The Boundary Behavior of a Quasi-Conformal Mapping

A. J. LOHWATER

Communicated by L. BERS

It is a well-known result of BEURLING [3] in the theory of conformal mapping that if $w = f(z)$ is analytic and univalent in $|z| < 1$ then the radial limits $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ exist for all points $e^{i\theta}$ on $|z| = 1$ except for a possible set of logarithmic capacity zero; furthermore, the set of points $e^{i\theta}$ on $|z| = 1$ on which $f(e^{i\theta})$ is constant cannot have positive capacity. In this note, we shall show that the theorem of BEURLING can be extended to the class of functions which give a quasi-conformal mapping of $|z| < 1$ onto a simply-connected region of the w -plane. We remark that E. SAKAI [6], and later S. AGMON [1], have shown that the radial limit values of such a mapping exist for all $e^{i\theta}$ of $|z| = 1$ with the possible exception of a set of linear measure zero; in both papers, the existence of such boundary values has been shown only for *radial* approach. (This result was applied by L. BERS [2] to a problem in the theory of non-linear elliptic partial differential equations, and, while it is not known whether the present result will have application in that theory, the result appears to be of independent interest.) BEURLING's proof of his theorem for the conformal case does not appear capable of extension to the quasi-conformal case, but a proof of BEURLING's theorem which is due to TSUJI [7] provides the idea for an extension.¹⁾ We prove first (Theorem 1) a simple corollary of a theorem of NOSHIRO [4] on the boundary values of quasi-conformal mappings, which is needed in the proof of the main result (Theorem 2). In Theorem 3, the requirement that a quasi-conformal mapping be univalent is removed, provided that the function map $|z| < 1$ onto a Riemann surface having a bounded spherical area.

¹⁾The author has been informed recently by Professor BERS that J. A. JENKINS has made a similar extension of BEURLING's theorem.

Editor's note: see the following paper.

A function $w = f(z) = u(x, y) + iv(x, y)$, whose derivatives u_x, u_y, v_x, v_y exist and are continuous in $|z| < 1$, and whose functional determinant

$$J(z) = \sqrt{EG - F^2} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is positive in $|z| < 1$, will be called *quasi-conformal* in $|z| < 1$ if there exists a constant K such that $E + G \leq 2K\sqrt{EG - F^2}$, where $E = u_x^2 + v_x^2$, $F = u_x u_y + v_x v_y$, $G = u_y^2 + v_y^2$. Whenever $K = 1$, the function $f(z)$ is analytic in $|z| < 1$. (We remark that such functions are called "pseudo-analytic" in [4] and [6], but, in order to avoid confusion with the pseudo-analytic functions of BERS, we shall use the term "quasi-conformal" even when the functions concerned are not univalent.)

Theorem 1. *Let $f(z)$ be quasi-conformal²⁾ and bounded, $|f(z)| < M$, in $|z| < 1$. Let L_1 and L_2 be two arcs lying in $|z| < 1$ except for a common endpoint z_0 on $|z| = 1$, and which are disjoint except for z_0 . If $\lim_{z \rightarrow z_0 \text{ along } L_1} f(z) = \alpha$ and $\lim_{z \rightarrow z_0 \text{ along } L_2} f(z) = \beta$, then $\alpha = \beta$, and $\lim_{z \rightarrow z_0} f(z) = \alpha$ exists uniformly as $z \rightarrow z_0$ along any path L which lies between L_1 and L_2 .*

Let z_1 and z_2 be two points of $|z| < 1$ lying on L_1 and L_2 respectively, and let γ be a simple arc joining z_1 and z_2 and forming a Jordan curve Γ which contains z_0 . Let $C_D(z_0)$ be the cluster set of $f(z)$ at z_0 for the domain D bounded by Γ , i.e., the set of all points w such that there exists a sequence $\{z_n\}$ lying in D such that, as $\lim_{n \rightarrow \infty} z_n = z_0$, $\lim_{n \rightarrow \infty} f(z_n) = w$. We define the boundary cluster set $C_\Gamma(z_0)$ of $f(z)$ at z_0 as follows: Let $M_r(z_0)$ denote the closure of the union $\bigcup_{\zeta} C_D(\zeta)$ for all ζ belonging to that part of $\Gamma - \{z_0\}$ lying in the circle $|z - z_0| < r$; then $C_\Gamma(z_0)$ is defined to be $\bigcap_{r>0} M_r(z_0)$. It is clear that both $C_D(z_0)$ and $C_\Gamma(z_0)$ are closed sets with $C_\Gamma(z_0) \subset C_D(z_0)$, and that $C_D(z_0)$ is connected. It has been shown by NOSHIO [4] that the set $\Omega = C_D(z_0) - C_\Gamma(z_0)$ is open, and that if Ω is not empty, the function $f(z)$ assumes infinitely often in any neighborhood of z_0 , every value—with two possible exceptions—belonging to each component Ω_n of Ω .

