On an Initial Value Problem for Flow of a Viscous Incompressible Fluid in an unbounded Region

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1. Introduction. The motion of an incompressible, viscous fluid is governed by the Navier-Stokes equation and the continuity equation. As is well known, the analysis of such fluid motion is greatly complicated by the fact that the Navier-Stokes equation is non-linear. It has been recognized that when a solution to the flow equations is sought in terms of a Fourier representation for the velocity field, the various Fourier components will interact with each other as a consequence of the non-linearity of the equation of motion. It is the purpose of this article to study this interaction and to apply the results of the investigation to an initial value problem.

For simplicity, we consider only an unbounded flow. We shall be concerned with a special case of the following problem: To investigate the subsequent development of the motion of an unbounded viscous incompressible fluid, when initially its motion is described by two spatially periodic waves. The solution of this problem will be carried out by successive approximation, and the existence of the solution will be proved. We shall first express the velocity and pressure fields in a Fourier representation and then transform the Navier-Stokes and continuity equations into equations for the Fourier components.

2. Transformation of the Navier-Stokes and Continuity Equations. To express the velocity and pressure field in a convenient trigonometrical representation, we introduce the following function:

$$K(\mathbf{r} \cdot \mathbf{x}) = \cos(\mathbf{r} \cdot \mathbf{x}) + \sin(\mathbf{r} \cdot \mathbf{x}),^* \quad (2.1)$$

where $\mathbf{r}$ is a wave number vector and $\mathbf{x}$ is a position vector in the physical space. By permitting $\mathbf{r}$ to be negative in (2.1), we see that a trigonometrical expansion

* $\mathbf{x}$ denotes a vector and x its magnitude.

in $\cos(\mathbf{r} \cdot \mathbf{x})$ and $\sin(\mathbf{r} \cdot \mathbf{x})$ can be expressed as an expansion in $K(\mathbf{r} \cdot \mathbf{x})$ and $K(-\mathbf{r} \cdot \mathbf{x})$, *i.e.*, as an expansion in $K(\mathbf{r} \cdot \mathbf{x})$ with the summation on $\mathbf{r}$ covering the whole $\mathbf{r}$ -space. A series $\sum_{\mathbf{r}} A_1(\mathbf{r})K(\mathbf{r} \cdot \mathbf{x})$ has the same generality as a series $\sum_{\mathbf{r}} A_2(\mathbf{r}) \exp(\mathbf{r} \cdot \mathbf{x})$ and is subject to the same restrictions; however, if both series represent real functions, then the coefficients $A_1(\mathbf{r})$ are real, whereas the coefficients $A_2(\mathbf{r})$ are in general complex. Thus we write

$$\mathbf{v}(\mathbf{x}, t) = \sum_{\mathbf{r}} \mathbf{A}(\mathbf{r}, t)K(\mathbf{r} \cdot \mathbf{x}), \quad (2.2)$$

$$p(\mathbf{x}, t) = \sum_{\mathbf{r}} \Phi(\mathbf{r}, t)K(\mathbf{r} \cdot \mathbf{x}), \quad (2.3)$$

where $\mathbf{v}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ denote the velocity and pressure of the fluid, and $\mathbf{A}(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, t)$ are the amplitudes of decomposition.

A decomposition of this nature permits the velocity and pressure fields to be spatially almost periodic, *i.e.*, the wave number vectors $\mathbf{r}$ that occur in the various wave components are not restricted to the case where their end-points form the vertices of a rectangular network as in the case when the fields are periodic. With this understanding, we can visualize that to every (almost periodic) velocity field there corresponds a denumerable infinity of vectors in the wave number space. Such a geometrical visualization is impossible if we use the exponential function $\exp(\mathbf{r} \cdot \mathbf{x})$ for the decomposition, since then the amplitude $\mathbf{A}(\mathbf{r}, t)$ would consist of complex components.

We note that in a flow of this kind the velocity does not decrease to zero at infinity and the kinetic energy of the flow is infinite. However, the average kinetic energy per unit volume is finite.*

We return to the subject of transforming the Navier-Stokes and continuity equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}, \quad (2.4)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.5)$$

where ∇ is the gradient operator and ρ and ν are the density and kinematic viscosity of the fluid, both assumed to be constant. By substituting (2.2) and (2.3) into (2.4), we have

$$\frac{\partial \mathbf{v}}{\partial t} = \sum_{\mathbf{r}} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} K(\mathbf{r} \cdot \mathbf{x}), \quad (2.6)$$

*This follows from BOHR's fundamental theorem on almost periodic functions. Actually, the average kinetic energy per unit volume is equal to $\frac{1}{2} \sum_{\mathbf{r}} \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r}, t)$.

$$\begin{aligned}
\mathbf{v} \cdot \nabla \mathbf{v} &= \sum_{\mathbf{r}} \mathbf{A}(\mathbf{r}, t) K(\mathbf{r} \cdot \mathbf{x}) \cdot \left\{ - \sum_{\mathbf{s}} \mathbf{s} \mathbf{A}(-\mathbf{s}, t) K(\mathbf{s} \cdot \mathbf{x}) \right\} \\
&= -\frac{1}{2} \sum_{\mathbf{s}} \{ \mathbf{A}(\mathbf{r} + \mathbf{s}, t) + \mathbf{A}(\mathbf{r} - \mathbf{s}, t) + \mathbf{A}(-\mathbf{r} + \mathbf{s}, t) - \mathbf{A}(-\mathbf{r} - \mathbf{s}, t) \} \\
&\quad \cdot \mathbf{s} \mathbf{A}(-\mathbf{s}, t) K(\mathbf{r} \cdot \mathbf{x}), \quad (2.7)
\end{aligned}$$

where we have used the following relation:

$$K(a)K(b) = \frac{1}{2} \{ K(a+b) + K(a-b) + K(-a+b) - K(-a-b) \}. \quad (2.8)$$

