to lack of space, we have avoided a detailed comparison of the results concerning these examples obtained here with the multitude of results which have already been obtained by other methods. This discussion, together with the stability problem, will be treated later in single papers.

§1. DESCRIPTION OF THE METHOD AND PROOF OF CONVERGENCE

1.1. A method of successive approximations. Consider the system of differential equations

$$(1.1.1) \quad \dot{\mathbf{y}} = A\mathbf{y} + \epsilon \mathbf{q}(\mathbf{y}, t, \epsilon),$$

where ϵ is a parameter, $A = \text{diag}(\rho_1, \rho_2, \dots, \rho_n)$, $\rho_1, \dots, \rho_n$ being complex numbers, $\mathbf{y} = (y_1, \dots, y_n)$, $\mathbf{q} = (q_1, \dots, q_n)$, and each $q_i(\mathbf{y}, t, \epsilon)$ is of the form

$$(1.1.2) \quad q_i(\mathbf{y}, t, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k q_{ik}(\mathbf{y}, t) = \sum_{k=0}^{\infty} \epsilon^k \sum_{n_1, \dots, n_n \geq 0} q_{ik}^{(n_1, \dots, n_n)}(t) y_1^{n_1} \dots y_n^{n_n}$$

convergent for $|\epsilon| < \epsilon_0$, $|y_j| < C$, $j = 1, 2, \dots, n$. The coefficients $q_{ik}^{(n_1, \dots, n_n)}(t)$ are periodic functions of period $2\pi/\omega$, integrable in $[0, 2\pi/\omega]$.

Let us consider the auxiliary system

$$(1.1.3) \quad \dot{\mathbf{y}} = B\mathbf{y} + \epsilon \mathbf{q}(\mathbf{y}, t, \epsilon)$$

where $B = \text{diag}(i\tau_1, i\tau_2, \dots, i\tau_n)$, $\tau_i = k_i\omega/m_i$, k_i, m_i positive integers, $j = 1, 2, \dots, n$ (k_i/m_i in lowest terms). In harmony with the general procedure of successive approximations used by L. CESARI [3] and J. K. HALE [11] (where the casting out of secular terms is somewhat more straightforward than in the classical method), we shall obtain first a periodic solution of period $T = 2\pi m/\omega$, $m = \prod_{i=1}^n m_i$, of the system

$$(1.1.4) \quad \dot{\mathbf{y}} = [B - \epsilon V(\boldsymbol{\tau}, \mathbf{a}, \epsilon)]\mathbf{y} + \epsilon \mathbf{q}(\mathbf{y}, t, \epsilon),$$

where $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$, $\mathbf{a} = (a_1, \dots, a_n)$, $a_1, \dots, a_n$ are non-zero complex constants, and $V = \text{diag}(v_1, \dots, v_n)$, $v_j = v_j(\boldsymbol{\tau}, \mathbf{a}, \epsilon)$, $j = 1, 2, \dots, n$. In order that this periodic solution of (1.1.4) is also a periodic solution of (1.1.1), we see that it is necessary to choose either $\rho_1, \dots, \rho_n$ or $a_1, \dots, a_n$ so that the equations

$$(1.1.5) \quad i\tau_j - \epsilon v_j = \rho_j, \quad j = 1, 2, \dots, n,$$

are satisfied. We shall discuss system (1.1.5) in detail in §2.

The following notations are similar to those introduced by S. LEFSCHETZ [14], and are the same as those used by J. K. HALE [11]. Let s_{mjk} denote the coefficient of ϵ^{m-1} , $m = 1, 2, \dots$, when $y_1, \dots, y_n$ in $q_{ik}(\mathbf{y}, t)$ are replaced by $\mathbf{y} = \mathbf{x}_0(t) + \epsilon \mathbf{x}_1(t) + \epsilon^2 \mathbf{x}_2(t) + \dots$, where each $\mathbf{x}_i(t)$ is independent of ϵ . Moreover, let the corresponding coefficient of ϵ^{m-1} in $q_i(\mathbf{y}, t, \epsilon)$ be denoted by s_{mi} , and we have

$$(1.1.6) \quad s_{mi} = s_{mi0} + s_{m-1,i,1} + \dots + s_{1,i,m-1}.$$

It is clear that each $s_{mjk} = s_{mjk}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{m-1}, t)$ and, hence, each s_{mi} is a power series in the components of $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{m-1}$ with periodic coefficients.

We shall consider only the case where each s_{mjk} is a periodic function of t , of period T . We then define

$$\begin{aligned} a_j S_{mjk} &= m[e^{-i\tau_j t} s_{mjk}] \\ (1.1.7) \quad a_j S_{mj} &= m[e^{-i\tau_j t} s_{mj}] \\ S_m &= \text{diag}(S_{m1}, \dots, S_{mn}), \quad j = 1, 2, \dots, n, \end{aligned}$$

where $a_1, \dots, a_n$ are non-zero complex constants, and $m[\dots]$ denotes the ordinary mean value for periodic functions. Thus,

$$(1.1.8) \quad S_{mjk} = (a_j T)^{-1} \int_0^T e^{-i\tau_j t} s_{mjk}(\mathbf{x}_0(t), \mathbf{x}_1(t), \dots, \mathbf{x}_{m-1}(t), t) dt.$$

From (1.1.6), it follows that

$$(1.1.9) \quad S_{mi} = S_{mi0} + S_{m-1,i,1} + \dots + S_{1,i,m-1}.$$

We now define the method of successive approximations as follows:

$$\begin{aligned} \mathbf{y}_0 &= \mathbf{x}_0 = (a_1 e^{i\tau_1 t}, a_2 e^{i\tau_2 t}, \dots, a_n e^{i\tau_n t}), \\ (1.1.10) \quad \dot{\mathbf{y}}_m &= B\mathbf{y}_m + \sum_{k=1}^m \epsilon^k \mathbf{s}_k - \left(\sum_{k=1}^m \epsilon^k S_k \right) \mathbf{y}_{m-1} \pmod{\epsilon^{m+1}}, \end{aligned}$$

where $\mathbf{s}_k = (s_{k1}, \dots, s_{kn})$ and

$$(1.1.11) \quad \mathbf{y}_m = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \dots + \epsilon^m \mathbf{x}_m$$

and the a_i are non-zero complex constants with $|a_i|$ small enough that we may substitute the zeroth approximation in $q_i(\mathbf{y}, t, \epsilon)$. By defining the method of successive approximations in this way, we are subtracting at least those terms which give rise to secular terms. Actually, we are subtracting more than necessary to delete the secular terms in order to satisfy an equation of the form (1.1.4). Also, notice that the congruence in (1.1.10) implies that we keep only the terms in $\epsilon^0, \epsilon^1, \dots, \epsilon^m$ in the m th approximation. If we now substitute the right member of (1.1.11) into (1.1.10) and equate coefficients of ϵ^m , we obtain

$$\begin{aligned} (1.1.12) \quad \mathbf{x}_0 &= (a_1 e^{i\tau_1 t}, a_2 e^{i\tau_2 t}, \dots, a_n e^{i\tau_n t}), \\ \dot{\mathbf{x}}_m &= B\mathbf{x}_m + \mathbf{s}_m - (S_1 \mathbf{x}_{m-1} + S_2 \mathbf{x}_{m-2} + \dots + S_m \mathbf{x}_0), \quad m = 1, 2, \dots \end{aligned}$$

If we put $e^{Bt} = \text{diag}(e^{i\tau_1 t}, \dots, e^{i\tau_n t})$, $e^{-Bt} = \text{diag}(e^{-i\tau_1 t}, \dots, e^{-i\tau_n t})$, then a particular solution of (1.1.12) is given by

$$\begin{aligned} \mathbf{x}_0 &= (a_1 e^{i\tau_1 t}, a_2 e^{i\tau_2 t}, \dots, a_n e^{i\tau_n t}), \\ (1.1.13) \quad \mathbf{x}_m &= e^{Bt} \int e^{-Bt} [(\mathbf{s}_m - S_m \mathbf{x}_0) - (S_1 \mathbf{x}_{m-1} + \dots + S_{m-1} \mathbf{x}_1)] dt, \quad m > 0, \end{aligned}$$

where the above integrations are always performed to give the unique periodic primitives of mean value zero. This is possible since each integrand is periodic and of mean value zero. Furthermore, if the integrands above are split into real and imaginary parts, then the primitives may be obtained as definite integrals with ranges of integration in $[0, T]$, as was shown in [10]. This last property will be used later in majorizing $|\mathbf{x}_m|$. In order to prove that the sequence $\mathbf{y}_m$ converges to a solution $\mathbf{y}$ of (1.1.4), we shall use the Weierstrass test, *i.e.*, we shall obtain majorants ξ_m of $|\mathbf{x}_m|$, $m = 0, 1, \dots$, and show that the sum of the constant vectors $\sum_{m=0}^{\infty} \epsilon^m \xi_m$ converges for $|\epsilon|$ sufficiently small, and, hence, that $\sum_{m=0}^{\infty} \epsilon^m \mathbf{x}_m$ converges absolutely and uniformly for all t , $-\infty < t < +\infty$, and $|\epsilon|$ sufficiently small.

1.2. Majorants for the $|\mathbf{x}_{mi}|$. We shall make use of the function

$$\varphi(x) = \left(1 - \frac{x}{C}\right)^{-n} = 1 + \binom{n}{1} \frac{x}{C} + \binom{n+1}{2} \left(\frac{x}{C}\right)^2 + \dots,$$

where $|x| < C$, $C > 0$. Let x be replaced by $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$ (x_i independent of ϵ), and order the result according to ascending powers of ϵ , to obtain

$$(1.2.1) \quad \varphi(x) = \sum_{m=1}^{\infty} \epsilon^{m-1} \psi_m(x_0, \dots, x_{m-1}),$$

where $\psi_m(x_0, \dots, x_{m-1})$ is a power series in $x_0, \dots, x_{m-1}$ with positive coefficients. The above substitution is permitted if x_0, ϵ are small enough. Since, by (1.1.2), the $q_{ik}^{(n_1, \dots, n_n)}(t)$ are coefficients in the power series expansion of a holomorphic function, there exists a constant M such that $|q_{ik}^{(n_1, \dots, n_n)}(t)| < MC^{-n_1 - \dots - n_n}$. Thus, we have

$$|q_{ik}(\mathbf{y}, t)| \leq \sum_{n_1, \dots, n_n \geq 0} |q_{ik}^{(n_1, \dots, n_n)}(t) y_1^{n_1} \dots y_n^{n_n}| < M \prod_{i=1}^n (1 - |y_i|/C)^{-1}.$$

If we put $Q(y_1, \dots, y_n) = M \prod_{i=1}^n (1 - |y_i|/C)^{-1}$, then $q_{ik}(\mathbf{y}, t) \ll Q(\mathbf{y})$, where we understand by " $\ll$ " that the coefficient of $y_1^{n_1} \dots y_n^{n_n}$ in $q_{ik}(\mathbf{y}, t)$ is smaller in absolute value than the coefficient of $y_1^{n_1} \dots y_n^{n_n}$ in $Q(\mathbf{y})$. Also, we denote by $[\ ]$ the effect on any number or function of replacing each coefficient in q_{ik} by the corresponding larger coefficient in Q .

Let ξ_m denote a common upper bound for $|\mathbf{x}_{mi}|$, $j = 1, \dots, n$, and put $\xi_m = (\xi_m, \dots, \xi_m)$. Since the coefficients of the components of $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{m-1}$ in s_{mjk} are products of positive integers and the periodic coefficients in q_{jk} , we have $|s_{mjk}(\xi_0, \xi_1, \dots, \xi_{m-1}, t)| < [s_{mjk}(\xi_0, \xi_1, \dots, \xi_{m-1})] = M\psi_m(\xi_0, \xi_1, \dots, \xi_{m-1})$. From (1.1.7), we see that $a_i e^{i\tau_i t} S_{mjk} = x_{0j} S_{mjk}$ is a term in s_{mjk} and, therefore, $\xi_0 [S_{mjk}]$ is a term in $[s_{mjk}]$, and $[S_{mjk}] \leq \xi_0^{-1} [s_{mjk}] = M\xi_0^{-1} \psi_m(\xi_0, \xi_1, \dots, \xi_{m-1})$. Thus, from (1.1.6) and (1.1.9), we see that $|s_{mi}| < M$

$(\psi_m + \psi_{m-1} + \cdots + \psi_1), |S_{m,i}| < M \xi_0^{-1} (\psi_m + \psi_{m-1} + \cdots + \psi_1)$, and from (1.1.13), we see that

$$\begin{aligned}
 |x_{m,i}| &\leq 2T\{[s_{m,i}] + |S_{1,i}| \xi_{m-1} + \cdots + |S_{m-1,i}| \xi_1\} \\
 (1.2.2) \quad &< 2TM\{\psi_m + \psi_{m-1} + \cdots + \psi_1\} \\
 &+ 2TM\xi_0^{-1}\{\psi_1\xi_{m-1} + (\psi_2 + \psi_1)\xi_{m-2} \\
 &+ \cdots + (\psi_{m-1} + \cdots + \psi_1)\xi_1\}, \quad j = 1, 2, \cdots, n.
 \end{aligned}$$

The factor "2" is needed in (1.2.2) since the integrands in (1.1.13) are split into two parts, real and imaginary. From (1.2.2), we may take the following recursive relations for the majorants ξ_m :

$$\begin{aligned}
 \xi_0 &= \xi_0 \\
 (1.2.3) \quad \xi_1 &= 2TM\psi_1 \\
 \xi_m &= 2TM\{\psi_m + \psi_{m-1} + \cdots + \psi_1\} + 2TM\xi_0^{-1}\{\psi_1\xi_{m-1} + (\psi_2 + \psi_1)\xi_{m-2} \\
 &+ \cdots + (\psi_{m-1} + \cdots + \psi_1)\xi_1\}, \quad m = 2, 3, \cdots.
 \end{aligned}$$

Except for a constant factor, these are the same recurrence relations as those obtained by J. K. HALE [11]. We refer the reader to his paper for the proof that $\sum_{m=0}^{\infty} \epsilon^m \xi_m$ converges, and, hence, that $\mathbf{y}_m$ converges uniformly to a function $\mathbf{y}$ which satisfies (1.1.4), where

$$(1.2.4) \quad V(\boldsymbol{\tau}, \mathbf{a}, \epsilon) = S_1(\mathbf{a}) + \epsilon S_2(\boldsymbol{\tau}, \mathbf{a}) + \epsilon^2 S_3(\boldsymbol{\tau}, \mathbf{a}) + \cdots.$$

We have thus exhibited a periodic solution $\mathbf{y} = (y_1(t), \cdots, y_n(t))$ of (1.1.4) of period T . We now turn to a discussion of the system of equations (1.1.5) to determine when this periodic solution of (1.1.4) is also a periodic solution of (1.1.1).

§2. EXISTENCE OF PERIODIC SOLUTIONS

In this section, we shall examine the system of equations (1.1.5), and we shall see that they may be interpreted in two ways. For convenience, we shall refer to system (1.1.5) as the *determining equations*.

