On Quasiconformal Mappings

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1. Very recently AGMON [1] has proved that for a univalent quasiconformal mapping $f(z)$ of $|z| < 1$ the radial limit $\lim_{r \rightarrow 1} f(re^{i\theta})$ exists for almost all θ and that the limit can have a given value at most on a set of measure zero. Now it is known that for conformal mappings much stronger results hold. Indeed BEURLING [4] has proved that for a function $f(z)$ regular and univalent or finitely multivalent in $|z| < 1$ the radial limit exists apart from sets of outer capacity zero and the limit values can be constant at most on such a set. In the present paper we shall show that analogues of Beurling's results hold when we deal with quasiconformal mappings rather than regular functions.

Some authors restrict the term *quasiconformal mapping* to refer to homeomorphic mappings from one plane domain to another. In the present paper we do not make this restriction, following the lead of GRÖRTZSCH [6], who first studied these mappings (under the name of mappings of bounded infinitesimal distortion). We shall then consider quasiconformal mappings of the unit circle $|z| < 1$. These are first of all interior transformations of the circle, the image being a Riemann surface $\mathcal{R}$ over the w -sphere. Further the ratio of the module M of a quadrangle in $|z| < 1$ and the module M' of its image on $\mathcal{R}$ is to be bounded and bounded from zero, uniformly for all quadrangles; *i.e.*, there exists a constant $D \geq 1$ such that

$$D^{-1}M \leq M' \leq DM.$$

The smallest constant which will serve is called the maximum dilation of the mapping; in particular for a regular function this is 1[6, 15, 2]. If the function $f(z)$ in question possesses partial derivatives it is well known that the condition just given is satisfied in the following circumstances. Suppose that $z = x + iy$ and the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are continuous. Set

$$p = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad q = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),$$

and assume that these do not vanish simultaneously except at isolated points. Then apart from these latter points the ratio $(|p| + |q|)/|p| - |q|$ gives the

dilation quotient of the mapping. It can be interpreted as the ratio of the maximum of $|dw/dz|$ to its minimum at the point in question. If it is bounded the mapping is quasiconformal, the least upper bound giving the same value of the maximum dilation as before. These conditions are frequently used to define quasiconformal mappings rather than the more geometrical one quoted above.

AGMON raised the natural question as to whether analogues of the theorems of FATOU and F. & M. RIESZ hold for bounded quasiconformal mappings. We are not at present able to answer this question, but we shall show such results to be equivalent to the measurability of the boundary mapping induced by a quasiconformal mapping of the unit circle $|z| < 1$ onto the unit circle $|w| < 1$.

Finally we shall give an interesting interpretation of the analogues for quasiconformal mappings of Beurling's results in terms of extremal length.

I wish to thank Professor W. SEIDEL for several useful references.

2. Let us consider first of all a univalent quasiconformal mapping $\zeta = \psi(z)$ of maximum dilation D of the unit circle $|z| < 1$ onto the unit circle $|\zeta| < 1$. Let us denote the inverse mapping by $z = \phi(\zeta)$. This mapping, in turn, is quasiconformal and of maximum dilation D . It is well known [6] that this correspondence can be extended to a homeomorphism between $|z| \leq 1$ and $|\zeta| \leq 1$. We shall denote the extended functions again by ϕ and ψ . We begin by showing that paths of non-tangential approach to boundary points correspond in the two cases.

Lemma 1. *Let the arc l in $|z| \leq 1$ have one endpoint at a on $|z| = 1$, otherwise lie in $|z| < 1$ and tend to a non-tangentially, i.e., so as to lie in the angle between two chords of $|z| = 1$ each with one endpoint at a . Then its image λ under ψ tends to $b = \psi(a)$ non-tangentially.*

Let us map each of $|z| < 1$, $|\zeta| < 1$ conformally onto the strip $S: 0 < t < 1$ in the $\omega = s + it$ -plane so that both a and b are mapped onto the boundary point P of S at infinity in whose neighborhood s has large positive values and so that another pair of corresponding points on $|z| = 1$ and $|\zeta| = 1$ respectively are mapped onto the other boundary point P' of S at infinity. To the mapping ψ corresponds a quasiconformal mapping χ of S onto itself leaving P and P' fixed (as boundary points) and of maximum dilation D . The statement of the lemma is seen at once to be a consequence of the fact that the image under χ of a line $t = t_0$, $0 < t_0 < 1$ lies in a strip $\epsilon \leq t \leq 1 - \epsilon$, $\epsilon > 0$. This latter result we now prove.