Also we have

$$-\frac{1}{\rho} \nabla p = \frac{\mathbf{r}}{\rho} \Phi(-\mathbf{r}, t) K(\mathbf{r} \cdot \mathbf{x}), \quad (2.9)$$

$$\nu \nabla^2 \mathbf{v} = -\nu \sum_{\mathbf{r}} r^2 \mathbf{A}(\mathbf{r}, t) K(\mathbf{r} \cdot \mathbf{x}), \quad (2.10)$$

$$\nabla \cdot \mathbf{v} = \sum_{\mathbf{r}} \mathbf{r} \cdot \mathbf{A}(\mathbf{r}, t) K(-\mathbf{r} \cdot \mathbf{x}). \quad (2.11)$$

From these relations we obtain the transformed equations corresponding to (2.4) and (2.5):

$$\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = \frac{1}{2} \mathbf{Q}(\mathbf{r}, t) + \frac{\mathbf{r}}{\rho} \Phi(-\mathbf{r}, t) - \nu r^2 \mathbf{A}(\mathbf{r}, t), \quad (2.12)$$

$$\mathbf{r} \cdot \mathbf{A}(\mathbf{r}, t) = 0, \quad (2.13)$$

where

$$\begin{aligned}
\mathbf{Q}(\mathbf{r}, t) &= \sum_{\mathbf{s}} \{ \mathbf{A}(\mathbf{r} + \mathbf{s}, t) + \mathbf{A}(\mathbf{r} - \mathbf{s}, t) + \mathbf{A}(-\mathbf{r} + \mathbf{s}, t) - \mathbf{A}(-\mathbf{r} - \mathbf{s}, t) \} \\
&\quad \cdot \mathbf{s} \mathbf{A}(-\mathbf{s}, t). \quad (2.14)
\end{aligned}$$

Multiplying (2.12) scalarly by $\mathbf{r}$ and using (2.13) we get

$$\Phi(-\mathbf{r}, t) = -\frac{\rho}{2r^2} \mathbf{Q}(\mathbf{r}, t) \cdot \mathbf{r}. \quad (2.15)$$

Thus (2.12) takes the alternative form:

$$\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = \mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\} - \nu r^2 \mathbf{A}(\mathbf{r}, t), \quad (2.16)$$

where

$$\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\} = \frac{1}{2} \left\{ \mathbf{Q}(\mathbf{r}, t) - \frac{\mathbf{r}}{r^2} \mathbf{Q}(\mathbf{r}, t) \cdot \mathbf{r} \right\}. \quad (2.17)$$

3. The Interaction of the Velocity Wave Components. We see that equation (2.13) can be interpreted very simply: The vector $\mathbf{A}(\mathbf{r}, t)$ is orthogonal to the corresponding wave number vector $\mathbf{r}$. An examination of (2.16) shows that the rate of change of the vector $\mathbf{A}(\mathbf{r}, t)$ has two parts: a non-linear part, $\frac{1}{2}\{\mathbf{Q}(\mathbf{r}, t) - \mathbf{r}\mathbf{Q}(\mathbf{r}, t) \cdot \mathbf{r}/r^2\} = \mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ due to the inertia and pressure terms, and a linear part, $-\nu r^2 \mathbf{A}(\mathbf{r}, t)$ due to viscosity. It is the non-linear part that is responsible for the interaction and we shall discuss it in some detail.

First, consider the simple case when there are only two waves corresponding to the wave number vectors $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\begin{aligned} \mathbf{A}(\mathbf{r}_1, t) &= \mathbf{A}_1, & \mathbf{A}_1 \cdot \mathbf{r}_1 &= 0 \\ \mathbf{A}(\mathbf{r}_2, t) &= \mathbf{A}_2, & \mathbf{A}_2 \cdot \mathbf{r}_2 &= 0 \\ \mathbf{A}(\mathbf{r}, t) &= 0, & \text{for } \mathbf{r} \neq \mathbf{r}_1, \mathbf{r}_2. \end{aligned} \quad (3.1)$$

By putting $\mathbf{s}$ first equal to $-\mathbf{r}_1$ and then equal to $-\mathbf{r}_2$ in (2.14), we have

$$\begin{aligned} \mathbf{Q}(\mathbf{r}, t) &= \{\mathbf{A}(\mathbf{r} - \mathbf{r}_1, t) + \mathbf{A}(\mathbf{r} + \mathbf{r}_1, t) + \mathbf{A}(-\mathbf{r} - \mathbf{r}_1, t) - \mathbf{A}(-\mathbf{r} + \mathbf{r}_1, t)\} \\ &\quad \cdot (-\mathbf{r}_1) \mathbf{A}_1 \\ &+ \{\mathbf{A}(\mathbf{r} - \mathbf{r}_2, t) + \mathbf{A}(\mathbf{r} + \mathbf{r}_2, t) + \mathbf{A}(-\mathbf{r} - \mathbf{r}_2, t) - \mathbf{A}(-\mathbf{r} + \mathbf{r}_2, t)\} \\ &\quad \cdot (-\mathbf{r}_2) \mathbf{A}_2. \end{aligned} \quad (3.2)$$