2.1. Determination of the ρ_i . From (1.1.5), (1.1.8) and (1.2.4), we have

$$\begin{aligned} \rho_i &= i\tau_i - \epsilon \{ S_{1i}(\mathbf{a}) + \epsilon S_{2i}(\mathbf{a}, \boldsymbol{\tau}) + \cdots \} \\ (2.1.1) \quad &= i\tau_i - \epsilon \left\{ (a_i T)^{-1} \int_0^T e^{-i\tau_i t} q_{i0}(\mathbf{x}_0, t) dt + \epsilon S_{2i}(\mathbf{a}, \boldsymbol{\tau}) + \cdots \right\}, \\ & \qquad \qquad \qquad j = 1, 2, \dots, n. \end{aligned}$$

Since the periodic coefficients in (1.1.2) are integrable, it is permissible to use the formal Fourier series of $q_{ik}^{(n_1, \dots, n_n)}(t) = \sum_{l=-\infty}^{+\infty} q_{ikl}^{(n_1, \dots, n_n)} e^{il\omega t}$ in calculating the mean values in (2.1.1) [12]. We may combine the above into the following

Theorem (2.1.i). *If $\rho_i, j = 1, 2, \dots, n$, is given by (2.1.1), then the differential system (1.1.1) has a periodic solution of the form*

$$(2.1.2) \quad y_i = a_i e^{\frac{ikh_j \omega t}{m_i}} + \epsilon W_i(t, \mathbf{a}, \epsilon), \quad j = 1, 2, \dots, n,$$

where $W_i(t, \mathbf{a}, \epsilon)$ is a holomorphic function of ϵ , whose power series expansion has coefficients periodic in t of period T . These coefficients are given explicitly by (1.1.13). In particular,

$$(2.1.3) \quad W_i(t, \mathbf{a}, 0) = e^{i\tau_i t} \int e^{-i\tau_i t} \left[q_{i0}(\mathbf{x}_0, t) - T^{-1} e^{i\tau_i t} \int_0^T e^{-i\tau_i t_1} q_{i0}(\mathbf{x}_0, t_1) dt_1 \right] dt.$$

2.2. Determination of the a_i . Assume ρ_i is given, and $\rho_i = i\tau_i + \epsilon\mu_i$. Since we may write system (1.1.1) in the form

$$(2.2.1) \quad \dot{y}_i = i\tau_i y_i + \epsilon q_i^*(\mathbf{y}, t, \epsilon), \quad j = 1, 2, \dots, n,$$

where $q_i^*(\mathbf{y}, t, \epsilon) = \mu_i y_i + q_i(\mathbf{y}, t, \epsilon)$, and apply the method of successive approximations described in section 1.1 to the system (2.2.1), it is no restriction to assume that $\rho_i = i\tau_i$. If $\rho_i = i\tau_i$, then the determining equations become

$$(2.2.2) \quad v_i(\boldsymbol{\tau}, \mathbf{a}, \epsilon) = 0, \quad j = 1, 2, \dots, n.$$

If the system

$$(2.2.3) \quad v_i(\boldsymbol{\tau}, \mathbf{a}, 0) = S_{1i}(\mathbf{a}) = 0, \quad j = 1, 2, \dots, n,$$

for some $\mathbf{a} = \mathbf{a}_0 = (a_{01}, \dots, a_{0n})$, $a_{01}, \dots, a_{0n}$ non-zero complex constants, and if the Jacobian

$$(2.2.4) \quad |\partial v_j(\boldsymbol{\tau}, \mathbf{a}_0, 0)/\partial a_i| = |\partial S_{1i}(\mathbf{a}_0)/\partial a_i| \neq 0,$$

then by the implicit function theorem, for $|\epsilon| < \epsilon_0$, there is a unique solution to (2.2.2) of the form

$$(2.2.5) \quad a_j = a_{0j} + \sum_{k=1}^{\infty} \epsilon^k b_{kj}(\boldsymbol{\tau}), \quad j = 1, \dots, n.$$

We may combine the above into the following

Theorem (2.2.i). *If $\rho_i = i\tau_i = ik_i\omega/m_i$ and if conditions (2.2.3), (2.2.4) are satisfied, then system (1.1.1) has a periodic solution of the form (2.1.2), where a_i is given by (2.2.5).*

According to J. J. STOKER [28], these solutions are classified as (α) subharmonics of order m_i if $\tau_i = \omega/m_i$; (β) ultraharmonics, if $\tau_i = k_i\omega$; (γ) ultra-subharmonics, if $\tau_i = k_i\omega/m_i$, $k_i, m_i \neq 1$.

§3. VARIATIONS OF THE METHOD

3.1. Let us consider the auxiliary system analogous to system (1.1.3)

$$(3.1.1) \quad \dot{\mathbf{y}} = B^* \mathbf{y} + \epsilon \mathbf{q}(\mathbf{y}, t, \epsilon)$$

where $B^* = \text{diag}(i\tau_1, \dots, i\tau_p, \rho_{p+1}, \dots, \rho_n)$, $\tau_i = k_i \omega / m_i$, k_i , m_i being positive integers, $j = 1, \dots, p$, and $\rho_{p+1}, \dots, \rho_n$ any complex numbers such that

$$(3.1.2) \quad \rho_i \not\equiv 0 \pmod{\omega i / m}, \quad j = p+1, \dots, n, \quad m = \prod_{i=1}^p m_i.$$

We shall use the method of successive approximations defined in Section 1.1 to obtain a periodic solution of period $T = 2\pi m_1 \dots m_p / \omega$, of the system

$$(3.1.3) \quad \dot{\mathbf{y}} = [B^* - \epsilon V^*(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n, \mathbf{a}, \epsilon)] \mathbf{y} + \epsilon \mathbf{q}(\mathbf{y}, t, \epsilon),$$

where $\mathbf{a} = (a_1, \dots, a_p)$, $a_1, \dots, a_p$ are non-zero complex constants, and $V^* = \text{diag}(v_1^*, \dots, v_p^*, 0, \dots, 0)$. The functions s_{m_i} are defined as in (1.1.7) and the numbers S_{m_i} are defined as in (1.1.8) with the only change that $S_{m_i} = 0$ for $j = p+1, \dots, n$, $m = 1, 2, \dots$. The method of successive approximations is defined as follows:

$$(3.1.4) \quad \begin{aligned} \mathbf{y}_0 &= \mathbf{x}_0 = (a_1 e^{i\tau_1 t}, \dots, a_p e^{i\tau_p t}, 0, \dots, 0) \\ \dot{\mathbf{y}}_m &= B^* \mathbf{y}_m + \sum_{k=1}^m \epsilon^k \mathbf{s}_k - \left(\sum_{k=1}^m \epsilon^k S_k \right) \mathbf{y}_{m-1} \pmod{\epsilon^{m+1}}. \end{aligned}$$

Substituting (1.1.11) in (3.1.4) and equating coefficients of ϵ^m , we obtain

$$(3.1.5) \quad \begin{aligned} \mathbf{x}_0 &= (a_1 e^{i\tau_1 t}, \dots, a_p e^{i\tau_p t}, 0, \dots, 0) \\ \dot{\mathbf{x}}_m &= B^* \mathbf{x}_m + \mathbf{s}_m - (S_1 \mathbf{x}_{m-1} + S_2 \mathbf{x}_{m-2} + \dots + S_m \mathbf{x}_0). \end{aligned}$$

If we put $e^{B^* t} = \text{diag}(e^{i\tau_1 t}, \dots, e^{i\tau_p t}, e^{\rho_{p+1} t}, \dots, e^{\rho_n t})$, $e^{-B^* t} = \text{diag}(e^{-i\tau_1 t}, \dots, e^{-i\tau_p t}, e^{-\rho_{p+1} t}, \dots, e^{-\rho_n t})$, then a particular solution of (3.1.5) is given by

$$(3.1.6) \quad \begin{aligned} \mathbf{x}_0 &= (a_1 e^{i\tau_1 t}, \dots, a_p e^{i\tau_p t}, 0, \dots, 0), \\ \mathbf{x}_m &= e^{B^* t} \int e^{-B^* t} [(\mathbf{s}_m - S_m \mathbf{x}_0) - (S_1 \mathbf{x}_{m-1} + \dots + S_{m-1} \mathbf{x}_1)] dt, \quad m > 0. \end{aligned}$$

Let us observe that the first p components of the integrand in (3.1.6) are periodic of period T , and have mean value zero. However, the $p+1, \dots, n$, components are not periodic but are functions of the form $f_i(t) = e^{-\rho_i t} p_i(t)$, where $p_i(t)$ is periodic of period T . Functions of this type have been studied in detail by J. K. HALÉ [10]. The function $f_i(t)$ is said to have mean value zero if $-\rho_i + ik\omega/m \neq 0$ for all integers k . By assumption (3.1.2), and since the $p+1, \dots, n$ components of the zeroth approximation (3.1.6) are zero, we see that the $p+1, \dots, n$ components of the integrand in (3.1.6) also have mean value zero. Hence, the

integrations may be performed to give the unique primitives of mean value zero [10]. We may now apply the following lemma, due to J. K. HALE [10], where we write $\rho_i = \mu_i + i\nu_i$, μ_i and ν_i being real.

Lemma. If $f_i(t) = e^{-(\mu_i + i\nu_i)t} p_i(t)$, where μ_i and ν_i are real, $\mu_i + i\nu_i \neq 0 \pmod{\omega i/m}$, and if $p_i(t)$ is a periodic integrable function of period $2\pi m/\omega$, then

$$(3.1.7) \quad \left| \int f_i(t) dt \right| \leq DN(\mu_i, \nu_i),$$

where $D = \int_0^{2\pi m/\omega} |p_i(t)| dt$ and $N(\mu_i, \nu_i)$ is a constant depending only on μ_i, ν_i , and $2\pi m/\omega$.

If we now carry out the evaluations of paragraph (2.1), we have the following inequalities, analogous to (1.2.2): $|x_{mi}| \leq 2T\{|s_{mi}| + |S_{1i}| \xi_{m-1} + \dots + |S_{m-1,i}| \xi_1\}$, $j = 1, 2, \dots, p$, $|x_{mj}| \leq N(\mu_j, \nu_j) |s_{mj}|$, $j = p+1, \dots, n$. Letting $K = \max(N(\mu_j, \nu_j), 2T)$, $j = p+1, \dots, n$, we have finally $|x_{mj}| \leq K\{|s_{mi}| + |S_{1i}| \xi_{m-1} + \dots + |S_{m-1,i}| \xi_1\}$. Thus, the proof that $\mathbf{y}_m$ converges to a solution $\mathbf{y}$ of (3.1.3), where

$$(3.1.8) \quad V^*(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n, \mathbf{a}, \epsilon) = S_1(\mathbf{a}) + \epsilon S_2(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n, \mathbf{a}) + \dots,$$

is carried out exactly as in paragraph (1.2).

We have thus exhibited a solution of (3.1.3) of the form

$$(3.1.9) \quad y_j = a_j e^{\frac{i k_j}{m_j} \omega t} + \epsilon W_j(t, \mathbf{a}, \epsilon), \quad j = 1, \dots, n,$$

where $a_{p+1}, \dots, a_n$ are zero, and $W_j(t, \mathbf{a}, \epsilon)$ is a holomorphic function of ϵ whose power series expansion has coefficients periodic in t of period T .

In order that this solution (3.1.9) of (3.1.3) is also a solution of (1.1.1), it is necessary to solve the equations

$$(3.1.10) \quad i\tau_j - \epsilon v_j^*(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n, \mathbf{a}, \epsilon) = \rho_j, \quad j = 1, \dots, p,$$

either for $\rho_1, \dots, \rho_p$, or $a_1, \dots, a_p$. If we fix $a_1, \dots, a_p$, then, by the implicit function theorem, there is a unique solution of (3.1.10) for $|\epsilon| < \epsilon_0$, $\epsilon_0 > 0$, of the form

$$(3.1.11) \quad \rho_j = i\tau_j + \sum_{k=1}^{\infty} \epsilon^k b_{jk}(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n, \mathbf{a}), \quad j = 1, \dots, p.$$

If ρ_j is given, $\rho_j = i\tau_j$, $j = 1, 2, \dots, p$, then the discussion of paragraph (2.2) is applied to determine $a_1, \dots, a_p$, i.e., if

$$(3.1.12) \quad v_j^*(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n, \mathbf{a}, 0) = S_{1j}(\mathbf{a}) = 0, \quad j = 1, \dots, p$$

for some $\mathbf{a} = \mathbf{a}_0 = (a_{01}, \dots, a_{0p})$, where $a_{01}, \dots, a_{0p}$ are non-zero complex constants, and if the Jacobian

$$(3.1.13) \quad |\partial S_{1j}(\mathbf{a}_0)/\partial a_i| \neq 0, \quad i, j = 1, \dots, p,$$

then, by the implicit function theorem, for $|\epsilon| < \epsilon_0$ there is a unique solution to $v_j^* = 0, j = 1, 2, \dots, p$, of the form

$$(3.1.14) \quad a_i = a_{0i} + \sum_{k=1}^{\infty} \epsilon^k b_{ik}(\tau_1, \dots, \tau_p, \rho_{p+1}, \dots, \rho_n), \quad j = 1, \dots, p.$$

The above may be combined in the following

Theorem (3.1.i). *If each $\rho_j, j = 1, \dots, p$, is given by (3.1.11), or if $\rho_j = i\tau_j, j = 1, \dots, n$ and (3.1.12), (3.1.13) are satisfied, and if (3.1.2) is satisfied, then the differential system (1.1.1) has a periodic solution of the form (3.1.9).*

3.2. Extension to systems with large forcing terms. Consider the differential system

$$(3.2.1) \quad \dot{\mathbf{y}} = B\mathbf{y} + \mathbf{f}(t) + \epsilon \mathbf{q}(\mathbf{y}, t, \epsilon),$$

where ϵ is a parameter, $B = \text{diag}(i\tau_1, \dots, i\tau_n)$, $\tau_j = k_j\omega/m_j$, k_j and m_j are positive integers, $\mathbf{y} = (y_1, \dots, y_n)$, $\mathbf{f} = (f_1, \dots, f_n)$, $\mathbf{q} = (q_1, \dots, q_n)$, and each $q_i(\mathbf{y}, t, \epsilon)$ is of the form (1.1.2), convergent for $|\epsilon| < \epsilon_0$, $|y_j| < C$. The coefficients $q_{ik}^{(n_1, \dots, n_n)}(t)$ and the functions $f_i(t)$ are periodic of period $2\pi/\omega$, integrable in $[0, 2\pi/\omega]$.

If we assume that $\int_0^{2\pi m_i/\omega} e^{-i\tau_i t} f_i(t) dt = 0, j = 1, \dots, n$, then

$$y_j^* = e^{i\tau_j t} \int e^{-i\tau_j t} f_j(t) dt, \quad j = 1, \dots, n,$$

is a periodic solution of $\dot{\mathbf{y}} = B\mathbf{y} + \mathbf{f}$, of period $2\pi/\omega$. Let us put $y_j = y_j^* + z_j$, $j = 1, \dots, n$, and (3.2.1) becomes

$$(3.2.2) \quad \dot{\mathbf{z}} = B\mathbf{z} + \epsilon \mathbf{q}(z_1 + y_1^*, \dots, z_n + y_n^*, t, \epsilon)$$

where $\mathbf{z} = (z_1, \dots, z_n)$. If we assume that $|y_j^*| < C_1 < C$, then we may write (3.2.2) in the form

$$(3.2.3) \quad \dot{\mathbf{z}} = B\mathbf{z} + \epsilon \mathbf{q}^*(\mathbf{z}, t, \epsilon),$$

where $\mathbf{q}^* = (q_1^*, \dots, q_n^*)$, and each q_j^* is of the form

$$q_j^*(\mathbf{z}, t, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \sum_{n_1, \dots, n_n \geq 0} q_{jk}^{*(n_1, \dots, n_n)}(t) z_1^{n_1} \dots z_n^{n_n},$$

convergent for $|\epsilon| < \epsilon_0$, $|z_j| < C - C_1, j = 1, \dots, n$. The coefficients $q_{jk}^{*(n_1, \dots, n_n)}(t)$ are periodic of period $2\pi/\omega$, integrable in $[0, 2\pi/\omega]$. Since the number $C - C_1 \neq 0$, then we may apply the algorithm of section (1.1) to system (3.2.3), where system (3.2.3) is taken as the auxiliary system (1.1.3). If (3.2.3) satisfies the conditions of theorem (2.2.i), then there is a periodic solution of period $2\pi m_1 \dots m_n/\omega$ of (3.2.3) of the form

$$(3.2.4) \quad y_j = e^{i\tau_j t} \int e^{-i\tau_j t} f_j(t) dt + a_j e^{i\tau_j t} + \epsilon W_j^*(t, \mathbf{a}, \epsilon), \quad j = 1, \dots, n.$$

§4. FURTHER CONSIDERATIONS FOR REAL SYSTEMS

4.1. Consistency of the determining equations. We shall now discuss the system of equations (1.1.1) with ϵ a real parameter > 0 and the matrix A defined by

$$(4.1.1) \quad A = \text{diag} (C_1, \dots, C_n, C), \quad C = \text{diag} (\beta_{2n+1}, \dots, \beta_N),$$

$$C_i = \begin{vmatrix} 0 & 1 \\ -\sigma_i^2 & -\alpha_i \end{vmatrix}, \quad j = 1, 2, \dots, n,$$

where $\sigma_i \geq 0$, $\alpha_i \geq 0$ and each q_k is a real function satisfying (1.1.2). Moreover, we assume that $4\sigma_i^2 - \alpha_i^2 > 0$, $j = 1, 2, \dots, n$ and the real numbers $\beta_\mu \not\equiv 0 \pmod{\omega i/k}$, $k = 0, 1, 2, \dots, \mu = 2n + 1, \dots, N$. In the following, we shall refer to the differential system satisfying these conditions as *system (4.1.1)*.