First we observe that the rectangle cut off on S by the lines $s = s_0$, $s = s_0 + h$, $h > 0$ has module $1/h$ for the class of curves joining its vertical sides. It is mapped by χ onto a quadrangle with one pair of opposite sides respectively on $t = 1$ and $t = 0$ and whose module M for the class of curves joining the other pair of opposite sides is at most D/h . Let the greatest value of s on the one of these sides to the left in S be denoted by u_1 , the least value of s on the one of these

sides to the right in S be denoted by u_2 . Then by TEICHMÜLLER's sharpening of the Ahlfors distortion theorem [15] we have

$$u_2 - u_1 \geq \Delta(1/M) \geq \Delta(h/D),$$

where Δ is such that for $y \geq 2$, $\Delta(y) \geq y - 4$. Thus for $h \geq 5D$ we have $u_2 - u_1 \geq 1$.

Suppose now that a point ω_1 on the line $t = t_0$ is mapped by χ into a point ω_2 whose vertical distance from the line $t = 1$ is d . Let us regard the rectangle T bounded by the lines $s = R\omega_1 - 5D$, $s = R\omega_1 + 5D$, $t = t_0$, $t = 1$. Its module for the class of curves joining its vertical sides is $(1 - t_0)/(10D)$. It is mapped by χ into a quadrangle Q whose module for the corresponding class of curves is at least $(1 - t_0)/(10D^2)$. We now evaluate the module of the image quadrangle from above in terms of d . For this we regard the ellipse with foci at the points ω_2 and $\omega_2 + 2di$ and semi-minor axis $1 - (\frac{1}{2}d)^2$. We obtain from the portion of the ellipse below the line $t = 1$ a quadrangle Q^* by assigning as sides the segment $R\omega_2 - 1 + (\frac{1}{2}d)^2 < s < R\omega_2 + 1 - (\frac{1}{2}d)^2$, $t = 1$, the segment $s = R\omega_2$, $S\omega_2 - (1 - \frac{1}{2}d)^2 < t < S\omega_2$ described twice and the two pieces into which the boundary arc of the ellipse below the line $t = 1$ is divided by $s = R\omega_2$. By a standard transformation this quadrangle can be mapped with preservation of vertices on the rectangle R in the $w = u + iv$ plane defined by $0 < u < 2 \log (2/d)$, $0 < v < \pi/2$ with the horizontal sides corresponding to the pair of opposite sides of Q first named above. The extremal metric for the module problem for the class of curves in R joining its vertical sides is $|dw|/[2 \log (2/d)]$. We carry this back to the ω -plane to obtain in Q^* the metric $\rho^*|d\omega| = |dw/d\omega| \cdot |d\omega|/(2 \log (2/d))$. Then the metric $\rho |d\omega|$ obtained in Q by taking $\rho \equiv \rho^*$ in $Q \cap Q^*$, $\rho \equiv 0$ in the remainder of Q is admissible in the module problem for the class of curves in Q which join its sides which are images by χ of the vertical sides of T . Thus this latter module is smaller than $\pi/(4 \log (2/d))$. Combining this with our previous estimate we have

$$\frac{1 - t_0}{10D^2} \leq \frac{\pi}{4 \log (2/d)},$$

which implies

$$d \geq 2e^{-5\pi D^2/[2(1-t_0)]}.$$

Thus the image of $t = t_0$ under χ lies below the line $t = 1 - 2e^{-5\pi D^2/[2(1-t_0)]}$. Similarly this image lies above the line $t = 2e^{-5\pi D^2/(2t_0)}$. This completes the proof of Lemma 1. It should be pointed out that here we have not tried to give best possible evaluations but only convenient explicit estimates. Some of the present estimates can be made best possible in a certain sense by the use of elliptic functions.