An examination of (3.2) immediately suggests that $\mathbf{Q}(\mathbf{r}, t)$ is not zero (in general) when $\mathbf{r} = \pm \mathbf{r}_1 \pm \mathbf{r}_2^*$ and is zero whenever $\mathbf{r} \neq \pm \mathbf{r}_1 \pm \mathbf{r}_2$. We shall evaluate the rate of change of $\mathbf{A}(\mathbf{r}, t)$ due to the non-linear part $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ when $\mathbf{r} = \pm \mathbf{r}_1 \pm \mathbf{r}_2$. Let us choose the coordinate axes (Fig. 1) such that $\mathbf{r}_1$ and $\mathbf{r}_2$ lie in the $R_1 - R_2$ plane. The vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are orthogonal to $\mathbf{r}_1$ and $\mathbf{r}_2$

FIG. 1

*We write $\pm \mathbf{r}_1 \pm \mathbf{r}_2$ to denote any of the following: $\mathbf{r}_1 + \mathbf{r}_2$, $\mathbf{r}_1 - \mathbf{r}_2$, $-\mathbf{r}_1 + \mathbf{r}_2$, $-\mathbf{r}_1 - \mathbf{r}_2$.

respectively; so we specify their directions by the angles α_1 and α_2 , measured in the counter-clockwise direction from a vector parallel to the R_3 -axis toward the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ when facing in the direction of the wave number vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Furthermore, let $\mathbf{r}_1$ and $\mathbf{r}_2$ make angles φ_1 and φ_2 with the R_1 -axis respectively, measured as shown in the figure. Then $0 \leq \alpha_1, \alpha_2, \varphi_1, \varphi_2 < 2\pi$. Using this choice of axes and the given data (3.1), we can calculate $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ when $\mathbf{r} = \pm\mathbf{r}_1 \pm \mathbf{r}_2$.

$$\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\} = \frac{1}{2}A_1A_2 \sin(\varphi_2 - \varphi_1) \left\{ - \frac{(r_1^2 - r_2^2)(r_1 \sin \varphi_1 + r_2 \sin \varphi_2) \sin \alpha_1 \sin \alpha_2}{r_1^2 + r_2^2 + 2r_1r_2 \cos(\varphi_2 - \varphi_1)}, \right. \\ \left. \frac{(r_1^2 - r_2^2)(r_1 \cos \varphi_1 + r_2 \cos \varphi_2) \sin \alpha_1 \sin \alpha_2}{r_1^2 + r_2^2 + 2r_1r_2 \cos(\varphi_2 - \varphi_1)}, \right. \\ \left. r_1 \cos \alpha_1 \sin \alpha_2 - r_2 \sin \alpha_1 \cos \alpha_2 \right\}; \quad (3.3)$$

$$|\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}| = \frac{1}{2}A_1A_2 |\sin(\varphi_2 - \varphi_1)| \left\{ \frac{(r_1^2 - r_2^2)^2 \sin^2 \alpha_1 \sin^2 \alpha_2}{r_1^2 + r_2^2 + 2r_1r_2 \cos(\varphi_2 - \varphi_1)} \right. \\ \left. + (r_1 \cos \alpha_1 \sin \alpha_2 - r_2 \sin \alpha_1 \cos \alpha_2)^2 \right\}^{\frac{1}{2}}; \quad (3.4)$$

for $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$.

$$\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\} = \frac{1}{2}A_1A_2 \sin(\varphi_2 - \varphi_1) \left\{ \frac{(r_1^2 - r_2^2)(r_1 \sin \varphi_1 - r_2 \sin \varphi_2) \sin \alpha_1 \sin \alpha_2}{r_1^2 + r_2^2 - 2r_1r_2 \cos(\varphi_2 - \varphi_1)}, \right. \\ \left. - \frac{(r_1^2 - r_2^2)(r_1 \cos \varphi_1 - r_2 \cos \varphi_2) \sin \alpha_1 \sin \alpha_2}{r_1^2 + r_2^2 - 2r_1r_2 \cos(\varphi_2 - \varphi_1)}, \right. \\ \left. - r_1 \cos \alpha_1 \sin \alpha_2 - r_2 \sin \alpha_1 \cos \alpha_2 \right\}; \quad (3.5)$$

$$|\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}| = \frac{1}{2}A_1A_2 |\sin(\varphi_2 - \varphi_1)| \left\{ \frac{(r_1^2 - r_2^2)^2 \sin^2 \alpha_1 \sin^2 \alpha_2}{r_1^2 + r_2^2 - 2r_1r_2 \cos(\varphi_2 - \varphi_1)} \right. \\ \left. + (r_1 \cos \alpha_1 \sin \alpha_2 + r_2 \sin \alpha_1 \cos \alpha_2)^2 \right\}^{\frac{1}{2}}; \quad (3.6)$$

for $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

Also

$$\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}_{\mathbf{r}=\mathbf{r}_1+\mathbf{r}_2} = \mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}_{\mathbf{r}=-\mathbf{r}_1-\mathbf{r}_2}, \quad (3.7)$$

$$\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}_{\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2} = -\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}_{\mathbf{r}=-\mathbf{r}_1+\mathbf{r}_2}. \quad (3.8)$$

Certain special configurations of the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ will result in $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\} = 0$, in which case we say that $\mathbf{A}_1$ and $\mathbf{A}_2$ do not interact. From (3.3)–(3.8) we deduce the following:

- (1) If $\mathbf{r}_1$ and $\mathbf{r}_2$ are collinear, then the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ do not interact.
- (2) If the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are both normal to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$, then $\mathbf{A}_1$ and $\mathbf{A}_2$ do not interact.
- (3) If either of the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ is normal to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$, then $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ will also be normal to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$.
- (4) If both the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ lie in the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$, then $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ will also lie in the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$. If, in addition, $r_1 = r_2$, then $\mathbf{A}_1$ and $\mathbf{A}_2$ do not interact.
- (5) If $r_1 = r_2$, then $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ is normal to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$.