We shall designate the roots of the equation $\rho^2 + \alpha_i \rho + \sigma_i^2 = 0$ by ρ_{i1} , $\rho_{i2} = \bar{\rho}_{i1}$. We then have

$$(4.1.2) \quad \rho_{i1} = (-\alpha_i/2) + i\gamma_i, \quad \gamma_i = \frac{1}{2}(4\sigma_i^2 - \alpha_i^2)^{\frac{1}{2}} > 0.$$

If we introduce new variables $z_1, \dots, z_N$ defined by

$$(4.1.3) \quad \mathbf{z} = P\mathbf{y}, \quad P = \text{diag} (P_1, \dots, P_n, I),$$

where

$$P_i = \begin{vmatrix} -\rho_{i2} & 1 \\ \rho_{i1} & -1 \end{vmatrix}, \quad j = 1, 2, \dots, n$$

and I is the identity matrix of dimension $N - 2n$, then system (4.1.1) is transformed into the canonical system

$$(4.1.4) \quad \dot{\mathbf{z}} = PAP^{-1}\mathbf{z} + \epsilon P\mathbf{q}(P^{-1}\mathbf{z}, t; \epsilon),$$

where $PAP^{-1} = \text{diag} (\rho_{11}, \rho_{12}, \dots, \rho_{n1}, \rho_{n2}, \beta_{2n+1}, \dots, \beta_N)$, $P^{-1} = (P_1^{-1}, \dots, P_n^{-1}, I)$, and

$$2i\gamma_i P_i^{-1} = \begin{vmatrix} 1 & 1 \\ \rho_{i1} & \rho_{i2} \end{vmatrix}, \quad j = 1, 2, \dots, n.$$

Let us replace ρ_{i1} , ρ_{i2} by $i\tau_i$, $-i\tau_i$, respectively, $\tau_i = k_i\omega/m_i$, k_i , m_i positive integers and consider the auxiliary system

$$(4.1.5) \quad \dot{\mathbf{z}} = B\mathbf{z} + \epsilon P\mathbf{q}(P^{-1}\mathbf{z}, t, \epsilon),$$

where $B = \text{diag} (i\tau_1, -i\tau_1, \dots, i\tau_n, -i\tau_n, \beta_{2n+1}, \dots, \beta_N)$. We may now apply the preceding algorithm to system (4.1.5), and we take the zeroth approximation to be

$$(4.1.6) \quad \mathbf{z}_0 = \mathbf{x}_0 = (a_1 e^{i\tau_1 t}, -\bar{a}_1 e^{-i\tau_1 t}, \dots, a_n e^{i\tau_n t}, -\bar{a}_n e^{-i\tau_n t}, 0, \dots, 0),$$

where the “ $-$ ” denotes complex conjugate and $a_1, \dots, a_n$ are complex numbers different from zero. (In terms of the preceding notation, we have let $a_{2i-1} = a_i$, $a_{2i} = -\bar{a}_i$). We shall also denote the m^{th} approximation by $\mathbf{z}_m = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \dots + \epsilon^m \mathbf{x}_m$. In order actually to obtain a periodic solution of (4.1.1), we see, from §1, that we must solve the system of equations

$$(4.1.7) \quad \begin{aligned} i\tau_j - \epsilon v_{2j-1}(\boldsymbol{\tau}, \mathbf{a}, \epsilon) &= \rho_{j1}, & j &= 1, 2, \dots, n, \\ -i\tau_j - \epsilon v_{2j}(\boldsymbol{\tau}, \mathbf{a}, \epsilon) &= \rho_{j2}, \end{aligned}$$

$[\boldsymbol{\tau} = (\tau_1, \dots, \tau_n), \mathbf{a} = (a_1, \dots, a_n)]$ for either the ρ_{j1}, ρ_{j2} or the numbers $a_1, \dots, a_n$. In the next few pages we shall show that $v_{2i-1}(\boldsymbol{\tau}, \mathbf{a}, \epsilon) \equiv \bar{v}_{2i}(\boldsymbol{\tau}, \mathbf{a}, \epsilon)$ for every permissible $\boldsymbol{\tau}, \mathbf{a}, \epsilon$. This identity is neither a necessary nor a sufficient condition for the solution of system (4.1.7). However, at a solution point of system (4.1.7), the relation $v_{2i-1} = \bar{v}_{2i}$ must hold. The particular choice of the first approximation $\mathbf{z}_0$ leads to the above identity, true for every system of the type (4.1.1).

Lemma (4.1.i). *If we apply the preceding algorithm to system (4.1.5) with zeroth approximation given by (4.1.6), we have*

$$x_{r,2j-1} = -\bar{x}_{r,2j}, \quad x_{r,\mu} = \bar{x}_{r,\mu}, \quad s_{r,2j-1} = -\bar{s}_{r,2j}, \quad s_{r,\mu} = \bar{s}_{r,\mu}, \quad S_{r,2j-1} = \bar{S}_{r,2j},$$

for $r = 0, 1, 2, \dots, j = 1, 2, \dots, n, \mu = 2n + 1, \dots, N$.

Proof. We shall first prove by induction that $x_{r,2j-1} = -\bar{x}_{r,2j}$, $x_{r,\mu} = \bar{x}_{r,\mu}$ for all r, j, μ . From (4.1.6), we see that the assertion is true for $r = 0$. Assume the assertion is true for $r = 0, 1, \dots, m-1$, and all j, μ . Then the expressions $\frac{x_{r,2j-1} + x_{r,2j}}{x_{r,2j-1} - \bar{x}_{r,2j-1}}$ and $\frac{\rho_{j1}x_{r,2j-1} + \rho_{j2}x_{r,2j}}{\rho_{j1}x_{r,2j-1} - \rho_{j2}x_{r,2j-1}}$ are purely imaginary. As a consequence, $P^{-1}\mathbf{x}_r$ ($r = 0, 1, \dots, m-1$) is real and, thus, $P^{-1}\mathbf{z}_r$ ($r = 0, 1, \dots, m-1$) is real. Therefore,

$$\bar{q}_k(P^{-1}\mathbf{z}_r, t; \epsilon) = q_k(\overline{P^{-1}\mathbf{z}_r}, t, \epsilon) = q_k(P^{-1}\mathbf{z}_r, t, \epsilon), \quad k = 1, 2, \dots, N.$$

Thus, by our definition of $s_{r,2j-1}, s_{r,2j}, s_{r,\mu}$, and formulas (4.1.5) and (4.1.3), we have

$$(4.1.8) \quad \begin{aligned} s_{r,2j-1} &= -\bar{s}_{r,2j}, & s_{r,\mu} &= \bar{s}_{r,\mu}, \\ r &= 1, 2, \dots, m, & j &= 1, 2, \dots, n, & \mu &= 2n + 1, \dots, N. \end{aligned}$$

We also know from (1.1.7) and (4.1.6) that $S_{r,2j-1} a_j e^{i\tau_j t}$ and $S_{r,2j} (-\bar{a}_j \cdot e^{-i\tau_j t})$ are terms in $s_{r,2j-1}$, $s_{r,2j}$, respectively. Therefore, using (4.1.8), we have $S_{r,2j-1} a_j e^{i\tau_j t} = -\bar{S}_{r,2j} (-a_j e^{i\tau_j t}) = \bar{S}_{r,2j} a_j e^{i\tau_j t}$, or, since $a_j \neq 0$,

$$(4.1.9) \quad S_{r,2j-1} = \bar{S}_{r,2j}, \quad r = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

Therefore, if we use all the previous results in expression (1.1.13), we obtain

$$\begin{aligned} \bar{x}_{m,2j-1} &= e^{-i\tau_j t} \int e^{i\tau_j t} [\bar{s}_{m,2j-1} - (\bar{S}_{1,2j-1} \bar{x}_{m-1,2j-1} + \dots + \bar{S}_{m,2j-1} \bar{x}_{0,2j-1})] dt \\ &= e^{-i\tau_j t} \int e^{i\tau_j t} [-s_{m,2j} + (S_{1,2j} x_{m-1,2j} + \dots + S_{m,2j} x_{0,2j})] dt \\ &= -x_{m,2j}, \quad j = 1, \dots, n, \\ \bar{x}_{m\mu} &= e^{\beta_{\mu} t} \int e^{-\beta_{\mu} t} \bar{s}_{m\mu} dt = e^{\beta_{\mu} t} \int e^{-\beta_{\mu} t} s_{m\mu} dt = x_{m\mu}, \quad \mu = 2n+1, \dots, N, \end{aligned}$$

and the induction on the x_{rk} is completed. Finally, (4.1.8) and (4.1.9) hold for all r , and the lemma is proved.

From (1.2.4), we know that $v_k(\tau, \mathbf{a}, \epsilon) = \sum_{m=1}^{\infty} \epsilon^{m-1} S_{mk}$, $k = 1, 2, \dots, 2n$, and, thus, from the preceding lemma, we necessarily have $\bar{v}_{2j-1} = v_{2j}$. Finally, we may state

Lemma (4.1.ii). *By applying the preceding algorithm to system (4.1.5) with the zeroth approximation given by (4.1.6), we have $\bar{v}_{2j-1} = v_{2j}$, $j = 1, 2, \dots, n$, and the equations (4.1.7) are consistent.*

4.2. Sufficient conditions for periodic solutions. We saw, in the preceding section, a general system of differential equations for which the determining equations (4.1.7) were consistent. In order to find a periodic solution for this particular system (4.1.1), we must solve the system of equations

$$(4.2.1) \quad i\tau_j - \epsilon v_{2j-1}(\tau, \mathbf{a}, \epsilon) = \rho_{j1}, \quad j = 1, 2, \dots, n$$

either for the complex numbers ρ_{j1} , $j = 1, 2, \dots, n$, or, for the complex numbers $a_1, \dots, a_n$. As we have already remarked in §2, this system of equations always has a solution $\rho_{11}, \dots, \rho_{n1}$ for any complex numbers $a_1, \dots, a_n$ provided that we allow $\rho_{11}, \dots, \rho_{n1}$ to be complex. On the other hand, if we wish to fix the numbers ρ_{j1} , we have seen that (§2) we may assume $\rho_{j1} = i\tau_j$, i.e., in (4.1.1), we have $\alpha_j = 0$, $\sigma_j = \tau_j$, $j = 1, 2, \dots, n$. In this case, since $v_{2j-1} = S_{1,2j-1} + \epsilon S_{2,2j-1} + \dots$, we see that, if the system of equations

$$(4.2.2) \quad \begin{aligned} I(S_{1,2j-1}) + \epsilon I(S_{2,2j-1}) + \dots &= 0, \\ R(S_{1,2j-1}) + \epsilon R(S_{2,2j-1}) + \dots &= 0, \end{aligned} \quad j = 1, 2, \dots, n$$

has a solution for some complex numbers $a_1, \dots, a_n$, then the system (4.2.1) will also have a solution for the same numbers $a_1, \dots, a_n$.

In order to find sufficient conditions for the existence of a solution of (4.2.2), we shall put $a_i = b_i e^{i\phi_i}$, $b_i > 0$, $0 \leq \phi_i < 2\pi$. Moreover, if we let $\mathbf{b} = (b_1, \dots, b_n)$, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$ and $P_i(\mathbf{b}, \boldsymbol{\phi}) = (b_i T) R(S_{1,2i-1})$, $Q_i(\mathbf{b}, \boldsymbol{\phi}) = (-b_i T) I(S_{1,2i-1})$, $j = 1, 2, \dots, n$, then it is a simple calculation to see that

$$(4.2.3) \quad \begin{aligned} P_i(\mathbf{b}, \boldsymbol{\phi}) &= \int_0^T q_{2i}(\mathbf{y}_0, t, 0) y_{0,2i} dt + \int_0^T q_{2i-1}(\mathbf{y}_0, t, 0) y_{0,2i-1} dt, \\ Q_i(\mathbf{b}, \boldsymbol{\phi}) &= \tau_i \int_0^T q_{2i}(\mathbf{y}_0, t, 0) y_{0,2i-1} dt - \tau_i \int_0^T q_{2i-1}(\mathbf{y}_0, t, 0) y_{0,2i} dt, \end{aligned}$$

where $y_{0,2i-1} = b_i \tau_i^{-1} \sin(\tau_i t + \phi_i)$, $y_{0,2i} = b_i \cos(\tau_i t + \phi_i)$, $j = 1, 2, \dots, n$, $y_{0\mu} = 0$, $\mu = 2n + 1, \dots, N$.

If there exist real numbers $b_{k0} > 0$, ϕ_{k0} , $k = 1, 2, \dots, n$, such that

$$(4.2.4) \quad P_i(\mathbf{b}_0, \boldsymbol{\phi}_0) = 0, \quad Q_i(\mathbf{b}_0, \boldsymbol{\phi}_0) = 0, \quad j = 1, 2, \dots, n,$$

and the Jacobian

$$(4.2.5) \quad \left| \frac{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)}{\partial(b_1, \dots, b_n, \phi_1, \dots, \phi_n)} \right| \neq 0$$

when $(\mathbf{b}, \boldsymbol{\phi}) = (\mathbf{b}_0, \boldsymbol{\phi}_0)$, then it follows, as in §2, that, for ϵ sufficiently small, there is a solution $(\mathbf{b}, \boldsymbol{\phi})$ of (4.2.2) of the form $b_i = b_{i0} + o(\epsilon)$, $\phi_i = \phi_{i0} + o(\epsilon)$.

Theorem (4.2.i). *If, in (4.1.1), $\sigma_i = k_i \omega / m_i$, k_i, m_i positive integers and if (4.2.4) and (4.2.5) are satisfied, then, for ϵ sufficiently small, there is a real periodic solution to system (4.1.1) of the form*

$$(4.2.6) \quad \mathbf{y} = \mathbf{y}_0 + \epsilon \mathbf{W}(t; \epsilon),$$

where $\mathbf{y}_0$ is defined as in (4.2.3) and each W_k is a real analytic function of ϵ with $W_k(t + 2\pi m_1 \dots m_n / \omega; \epsilon) = W_k(t; \epsilon)$.

Proof. The existence of a periodic solution is a result of (4.2.4) and (4.2.5) and the form of the solution (4.2.6) is obtained from the transformation formulas (4.1.3) and lemma (4.1.i).

§5. CASES WHERE THE JACOBIAN IS IDENTICALLY ZERO

5.1. Expressions for the s_{rjk} . Consider the system of equations

$$(5.1.1) \quad \ddot{y}_j + \sigma_j^2 y_j = \epsilon f_j(\mathbf{y}, \dot{\mathbf{y}}, t, \epsilon), \quad j = 1, 2, \dots, n,$$

where $\mathbf{y} = (y_1, \dots, y_n)$, $\dot{\mathbf{y}} = (\dot{y}_1, \dots, \dot{y}_n)$, $\epsilon > 0$, $\sigma_1, \dots, \sigma_n$ are real numbers and each f_j is a real analytic function for $\epsilon < \epsilon_0$, $|y_j|, |\dot{y}_j| < A$, $j = 1, 2, \dots, n$, and the coefficients in the expansion of f_j are real functions of t , periodic of period $2\pi/\omega$.

System (5.1.1) is equivalent to the system

$$(5.1.1)' \quad \begin{aligned} \dot{u}_{2j-1} &= u_{2j} \\ \dot{u}_{2j} &= -\sigma_j^2 u_{2j-1} + \epsilon f_j(u_1, \dots, u_{2n-1}, u_{2j}, \dots, u_{2n}, t, \epsilon), \end{aligned} \quad j = 1, 2, \dots, n,$$

and is therefore a special case of system (4.1.1). Also, this system has been discussed for the case where the Jacobian (4.2.5) is different from zero. In the present section, we will determine some general systems (5.1.1) for which the Jacobian (4.2.5) may vanish identically and, in such cases, we shall also give sufficient conditions for periodic solutions. In order to achieve these results, we shall first need a general expression for the functions s_{rjk} .