Lemma 2. *Let φ be a univalent quasiconformal mapping of $|\zeta| < 1$ onto $|z| < 1$ satisfying $\varphi(0) = 0$ and extended to a homeomorphism of $|\zeta| \leq 1$ onto $|z| \leq 1$. Then φ carries a set E of capacity zero on $|\zeta| = 1$ into a set of capacity zero on $|z| = 1$.*

Capacity of a non-closed set is here understood in the sense of FROSTMAN [5] *i.e.*, the least upper bound of the capacities of closed sets contained in the set in question. Hence it is sufficient to prove the result for a closed set E . If we extend the definition of φ to $|\zeta| > 1$ by setting $\varphi(\zeta) = [\bar{\varphi}(\bar{\zeta}^{-1})]^{-1}$ for $|\zeta| > 1$ we obtain a homeomorphism of the ζ -plane onto the z -plane which is quasiconformal for $|\zeta| < 1$ and for $|\zeta| > 1$. It is then quasiconformal on the whole plane [2, p. 11]. In particular it maps the complement of E quasiconformally onto the complement of $\varphi(E)$. By a result of PFLUGER [12] it then follows that if E has capacity zero so does $\varphi(E)$.

3. BEURLING [4] has shown that much stronger statements can be made to replace the theorems of Fatou and F. & M. Riesz when the functions involved instead of being regular and bounded are assumed to be regular and univalent or finitely multivalent. We turn now to the corresponding questions for quasiconformal mappings and state our first main result.

Theorem 1. *Let $f(z)$ be a quasiconformal mapping of the unit circle $|z| < 1$ which is univalent or finitely multivalent. Then the limit of $f(z)$ exists as z tends to a boundary point along any path of non-tangential approach apart at most from the points of a set of capacity zero.*

The function f maps $|z| < 1$ onto a simply-connected Riemann surface $\mathfrak{R}$ which is of hyperbolic type since quasiconformal mappings preserve type. This surface can then be mapped conformally onto $|\zeta| < 1$ so that the image on $\mathfrak{R}$ of $z = 0$ goes into $\zeta = 0$. Let the combined mappings between z and ζ be denoted by $z = \varphi(\zeta)$, $\zeta = \psi(z)$. The function $f(\varphi(\zeta))$ is regular and univalent or finitely multivalent for $|\zeta| < 1$. Thus by Beurling's result its radial limit exists for all boundary points of $|\zeta| < 1$ apart at most from a set E (possibly void) of capacity zero. For univalent or finitely multivalent functions this is equivalent to the existence of the non-tangential limit at the same points. As z tends to a point $e^{i\theta}$ on $|z| = 1$ not in the set $\varphi(E)$ along a path γ of non-tangential approach the value $f(z)$ exhibits the same limiting behavior as does $f(\varphi(\zeta))$ as ζ tends to the point $\psi(e^{i\theta})$ along the path $\psi(\gamma)$. By Lemma 1 this latter path is a path of non-tangential approach. Thus $f(\varphi(\zeta))$ and so $f(z)$ tend to limits depending only on the point $e^{i\theta}$. By Lemma 2 the set $\varphi(E)$ has capacity zero. This proves Theorem 1.

Theorem 2. *Let $f(z)$ be a quasiconformal mapping of the unit circle $|z| < 1$ which is univalent or finitely multivalent. Let A be a set on $|z| = 1$ such that the limit of $f(z)$ exists as z tends radially to each point of A and has the same value c for all these points*

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = c, \quad e^{i\theta} \in A.$$

Then A has capacity zero.

In the same notation as before the function $f(\varphi(\zeta))$ is regular and univalent or finitely multivalent for $|\zeta| < 1$. By Lemma 1 it has the limit c as ζ tends to each point of $\psi(A)$ along a certain path of non-tangential approach. Thus it also has the limit c as ζ tends to these points radially. By Beurling's result the set $\psi(A)$ has capacity zero. Hence by Lemma 2 A has capacity zero.

It will be observed that in Beurling's statement the exceptional sets are stated to be of outer capacity zero *i.e.*, the greatest lower bound of the capacities of open sets containing such a set is zero rather than just the set itself having capacity zero. The same result can be obtained for quasiconformal mappings if we assume them to satisfy the stronger condition of possessing continuous partial derivatives. However we shall not go further into this matter here.