By following a similar line of reasoning, the above results can be generalized to the case when the velocity field consists of more than two waves. This results in the following: *At any time, t , let the velocity field be composed of the wave components $\mathbf{A}_1, \mathbf{A}_2, \dots$ corresponding to the wave number vectors $\mathbf{r}_1, \mathbf{r}_2, \dots$. Then the vectors $\mathbf{A}_1, \mathbf{A}_2, \dots$ interact in pairs, so that $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ is not zero (in general) when $\mathbf{r} = \pm\mathbf{r}_i \pm \mathbf{r}_j$, $i \neq j$, and is zero whenever $\mathbf{r} \neq \pm\mathbf{r}_i \pm \mathbf{r}_j$, $i, j = 1, 2, \dots$. Each pair of vectors $\mathbf{A}_i$ and $\mathbf{A}_j$ interacts in this manner independently of every other pair, and formulæ (3.3)–(3.8) may be used to calculate the values of $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ for each pair as if the remaining vectors were not present.*

We have used the notation $\mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}$ to denote the part contributed to $\partial\mathbf{A}(\mathbf{r}, t)/\partial t$ due to the interaction of vectors by taking each pair of $\mathbf{A}(\mathbf{r}, t)$. Suppose $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_1(\mathbf{r}, t) + \mathbf{A}_2(\mathbf{r}, t)$, then we shall denote the contribution to $\partial\mathbf{A}(\mathbf{r}, t)/\partial t$ by the interaction of each vector of $\mathbf{A}_1(\mathbf{r}, t)$ with each vector of $\mathbf{A}_2(\mathbf{r}, t)$ by $\mathfrak{F}\{\mathbf{A}_1(\mathbf{r}, t) * \mathbf{A}_2(\mathbf{r}, t)\}$. Here we have

$$\begin{aligned} & \mathfrak{F}\{\mathbf{A}_1(\mathbf{r}, t) * \mathbf{A}_2(\mathbf{r}, t)\} \\ &= \frac{1}{2} \left\{ \mathbf{Q}_{12}(\mathbf{r}, t) - \frac{\mathbf{r}}{r^2} \mathbf{Q}_{12}(\mathbf{r}, t) \cdot \mathbf{r} + \mathbf{Q}_{21}(\mathbf{r}, t) - \frac{\mathbf{r}}{r^2} \mathbf{Q}_{21}(\mathbf{r}, t) \cdot \mathbf{r} \right\}, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \mathbf{Q}_{ij} = \sum_{\mathbf{s}} \{ & \mathbf{A}_i(\mathbf{r} + \mathbf{s}, t) + \mathbf{A}_i(\mathbf{r} - \mathbf{s}, t) + \mathbf{A}_i(-\mathbf{r} + \mathbf{s}, t) - \mathbf{A}_i(-\mathbf{r} - \mathbf{s}, t) \\ & \cdot \mathbf{s} \mathbf{A}_j(-\mathbf{s}, t) \}. \end{aligned} \quad (3.10)$$

It follows that if $\mathbf{A}_1(\mathbf{r}, t) = \mathbf{A}_2(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t)$, (3.9) and (3.10) reduce to (2.17) and (2.14) respectively. Obviously, $\mathfrak{F}\{\mathbf{A}_1(\mathbf{r}, t) * \mathbf{A}_2(\mathbf{r}, t)\}$ is symmetrical with respect to the indices 1 and 2. We note further that

$$\begin{aligned} & \mathfrak{F}\{\mathbf{A}_1(\mathbf{r}, t) + \mathbf{A}_2(\mathbf{r}, t)\} \\ &= \mathfrak{F}\{\mathbf{A}_1(\mathbf{r}, t)\} + \mathfrak{F}\{\mathbf{A}_2(\mathbf{r}, t)\} + \mathfrak{F}\{\mathbf{A}_1(\mathbf{r}, t) * \mathbf{A}_2(\mathbf{r}, t)\}, \end{aligned} \quad (3.11)$$

$$\mathfrak{F}\{c\mathbf{A}(\mathbf{r}, t)\} = c^2 \mathfrak{F}\{\mathbf{A}(\mathbf{r}, t)\}, \quad (3.12)$$

c being a constant.

4. The General Problem. Let $t = 0$ be the initial instant, and let $\mathbf{A}(\mathbf{r}, t)$ be given at $t = 0$, thus $\mathbf{A}(\mathbf{r}, 0) = \mathbf{A}_0(\mathbf{r})$. It is proposed to determine for the time interval $0 \leq t \leq T$ the unique solution $\mathbf{A}(\mathbf{r}, t)$ which satisfies the transformed Navier-Stokes equation (2.16) and continuity equation (2.13), and takes the initial value $\mathbf{A}_0(\mathbf{r})$ at $t = 0$.