Since $f(z)$ tends asymptotically to α and β along L_1 and L_2 , respectively, it is clear that the set $C_\Gamma(z_0)$ consists of the two points α and β . Because $|f(z)| < M$ in $|z| < 1$, the set $C_D(z_0)$ is contained in the circle $|w| \leq M$, so that $\Omega = C_D(z_0) - \{\alpha \cup \beta\}$ cannot be an open set, unless it is empty and $C_D(z_0)$ consists of the two points α and β . Since $C_D(z_0)$ is connected, the set Ω can be empty only if $\alpha = \beta$ and $C_D(z_0)$ reduces to a single point α . Hence $\lim_{z \rightarrow z_0} f(z) = \alpha$ as $z \rightarrow z_0$ along any path lying between L_1 and L_2 ; the uniformity of the limit is trivial. Hence Theorem 1 is proved.

²⁾But not necessarily univalent.

Theorem 2. Let $w = f(z)$ be a quasi-conformal mapping of $|z| < 1$ onto a simply-connected region G of the w -plane. Then, for all $e^{i\theta}$ on $|z| = 1$ except for a possible set of logarithmic capacity zero, $\lim_{z \rightarrow e^{i\theta}} f(z)$ exists uniformly inside any angular region inside $|z| < 1$ with vertex at $e^{i\theta}$. Furthermore, $f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ cannot be constant on any set of positive capacity on $|z| = 1$.

We remark first that it is no loss of generality to assume that $|f(z)| < M$ in $|z| < 1$. Let $e^{i\theta}$ be an arbitrary point of $|z| = 1$, and let l_t ($-\frac{1}{2}\pi < t < \frac{1}{2}\pi$) be a rectilinear segment of length $\cos t$ lying in $|z| < 1$ and forming an angle t at $e^{i\theta}$ with the radius drawn to $e^{i\theta}$. We form the integral

$$(1) \quad L(\theta, t) = \int_{l_t} \sqrt{E dx^2 + 2F dx dy + G dy^2},$$

which represents the length, whenever it exists, of the image of the segment l_t under the mapping $w = f(z)$. Since

$$\frac{E dx^2 + 2F dx dy + G dy^2}{dx^2 + dy^2} \leq \frac{1}{2} \{ (E + G) + \sqrt{(E + G)^2 - 4(EG - F^2)} \}$$

and

$$E + G \leq 2K \sqrt{EG - F^2},$$

we have

$$(2) \quad L(\theta, t) \leq (K + \sqrt{K^2 - 1})^{\frac{1}{2}} \int_{l_t} \sqrt{EG - F^2} |dz|.$$

We set $\lambda(\theta, t) = \int_{l_t} \sqrt{EG - F^2} |dz|$ and consider the integral

$$(3) \quad \Lambda(\theta) = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \lambda(\theta, t) \cos t dt.$$

It is clear that whenever $\lambda(\theta, t)$ is finite, so is $L(\theta, t)$, and that whenever $\Lambda(\theta)$ is finite, then $\lambda(\theta, t)$ is finite for all t in the interval $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$ except for a set of linear measure zero. Our method of proof will be to show that $\Lambda(\theta)$ is finite for all $e^{i\theta}$ on $|z| = 1$ except for a possible set of logarithmic capacity zero. Thus, for any $e^{i\theta}$ not belonging to a certain set of capacity zero, $L(\theta, t)$ will be finite for almost all t in $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$, so that for all those values of t for which $L(\theta, t)$ is finite, the limit of $f(z)$ must exist as $z \rightarrow e^{i\theta}$ along a rectilinear segment making an angle t with the radius of $|z| < 1$ drawn to $e^{i\theta}$. Since $f(z)$ is assumed to be bounded in $|z| < 1$, it follows from Theorem 1 that $f(z)$ tends uniformly to a unique limit as $z \rightarrow e^{i\theta}$ along any path lying inside an angular region of opening less than π lying in $|z| < 1$ and with vertex at $e^{i\theta}$. In order to

complete the first part of the proof of Theorem 2, we must prove that $\Lambda(\theta)$ is finite for all $e^{i\theta}$ on $|z| = 1$ with the possible exception of a set of capacity zero.