If we make the transformation of variables given by (4.1.3) and put

$$F_i(\mathbf{z}, t; \epsilon) \equiv f_i[(2i\sigma_1)^{-1}(z_1 + z_2), \dots, (2i\sigma_n)^{-1}(z_{2n-1} + z_{2n}), \\ 2^{-1}(z_1 - z_2), \dots, 2^{-1}(z_{2n-1} - z_{2n}), t; \epsilon],$$

then system (5.1.1)' takes the form

$$(5.1.2) \quad \begin{aligned} \dot{z}_{2j-1} &= i\sigma_j z_{2j-1} + \epsilon F_i(\mathbf{z}, t; \epsilon), \\ \dot{z}_{2j} &= -i\sigma_j z_{2j} - \epsilon F_i(\mathbf{z}, t; \epsilon), \end{aligned} \quad j = 1, 2, \dots, n.$$

Since we have assumed that each $f_j(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon)$ is an analytic function of ϵ , $\mathbf{y}, \dot{\mathbf{y}}$, we have

$$(5.1.3) \quad f_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \sum_{(\nu+\mu) \geq 0} C_{jk}^{(\nu+\mu)}(t) y_1^{\nu_1} \dots y_n^{\nu_n} \dot{y}_1^{\mu_1} \dots \dot{y}_n^{\mu_n},$$

where $(\nu + \mu) = (\nu_1, \dots, \nu_n, \mu_1, \dots, \mu_n)$ and the $C_{jk}^{(\nu+\mu)}(t)$ are periodic of period $2\pi/\omega$. If we let $2^{-(\nu_1 + \dots + \nu_n + \mu_1 + \dots + \mu_n)} C_{jk}^{(\nu+\mu)}(t) = D_{jk}^{(\nu+\mu)}(t)$, then we have

$$(5.1.4) \quad \begin{aligned} F_i(\mathbf{z}, t; \epsilon) \\ = \sum_{k=0}^{\infty} \epsilon^k \sum_{(\nu+\mu) \geq 0} D_{jk}^{(\nu+\mu)}(t) \prod_{l=1}^n (i\sigma_l)^{-\nu_l} (z_{2l-1} + z_{2l})^{\nu_l} (z_{2l-1} - z_{2l})^{\mu_l}. \end{aligned}$$

Let us replace σ_i by $\tau_i = k_i\omega/m_i$, k_i , m_i positive integers, and consider the auxiliary system

$$(5.1.5) \quad \begin{aligned} \dot{z}_{2j-1} &= i\tau_j z_{2j-1} + \epsilon F_j(\mathbf{z}, t; \epsilon), \\ \dot{z}_{2j} &= -i\tau_j z_{2j} - \epsilon F_j(\mathbf{z}, t; \epsilon), \quad j = 1, 2, \dots, n, \end{aligned}$$

where $F_j(\mathbf{z}, t; \epsilon)$ is given by (5.1.4).

We shall again take the zeroth approximation to be

$$(5.1.6) \quad \mathbf{z}_0 = \mathbf{x}_0 = (a_1 e^{i\tau_1 t}, -\bar{a}_1 e^{-i\tau_1 t}, \dots, a_n e^{i\tau_n t}, -\bar{a}_n e^{-i\tau_n t})$$

and shall denote the m^{th} approximation by $\mathbf{z}_m = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \dots + \epsilon^m \mathbf{x}_m$, $m = 1, 2, \dots$. We then have

$$(5.1.7) \quad z_{s, 2j-1} \pm z_{s, 2j} = \sum_{p=0}^s \epsilon^p (x_{p, 2j-1} \pm x_{p, 2j})$$

and

$$(5.1.8) \quad \begin{aligned} (z_{s, 2l-1} + z_{s, 2l})^{\nu_l} &= \sum_{\alpha_1, \dots, \alpha_{\nu_l}=0}^s \epsilon^{\alpha_1 + \dots + \alpha_{\nu_l}} \prod_{\delta=1}^{\nu_l} (x_{\alpha_\delta, 2l-1} + x_{\alpha_\delta, 2l}), \\ (z_{s, 2l-1} - z_{s, 2l})^{\mu_l} &= \sum_{\beta_1, \dots, \beta_{\mu_l}=0}^s \epsilon^{\beta_1 + \dots + \beta_{\mu_l}} \prod_{\lambda=1}^{\mu_l} (x_{\beta_\lambda, 2l-1} - x_{\beta_\lambda, 2l}). \end{aligned}$$

From the definition in §1, we know that $s_{r, 2j-1, k}$ is the coefficient of ϵ^{r-1} in f_{jk} when $(z_{2l-1} + z_{2l})$, $(z_{2l-1} - z_{2l})$ are replaced by the expressions (5.1.7). Thus, if we substitute (5.1.8) in the coefficient of ϵ^{k+1} in (5.1.5) and keep only the terms in ϵ^{r-1} , we obtain

$$(5.1.9) \quad \begin{aligned} s_{r, 2j-1, k} &= \sum D_{2j-1, k}^{(\nu+\mu)}(t) \prod_{l=1}^n (i\sigma_l)^{-\nu_l} \\ &\quad \cdot \left[\sum \prod_{\delta=1}^{\nu_l} (x_{\alpha_\delta, 2l-1} + x_{\alpha_\delta, 2l}) \prod_{\lambda=1}^{\mu_l} (x_{\beta_\lambda, 2l-1} - x_{\beta_\lambda, 2l}) \right], \end{aligned}$$

where the last sum is to be taken over all values of $\alpha_1, \dots, \alpha_{\nu_l}$ and $\beta_1, \dots, \beta_{\mu_l}$ such that

$$(5.1.10) \quad \alpha_1 + \dots + \alpha_{\nu_l} + \beta_1 + \dots + \beta_{\mu_l} = r - 1,$$

and the first sum is taken over all values of $\nu_1, \dots, \nu_n, \mu_1, \dots, \mu_n$ which make (5.1.10) possible.

5.2. Some preliminary statements. We know that x_{m_i} , s_{m_i} are periodic of period $T = 2\pi m_1 \dots m_n / \omega = 2\pi / \omega^*$, and, therefore, we may write

$$(5.2.1) \quad x_{m_i} = \sum_{k=-\infty}^{+\infty} \sigma_{m_i}^{(k)} e^{ik\omega^* t}, \quad s_{m_i} = \sum_{k=-\infty}^{+\infty} \rho_{m_i}^{(k)} e^{ik\omega^* t},$$

$j = 1, 2, \dots, n, m = 1, 2, \dots$, where the $\sigma_{mi}^{(k)}, \rho_{mi}^{(k)}$ are constants. Let us observe, from (5.1.5) and the definition of s_{rik} in Section 1.1, that $s_{r,2i-1,k} = -s_{r,2i,k}$ for every r, j, k .

Lemma (5.2.i). *If we assume that f_i in (5.1.1) is of the form*

$$(\alpha) \quad f_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon) = g_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon) + h_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon), \quad \text{where}$$

$$(\beta) \quad g_i(\mathbf{y}, \dot{\mathbf{y}}, -t; \epsilon) = g_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon), \quad h_i(\mathbf{y}, \dot{\mathbf{y}}, -t; \epsilon) = -h_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon),$$

$$(\gamma) \quad g_i(\mathbf{y}, -\dot{\mathbf{y}}, t; \epsilon) = g_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon), \quad h_i(\mathbf{y}, -\dot{\mathbf{y}}, t; \epsilon) = -h_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon),$$

$j = 1, 2, \dots, n$, and if we apply the preceding algorithm to the auxiliary system (5.1.5) with zeroth approximation given by (5.1.6), where the a_i in (5.1.6) are purely imaginary, then we have $\sigma_{m,2i-1}^{(k)} = \sigma_{m,2i}^{(-k)}, \rho_{ml}^{(k)} = \rho_{ml}^{(-k)}, S_{m,2i-1} = -S_{m,2i}$, for every $m = 0, 1, 2, \dots, k = 0, \pm 1, \pm 2, \dots, j = 1, 2, \dots, n, l = 1, 2, \dots, 2n$.

Proof. Since the a_i are purely imaginary, we see, from (5.1.6), that $\sigma_{0,2i-1}^{(k)} = \sigma_{0,2i}^{(-k)}$ for every j, k . Assume that $\sigma_{p,2i-1}^{(k)} = \sigma_{p,2i}^{(-k)}$ for $k = 0, \pm 1, \pm 2, \dots, p = 0, 1, \dots, m-1, j = 1, 2, \dots, n$. We shall show that this relation holds for $p = m$ and every k . We have

$$x_{p,2i-1} = \sum_{k=-\infty}^{+\infty} \sigma_{p,2i-1}^{(k)} e^{ik\omega^*t},$$

$$x_{p,2i} = \sum_{k=-\infty}^{+\infty} \sigma_{p,2i}^{(k)} e^{ik\omega^*t} = \sum_{k=-\infty}^{+\infty} \sigma_{p,2i-1}^{(k)} e^{-ik\omega^*t},$$

for $j = 1, 2, \dots, n, p = 0, 1, \dots, m-1$, and, thus,

$$(5.2.2) \quad x_{p,2i-1} + x_{p,2i} = \sum_{k=-\infty}^{+\infty} \sigma_{p,2i-1}^{(k)} (e^{ik\omega^*t} + e^{-ik\omega^*t}),$$

$$x_{p,2i-1} - x_{p,2i} = \sum_{k=-\infty}^{+\infty} \sigma_{p,2i-1}^{(k)} (e^{ik\omega^*t} - e^{-ik\omega^*t}).$$

If we substitute expression (5.2.2) in (5.1.9) for $s_{p,2i-1,k}$, we see that the second sum in (5.1.9) consists of a sum of exponential terms (multiplied by appropriate factors) of the form

$$(5.2.3) \quad \prod_{\delta=1}^{\nu l} (e^{im_\delta \omega^*t} + e^{-im_\delta \omega^*t}) \prod_{\lambda=1}^{\mu l} (e^{in_\lambda \omega^*t} - e^{-in_\lambda \omega^*t}),$$

where $e^{im_\delta \omega^*t} + e^{-im_\delta \omega^*t}, e^{in_\lambda \omega^*t} - e^{-in_\lambda \omega^*t}$ are terms in $x_{\alpha_\delta,2l-1} + x_{\alpha_\delta,2l}, x_{\beta_\lambda,2l-1} - x_{\beta_\lambda,2l}$, respectively. Let us consider any term in the development of (5.2.3), say $e^{ik\omega^*t}$. The corresponding term $e^{-ik\omega^*t}$ will also be in the development of (5.2.3) and the coefficients will differ by a factor $(+1)^{\nu l} (-1)^{\mu l} = (-1)^{\mu l}$. Consequently, the expression $\prod_{l=1}^n (i\sigma_l)^{-\nu l} [\sum \dots]$ in (5.1.9) will consist of pairs

of terms of the form $c_k e^{ik\omega^*t}$, $c_k(-1)^\mu e^{-ik\omega^*t}$, where $\mu = \mu_1 + \dots + \mu_n$ and c_k is a constant.

Now, in (5.1.9), each of the above terms in $e^{ik\omega^*t}$, $e^{-ik\omega^*t}$ must be multiplied by a Fourier series of one of the $C_{jk}^{(\nu+\mu)}(t)$ and, thus, we obtain terms of the form $e^{i(k+l)\omega^*t}$, $e^{-i(k+l)\omega^*t}$. But, since the coefficients of $e^{il\omega^*t}$, $e^{-il\omega^*t}$ in a Fourier series of a real function are complex conjugate, we see that these terms in $e^{i(k+l)\omega^*t}$, $e^{-i(k+l)\omega^*t}$ are of the form $b_l c_k e^{i(k+l)\omega^*t}$, $\bar{b}_l c_k (-1)^\mu e^{-i(k+l)\omega^*t}$, where b_l , c_k are constants.

Since we assume $f_i = g_i + h_i$, then (5.1.9) can be considered as a sum of two parts, one corresponding to g_i and the other to h_i . Since the above reasoning could be applied to each part of (5.1.9), we conclude that each part of (5.1.9) consists of pairs of terms of the form $b_l c_k e^{i(k+l)\omega^*t}$, $\bar{b}_l c_k (-1)^\mu e^{-i(k+l)\omega^*t}$. If these terms are contained in the part belonging to g_i , then b_l is real and μ is even and the coefficients of these two terms are the same. If these terms are contained in the part belonging to h_i , then b_l is purely imaginary and μ is odd and we obtain the same result. Since this is true for *every* pair of terms in (5.1.9), we know that the coefficient of $e^{il\omega^*t}$ in $s_{p,2j-1,k}$ is the same as the coefficient of $e^{-il\omega^*t}$ in $s_{p,2j-1,k}$ for every l . Furthermore, from (1.1.6), we have $s_{p,2j-1} = s_{p,2j-1,0} + s_{p-1,2j-1,1} + \dots + s_{1,2j-1,p-1}$, and, finally,

$$(5.2.4) \quad \rho_{p,2j-1}^{(k)} = \rho_{p,2j-1}^{(-k)},$$

$$p = 1, 2, \dots, m, \quad j = 1, 2, \dots, n, \quad k = 0, \pm 1, \pm 2, \dots$$

From the definition (1.1.7), we have $S_{p,2j-1}x_{0,2j-1}$, $S_{p,2j}x_{0,2j}$ are terms in $s_{p,2j-1}$, $s_{p,2j}$, respectively, and since $s_{p,2j-1} = -s_{p,2j}$, $\bar{a}_j = -a_j$, we must have $S_{p,2j-1}x_{0,2j-1} - S_{p,2j}x_{0,2j} = S_{p,2j-1}a_j e^{i\tau_j t} - S_{p,2j}(-\bar{a}_j)e^{-i\tau_j t} = S_{p,2j-1}a_j \cdot e^{i\tau_j t} - S_{p,2j}a_j e^{-i\tau_j t}$ is a term in $s_{p,2j-1}$. But, from (5.2.4), $\rho_{p,2j-1}^{(k;m/m_j)} = \rho_{p,2j-1}^{(-k;m/m_j)}$, $m = m_1 \dots m_n$, and this implies

$$(5.2.5) \quad S_{p,2j-1} = -S_{p,2j}, \quad p = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

If we consider the expression (1.1.13), and remember that we always choose the primitive of mean value zero, we see that

$$\begin{aligned} x_{m,2j-1} &= e^{i\tau_j t} \int \sum_{k=-\infty}^{+\infty} [\rho_{m,2j-1}^{(k)} - S_{1,2j-1}\sigma_{m-1,2j-1}^{(k)} \\ &\quad - \dots - S_{m-1,2j-1}\sigma_{1,2j-1}^{(k)}] e^{i(k\omega^* - \tau_j)t} dt \\ &= \sum_{k=-\infty}^{+\infty} [\rho_{m,2j-1}^{(k)} - S_{1,2j-1}\sigma_{m-1,2j-1}^{(k)} \\ &\quad - \dots - S_{m-1,2j-1}\sigma_{1,2j-1}^{(k)}] [i(k\omega^* - \tau_j)]^{-1} e^{ik\omega^*t}, \\ x_{m,2j} &= \sum_{k=-\infty}^{+\infty} [\rho_{m,2j}^{(k)} - S_{1,2j}\sigma_{m-1,2j}^{(k)} - \dots - S_{m-1,2j}\sigma_{1,2j}^{(k)}] [i(k\omega^* + \tau_j)]^{-1} e^{ik\omega^*t}. \end{aligned}$$

Since $s_{r,2j-1} = -s_{r,2j}$, we have $\rho_{m,2j}^{(k)} = -\rho_{m,2j-1}^{(k)}$. If we use this fact and relations (5.2.4) and (5.2.5) and our assumption on the $\sigma_{pi}^{(k)}$, we obtain

$$\begin{aligned} x_{m,2j} &= - \sum_{k=-\infty}^{+\infty} [\rho_{m,2j-1}^{(k)} - S_{1,2j-1} \sigma_{m-1,2j-1}^{(-k)} \\ &\quad - \dots - S_{m-1,2j-1} \sigma_{1,2j-1}^{(-k)}] [i(k\omega^* + \tau_j)]^{-1} e^{ik\omega^* t} \\ &= \sum_{k=-\infty}^{+\infty} [\rho_{m,2j-1}^{(k)} - S_{1,2j-1} \sigma_{m-1,2j-1}^{(k)} \\ &\quad - \dots - S_{m-1,2j-1} \sigma_{1,2j-1}^{(k)}] [i(k\omega^* - \tau_j)]^{-1} e^{-ik\omega^* t} \end{aligned}$$

or $\sigma_{m,2j-1}^{(k)} = \sigma_{m,2j}^{(-k)}$ and the induction on the $\sigma_{mj}^{(k)}$ is completed. Thus, formulas (5.2.4) and (5.2.5) hold for all p and the lemma is proved.