4. We now consider the following three statements:

X. Let φ be a univalent quasiconformal mapping of $|\zeta| < 1$ onto $|z| < 1$ extended to a homeomorphism of $|\zeta| \leq 1$ onto $|z| \leq 1$. Then φ carries a set E of angular measure zero on $|\zeta| = 1$ into a set of angular measure zero on $|z| = 1$.

Y. Let $f(z)$ be a quasiconformal mapping of the unit circle $|z| < 1$ which is bounded, $|f(z)| \leq M$. Then the limit of $f(z)$ exists as z tends to a boundary point $e^{i\theta}$ along any path of non-tangential approach for almost all boundary points $e^{i\theta}$ (θ real).

Z. Let $f(z)$ be a quasiconformal mapping of the unit circle $|z| < 1$. Let A be a set on $|z| = 1$ such that the limit of $f(z)$ exists as z tends to each point of A along any path of non-tangential approach and has the same value c for all these points. Then the angular measure of A is zero unless $f(z)$ reduces to the constant c .

It is not at present known whether or not any of these statements is true. A purported proof of *X* due to PFLUGER [11] is defective. If true, *Y* would be the analogue of Fatou's Theorem and *Z* would be the analogue of the Lusin-Priwaloff Theorem for regular functions. However we can prove

Theorem 3. *The statements X, Y and Z are simultaneously true or false.*

We prove this by showing that if *X* is true *Y* and *Z* are true and if *X* is false *Y* and *Z* are false.

Lemma 3. *If statement X is true statement Y is true.*

The function f maps $|z| < 1$ onto a simply-connected Riemann surface of hyperbolic type. This can be mapped conformally onto $|\zeta| < 1$. Let the combined mappings between z and ζ be denoted by $z = \varphi(\zeta)$ and $\zeta = \psi(z)$ and extended as before. Then $f(\varphi(\zeta))$ is regular for $|\zeta| < 1$ and bounded. Thus by Fatou's Theorem the limit of $f(\varphi(\zeta))$ exists as ζ tends to a point on $|\zeta| = 1$ along any path of non-tangential approach for all boundary points except those of a set E (possibly void) of measure zero. By Lemma 1 paths of non-tangential approach correspond in $|z| < 1$ and $|\zeta| < 1$. Thus the non-tangential

limit of $f(z)$ exists apart from the points of $\varphi(E)$. If statement X is true the latter set has measure zero.

Lemma 4. *If statement X is true statement Z is true.*

In the same notation as before the function $f(\varphi(\zeta))$ is regular for $|\zeta| < 1$. It has by Lemma 1 the limit c as ζ tends to each point of $\psi(A)$ along any path of non-tangential approach. Thus by the Lusin-Priwaloff Theorem the set $\psi(A)$ has angular measure zero unless $f(\varphi(\zeta))$ reduces to the constant c . If statement X is true the set A has angular measure zero unless $f(z)$ reduces to the constant c .

Corollary 1. *Let $f(z)$ be a quasiconformal mapping of the unit circle $|z| < 1$ which is bounded, $|f(z)| \leq M$. Let A be a set on $|z| = 1$ such that the limit of $f(z)$ exists as z tends radially to each point of A and has the same value c for all these points,*

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = c, \quad e^{i\theta} \in A.$$

Then if statement X is true the angular measure of A is zero unless $f(z)$ reduces to the constant c .

This corresponds to the Theorem of F. & M. Riesz for regular functions. It is an immediate consequence of Lemma 4 and the fact that for bounded regular functions the existence of radial and non-tangential limits are equivalent.

Lemma 5. *If statement X is false there exists a quasiconformal mapping $f(z)$ of the unit circle $|z| < 1$ which is bounded and such that the limit of $f(z)$ fails to exist as z tends radially to each point of a set of positive angular measure.*

If statement X is false there exists a univalent quasiconformal mapping $\varphi(\zeta)$ of $|\zeta| < 1$ onto $|z| < 1$ whose natural extension carries a set E of angular measure zero on $|\zeta| = 1$ into a set of positive angular measure on $|z| = 1$. Let $\psi(z)$ be the inverse mapping. LUSIN & PRIWALOFF [10] have given an example of a function $g(\zeta)$ regular and bounded for $|\zeta| < 1$ and such that the limit of $g(\zeta)$ fails to exist as ζ tends to each point of E along any path of non-tangential approach. Then $g(\psi(z))$ is a quasiconformal mapping of $|z| < 1$ which is bounded and whose limit fails to exist as z tends radially to each point of the set $\varphi(E)$ which has positive angular measure.