I. Series expansion in time t .

Assuming that $\mathbf{A}(\mathbf{r}, t)$ can be developed into a Taylor series in t for $0 \leq t \leq T$, we write

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}) + \mathbf{A}_1(\mathbf{r}) \frac{t}{1!} + \mathbf{A}_2(\mathbf{r}) \frac{t^2}{2!} + \dots \quad (4.1)$$

Substituting (4.1) in (2.16), making use of (3.11) and (3.12) and equating the coefficients of $t, t^2, \dots$, successively, we obtain

$$\mathbf{A}_1(\mathbf{r}) = \mathfrak{F}\{\mathbf{A}_0(\mathbf{r})\} - \nu r^2 \mathbf{A}_0(\mathbf{r}), \quad (4.2)$$

$$\mathbf{A}_2(\mathbf{r}) = \mathfrak{F}\{\mathbf{A}_0(\mathbf{r}) * \mathbf{A}_1(\mathbf{r})\} - \nu r^2 \mathbf{A}_1(\mathbf{r}), \quad (4.3)$$

$\dots$

$$\mathbf{A}_{2k}(\mathbf{r}) = \sum_{j=0}^{k-1} \binom{2k-1}{j} \mathfrak{F}[\mathbf{A}_j(\mathbf{r}) * \mathbf{A}_{2k-j-1}(\mathbf{r})] - \nu r^2 \mathbf{A}_{2k-1}(\mathbf{r}), \quad (4.4)$$

$$\mathbf{A}_{2k+1}(\mathbf{r}) = \sum_{j=0}^k \binom{2k}{j} \mathfrak{F}[\mathbf{A}_j(\mathbf{r}) * \mathbf{A}_{2k-j}(\mathbf{r})] - \nu r^2 \mathbf{A}_{2k}(\mathbf{r}). \quad (4.5)$$

A physical picture regarding the determination of the successive coefficients $\mathbf{A}_1(\mathbf{r}), \mathbf{A}_2(\mathbf{r}), \dots$, is clear. For, disregarding viscosity, we see from (4.2)–(4.5) that the interaction of $\mathbf{A}_0(\mathbf{r})$ yields $\mathbf{A}_1(\mathbf{r})$; the interaction of $\mathbf{A}_0(\mathbf{r})$ with $\mathbf{A}_1(\mathbf{r})$ yields $\mathbf{A}_2(\mathbf{r})$; the interaction of $\mathbf{A}_0(\mathbf{r})$ with $\mathbf{A}_2(\mathbf{r})$ and that of $\mathbf{A}_1(\mathbf{r})$ yields $\mathbf{A}_3(\mathbf{r}), \dots$; and so on. The various coefficients $\mathbf{A}_1(\mathbf{r}), \mathbf{A}_2(\mathbf{r}), \dots$, can be determined successively by using (3.3)–(3.8).

II. Successive approximations.

Upon integration, the transformed Navier-Stokes equation (2.16) can be written in the following form:

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}) \exp(-\nu r^2 t) + \exp(-\nu r^2 t) \int_0^t \exp(\nu r^2 \tau) \mathfrak{F}\{\mathbf{A}(\mathbf{r}, \tau)\} d\tau. \quad (4.6)$$

We propose to solve the integral equation (4.6) by the following formal iteration procedure:

$$\mathbf{A}_1(\mathbf{r}, t) = \exp(-\nu r^2 t) \mathbf{A}_0(\mathbf{r}), \quad (4.7)$$

$$\mathbf{A}_n(\mathbf{r}, t) = \exp(-\nu r^2 t) \mathbf{A}_0(\mathbf{r}) + \exp(-\nu r^2 t) \int_0^t \exp(\nu r^2 \tau) \mathfrak{F}\{\mathbf{A}_{n-1}(\mathbf{r}, \tau)\} d\tau, \quad (4.8)$$

$n = 2, 3, 4, \dots$

5. A Special Problem. It is our object to prove the following: *Suppose the following initial data are given: $\mathbf{A}(\mathbf{r}_{10}, 0) = \mathbf{a}_{10}$, $\mathbf{A}(\mathbf{r}_{01}, 0) = \mathbf{a}_{01}$ and $\mathbf{A}(\mathbf{r}, 0) = 0$ for $\mathbf{r} \neq \mathbf{r}_{10}, \mathbf{r}_{01}$. Let $\mathbf{a}_{10} \cdot \mathbf{r}_{10} = 0$ and $\mathbf{a}_{01} \cdot \mathbf{r}_{01} = 0$ so as to satisfy the continuity equation, and furthermore let $\mathbf{a}_{01}$ be perpendicular to the plane of $\mathbf{r}_{10}$ and $\mathbf{r}_{01}$. Then the solution of the transformed Navier-Stokes equation (2.16) corresponding to the initial data exists for all time, $t \geq 0$.*

Before going into the details of the proof, we note that initially there are two waves, the components of which are given by

$$u = 0,$$

$$v = \sqrt{2} a_{10y} \sin(r_{10x} + \pi/2),$$

$$w = \sqrt{2} a_{10z} \sin(r_{10x} + \pi/2) + \sqrt{2} a_{01} \sin(r_{10x}x + r_{01y}y + \pi/2),$$

where the subscripts x, y, z indicate the components along the x, y, z axes, and the coordinate axes are so chosen that the x -axis coincides with $\mathbf{r}_{10}$. As will be shown, the solution of the problem will be of the form

$$u = 0,$$

$$v = \sqrt{2} a_{10y} \exp(-\nu r_{10}^2 t) \sin(r_{10x} + \pi/2),$$

$$w = \sqrt{2} a_{10z} \exp(-\nu r_{10}^2 t) \sin(r_{10x} + \pi/2)$$

$$+ \sum_{m=-\infty}^{\infty} \sum_{n=-1}^{+1} A_{mn} \sin(r_{mnx}x + r_{mny}y + \pi/2),$$

where the coefficients A_{mn} are functions of t and are determined by substituting these expressions into the equation of motion. It will be proved that all these coefficients A_{mn} remain finite for all t . Furthermore, by making use of the results in the proof, it can be shown that $\sum_{m,n} |A_{mn}|$ and $\sum_{m,n} |A_{mn} r_{mnx}|$ remain finite for all t . Thus not only the velocity components u, v, w but also the vorticity components $\omega_x, \omega_y, \omega_z$ will remain finite. It is interesting to observe that the vorticity components satisfy the following equations:

$$\frac{D\omega_x}{Dt} = \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \omega_x,$$

$$\frac{D\omega_y}{Dt} = \omega_x \omega_z + \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \omega_y,$$

$$\frac{D\omega_z}{Dt} = \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \omega_z,$$

D/Dt being the material derivative, so that if $\nu = 0$, the components ω_x, ω_z for a fluid particle will remain constant while ω_y will grow indefinitely with time.