Let us suppose, to the contrary, that $\Lambda(\theta)$ is infinite on a set of positive capacity; such a set must contain a closed subset E of positive capacity. There exists a distribution $\mu(\zeta)$ of unit mass on E such that the potential

$$(4) \quad U(z) = \int_{\zeta \in E} \log \frac{1}{|z - \zeta|} d\mu(\zeta)$$

is bounded, $|U(z)| < V_0 < \infty$, for $|z| < \infty$. In particular, $U(z) = U(r, \theta)$ is harmonic and bounded in $|z| < 1$, and has a trigonometrical series representation

$$(5) \quad U(r, \theta) = \sum_{n=1}^{\infty} \frac{r^n}{n} (a_n \cos n\theta + b_n \sin n\theta)$$

in terms of its Fourier-Stieltjes coefficients

$$(6) \quad a_n = \int_E \cos n\theta d\mu(\theta); \quad b_n = \int_E \sin n\theta d\mu(\theta).$$

We have that

$$\sum_{n=1}^{\infty} \frac{r^n}{n} (a_n^2 + b_n^2) = \int_E U(r, \theta) d\mu(\theta) \leq V_0,$$

whence

$$(7) \quad \sum_{n=1}^{\infty} \frac{1}{n} (a_n^2 + b_n^2) \leq V_0.$$

Thus it follows that

$$(8) \quad \iint_{|z| < 1} \left(\frac{\partial U}{\partial r} \right)^2 r dr d\theta = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} (a_n^2 + b_n^2) \leq \frac{\pi}{2} V_0.$$

If we define the area integral

$$(9) \quad S = \iint_{|z| < 1} \sqrt{EG - F^2} \frac{\partial U}{\partial r} r dr d\theta,$$

it follows from Schwarz's inequality that

$$(10) \quad S^2 \leq \iint_{|z| < 1} \sqrt{EG - F^2} r dr d\theta \cdot \iint_{|z| < 1} \left(\frac{\partial U}{\partial r} \right)^2 r dr d\theta \leq \pi M^2 \cdot \frac{\pi}{2} V_0 < \infty,$$

since the image region of $|z| < 1$ under $w = f(z)$ is contained in the circle $|w| \leq M$. Because $U(r, \theta)$ is harmonic, we may write

$$\frac{\partial U}{\partial r} r d\theta = - \int_E d \arg (re^{i\theta} - \zeta) d\mu(\zeta),$$

so that

$$S = - \int_{\zeta \in E} d\mu(\zeta) \iint_{|z| < 1} \sqrt[4]{EG - F^2} d \arg (re^{i\theta} - \zeta) dr.$$

If we set

$$\sigma(\zeta) = - \iint_{|z| < 1} \sqrt[4]{EG - F^2} d \arg (re^{i\theta} - \zeta) dr,$$

we have

$$(11) \quad S = \int_{\zeta \in E} \sigma(\zeta) d\mu(\zeta).$$

We shall show that $\sigma(\zeta) = \infty$ for each point $\zeta \in E$; from the fact that S in (11) is finite it will follow that E cannot have positive capacity. We may assume without loss of generality that $\zeta = 1$; we have that

$$(12) \quad t = -\arg (re^{i\theta} - 1), \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi)$$

so that

$$\frac{dt}{d\theta} = \frac{r \cos \theta - r^2}{1 + r^2 - 2r \cos \theta}.$$

We fix $r = \cos \theta_0$, so that the point $z = re^{i\theta}$ lies outside the circle $r = \cos \theta$ whenever $\theta_0 \leq |\theta| \leq \pi$, and we observe that $dt \geq 0$ for $|\theta| \leq \theta_0$ and $dt \leq 0$ for $\theta_0 \leq |\theta| \leq \pi$. Now, for $\theta_0 \leq |\theta| \leq \pi$ we have that

$$0 \leq \frac{r - \cos \theta}{1 + r^2 - 2r \cos \theta} \leq \frac{1}{1 + r} \leq 1,$$

so that $|dt| \leq r d\theta$. From these estimates, it follows that

$$\begin{aligned} \int_0^1 dr \int_{\theta_0 \leq |\theta| \leq \pi} \sqrt[4]{EG - F^2} |dt| &\leq \iint_{|z| < 1} \sqrt[4]{EG - F^2} r dr d\theta \\ &\leq \left(\iint_{|z| < 1} \sqrt[4]{EG - F^2} r dr d\theta \right)^{\frac{1}{4}} \left(\iint_{|z| < 1} r dr d\theta \right)^{\frac{3}{4}} \leq \sqrt{\pi M^2} \sqrt{\pi} = \pi M. \end{aligned}$$

Now, for $\zeta = 1$, we have

$$\sigma(1) = \iint_{|z| < 1} \sqrt[4]{EG - F^2} \, dt \, dr,$$

so that

$$\begin{aligned} (13) \quad \sigma(1) &\geq \int_0^1 dr \int_{|\theta| \leq \theta_0} \sqrt[4]{EG - F^2} \, dt - \pi M \\ &= \iint_{|z - \frac{1}{2}| \leq \frac{1}{2}} \sqrt[4]{EG - F^2} \, dr \, dt - \pi M. \end{aligned}$$