Lemma (5.2.ii). *If we assume that f_j in (5.1.1) satisfies (α) and (β) of lemma (5.2.i), and if (δ) $g_i(-\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon) = -g_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon)$, $h_i(-\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon) = h_i(\mathbf{y}, \dot{\mathbf{y}}, t; \epsilon)$, $j = 1, 2, \dots, n$, and if we apply the preceding algorithm to the auxiliary system (5.1.5) with zeroth approximation given by (5.1.6), where the a_i in (5.1.6) are real, then we have $\sigma_{m,2j-1}^{(k)} = -\sigma_{m,2j}^{(-k)}$, $\rho_{ml}^{(k)} = -\rho_{ml}^{(-k)}$, $S_{m,2j-1} = -S_{m,2j}$, for every $m = 0, 1, \dots, k = 0, \pm 1, \pm 2, \dots, j = 1, 2, \dots, n, l = 1, 2, \dots, 2n$.*

Proof. Since the a_i are real, we see from (5.1.6) that $\sigma_{0,2j-1}^{(k)} = -\sigma_{0,2j}^{(-k)}$ for every j, k . Assume that $\sigma_{p,2j-1}^{(k)} = -\sigma_{p,2j}^{(-k)}$ for $k = 0, \pm 1, \pm 2, \dots, p = 0, 1, \dots, m-1, j = 1, 2, \dots, n$. We shall show that this relation holds for $p = m$ and every k . Proceeding exactly as in the preceding lemma, we find that (5.1.9) consists of pairs of terms of the form $b_i c_k e^{i(k+l)\omega^* t}$, $\bar{b}_i c_k (-1)^\nu e^{-i(k+l)\omega^* t}$, where $\nu = \nu_1 + \dots + \nu_n$ and b_i, c_k are constants and the constant b_i corresponds to one of the Fourier coefficients of $C_{jk}^{(\nu+\mu)}(t)$. If these terms belong to that part of (5.1.9) belonging to g_j , then each b_i is real and each ν is odd and the coefficients of these two terms are opposite in sign. If these terms belong to that part of (5.1.9) belonging to h_j , then each b_i is purely imaginary and each ν is even and we obtain the same result. Therefore, using the same reasoning as in the preceding lemma, we have $\rho_{p,2j-1}^{(k)} = -\rho_{p,2j}^{(-k)}$, $S_{p,2j-1} = -S_{p,2j}$, for $p = 1, 2, \dots, m, k = 0, \pm 1, \pm 2, \dots, j = 1, 2, \dots, n$. The remaining part of the proof is exactly the same as the proof of lemma (5.2.i).

Since we have already shown in lemma (4.1.i) that $S_{r,2j-1} = \bar{S}_{r,2j}$, we can combine this result with the preceding lemmas to obtain

Lemma (5.2.iii). *If either the conditions of lemma (5.2.i) or the conditions of lemma (5.2.ii) are satisfied, and if we apply the preceding algorithm to system (5.1.5) with zeroth approximation given by (5.1.6), then the function $v_{2j-1}(\tau, \mathbf{a}; \epsilon) = S_{1,2j-1} + \epsilon S_{2,2j-1} + \dots$ is purely imaginary.*

Note. If either the conditions of lemma (5.2.i) or lemma (5.2.ii) are satisfied, then the Jacobian discussed in section 4.2 may be identically zero. In such

cases, we need new criteria for the existence of periodic solutions, and such criteria shall be obtained in the next section.

5.3. Applications. Existence theorems for periodic solutions.

Theorem (5.3.i). *Assume system (5.1.1) satisfies the conditions of lemma (5.2.i). If $b_j, j = 1, 2, \dots, n$ are arbitrary real numbers, if $k_j, m_j, j = 1, 2, \dots, n$ are positive integers, then the numbers $\sigma_1, \dots, \sigma_n$ can always be chosen as real functions of ω, b_j, k_j, m_j , such that, for ϵ sufficiently small, there is a periodic real solution of (5.1.1) of the form*

$$(5.3.1) \quad y_j = b_j \sigma_j^{-1} \cos(k_j \omega t / m_j) + \epsilon W_j(t; \epsilon), \quad j = 1, 2, \dots, n,$$

where each W_j is a real, analytic function of ϵ whose power series expansion has coefficients periodic in t of period $2\pi m_1, \dots, m_n / \omega$, and $W_j(-t; \epsilon) = W_j(t; \epsilon)$, $j = 1, 2, \dots, n$. In fact, $\sigma_j = k_j \omega / m_j + o(\epsilon)$, $j = 1, \dots, n$.

Proof. If we apply the preceding algorithm to system (5.3.i) with $a_j = ib_j$ in the zeroth approximation, then the conditions of lemma (5.2.i) are satisfied, and, thus, the conditions of lemma (5.2.iii) are satisfied. Since each $v_{2j-1}, j = 1, 2, \dots, n$, is purely imaginary, we see that the system of equations $ik_j \omega / m_j - \epsilon v_{2j-1} = i\sigma_j, j = 1, 2, \dots, n$, has a real solution $\sigma_1, \dots, \sigma_n$ for every $b_1, \dots, b_n$. Therefore, there is a periodic solution of system (5.1.1), and by using the transformation formulas (4.1.3) and lemma (5.2.i), it is easily shown that the solution is of the form (5.3.1).

Theorem (5.3.ii). *If $a_j, j = 1, 2, \dots, n$, are arbitrary, real numbers, if $k_j, m_j, j = 1, 2, \dots, n$, are positive integers, and if system (5.1.1) satisfies the conditions of lemma (5.2.ii), then the numbers $\sigma_1, \dots, \sigma_n$ can always be chosen as real functions of $\omega, a_j, k_j, m_j, j = 1, 2, \dots, n$ such that, for ϵ sufficiently small, there is a periodic real solution of (5.1.1) of the form*

$$(5.3.2) \quad y_j = a_j \sigma_j^{-1} \sin(k_j \omega t / m_j) + \epsilon W_j(t; \epsilon), \quad j = 1, \dots, n,$$

where each W_j is a real analytic function of ϵ whose power series expansion has coefficients periodic in t of period $2\pi m_1 \dots m_n / \omega$, and $W_j(-t; \epsilon) = -W_j(t; \epsilon)$, $j = 1, 2, \dots, n$. In fact, $\sigma_j = k_j \omega / m_j + o(\epsilon)$, $j = 1, \dots, n$.

Proof. The proof is exactly the same as the proof of theorem (5.3.i) except that we use lemma (5.2.ii).

In the following, we shall let $\mathbf{y}_0 = [b_1 \sigma_1^{-1} \sin(\sigma_1 t + \phi_1), \dots, b_n \sigma_n^{-1} \sin(\sigma_n t + \phi_n)]$ and

$$(5.3.3) \quad Q_i(\mathbf{b}, \boldsymbol{\phi}) = \int_0^T [g_i(\mathbf{y}_0, \dot{\mathbf{y}}_0, t; 0) + h_i(\mathbf{y}_0, \dot{\mathbf{y}}_0, t; 0)] y_{0i} dt,$$

$j = 1, 2, \dots, n.$

(This expression corresponds to expression (4.2.3) for the system (5.1.1) with $f_i = g_i + h_i$.) Also, we shall let $\frac{1}{2}\pi = (\frac{1}{2}\pi, \dots, \frac{1}{2}\pi)$, $\mathbf{0} = (0, \dots, 0)$.

Theorem (5.3.iii). *If the conditions of theorem (5.3.i) are satisfied, if $\sigma_i = k_i \cdot \omega/m_i$, k_i , m_i positive integers, $j = 1, 2, \dots, n$, and if there exist real numbers b_{10} , $\dots$, b_{n0} such that $Q_i(\mathbf{b}_0, \frac{1}{2}\pi) = 0$, $j = 1, 2, \dots, n$ and the Jacobian $|\partial Q_i(\mathbf{b}_0, \frac{1}{2}\pi)/\partial b_i| \neq 0$, then, for ϵ sufficiently small, there is a periodic solution of system (5.1.1) of the form (5.3.1) with $b_j = b_{j0}$, $j = 1, 2, \dots, n$.*

Proof. Since we assume the conditions of theorem (5.3.i) are satisfied, we may apply the preceding algorithm to system (5.1.1) with $a_i = ib_i = b_i e^{i\pi/2}$, and find that the functions $v_{2j-1} = S_{1,2j-1} + \epsilon S_{2,2j-1} + \dots$, $j = 1, 2, \dots, n$ are purely imaginary. Therefore, since $\sigma_i = k_i \omega/m_i = \tau_i$, $j = 1, 2, \dots, n$, we need only show that the system of equations $S_{1,2j-1} + \epsilon S_{2,2j-1} + \dots = 0$, $j = 1, 2, \dots, n$, has a solution for some $b_1, \dots, b_n$. But, from (4.2.3) and the fact that $a_i = b_i e^{i\pi/2}$, we know that $(-b_i T) S_{1,2j-1} = Q_i(\mathbf{b}, \frac{1}{2}\pi)$, $j = 1, 2, \dots, n$. Thus, if the conditions of this theorem are satisfied, we know from the implicit function theorem that there is a solution to these equations of the form $b_i = b_{i0} + o(\epsilon)$, $j = 1, 2, \dots, n$.

Theorem (5.3.iv). *If the conditions of theorem (5.3.ii) are satisfied, and if $\sigma_i = k_i \omega/m_i$, k_i , m_i positive integers, $j = 1, 2, \dots, n$, and if there exist real numbers a_{10} , $\dots$, a_{n0} such that $Q_i(\mathbf{a}_0, \mathbf{0}) = 0$, $j = 1, 2, \dots, n$ and the Jacobian $|\partial Q_i(\mathbf{a}_0, \mathbf{0})/\partial a_i| \neq 0$, then, for ϵ sufficiently small, there is a periodic solution of (5.1.1) of the form (5.3.2) with $a_j = a_{j0}$, $j = 1, 2, \dots, n$.*

Proof. The proof is the same as the proof of theorem (5.3.iii), except we put $a_i = a_i e^{i0}$, $j = 1, 2, \dots, n$.

§6. APPLICATIONS TO SECOND ORDER EQUATIONS

6.1. Consider the equation

$$(6.1.1) \quad \ddot{y} + c\dot{y} + \sigma^2 y = B \cos 2\omega t + \epsilon \alpha \cos 2\omega t \cdot y + \epsilon b y^3,$$

where $c \geq 0$, $\omega > 0$, B, α, b are constants, and $\epsilon > 0$ is a small parameter. This equation contains as special cases the non-linear Mathieu equation with large forcing term ($\alpha \neq 0, b \neq 0, c = 0$), the Duffing equation ($\alpha = 0, c = 0, b \neq 0$), and the Duffing equation with damping ($\alpha = 0, c \neq 0, b \neq 0$). We shall discuss each of these cases in the following sections. However, we first write (6.1.1) in a more convenient form.

Following the procedure of §3, put $y = y_0 + z$, where $y_0 = A \cos(2\omega t + \psi)$, $A = B[\sigma^2 - 4\omega^2]^2 + 4\omega^2 c^2]^{-\frac{1}{2}}$, $\tan \psi = 2c\omega(\sigma^2 - 4\omega^2)^{-1}$, and obtain

$$(6.1.2) \quad \ddot{z} + c\dot{z} + \sigma^2 z = \epsilon \alpha \cos 2\omega t \cdot (y_0 + z) + \epsilon b(y_0 + z)^3.$$

If we make the transformation (cf. (4.1.3)) $z = (2i\gamma)^{-1}(x_1 + x_2)$, $\dot{z} = (2i\gamma)^{-1}(\rho_1 x_1 + \rho_2 x_2)$, where $\gamma = \frac{1}{2}(4\sigma^2 - c^2)^{\frac{1}{2}} > 0$, $\rho_1 = -\frac{1}{2}c + i\gamma$, $\rho_2 = \bar{\rho}_1$, then (6.1.2) becomes

$$(6.1.3) \quad \begin{aligned} \dot{x}_1 &= \rho_1 x_1 + \epsilon \{ \alpha \cos 2\omega t \cdot [y_0 + (2i\gamma)^{-1}(x_1 + x_2)] \\ &\quad + b[y_0 + (2i\gamma)^{-1}(x_1 + x_2)]^3 \}, \\ \dot{x}_2 &= \rho_2 x_2 - \epsilon \{ \dots \}. \end{aligned}$$

6.2. Non-linear Mathieu equation with large forcing term. In this section, we consider the equation

$$(6.2.1) \quad \ddot{y} + \sigma^2 y = B \cos 2\omega t + \epsilon \alpha \cos 2\omega t \cdot y + \epsilon b y^3,$$

which may be obtained from (6.1.1) by putting $c = 0$. The physical interpretation of this equation with $B = 0$, $\omega = 1$, $\alpha = 1$, has been discussed by N. W. MacLACHLAN [17, p. 123 ff.] for the subharmonics of order 2. In the present section, for this equation we give sufficient conditions for the existence of a periodic solution, whose dominant term is a sum of two terms, one of period π/ω (due to the "large" forcing term $B \cos 2\omega t$), and the other of period (m/k) ($2\pi/\omega$), m, k positive integers. These conditions will depend on the values of the constants $m, k, \sigma, \omega, b, B, \alpha$. Also, we give the explicit expression for the solution up to terms of order ϵ^2 . Furthermore, it should be remarked that from one

periodic solution we can obtain others simply by a translation in t . (The number of such periodic solutions obtained in this way will depend upon k, m .)

Since $c = 0$ in (6.1.1), we have

$$\rho_1 = i\sigma, \quad \rho_2 = -i\sigma, \quad \gamma = \sigma, \quad \text{and} \quad y_0 = A \cos 2\omega t, \quad A = B[\sigma^2 - 4\omega^2]^{-1}.$$

Therefore, if we replace ρ_1, ρ_2 in (6.1.3) by $i\tau, -i\tau, \tau = k\omega/m, k, m$ positive integers, we may apply the algorithm described in §1, where we take the zeroth approximation to be

$$(6.2.2) \quad x_{01} = iae^{i\tau t}, \quad x_{02} = iae^{-i\tau t},$$

where a is a positive or negative real number, different from zero. From the definition of the algorithm in (1.1.13), and lemma (4.1.i), we obtain as the first approximation

$$(6.2.3) \quad \begin{aligned} x_{11} = e^{i\tau t} \int e^{-i\tau t} & \left\{ \frac{A\alpha}{4} C^2(2\omega t) + \frac{\alpha a}{4\sigma^2} C(\tau t) C(2\omega t) \right. \\ & + \frac{bA^3}{8} C^3(2\omega t) + \frac{3bA^2a}{8\sigma} C(\tau t) C^2(2\omega t) \\ & \left. + \frac{3bAa^2}{8\sigma^2} C(2\omega t) C^2(\tau t) + \frac{ba^3}{8\sigma^3} C^3(\tau t) - iae^{i\tau t} S_{11} \right\} dt, \\ x_{12} = -\bar{x}_{11}, \end{aligned}$$

where $C(t) = e^{it} + e^{-it}$. The explicit expression of S_{11} is obtained from (1.1.8). We see that there are several particular numerical values of k/m which have a definite influence upon S_{11} . That is, S_{11} will take on some different values depending on the ratio k/m . There are six cases to be considered.