We note also that in this problem the pressure gradient is always zero. To prove this, we consider two velocity wave components $\mathbf{A}_1$, $\mathbf{A}_2$ at t , and from (2.15) derive formulæ for the pressure wave components in the notation of Section 3:

$$\Phi(\mathbf{r}_1 + \mathbf{r}_2, t) = -\Phi(-\mathbf{r}_1 - \mathbf{r}_2, t) = \frac{\rho A_1 A_2 r_1 r_2 \sin \alpha_1 \sin \alpha_2 \sin^2 (\varphi_1 - \varphi_2)}{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\varphi_1 - \varphi_2)},$$

$$\Phi(\mathbf{r}_1 - \mathbf{r}_2, t) = \Phi(-\mathbf{r}_1 + \mathbf{r}_2, t) = -\frac{\rho A_1 A_2 r_1 r_2 \sin \alpha_1 \sin \alpha_2 \sin^2 (\varphi_1 - \varphi_2)}{r_1^2 + r_2^2 - 2r_1 r_2 \cos (\varphi_1 - \varphi_2)},$$

while $\Phi(\mathbf{r}, t) = 0$ for $\mathbf{r} \neq \pm \mathbf{r}_1 \pm \mathbf{r}_2$. As will be shown, all the velocity wave components are perpendicular to the plane of $\mathbf{r}_{10}$, $\mathbf{r}_{01}$ except $a_{10} \exp(-\nu r_{10}^2 t)$ for all t , so that $\Phi(\mathbf{r}, t) = 0$ and the pressure gradient is zero.

Having described some physical details of the problem*, we now come to the main existence proof.

Proof. We first introduce some notation that will be followed in this section. Consider the plane of $\mathbf{r}_{10}$ and $\mathbf{r}_{01}$, and all the vectors $i\mathbf{r}_{10} + j\mathbf{r}_{01}$, $i = 0, \pm 1, \pm 2, \dots$; $j = \pm 1$, Fig. 2. We denote the vectors $i\mathbf{r}_{10} + j\mathbf{r}_{01}$ by $\mathbf{r}_{ij}$

FIG. 2

and their end-points by P_{ij} . We also denote the angle measured from $\mathbf{r}_{10}$ to $\mathbf{r}_{ij}$, in the counter-clockwise direction, by φ_{ij} . Let a point wave from P_{0i} to the right, passing all the points P_{1i} , P_{2i} , $\dots$ $P_{i-1,i}$, successively, to P_{ii} and construct the following functions. (The subscript of the function indicates the number of points passed over, including the last, or the number of repeated integrals.)

$$F\{r_{1i}\} = \exp\{-\nu r_{1i}^2 t\} \int_0^t \exp\{\nu(r_{1i}^2 - r_{0i}^2 - r_{10}^2)\tau_1\} d\tau_1, \quad (5.1)$$

*I am indebted to Professor LIN for calling my attention to the various physical aspects of this problem as discussed above.

$$\begin{aligned}
F\{r_{2i}, r_{1i}\}_2 &= \exp\{-\nu r_{2i}^2 t\} \int_0^t \exp\{\nu(r_{2i}^2 - r_{1i}^2 - r_{10}^2)\tau_2\} d\tau_2 \\
&\quad \cdot \int_0^{\tau_2} \exp\{\nu(r_{1i}^2 - r_{0i}^2 - r_{10}^2)\tau_1\} d\tau_1, \quad (5.2) \\
&\dots,
\end{aligned}$$

$$\begin{aligned}
F\{r_{ij}, r_{i-1,i}, \dots, r_{1i}\}_i &= \exp\{-\nu r_{ij}^2 t\} \int_0^t \exp\{\nu(r_{ij}^2 - r_{i-1,i}^2 - r_{10}^2)\tau_i\} d\tau_i \\
&\quad \cdot \int_0^{\tau_{i-1}} \exp\{\nu(r_{i-1,i}^2 - r_{i-2,i}^2 - r_{10}^2)\tau_{i-1}\} d\tau_{i-1} \int_0^{\tau_{i-2}} \dots \\
&\quad \int_0^{\tau_2} \exp\{\nu(r_{1i}^2 - r_{0i}^2 - r_{10}^2)\tau_1\} d\tau_1. \quad (5.3)
\end{aligned}$$

We now assert that

$$F\{r_{ij}, r_{i-1,i}, \dots, r_{1i}\}_i \leq \left\{ \frac{1 - \exp(-\nu r_{10}^2 t)}{\nu r_{10}^2} \right\}^i \frac{\exp(-\nu r_m^2 t)}{i!}, \quad (5.4)$$

where $r_m = r_{mi} = \min\{r_{ij}, r_{i-1,i}, \dots, r_{1i}, r_{0i}\}$.