If, in this last integral, we set

$$\rho e^{it} = 1 - re^{-i\theta}, \quad dr \, dt = \frac{\cos t - \rho}{\sqrt{1 + \rho^2 - 2\rho \cos t}} \, d\rho \, dt,$$

and observe that $1 + \rho^2 - 2\rho \cos t \leq 1$ in $|z - \frac{1}{2}| \leq \frac{1}{2}$, we have that

$$\begin{aligned} \iint_{|z - \frac{1}{2}| \leq \frac{1}{2}} \sqrt[4]{EG - F^2} \, dr \, dt &= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dt \int_0^{\cos t} \frac{\sqrt[4]{EG - F^2} (\cos t - \rho)}{\sqrt{1 + \rho^2 - 2\rho \cos t}} \, d\rho \\ &\geq \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dt \int_0^{\cos t} \sqrt[4]{EG - F^2} (\cos t - \rho) \, d\rho. \end{aligned}$$

If $0 \leq \rho \leq \frac{1}{2} \cos t$, then $\cos t - \rho \geq \frac{1}{2} \cos t$, so that the last integral is greater than

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos t}{2} \, dt \int_0^{\cos t} \sqrt[4]{EG - F^2} \, d\rho = \frac{1}{2} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \lambda(0, t) \cos t \, dt = \frac{1}{2} \Lambda(0).$$

But $\Lambda(\theta)$ was assumed to be infinite for $\theta = 0$; this contradiction shows that E has capacity zero, so that the first part of the theorem is proved.

To prove the second part of the theorem, we assume that $\lim_{r \rightarrow 1} f(re^{i\theta}) = 0$ for a closed set E of positive capacity on $|z| = 1$. If we denote by $\xi = \varphi(w)$, $\varphi(f(0)) = 0$, a function which maps the simply-connected region G of the w -plane *conformally* onto $|\xi| < 1$, the function $\xi = F(z) = \varphi(f(z))$ maps $|z| < 1$ *quasi-conformally* onto $|\xi| < 1$ and can be extended so as to give a topological mapping of $|z| \leq 1$ onto $|\xi| \leq 1$. Under the mapping $w = f(z)$ each radius of $|z| < 1$ drawn to a point of E goes into a Jordan arc of G terminating at the point $w = 0$. Under the conformal mapping $\xi = \varphi(w)$ each of these arcs of G terminating at $w = 0$ corresponds to a curve lying in $|\xi| < 1$ and terminating at some point of

$|\xi| = 1$. If we denote the set of these terminal points on $|\xi| = 1$ by E' , it is easy to see that E' is the image of the set E under the topological extension of the mapping $\xi = F(z)$. By the theorem of Beurling cited in the introduction, the set E' must have capacity zero, since, by Lindelöf's theorem, the inverse function of $\xi = \varphi(w)$ must have radial limit 0 at each point of E' . However, it is a known result [5] that, under a quasi-conformal mapping of $|z| < 1$ onto $|\xi| < 1$, a closed set of capacity zero on $|z| = 1$ corresponds to a closed set of capacity zero on $|\xi| = 1$, and vice versa. Hence, E must have capacity zero, and the proof of Theorem 2 is complete.

Theorem 2 can be extended, by considering the spherical metric, to

Theorem 3. *Let $f(z)$ be quasi-conformal in $|z| < 1$ and map $|z| < 1$ onto a region (on the Riemann sphere) having bounded spherical area. Then for every point $e^{i\theta}$ on $|z| = 1$, with the possible exception of a set of logarithmic capacity zero, $\lim_{z \rightarrow e^{i\theta}} f(z)$ exists uniformly inside every angular region of $|z| < 1$ with vertex at $e^{i\theta}$.*

Mathematical Institute, Helsinki University
and
University of Michigan, Ann Arbor, Michigan

BIBLIOGRAPHY

1. S. AGMON, *A property of quasi-conformal mappings*, Journal of Rational Mechanics and Analysis, vol. 3 (1954), 763–765.
2. L. BERS, *Non-linear elliptic equations without non-linear entire solutions*, Journal of Rational Mechanics and Analysis, vol. 3 (1954) 767–787.
3. A. BEURLING, *Ensembles exceptionnels*, Acta Math., vol. 72 (1940), 1–13.
4. K. NOSHIRO, *A theorem on the cluster sets of pseudo-analytic functions*, Nagoya Math. Journ. vol. 1 (1950), 83–89.
5. A. PFLUGER, *Extremallängen und Kapazität*, Comm. Math. Helv. vol. 29 (1955), 120–131.
6. E. SAKAI, *Note on pseudo-analytic functions*, Proc. Japan Acad., vol. 25 (1950), 12–17.
7. M. TSUJI, *Beurling's theorem on exceptional sets*, Tohoku Math. Journ. (Series 2) vol. 2 (1950), 113–125.