I. $k/m \neq 2/3, 1, 2, 4, 6$. Since system (6.2.1) satisfies the conditions of lemma (5.2.i), and since the amplitudes of the zeroth approximation (6.2.2) are chosen purely imaginary, we know that the real part of the determining equations $i\tau - \epsilon S_{11} - \epsilon^2 S_{21} - \dots = i\sigma$ is identically zero. Therefore, in this case *I*, the determining equation is $k\omega/m - \sigma + (3\epsilon b/4\sigma)(A^2 + a^2/2\sigma^2) + O(\epsilon^2) = 0$. It is clear that if we choose $\sigma = k\omega/m$, then there is no real solution " a ". However, if we choose any real value for " a ", then there is a solution of (6.2.1) for $\sigma = k\omega/m + (3\epsilon b m^3/4k\omega^3) [B^2\omega^{-2}m^2 (k^2 - 4m^2)^{-2} + a^2/2k^2] + O(\epsilon^2)$. From the transformation formulas in Section 6.1 and (6.2.3), this solution of (6.2.1) is given immediately as

$$\begin{aligned}
(6.2.4) \quad y = & A \cos 2\omega t + a\sigma^{-1} \cos p\omega t + \epsilon(2\sigma\omega)^{-1} \{ Ap\alpha \\
& - 3bAp(A\sigma^2 + 2a^2)[2\sigma^2(4 - p^2)]^{-1} \cos 2\omega t \\
& - Ap\alpha(16 - p^2)^{-1} \cos 4\omega t - bA^3p2^{-1}(36 - p^2)^{-1} \cos 6\omega t \\
& - 3bAa^2p[2\sigma^2(2 + p)(2 + 3p)]^{-1} \cos [2(p + 1)\omega t] \\
& - 3bAa^2p[2\sigma^2(2 - p)(2 - 3p)]^{-1} \cos [2(p - 1)\omega t] \\
& - 3bA^2ap[16\sigma(2 - p)]^{-1} \cos [(p - 4)\omega t] \\
& - a p\alpha[4\sigma(1 - p)]^{-1} \cos [(p - 2)\omega t] \\
& + 3ba(4\sigma p)^{-1}[A^2 + a^2(2\sigma^2)^{-1}] \cos p\omega t \\
& - ba^3(16\sigma^3p)^{-1} \cos 3p\omega t - 2p\alpha[4\sigma(1 + p)]^{-1} \cos [(p + 2)\omega t] \\
& - 3bA^2ap[16\sigma(2 + p)]^{-1} \cos [(p + 4)\omega t] \} + O(\epsilon^2),
\end{aligned}$$

where $p = k/m$.

For $B = 0$, $\alpha = 1$, (6.2.4) reduces to

$$\begin{aligned}
(6.2.5) \quad y = & a\sigma^{-1} \cos p\omega t + (a\epsilon/8\sigma^2\omega) \{ -p(1 - p)^{-1} \cos [(p - 2)\omega t] \\
& + 3ba^2(2\sigma^2p)^{-1} \cos p\omega t - ba^2(4\sigma^2p)^{-1} \cos 3p\omega t \\
& - p(1 + p)^{-1} \cos [(p + 2)\omega t] \} + O(\epsilon^2),
\end{aligned}$$

which solves the non-linear Mathieu equation

$$(6.2.6) \quad \ddot{y} + \sigma^2 y = \epsilon \cos 2\omega t \cdot y + \epsilon b y^3,$$

if $k/m \neq 1$. In (6.2.5), σ is given by $\sigma = k\omega/m + 32\epsilon ba^2m^2/8k^2\omega^2 + O(\epsilon^2)$.

II. $k/m = 2/3$. The determining equation $i\tau - \epsilon S_{11} - \dots = i\sigma$ is $2\omega/3 - \sigma + (3\epsilon b/4\sigma)(A^2 + a^2/2\sigma^2 + aA/2\sigma) = 0$. If we choose $\sigma = 2\omega/3$, $b \neq 0$, then this equation has no real solution a since the discriminant is $A^2\sigma^2 - 8A^2\sigma^2 < 0$. We conclude, here, that there is no subharmonic vibration for which $\sigma = 2\omega/3$. However, if we choose " a " arbitrary, then there is a solution of (6.2.1) if $\sigma = 2\omega/3 + (81\epsilon b/64\omega^3)(9/128 + a^2 - 3a/k\omega)$. In case *II*, the solution, up to terms of order ϵ^2 , takes the form (6.2.4), and can be obtained from (6.2.4) by putting $k/m = 2/3$ and replacing the coefficient of $\cos [2(p - 1)\omega t]$ in (6.2.4) by $9bAa^2(16\sigma^2)^{-1}$.

It should also be remarked that in this case *II*, if $y(t)$, given by (6.2.4), is a solution of (6.2.1), then so is $y(t + k\pi/\omega)$, $k = 1, 2$. If we had chosen our zeroth approximation to be $ae^{i\tau t}$, $-\bar{a}e^{-i\tau t}$, $a = |a|e^{i\phi}$, then all of these solutions would be obtained directly from the determining equations.

III. $k/m = 1$ (subharmonic resonance of order 2). The determining equation $i\tau - \epsilon S_{11} - \dots = i\sigma$ is $\omega - \sigma + (\epsilon/4\sigma)(3bA^2 + 3ba^2/2\sigma^2 + \alpha) = 0$. If we choose $\sigma = \omega$, then $a^2 = (2\omega^2/3b)(-\alpha - bB^2/3\omega^4)$. If $\alpha b < 0$, and $|b/\alpha| < 3\omega^4/B^2$, there are two solutions $\pm a$. Otherwise, there are no real solutions a .

If we choose a arbitrarily, then there will be a subharmonic of order 2 if $\sigma = \omega + (\epsilon/4\omega)(bB^2/3\omega^2 + 3ba^2/2\omega^2 + \alpha)$. In case *III*, the solution up to terms of order ϵ^2 takes the same form as (6.2.4), and can be obtained from (6.2.4) by putting $k/m = 1$, and replacing the coefficient of $\cos(p - 2)\omega t$ in (6.2.4) by $a(4\sigma)^{-1}$.

If $B = 0$, $k/m = 1$, $\alpha = 1$, then the solution of the non-linear Mathieu equation (6.2.6) takes the same form as (6.2.5), and can be obtained from (6.2.5) by replacing the coefficient of $\cos(p - 2)\omega t$ in (6.2.5) by 1. In this case, the determining equation is $\omega - \sigma + (\epsilon/4\sigma)(3ba^2/2\sigma^2 + 1) = 0$, and if we choose $\sigma = \omega$, then $a^2 = -2\sigma^2/3b$, and b must be chosen negative in order to get a solution.

In this case with $B = 0$, $k/m = 1$, $\alpha = 1$, the conditions of lemma (5.2.ii) are satisfied and we could choose our zeroth approximation (6.2.2) such that " ia " is real, *i.e.*, " a " purely imaginary. Then, from the analysis in the preceding paragraph, we should have to take b positive in order to get a subharmonic of order 2. These are the conclusions of N. W. MACLACHLAN [17, *p.* 123–125].

IV. $k/m = 4$. The determining equation is $4\omega - \sigma + (3\epsilon/4\sigma)(bA^2 + ba^2/2\sigma^2 + A\alpha/4a) = 0$. If we choose $\sigma = 4\omega$, then $a^3 + 2\sigma^2 A^2 a + A\alpha\sigma^2/2b = 0$. If $Bab^{-1} < 0$, then there is a positive solution " a " and if $Bab^{-1} > 0$, there is a negative solution " a ".

V. $k/m = 6$. This case is similar to *IV* and will not be discussed here.

6.3. The Duffing equation. In this section, we consider the equation

$$(6.3.1) \quad \ddot{y} + \sigma^2 y = B \cos \omega t + \epsilon y^3,$$

which may be obtained from (6.2.1) by putting $\alpha = 0$, $b = 1$, $2\omega = \omega'$. Some of the periodic solutions of this equation are discussed in detail in the book of J. J. STOKER [28, *p.* 83 *ff.*]. For the harmonic case (σ close to ω) he takes $B = \epsilon F_1$, and for a subharmonic of order 3 (σ close to $\omega/3$), he takes B "large" as we do here. In this section, we shall discuss the periodic solution of (6.3.1) whose dominant term is a sum of two terms, one of period $2\pi/\omega$ (due to the "large" forcing term $B \cos \omega t$), the other of period $(m/k)(2\pi/\omega)$, m, k positive integers.

Since (6.3.1) is a special case of (6.2.1), we may use the formulas (6.2.2) and (6.2.3) with the changes mentioned above. As in Section 6.2, there are again several cases to consider.

I. $k/m \neq \frac{1}{3}, 1, 3$. As was observed in section 6.2, case *I*, the real part of the determining equations $i\tau - \epsilon S_{11} - \epsilon^2 S_{21} - \dots = i\sigma$ is zero, and the actual

determining equation is $k\omega/m - \sigma + (3\epsilon/4\sigma)(A^2 + a^2/2\sigma^2) = 0$, $A = B(\sigma^2 - \omega^2)^{-1}$. This is the same equation as the one in section 6.2, case *I*, with $b = 1$, and we refer the reader to that discussion. If we put $k/m = p$, the solution of (6.3.1) is given as

$$\begin{aligned}
 (6.3.2) \quad y = & A \cos \omega t + a\sigma^{-1} \cos p\omega t \\
 & + \epsilon(4\sigma\omega)^{-1} \{3A \cdot [a^2 - A^2\sigma^2 p + a^2 p][\sigma^2(1 - p^2)]^{-1} \cos \omega t \\
 & - A^3 p(9 - p^2)^{-1} \cos 3\omega t \\
 & - 3Aa^2 p[\sigma^2(1 + p)(1 + 3p)]^{-1} \cos [(2p + 1)\omega t] \\
 & - 3Aa^2 p[\sigma^2(1 - p)(1 - 3p)]^{-1} \cos [(2p - 1)\omega t] \\
 & - 3Aap[4\sigma(1 + p)]^{-1} \cos [(p + 2)\omega t] \\
 & + 3a(2A^2\sigma^2 + a^2)(4\sigma^3 p)^{-1} \cos p\omega t - a^3(8\sigma^3 p)^{-1} \cos 3p\omega t \\
 & - 3A^2 ap[4\sigma(1 - p)]^{-1} \cos [(p - 2)\omega t] + O(\epsilon^2).
 \end{aligned}$$

II. $k/m = \frac{1}{3}$. This interesting case, the subharmonic of order 3, is the only other case to be discussed here. First, it should be noted that, in this case, if $y(t)$ is a solution of (6.3.1), then so is $y(t + 2\pi k/\omega)$, $k = 1, 2$. Thus, there will be three solutions corresponding to every " a " in (6.2.2) (three for " a " negative and three for " a " positive).

The determining equation in this case is $\omega/3 - \sigma + (3\epsilon/4\sigma)(A^2 + a^2/2\sigma^2 + Aa/2\sigma) = 0$. If we choose $\sigma = \omega/3$, then $a^2 + A\sigma a + 2\sigma^2 A^2 = 0$. For this equation, there is no real solution " a ", and, thus, there is no subharmonic vibration with $\sigma = \omega/3$. However, if we choose " a " arbitrarily, then $\sigma = \omega/3 + (81\epsilon/8\omega^3)(9B^2/32\omega^2 + a^2 + 3Ba/8\omega)$.

In case *II*, the solution takes the same form as (6.3.2), and can be obtained from (6.3.2) by putting $k/m = \frac{1}{3}$, and replacing the coefficient of $\cos(p - 1)\omega t$ in (6.3.2) by $9Aa^2(4\sigma^2)^{-1}$.

6.4. Duffing equation with damping. Before discussing this equation, we need a general lemma. Consider the equation

$$(6.4.1) \quad \ddot{y} + c\dot{y} + \sigma^2 y = \epsilon f(y, \omega t + \phi; \epsilon),$$

where $\epsilon > 0$ is a small parameter, $c \geq 0$, $\sigma > 0$, ϕ are constants, and

$$f(y, -(\omega t + \phi); \epsilon) = f(y, \omega t + \phi; \epsilon)$$

is an analytic function of y , ϵ whose power series expansion has coefficients periodic of period $2\pi/\omega$. If we make the same transformations as in Section 6.1, then we obtain the canonical system

$$(6.4.2) \quad \begin{aligned} \dot{z}_1 &= \rho_1 z_1 + \epsilon f[(2i\gamma)^{-1}(z_1 + z_2), \omega t + \phi; \epsilon], \\ \dot{z}_2 &= \rho_2 z_2 - \epsilon f[\dots]. \end{aligned}$$

Let us replace ρ_1, ρ_2 by $i\tau, -i\tau$, $\tau = k\omega/m$, k, m positive integers, and apply the algorithm in §1, taking the zeroth approximation to be $x_{01} = iae^{i\tau t}$, $x_{02} = iae^{-i\tau t}$, where a is real. If we denote the m th approximation by $z_{mk} = x_{0k} + \epsilon x_{1k} + \dots + \epsilon^m x_{mk}$, $k = 1, 2$, $m = 0, 1, 2, \dots$, then we certainly have

$$(6.4.3) \quad \begin{aligned} x_{rk} &= \sum \sigma_{rk}^{(\mu, \gamma)} e^{i[\mu(\omega t + \phi) + \gamma \tau t]} \\ s_{rk} &= \sum \rho_{rk}^{(\mu, \gamma)} e^{i[\mu(\omega t + \phi) + \gamma \tau t]}, \quad k = 1, 2, \end{aligned}$$

where μ, γ are integers and the sum is over certain values of μ, γ .

Lemma (6.4.i). *If $\mu\omega + \gamma\tau \neq \tau$, $\mu \neq 0$ in each x_{rk} , $r = 1, 2, \dots, m$, $k = 1, 2$, then $\sigma_{r1}^{(\mu, \gamma)} = \sigma_{r2}^{(-\mu, -\gamma)}$, $\rho_{rk}^{(\mu, \gamma)} = \rho_{rk}^{(-\mu, -\gamma)}$, $S_{r1} = -S_{r2}$ for $r = 0, 1, 2, \dots, m$, $k = 1, 2$, where the numbers S_{r1}, S_{r2} are defined by (1.1.8). Furthermore, each S_{r1} , $r = 1, 2, \dots, m$, is purely imaginary.*

Proof. The proof is similar to the proof of lemma (5.2.i) when we replace formulas (5.2.1) by (6.4.3). That the numbers S_{r1} are purely imaginary follows from lemma (4.1.i).

Remark. This lemma can be generalized to systems of n equations of the type (6.4.1), but we apply the lemma only to one second order equation.

Now, let us consider the equation

$$(6.4.4) \quad \ddot{y} + c\dot{y} + \sigma^2 y = B \cos \omega t + \epsilon by^3,$$

which may be obtained from (6.1.1) by putting $\alpha = 0$. This equation has been discussed by J. J. STOKER [28, p. 108] for subharmonics of order 3. He has shown that if $c \neq 0$ is of order ϵ , then there is a subharmonic of order 3 for (6.4.4). He also conjectures that $c \neq 0$ must be chosen of higher order in ϵ in order to obtain subharmonics of (6.4.4) of order greater than 3. We will show here that $c \neq 0$ must be taken at least of order ϵ^n in order that there is a subharmonic of (6.4.4) of order $2n + 1$, $n \geq 1$.

If we make the transformations in section 6.1, then the equations corresponding to (6.1.3) are

$$(6.4.5) \quad \begin{aligned} \dot{z}_1 &= \rho_1 z_1 + \epsilon b[y_0 + (2i\gamma)^{-1}(z_1 + z_2)]^3, \\ \dot{z}_2 &= \rho_2 z_2 - \epsilon b[y_0 + (2i\gamma)^{-1}(z_1 + z_2)]^3, \end{aligned}$$

where $y_0 = A \cos(\omega t + \psi)$, $A = B[(\sigma^2 - \omega^2)^2 + \omega^2 c^2]^{-\frac{1}{2}}$, $\tan \psi = \omega c[\sigma^2 - \omega^2]^{-1}$. If we replace ρ_1, ρ_2 by $i\tau, -i\tau$, $\tau = \omega/(2n + 1)$, n a positive integer, we may apply the algorithm in §1, taking the zeroth approximation to be $|a| e^{i(\tau t + \phi)}$, $-|a| e^{-i(\tau t + \phi)}$. If we put $t = t' - (\phi - \pi/2)(2n + 1)/\omega$ in (6.4.5), then the

system is the same except that $y_0 = A \cos (\omega t' + \phi^*)$, where $\phi^* = \psi - (2n + 1) \cdot \phi + (2n + 1) (\pi/2)$, and the zeroth approximation is reduced to $i |a| e^{i\tau t'}, i |a| e^{-i\tau t'}$.