First of all, we observe

$$\begin{aligned}
&\int_0^{\tau_{s+1}} \exp\{\nu(r_{sj}^2 - r_{s-1,i}^2 - r_{10}^2)\tau_s\} d\tau_s \\
&\leq \exp\{\nu(r_{sj}^2 - r_{s-1,i}^2)\tau_{s+1}\} \int_0^{\tau_{s+1}} \exp(-\nu r_{10}^2 \tau_s) d\tau_s \quad (5.5)
\end{aligned}$$

if $r_{sj} \geq r_{s-1,i}$, and

$$\int_0^{\tau_{s+1}} \exp\{\nu(r_{sj}^2 - r_{s-1,i}^2 - r_{10}^2)\tau_s\} d\tau_s \leq \int_0^{\tau_{s+1}} \exp(-\nu r_{10}^2 \tau_s) d\tau_s \quad (5.6)$$

if $r_{sj} \leq r_{s-1,i}$. Now since r_{mi} is the minimum of $r_{ij}, r_{i-1,i}, \dots, r_{1i}, r_{0i}$, we have

$$r_{ij} > r_{i-1,i} > \dots > r_{m+1,i} \geq r_{mi} \leq r_{m-1,i} < \dots < r_{1i} < r_{0i},$$

and

$$\begin{aligned}
&F\{r_{ij}, r_{i-1,i}, \dots, r_{mi}, \dots, r_{1i}\}_i \\
&= \exp\{-\nu r_{ij}^2 t\} \int_0^t \exp\{\nu(r_{ij}^2 - r_{i-1,i}^2 - r_{10}^2)\tau_i\} d\tau_i \\
&\quad \cdot \int_0^{\tau_i} \exp\{\nu(r_{i-1,i}^2 - r_{i-2,i}^2 - r_{10}^2)\tau_{i-1}\} d\tau_{i-1} \int_0^{\tau_{i-1}} \dots \\
&\quad \int_0^{\tau_{m+1}} \exp\{\nu(r_{m+1,i}^2 - r_{mi}^2 - r_{10}^2)\tau_{m+1}\} d\tau_{m+1}
\end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^{\tau_{m+1}} \exp \{ \nu(r_{mi}^2 - r_{m-1,i}^2 - r_{10}^2) \tau_m \} d\tau_m \int_0^{\tau_m} \cdots \\
& \int_0^{\tau_s} \exp \{ \nu(r_{2i}^2 - r_{1i}^2 - r_{10}^2) \tau_2 \} d\tau_2 \int_0^{\tau_s} \exp \{ (\nu r_{1i}^2 - r_{0i}^2 - r_{10}^2) \tau_1 \} d\tau_1 \\
& < \exp \{ -\nu r_{ii}^2 t \} \int_0^t \exp \{ \nu(r_{ii}^2 - r_{i-1,i}^2 - r_{10}^2) \tau_i \} d\tau_i \\
& \quad \cdot \int_0^{\tau_i} \exp \{ \nu(r_{i-1,i}^2 - r_{i-2,i}^2 - r_{10}^2) \tau_{i-1} \} d\tau_{i-1} \int_0^{\tau_{i-1}} \cdots \\
& \int_0^{\tau_{m+2}} \exp \{ \nu(r_{m+1,i}^2 - r_{mi}^2 - r_{10}^2) \tau_{m+1} \} d\tau_{m+1} \int_0^{\tau_{m+1}} \exp \{ -\nu r_{10}^2 \tau_m \} d\tau_m \int_0^{\tau_m} \cdots \\
& \int_0^{\tau_s} \exp \{ -\nu r_{10}^2 \tau_2 \} d\tau_2 \int_0^{\tau_s} \exp \{ -\nu r_{10}^2 \tau_1 \} d\tau_1, \\
& < \exp \{ -\nu r_{ii}^2 t \} \int_0^t \exp \{ -\nu r_{10}^2 \tau_i \} d\tau_i \int_0^{\tau_i} \exp \{ -\nu r_{10}^2 \tau_{i-1} \} d\tau_{i-1} \int_0^{\tau_{i-1}} \cdots \\
& \int_0^{\tau_{m+2}} \exp \{ -\nu r_{10}^2 \tau_{m+1} \} d\tau_{m+1} \int_0^{\tau_{m+1}} \exp \{ -\nu r_{10}^2 \tau_m \} d\tau_m \int_0^{\tau_m} \cdots \\
& \int_0^{\tau_s} \exp \{ -\nu r_{10}^2 \tau_2 \} d\tau_2 \int_0^{\tau_s} \exp \{ -\nu r_{10}^2 \tau_1 \} d\tau_1 \\
& = \left\{ \frac{1 - \exp(-\nu r_{10}^2 t)}{\nu r_{10}^2} \right\}^i \frac{\exp(-\nu r_m^2 t)}{i!},
\end{aligned}$$

where use has been made of (5.6) and (5.5).

This proves (5.4). Next consider a more general case: let the point start from P_{0i} and move along the line $P_{1i}P_{2i}$ to P_{ii} in any manner (*i.e.*, either to the left or the right), and construct similar integrals as defined before. Let there be k points passed over, including the last (the same point may be passed over several times), or k repeated integrals in this case. Then, by following a similar line of reasoning, we can prove that

$$F\{r_{ii}, \cdots, r_{m-1,i}\}_k \leq \left\{ \frac{1 - \exp(-\nu r_{10}^2 t)}{\nu r_{10}^2} \right\}^k \frac{\exp(-\nu r_m^2 t)}{k!}, \quad (5.7)$$

where r_m is the minimum of all r 's that appear in the repeated integrals.