Lemma 6. *If statement X is false there exists a non-constant quasiconformal mapping $f(z)$ of the unit circle $|z| < 1$ which is bounded and such that the limit of $f(z)$ exists as z tends radially to each point of a set of positive angular measure and has the same value for all these points.*

In the same notation as before, again let E be a set of angular measure zero on $|\zeta| = 1$ carried by φ into a set of positive measure on $|z| = 1$. PRIWALOV (PRIWALOFF) [13, pp. 294–298] has given an example of a function $g(\zeta)$ non-constant, regular and bounded for $|\zeta| < 1$ and such that the limit of $g(\zeta)$ exists

as ζ tends to each point of E along any path of non-tangential approach and has the same value for all these points. Then $g(\psi(z))$ is a quasiconformal mapping of $|z| < 1$ which satisfies the conditions of Lemma 6.

Lemmas 3, 4, 5 and 6 comprise the proof of Theorem 3.

5. We shall now give an interesting interpretation and at the same time a new proof of the results of Theorem 1 and Theorem 2 in the case where the function involved is univalent. We have just seen that the result for quasiconformal mappings follows from the result for regular functions, and so for simplicity we shall confine ourselves to the latter case although the argument could be extended to quasiconformal mappings also.

We begin by recalling the concept of the extremal length of a family of curves. Let Γ be a family of locally rectifiable curves lying in a domain Ω of the z -plane ($z = x + iy$). By locally rectifiable we mean here that every closed subarc is rectifiable. Let ρ be a non-negative real valued function defined in Ω , of integrable square and such that $\int_{\gamma} \rho |dz|$ exists for all $\gamma \in \Gamma$ and satisfies

$$\int_{\gamma} \rho |dz| \geq 1.$$

Let m be the greatest lower bound of $\iint_{\Omega} \rho^2 dx dy$ for all functions ρ as above. Then the extremal length of the family Γ is given by m^{-1} . It is usually more convenient to work with the quantity m which we call the module of the domain Ω for the class of curves Γ .

Now let C be any compact set of the z -plane and D the (open) component of its complement which contains the point at infinity. Let R be large enough that C lies in $|z| < R$. Let Ω_1 be the domain $D \cap \{|z| < R\}$ and let Γ_1 be the class of locally rectifiable open arcs lying in Ω_1 and running from a point of C to a point of $|z| = R$. We denote the module of Ω_1 for the class of curves Γ_1 by $m(C, R)$. It is well known that this vanishes if and only if C has capacity zero.

Now let E be a closed set on $|z| = 1$ and for $r < 1$ let Ω_2 be the domain $r < |z| < 1$ and let Γ_2 be the class of locally rectifiable open arcs lying in Ω_2 and running from a point of E to a point of $|z| = r$. We denote the module of Ω_2 for the class of curves Γ_2 by $m^*(E, r)$. We shall prove

Lemma 7. *The module $m^*(E, r)$ vanishes if and only if E has capacity zero.*

First let ρ be an admissible function in the domain Ω_1 where $R = r^{-1}$ for the problem determining $m(E, r^{-1})$. Define the function ρ' in Ω_2 by

$$\rho'(z) = |z|^{-2} \rho(\bar{z}^{-1}), \quad z \in \Omega_2.$$

Then it is verified at once that ρ' is admissible for the problem determining $m^*(E, r)$ and

$$\iint_{\Omega_2} (\rho'(z))^2 dx dy \leq \iint_{\Omega_1} (\rho(z))^2 dx dy.$$

Thus

$$m^*(E, r) \leq m(E, r^{-1})$$

and $m^*(E, r)$ vanishes if E has capacity zero.