We shall now show that $S_{r,1}$ is purely imaginary for $r = 1, 2, \dots, n - 1$. Since system (6.4.5) is a special case of system (6.4.2) and since the coefficients of the zeroth approximation are purely imaginary, then, by lemma (6.4.i), we need only show that $\mu\omega + \nu\tau \neq \tau, \mu \neq 0$ in $x_{r,1}, x_{r,2}, r = 1, 2, \dots, n - 1$. We know that each approximation is composed of sums of exponentials of the form $\exp [k_1(i\omega) + k_2(-i\omega) + k_3(i\tau) + k_4(-i\tau)]t$, where k_1, k_2, k_3, k_4 are positive integers, and it is readily observed in the m^{th} approximation we have $k_1 + k_2 + k_3 + k_4 = 2m + 1$. Since $\mu = k_1 - k_2, \nu = k_3 - k_4$, it is then clear that, in all approximations up through the $(n - 1)^{\text{th}}$, the equation $\mu + \nu/(2n + 1) = 1/(2n + 1)$ has no solution for $\mu \neq 0$. Hence, $S_{r,1}$ is purely imaginary, $r = 1, 2, \dots, n - 1$, and the approximate determining equations $i\tau - \rho_1 - \epsilon[S_{11} + \epsilon S_{21} + \dots + \epsilon^{n-1} S_{n-1,1}] = 0, \rho_1 = -c/2 + i\gamma$ for system (6.4.5) have no solution for $c \neq 0$. From this, we conclude that $c \neq 0$ must be taken at least of order ϵ^n in order that (6.4.4) has a subharmonic of order $2n + 1$. The question of the actual existence of a subharmonic $y(t)$ of order $2n + 1$ involves the computation of all terms of order n in ϵ . For $n = 1$, and $n = 2$, the computations, which we omit here for the sake of brevity, show that the solution $y(t)$ actually exists. Thus, for $n = 1, n = 2$, we conclude that given $c = a_n \epsilon^n + \dots, a_n \neq 0, \sigma = \omega/(2n + 1) + b_1 \epsilon + \dots$, then a periodic solution $y(t)$ of period $(2n + 1) 2\pi/\omega$ exists (or, *vice versa*, given numerically $|a|$ and ϕ, c and σ of the form above can be chosen in such a way that the solution $y(t)$ exists).

§7. A LIÉNARD TYPE EQUATION WITH A FORCING TERM

7.1. Consider the differential equation

$$(7.1.1) \quad \ddot{y} + c\dot{y} + \sigma^2 y = \epsilon p_0(y) + \epsilon p_1(y)\dot{y} + \epsilon b \cos(\omega t + \alpha)$$

where $\epsilon > 0$ is a small parameter, $c \geq 0$, $\sigma > 0$, $b, \omega > 0$, $\alpha > 0$ are constants, $p_0(y)$, $p_1(y)$ are polynomials of degree n_1 , n_2 , respectively, $p_0(y) = \sum_{k=0}^{n_1} b_k y^k$, $p_1(y) = \sum_{k=0}^{n_2} c_k y^k$.

For simplicity, we shall give only the determining equations and not the solution. We shall give both the determining equations and the approximate solution for a particular case (7.2.1).

Following the procedure of §4, put $y = (2i\gamma)^{-1}(z_1 + z_2)$, $\dot{y} = (2i\gamma)^{-1}(\rho_1 z_1 + \rho_2 z_2)$, where $\rho_1 = -c/2 + i\gamma$, $\gamma = 2^{-1}(4\sigma^2 - c^2)^{\frac{1}{2}} > 0$, $\rho_2 = \bar{\rho}_1$ are the roots of the equation $\rho^2 + c\rho + \sigma^2 = 0$. Then (7.1.1) becomes

$$(7.1.2) \quad \begin{aligned} \dot{z}_1 &= \rho_1 z_1 + \epsilon \{p_0[(2i\gamma)^{-1}(z_1 + z_2)] \\ &\quad + p_1[(2i\gamma)^{-1}(z_1 + z_2)][(2i\gamma)^{-1}(\rho_1 z_1 + \rho_2 z_2)] + b \cos(\omega t + \alpha)\}, \\ \dot{z}_2 &= \rho_2 z_2 - \epsilon \{\dots\}. \end{aligned}$$

We are interested in determining harmonics and subharmonics of equation (7.1.1). Therefore, let us replace system (7.1.2) by the auxiliary system

$$(7.1.3) \quad \begin{aligned} \dot{z}_1 &= i\tau z_1 + \epsilon \{\dots\} \\ \dot{z}_2 &= -i\tau z_2 - \epsilon \{\dots\} \end{aligned}$$

where $\tau = \omega/m$, m a positive integer. We may now apply the algorithm of §1 to (7.1.3), taking as the zeroth approximation $x_{01} = ae^{i\tau t}$, $x_{02} = -\bar{a}e^{-i\tau t}$, $a = |a|e^{i\phi}$, then

$$\begin{aligned} p_0[(2i\gamma)^{-1}(x_{01} + x_{02})] &= p_0[(2i\gamma)^{-1}(ae^{i\tau t} - \bar{a}e^{-i\tau t})] \\ &= \sum_{r=0}^{n_1} b_r \sum_{k=0}^r (2i\gamma)^{-r} (-1)^k \binom{r}{k} a^{r-k} \bar{a}^k e^{i(r-2k)\tau t}, \quad p_1[(2i\gamma)^{-1}(x_{01} + x_{02})] \\ &= p_1[(2i\gamma)^{-1}(ae^{i\tau t} - \bar{a}e^{-i\tau t})] = \sum_{r=0}^{n_2} c_r \sum_{k=0}^r (2i\gamma)^{-r} (-1)^k \binom{r}{k} a^{r-k} \bar{a}^k e^{i(r-2k)\tau t}, \end{aligned}$$

and from the definition of the algorithm in §1, we obtain as the first approximation,

$$\begin{aligned}
(7.1.4) \quad x_{11} &= e^{i\tau t} \int e^{-i\tau t} \left\{ \sum_{r=0}^{n_1} \sum_{k=0}^r b_r (2i\gamma)^{-r} (-1)^k \binom{r}{k} a^{r-k} \bar{a}^k e^{i(r-2k)\tau t} \right. \\
&\quad + [(2i\gamma)^{-1} (\rho_1 a e^{i\tau t} - \rho_2 \bar{a} e^{-i\tau t})] \left[\sum_{r=0}^{n_2} \sum_{k=0}^r c_r (2i\gamma)^{-r} (-1)^k \right. \\
&\quad \left. \left. \cdot \binom{r}{k} a^{r-k} \bar{a}^k e^{i(r-2k)\tau t} \right] + \frac{b}{2} (e^{i(\omega t + \alpha)} + e^{-i(\omega t + \alpha)}) - a e^{i\tau t} S_{11} \right\} dt, \\
x_{12} &= -\bar{x}_{11}.
\end{aligned}$$

From the definition of S_{11} , formula (1.1.8), we obtain, after putting $(|a|/2\gamma) = A$,

$$\begin{aligned}
(7.1.5) \quad S_{11}^{(m+1)} &= -i(2\gamma)^{-1} \left\{ \sum_{k=0}^{n_1} b_{2k+1} \binom{2k+1}{k} A^{2k} \right. \\
&\quad \left. + \sum_{k=0}^{n_2} A^{2k} \left[\rho_1 c_{2k} \binom{2k}{k} + \rho_2 c_{2k+2} \binom{2k+2}{k} A^2 \right] \right\}, \\
S_{11}^{(m-1)} &= S_{11}^{(m+1)} + \frac{b}{2a} e^{i\alpha},
\end{aligned}$$

with $b_j = 0, j > n_1, c_j = 0, j > n_2$. Hence, the determining equations are

$$\begin{aligned}
(7.1.6) \quad \frac{\omega}{m} - \gamma + (2\gamma)^{-1} \epsilon \left\{ \sum_{k=0}^{n_1} b_{2k+1} \binom{2k+1}{k} A^{2k} \right. \\
\left. - \frac{c}{2} \sum_{k=0}^{n_2} A^{2k} \left[c_{2k} \binom{2k}{k} + c_{2k+2} \binom{2k+2}{k} A^2 \right] \right\} = 0 \\
c - \epsilon \left\{ \sum_{k=0}^{n_2} A^{2k} \left[c_{2k} \binom{2k}{k} - c_{2k+2} \binom{2k+2}{k} A^2 \right] \right\} = 0,
\end{aligned}$$

for $m \neq 1$, and

$$\begin{aligned}
(7.1.7) \quad \frac{\omega}{m} - \gamma + (2\gamma)^{-1} \epsilon \{\dots\} - \frac{\epsilon b}{4\gamma A} \sin(\alpha - \phi) &= 0 \\
c - \epsilon \{\dots\} - \frac{\epsilon b}{2\gamma A} \cos(\alpha - \phi) &= 0
\end{aligned}$$

for $m = 1$, where the terms in $\{\dots\}$ in (7.1.7) refer to the corresponding terms in (7.1.6).

If we choose $c = 0, \sigma = \omega/m$ in (7.1.6), then, in general, the equations will have no solution $A > 0$. Hence, we may say that, in general, there is no subharmonic of order $m > 1$ for $c = 0, \sigma = \omega/m$. However, if we choose $A > 0$, then we may always determine c and σ to satisfy (7.1.6).

If we choose $c = 0$, $\sigma = \omega/m$ in (7.1.7), then it is probable that system (7.1.7) will have a real solution (A, ϕ) . If we choose any $A > 0$, ϕ , then we may always determine c , σ to satisfy (7.1.7).

7.2. Generalized Van der Pol equation with a forcing term. Consider the equation

$$(7.2.1) \quad \ddot{y} + y = \epsilon(1 - y^{2n})\dot{y} + \epsilon p\omega \cos(\omega t + \alpha),$$

obtained from (7.1.1) by putting $p_0(y) \equiv 0$, $p_1(y) = 1 - y^{2n}$, $\sigma = 1$, $b = p\omega$, $n \geq 1$, an integer. Equation (7.2.1), with $n = 1$, has been studied extensively by, among others, M. L. CARTWRIGHT [2, p. 202]. She has considered questions of existence and stability for the periodic and almost periodic solutions of (7.2.1) with $n = 1$. In equation (7.2.1), the function $y^{2n} - 1$, which occurs there, is nearly constant for large n , $-1 + \delta < y < 1 - \delta$, $\delta > 0$. As CARTWRIGHT points out [2, p. 214], this is the type of function which is likely to occur in radio problems.

In this section, we shall consider only the existence of harmonic solutions of (7.2.1). If we apply the procedure of the last section, with $m = 1$, the determining equations (7.1.7) become

$$(7.2.2) \quad \begin{aligned} \omega - 1 - (\epsilon p\omega/2 |a|) \sin(\alpha - \phi) &= 0 \\ (p\omega/|a|) \cos(\alpha - \phi) + 1 - |a|^{2n} N/2^{2n} &= 0 \end{aligned}$$

where $N = (2n)!/(n+1)!(n)!$. If we solve the first equation in (7.2.2) for $|a| = \epsilon p\omega [2(\omega - 1)]^{-1} \sin(\alpha - \phi)$, and substitute into the second equation of (7.2.2), we obtain

$$(7.2.3) \quad \cotn(\alpha - \phi) + \frac{\epsilon}{2(\omega - 1)} = \frac{\epsilon N}{2(\omega - 1)} \left[\frac{\epsilon p\omega}{4(\omega - 1)} \right]^{2n} \sin^{2n}(\alpha - \phi).$$

We wish to solve this equation for ϕ , for given α , ϵ , ω , p . If we suppose $\omega - 1 > 0$, then, for $0 \leq \alpha - \phi \leq \pi/2$, the left member, $L(\alpha - \phi)$, of (7.2.3) is strictly decreasing, and the right member, $R(\alpha - \phi)$, is strictly increasing, and $L(\pi/2) = \epsilon[2(\omega - 1)]^{-1}$, $R(\pi/2) = \epsilon N[2(\omega - 1)]^{-1} [\epsilon p\omega]^{2n} [4(\omega - 1)]^{-2n}$. Therefore, if $N(\epsilon p\omega)^{2n} [4(\omega - 1)]^{-2n} \geq 1$, there is a simple root $\alpha - \phi$, $0 < \alpha - \phi \leq \pi/2$, of (7.2.3), and we see from the expression for $|a|$ above, that for any $p > 0$, there is a simple root $|a| > 0$. There may also be a solution $\alpha - \phi$, $-(\pi/2) \leq \alpha - \phi < 0$, of (7.2.3), depending on ω , α , ϵ , p , n , and in this case, we must choose $p < 0$ to obtain a root $|a| > 0$. A similar analysis holds for $\omega - 1 < 0$.

If $\omega = 1$, $p \neq 0$, $\epsilon \neq 0$, then we see directly from (7.2.2) that $\alpha = \phi$, or $\phi = \alpha - \pi$, and if p is chosen to have the same sign as $\cos(\alpha - \phi)$, there is exactly one positive simple root $|a|$ for a given n .

Let us now write the approximate solution of (7.2.1). From (7.1.4), with the proper change in notations and the transformations in section 7.1, we have

$$\begin{aligned}
 y = & |a| \sin(\omega t + \phi) + \frac{\epsilon}{4} \left\{ |a| \omega^{-1} \cos(\omega t + \phi) + p \cos(\omega t + \alpha) \right. \\
 & + (-1)^n 2^{-2n} |a|^{2n+1} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} [\omega^{-1}(n-k)^{-1} \cos[(2n-2k+1)(\omega t + \phi)] \\
 & \left. + \omega^{-1}(n-k-1)^{-1} \cos[(2n-2k-1)(\omega t + \phi)]] \right\} + O(\epsilon^2),
 \end{aligned}$$

where in the sum the terms with zero denominators are excluded.

§8. THE REUTER EQUATION

R. REUTER [24] has discussed the equation

$$(8.1) \quad \ddot{y} + \sigma^2 y = \epsilon[3\nu \cos 2t - y^2] - \epsilon^2(\lambda \dot{y} + \mu y) - \epsilon^3 \mu y^2,$$

where $\sigma = 1$, λ , μ , ν are constants and $\epsilon > 0$ is a small parameter. Here, we discuss this equation for $\sigma = k/m$, k , m positive integers. We do not give all the computations, since they are analogous to those in §6, §7. The harmonic case ($\sigma = 2$) will not be discussed here.

First, we discuss the case where $\sigma = 1$. If we wish to find a periodic solution close to $a \sin t + b \cos t$, we may apply the preceding algorithm to this equation. In terms of our notation, if the zeroth approximation is $(ae^{it}, -\bar{a}e^{-it})$, $a = |a|e^{i\phi}$, the first secular terms are found to be of order ϵ^2 , and, in fact, $12S_{21} = (-6\lambda - 6\nu \sin 2\phi) + i(6\mu + 6\nu \cos 2\phi - 5|a|^2)$. In order to have a periodic solution of (8.1), we must have this expression equal zero, *i.e.*,

$$(8.2) \quad \sin 2\phi = \frac{\lambda}{\nu},$$

$$|a|^2 = \frac{8}{3}\mu + \nu \cos 2\phi = \frac{8}{3}(\mu \pm \sqrt{\nu^2 - \lambda^2}),$$

which implies first of all that $\lambda \leq \nu$. Thus, for system (8.1) with $\sigma = 1$, there is a periodic solution given by

$$(8.3) \quad y = |a| \sin(t + \phi) - \nu \epsilon \cos 2t - (|a|^2 \epsilon/6) \cos 2(t + \phi) - (|a| \epsilon/2) \\ - (\nu |a| \epsilon^2/8) \sin(3t + \phi) - (|a|^3 \epsilon^2/48) \sin 3(t + \phi) + O(\epsilon^3),$$

where $|a|$, ϕ are given by (8.2). These are the results obtained by REUTER.

We now discuss the question of the existence of subharmonics of (8.1) of order higher than 2, with $\lambda \neq 0$, in a manner analogous to that of section 6.4. We will show that if $\lambda \neq 0$, then λ must be at least of order ϵ^{n-2} in order to obtain a subharmonic of order n .

Proceeding as in (6.4), we make a convenient translation in t so that the zeroth approximation may be chosen as $(iae^{2it/n}, iae^{-2it/n})$, a real. We know that each approximation is composed of sums of exponentials of the form $\exp[k_1(2i) + k_2(-2i) + k_3(2i/n) + k_4(-2i/n)]t$, where k_1, k_2, k_3, k_4 are positive integers, and it is easy to show that in the m th approximation we have $2k_1 + 2k_2 + k_3 + k_4 \leq m + 1$. Hence, in all approximations up through the $(n-1)$ th, $(k_1 - k_2) \cdot 2 + (k_3 - k_4)(2/n)$ is never equal to $2/n$, if $k_1 - k_2 \neq 0$. From lemma (6.4.i), and the final reasoning in section 6.4, it follows that $\lambda \neq 0$ must be taken at least of order ϵ^{n-2} to obtain a subharmonic of order n . Concerning the actual existence of these periodic solutions, results analogous to the ones mentioned in §6 will be given in a later paper.

Now, let us consider (8.1) without damping ($\lambda = 0$) and $\sigma = k/m \neq 1, \neq 2$, k, m positive integers. Since this equation satisfies the conditions of theorem (5.3.iii), we may take the zeroth approximation to be $(iae^{i(k/m)t}, iae^{-i(k/m)t})$, where a is a real number (positive or negative) different from zero. We obtain $2\sigma S_{21} = (\mu - 5a^2/6\sigma^4)i$, $\sigma \neq 1, \neq 2$, and if we choose $a^2 = (6/5)\mu\sigma^4$, then there is a periodic solution of (8.1) with $\lambda = 0$ of the form

$$\begin{aligned} y = & a\sigma^{-1} \cos \sigma t - 3\nu\epsilon(4 - \sigma^2)^{-1} \cos 2t + (a^2\epsilon/6\sigma^4)^{-1} \cos 2\sigma t \\ (8.4) \quad & - (a^2\epsilon/2\sigma^4) - 3\nu a\epsilon^2[4\sigma(1 + \sigma)(4 + \sigma^2)]^{-1} \cos(2t + \sigma t) \\ & - 3\nu a\epsilon^2[4\sigma(1 - \sigma)(4 - \sigma^2)]^{-1} \cos(2t - \sigma t) + (a^3\epsilon^2/48\sigma^4) \cos 3\sigma t, \end{aligned}$$

where $a^2 = (6/5)\mu\sigma^4$ and $\sigma = k/m \neq 1$. It is also worth noticing that system (8.1) with $\lambda = 0$, $\sigma \neq 1, \neq 2$, is a case where the classical Jacobian is identically zero.

We may also find a solution of (8.1) with $\lambda = 0$, $\sigma = 1$, and, in this case, we find, from (8.2), that either $\phi = \pi/2, 3\pi/2$, $|a|^2 = \frac{8}{3}\mu - \nu$ or $\phi = 0, \pi$, $|a|^2 = \frac{8}{3}\mu + \nu$, and the solution (8.3) takes a much simpler form.

§9. APPLICATIONS TO SYSTEMS

9.1. Coupling of non-linear Mathieu equations. Let us consider the system of differential equations

$$(9.1.1) \quad \begin{aligned} \ddot{x} + \sigma_1^2 x &= \epsilon \alpha x + \epsilon A \cos t \cdot x + \epsilon \beta x^3 + \epsilon \gamma x y^2 \\ \ddot{y} + \sigma_2^2 y &= \epsilon \delta y + \epsilon B \cos \omega t \cdot y + \epsilon \mu y^3 + \epsilon \nu x^2 y, \end{aligned}$$

where $\epsilon > 0$ is a small parameter, and $\alpha, \beta, \gamma, \delta, \mu, \nu, \sigma_1, \sigma_2$ are constants, and ω is a rational number. In the present section, we obtain conditions for the existence of periodic solutions of (9.1.1), and, in particular, we discuss the non-linear Mathieu equation (the first equation with $\gamma = 0$). This example illustrates how the preceding theory may be applied to systems.

If we let $x = (2i\sigma_1)^{-1} (x_1 + x_2)$, $\dot{x} = \frac{1}{2}(x_1 - x_2)$, $y = (2i\sigma_2)^{-1} (x_3 + x_4)$, $\dot{y} = \frac{1}{2} (x_3 - x_4)$, then we obtain the canonical system

$$(9.1.2) \quad \begin{aligned} \dot{x}_1 &= i\sigma_1 x_1 + \epsilon(2i\sigma_1)^{-1} [\alpha(x_1 + x_2) - \beta(4\sigma_1^2)^{-1}(x_1 + x_2)^3 \\ &\quad - \gamma(4\sigma_2^2)^{-1}(x_1 + x_2)(x_3 + x_4)^2 + (A/2)(e^{it} + e^{-it})(x_1 + x_2)], \\ \dot{x}_2 &= -i\sigma_1 x_2 - \epsilon(2i\sigma_1)^{-1} [\dots], \\ \dot{x}_3 &= i\sigma_2 x_3 + \epsilon(2i\sigma_2)^{-1} [\delta(x_3 + x_4) - \mu(4\sigma_2^2)^{-1}(x_3 + x_4)^3 \\ &\quad - \nu(4\sigma_1^2)^{-1}(x_1 + x_2)^2(x_3 + x_4) + (B/2)(e^{i\omega t} + e^{-i\omega t})(x_3 + x_4)], \\ \dot{x}_4 &= -i\sigma_2 x_4 - \epsilon(2i\sigma_2)^{-1} [\dots], \end{aligned}$$

where $[\dots]$ denotes the expression in the brackets of the equation preceding it.

We replace σ_1, σ_2 by $\tau_1 = k_1/m_1, \tau_2 = k_2\omega/m_2$, respectively, k_i, m_i positive integers, $j = 1, 2$, and apply the algorithm of §1, taking the zeroth approximation to be $(ia_1 e^{i\tau_1 t}, ia_1 e^{-i\tau_1 t}, ia_2 e^{i\tau_2 t}, ia_2 e^{-i\tau_2 t})$, where a_1, a_2 are non-zero real numbers, positive or negative. Proceeding with the calculations, the first approximation is determined from the equations

$$(9.1.3) \quad \begin{aligned} x_{11} &= e^{i\tau_1 t} \int e^{-i\tau_1 t} \{ a_1(2\sigma_1)^{-1} [\alpha C(\tau_1 t) + \beta a_1^2(4\sigma_1^2)^{-1} C^3(\tau_1 t) \\ &\quad + \gamma a_2^2(4\sigma_2^2)^{-1} C(\tau_1 t) C^2(\tau_2 t) + (A/2) C(t) C(\tau_1 t)] - S_{11}(ia_1 e^{i\tau_1 t}) \} dt, \\ x_{12} &= -\bar{x}_{11}, \\ x_{13} &= e^{i\tau_2 t} \int e^{-i\tau_2 t} \{ a_2(2\sigma_2)^{-1} [\delta C(\tau_2 t) + \mu a_2^2(4\sigma_2^2)^{-1} C^3(\tau_2 t) \\ &\quad + \nu a_1^2(4\sigma_1^2)^{-1} C(\tau_2 t) C^2(\tau_1 t) + (B/2) C(\omega t) C(\tau_2 t)] - S_{13}(ia_2 e^{i\tau_2 t}) \} dt, \\ x_{14} &= -\bar{x}_{13}, \end{aligned}$$

where $C(t) = e^{it} + e^{-it}$ and the numbers S_{11} , S_{13} are the coefficients of $ia_1e^{i\tau_1 t}$, $ia_2e^{i\tau_2 t}$, contained in the brackets [ ] in the first and third equations, respectively. It is clear that the number of terms in S_{11} , S_{13} will depend upon the values of τ_1 , τ_2 . In fact, if $\tau_1 \neq \tau_2$, $\tau_1 \neq \frac{1}{2}$, $\tau_2 \neq \frac{1}{2}\omega$, then

$$(9.1.4) \quad \begin{aligned} S_{11} &= (2i\sigma_1)^{-1}[\alpha + 3\beta(4\sigma_1^2)^{-1}a_1^2 + \gamma(2\sigma_2^2)^{-1}a_2^2], \\ S_{13} &= (2i\sigma_2)^{-1}[\delta + \nu(2\sigma_1^2)^{-1}a_1^2 + 3\mu(4\sigma_2^2)^{-1}a_2^2]. \end{aligned}$$

Since system (9.1.1) satisfies the conditions of theorem (5.3.iii), we know that the real parts of the determining equations $i\tau_j - \epsilon S_{1,2j-1} - \epsilon^2 S_{2,2j-1} - \dots = i\sigma_j$, $j = 1, 2$, are identically zero and, thus, if $\sigma_j = \tau_j$, $j = 1, 2$, then the determining equations are $S_{11} = 0$, $S_{13} = 0$. These equations have a solution

$$(9.1.5) \quad \begin{aligned} (a_1\sigma_1^{-1})^2 &= -4(3\mu\alpha - 4\gamma\delta)(9\mu\beta - 16\gamma\nu)^{-1}, \\ (a_2\sigma_2^{-1})^2 &= -4(3\beta\delta - 4\nu\alpha)(9\mu\beta - 16\gamma\nu)^{-1}, \end{aligned}$$

provided that the right hand members are > 0 . This condition is certainly satisfied for some values of the constants α , β , γ , δ , μ , ν . Moreover, the periodic solution of (9.1.1) is given by

$$(9.1.6) \quad x = a_1 \cos \sigma_1 t + \epsilon W_1(t; \epsilon), \quad y = a_2 \cos \sigma_2 t + \epsilon W_2(t; \epsilon),$$

where $W_i(-t; \epsilon) = W_i(t; \epsilon)$, $j = 1, 2$, and the numbers a_1 , a_2 are given by (9.1.5). The first terms in W_1 , W_2 may be obtained from (9.1.3) and the transformation formulas.

Since system (9.1.1) also satisfies the conditions of theorem (5.3.iv), we could choose our zeroth approximation with a_1 , a_2 purely imaginary and obtain a periodic solution of (9.1.1) which is odd in t . In all the cases we have discussed, it is interesting to note that the classical Jacobian is identically zero.

A case of particular interest is when $\gamma = \nu = 0$. Then the two equations of (9.1.1) have no influence upon each other and, also, by a transformation of variables, the second equation may be obtained from the first. Therefore, we need only discuss the first equation with $\gamma = 0$. In order to do this, we merely put $\gamma = 0$ in all the preceding formulas and conclude that there will be a periodic solution of the non-linear Mathieu equation

$$\ddot{x} + \sigma_1^2 x = \epsilon \alpha x + \epsilon A \cos t \cdot x + \epsilon \beta x^3$$

of the form (9.1.6) with $\sigma_1 = k_1/m_1 \neq \frac{1}{2}$ provided that $(a_1\sigma_1^{-1})^2 = -4\alpha(3\beta)^{-1}$, i.e., provided that $\alpha\beta < 0$.

The exceptional cases (either $\tau_1 = \tau_2$ or $\tau_1 = \frac{1}{2}$ or $\tau_2 = \omega/2$) may be discussed in a similar manner.

9.2. System of two non-linear Mathieu equations in which one has a large forcing term. Consider the system

$$(9.2.1) \quad \begin{aligned} \ddot{y}_1 + 4y_1 &= \cos t + \epsilon \{ \cos 3t \cdot y_1 + ky_2^3 + by_1 \}, \\ \ddot{y}_2 + y_2 &= \epsilon \{ \cos 2t \cdot y_2 + cy_1^3 \}, \end{aligned}$$

where $\epsilon > 0$ is a small parameter, k, b, c , constants. We give here the explicit expression, up to terms of order ϵ^2 , of a periodic solution of (9.2.1) whose dominant terms have period 2π .

Following the procedure of §3, put $y_1 = y_{10} + z_1$, $y_2 = z_2$, where $y_{10} = \frac{1}{3} \cos t$, obtaining

$$(9.2.2) \quad \begin{aligned} \ddot{z}_1 + 4z_1 &= \epsilon \{ \cos 3t(y_{10} + z_1) + kz_2^3 + b(y_{10} + z_1) \}, \\ \ddot{z}_2 + z_2 &= \epsilon \{ \cos 2t \cdot z_2 + c(y_{10} + z_1)^3 \}. \end{aligned}$$

If we make the transformation $z_1 = (4i)^{-1} (x_1 + x_2)$, $\dot{z}_1 = \frac{1}{2}(x_1 - x_2)$, $z_2 = (2i)^{-1} (x_3 + x_4)$, $\dot{z}_2 = \frac{1}{2}(x_3 - x_4)$, then (9.2.2) becomes

$$(9.2.3) \quad \begin{aligned} \dot{x}_1 &= 2ix_1 + \epsilon \{ y_{10} \cos 3t + (4i)^{-1} \cos 3t(x_1 + x_2) - k(8i)^{-1}(x_3 + x_4)^3 \\ &\quad + by_{10} + b(4i)^{-1}(x_1 + x_2) \}, \\ \dot{x}_2 &= -2ix_2 - \epsilon \{ \dots \}, \\ \dot{x}_3 &= ix_3 + \epsilon \{ (2i)^{-1} \cos 2t(x_3 + x_4) + c[y_{10} + (4i)^{-1}(x_1 + x_2)]^3 \}, \\ \dot{x}_4 &= -ix_4 - \epsilon \{ \dots \}. \end{aligned}$$

As in Section 9.1, system (9.2.2) satisfies the conditions of theorem (5.3.iii). Hence, we may apply the algorithm of §1, taking as the zeroth approximation $(ia_1 e^{2it}, ia_1 e^{-2it}, ia_2 e^{it}, ia_2 e^{-it})$, where a_1, a_2 , are non-zero real numbers. We obtain as the first approximation

$$(9.2.4) \quad \begin{aligned} x_{11} &= e^{2it} \int e^{-2it} \{ \frac{1}{12} C(t) C(3t) + \frac{1}{8} a_1 C(3t) C(2t) + \frac{1}{8} k a_2^3 C^3(t) \\ &\quad + \frac{1}{6} b C(t) + \frac{1}{4} b a_1 C(2t) - i a_1 e^{2it} S_{11} \} dt, \\ x_{12} &= -\bar{x}_{11}, \\ x_{13} &= e^{it} \int e^{-it} \{ \frac{1}{4} a_2 C(2t) C(t) + c [\frac{1}{6} C(t) + \frac{1}{4} a_1 C(2t)]^3 - i a_2 e^{it} S_{13} \} dt, \\ x_{14} &= -\bar{x}_{13}, \end{aligned}$$

where $C(t) = e^{it} + e^{-it}$. We see immediately that $S_{11} = (12ia_1)^{-1} + b(4i)^{-1}$, $S_{13} = (4i)^{-1} + 3c(6^3 ia_2)^{-1} + ca_1^2 (16ia_2)^{-1}$. As is expected, (theorem (5.3.iii)), S_{11}, S_{13} are purely imaginary. If $b \neq 0$, then the Jacobian $|\partial(S_{11}, S_{13})/\partial(a_1, a_2)| \neq 0$, and there is a periodic solution of (9.2.2) with approximate amplitude given by $a_1 = -(3b)^{-1}$, $a_2 = -(c/18) [1 + (2b^3)^{-1}]$. Integrating (9.2.4) and applying the transformation above, one can easily obtain an approximate expression for y_1, y_2 .

§10. REMARKS ON AUTONOMOUS SYSTEMS

In the preceding sections, all differential systems were assumed to contain the time t explicitly. In case the systems are autonomous, the method of successive approximations described remains valid provided only that the numbers τ_i are chosen such that $\tau_i = k_i\omega/m_i$, where k_i, m_i are positive integers and ω is a real constant. We assume ω real, since this case is the only concern of the applications. In §1, if one assumes the numbers ρ_i to be known, then one may think of equations (1.1.5) as the determining equations for $\omega, a_1, a_2, \dots, a_n$, where $a_1, \dots, a_n$ are the complex numbers defined in formula (1.1.10). If we put $a_i = |a_i| e^{i\varphi_i}$, φ_i real, and take into account the real and imaginary parts of each equation in (1.1.5), then it is clear that the $2n + 1$ unknowns $\omega, |a_1|, \dots, |a_n|, \varphi_1, \dots, \varphi_n$ are restricted by only $2n$ relations. Therefore, if one of these $2n + 1$ unknowns is chosen arbitrarily, there remains only $2n$ relations and $2n$ unknowns. If some obvious conditions are satisfied, then a theorem similar to theorem (2.2.i) may be obtained. The other results of the succeeding sections may be extended in an analogous manner. Of course, in the theorems similar to theorems (5.3.i)–(5.3.iv), for the autonomous case, the function h_i is identically zero. It should also be remarked that these same theorems for autonomous systems have been obtained in a somewhat less general form by a similar method by J. K. HALE [11].

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