Next let ρ^* be an admissible function in the domain Ω_2 for the problem determining $m^*(E, r)$. Let Ω'_2 be the image of Ω_2 under reflection in the circle $|z| = 1$. We extend the function ρ^* to Ω'_2 by setting

$$\rho^*(z) = |z|^{-2} \rho^*(\bar{z}^{-1}), \quad z \in \Omega'_2$$

Next we define $\rho(z)$ for $z \in \Omega_1$ by setting

$$\begin{aligned} \rho(z) &= \rho^*(z), & z \in \Omega_2 \cup \Omega'_2 \\ \rho(z) &\equiv 0, & \text{elsewhere in } \Omega_1. \end{aligned}$$

It can be verified readily that ρ is admissible in the problem determining $m(E, r^{-1})$. Moreover

$$\iint_{\Omega_1} [\rho(z)]^2 dx dy = 2 \iint_{\Omega_2} [\rho^*(z)]^2 dx dy.$$

Thus

$$m(E, r^{-1}) \leq 2m^*(E, r)$$

so that if $m^*(E, r)$ vanishes, E has capacity zero.

We now turn to the interpretation of Theorem 1 in the case where the function $f(z)$ is regular and univalent. Let U denote the set of points on $|z| = 1$ for which the non-tangential limit of $f(z)$ does not exist. Then $f(z)$ cannot tend to a limit as z approaches a point of U along any curve, as follows by a well known result of LINDELÖF. Let E be any closed subset of U . Let Ω_2 and Γ_2 be as above. The image by $w = f(z)$ of any curve of Γ_2 is then a curve of infinite length in the spherical metric on the w -sphere. Let ds denote the element of spherical length and take in the image Λ_2 of Ω_2 by $f(z)$ the metric ϵds for any $\epsilon > 0$. In this metric the image curves of the curves in Γ_2 have infinite length, hence surely length ≥ 1 . On the other hand the area of Λ_2 in this metric tends to zero with ϵ . Recalling that all module problems are conformally invariant, we see $m^*(E, r) = 0$. Thus E has capacity zero and so does U .

Next we deal with Theorem 2, again assuming $f(z)$ regular and univalent. Let A denote the set on which $f(z)$ has the radial limit c . Since $f(z)$ is univalent there are points in $|z| < 1$ where it takes other values and indeed the value c is not taken at any point of $|z| < 1$. The circle $|z| = r$ is mapped into a curve L in the w -plane by $w = f(z)$. Let $c(\alpha)$ be the circle of radius $\alpha > 0$ about the point c where we assume α so small that $c(\alpha)$ does not meet L . Then $c(\alpha)$ and L bound a doubly-connected domain $D(\alpha)$ on the w -sphere. Let $\mu(\alpha)$ be the module for $D(\alpha)$ for the class of curves in $D(\alpha)$ joining its boundaries and let $\sigma(w)$ be

the extremal function in this module problem. Let E be any closed subset of A . Let Ω_2 and Γ_2 be the corresponding entities defined as above. Let us define in the image Λ_2 of Ω_2 by $w = f(z)$ the function $\rho(w)$ by

$$\begin{aligned}\rho(w) &= \sigma(w) & w \in D(\alpha) \cap \Lambda_2 \\ \rho(w) &= 0 & \text{in the remainder of } \Lambda_2.\end{aligned}$$

The image under $w = f(z)$ of any path running in Ω_2 from $|z| = r$ to a point of E non-tangentially goes into a curve running from L to $w = c$ which thus has length at least 1 in the metric $\rho(w) |dw|$. For a path in Ω_2 tending to a point of E but not non-tangentially the value c is still a limiting value of $f(z)$ [14] as we approach the endpoint on the set E . Thus the image of any open arc in Γ_2 by $w = f(z)$ has length at least 1 in the metric $\rho(w) |dw|$. On the other hand Λ_2 has area at most $\mu(\alpha)$ in this metric. Again using the conformal invariance of the module problem we have

$$m^*(E, r) \leq \mu(\alpha).$$

Since $\mu(\alpha)$ tends to zero with α it follows that $m^*(E, r) = 0$. Thus E has capacity zero and so does A .

It should be remarked again that these interpretations can be extended also to quasiconformal mappings and functions which are finitely multivalent rather than univalent.

Added July 29, 1955. Professor BERS has informed me that BEURLING'S results for quasiconformal mappings have been proved also by A. J. LOHWATER, the approach being quite different.

Editor's note: See the preceding paper.

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