To prove the existence of solutions, we first examine the inviscid case, for which $\nu = 0$. Let us denote the successive solutions obtained by following the iterative procedure outlined in the previous section by $\mathbf{a}_1(\mathbf{r}, t)$, $\mathbf{a}_2(\mathbf{r}, t)$, $\cdots$. Then from (4.7), (4.8), by putting $\nu = 0$, we have

$$\mathbf{a}_1(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}), \quad (5.8)$$

$$\mathbf{a}_n(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}) + \int_0^t \mathcal{F}\{\mathbf{a}_{n-1}(\mathbf{r}, \tau)\} d\tau, \quad (5.9)$$

$$n = 2, 3, \dots$$

Since $\mathbf{a}_{01}$ is normal to the plane of $\mathbf{r}_{10}, \mathbf{r}_{01}$, the vectors $\mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\}$ are normal to this plane (section 3). Thus $\mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\}$ and $\mathbf{a}_{01}$ do not interact and the vectors $\mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\}$ do not interact among themselves (Section 3). Furthermore, $\mathcal{F}\{\mathbf{a}_{10} * \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\}_2, \mathcal{F}\{\mathbf{a}_{10} * \mathcal{F}[\mathbf{a}_{10} * \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01})]\}_3^*, \dots$, are all normal to the plane of $\mathbf{r}_{10}, \mathbf{r}_{01}$. From these considerations we have, using (3.11),

$$\begin{aligned} \mathbf{a}_2(\mathbf{r}, t) &= \mathbf{A}_0(\mathbf{r}) + \int_0^t \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) d\tau \\ &= \mathbf{A}_0(\mathbf{r}) + t\mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\}, \\ \mathbf{a}_3(\mathbf{r}, t) &= \mathbf{A}_0(\mathbf{r}) + \int_0^t \mathcal{F}\{\mathbf{A}_0(\mathbf{r}) + \tau\mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\} d\tau \\ &= \mathbf{A}_0(\mathbf{r}) + t\mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\} + \frac{t^2}{2!} \mathcal{F}\{\mathbf{a}_{10} * \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\}_2, \end{aligned}$$

and it can be shown by induction that

$$\begin{aligned} \mathbf{a}_n(\mathbf{r}, t) &= \mathbf{A}_0(\mathbf{r}) + \frac{t}{1!} \mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\} + \frac{t^2}{2!} \mathcal{F}\{\mathbf{a}_{10} * \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\}_2 \\ &+ \dots + \frac{t^{n-1}}{(n-1)!} \mathcal{F}\{\mathbf{a}_{10} * \mathcal{F}[\mathbf{a}_{10} * \dots * \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{n-1}. \end{aligned} \quad (5.10)$$

We wish to find a dominant for $\mathcal{F}\{\mathbf{a}_{10} * \dots\}$; by making use of (3.4); (3.6); (3.7), (3.8). If the direction of $\mathbf{a}_{10}$ is denoted by α (same convention as in Section 3), then we obtain

$$\begin{aligned} |\mathcal{F}\{\mathbf{a}_{10} \mathcal{F} \mathbf{a}_{01}\}| &= \frac{1}{2} a_{10} a_{01} r_{01} |\sin \varphi_{01} \sin \alpha| \\ &= \frac{1}{2} a_{10} a_{01} d |\sin \alpha|, \end{aligned}$$

where d is the vertical distance from P_{00} to the line $P_{01}P_{11}$ or $P_{0-1}P_{1-1}$. Moreover,

$$r_{11} \sin \varphi_{11} = r_{21} \sin \varphi_{21} = \dots = d.$$

The results are arranged as follows:

$$\mathbf{r} = \pm \mathbf{r}_{10} \pm \mathbf{r}_{01} : |\mathcal{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\}| = a_{01} \left\{ \frac{1}{2} a_{10} d |\sin \alpha| \right\};$$

*The subscript indicates the number of interactions.

$$\begin{cases} \mathbf{r} = \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\}_2| \leq 2.2a_{01}\{\frac{1}{2}a_{10}d|\sin \alpha|\}^2, \\ \mathbf{r} = \pm 2\mathbf{r}_{10} \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\}_2| \leq 2a_{01}\{\frac{1}{2}a_{10}d|\sin \alpha|\}^2; \\ \mathbf{r} = \pm \mathbf{r}_{10} \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01})]\}_3| \leq 3.2^2a_{01}\{\frac{1}{2}a_{10}d|\sin \alpha|\}^3, \\ \mathbf{r} = \pm 3\mathbf{r}_{10} \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01})]\}_3| \leq 2^2a_{01}\{\frac{1}{2}a_{10}d|\sin \alpha|\}^3. \end{cases}$$

By continuing in this way it can be proved by induction that the magnitude of $\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{2k}$ is dominated by:

$$\begin{aligned} \mathbf{r} = \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{2k}| \\ & \leq \binom{2k}{k} 2^{2k-1} a_{01} \{\frac{1}{2}a_{10}d|\sin \alpha|\}^{2k}, \\ \mathbf{r} = \pm 2\mathbf{r}_{10} \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{2k}| \\ & \leq \binom{2k}{k-1} 2^{2k-1} a_{01} \{\frac{1}{2}a_{10}d|\sin \alpha|\}^{2k}, \\ \dots & \\ \mathbf{r} = \pm 2k\mathbf{r}_{10} \pm \mathbf{r}_{01} : & |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{2k}| \\ & \leq 2^{2k-1} a_{01} \{\frac{1}{2}a_{10}d|\sin \alpha|\}^{2k}. \end{aligned}$$

We have, then,

$$\begin{aligned} \sum_{\mathbf{r}} |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{2k}| \\ \leq 4.2^{2k-1} a_{01} \{\frac{1}{2}a_{10}d|\sin \alpha|\}^{2k} \left\{ \sum_{j=0}^{k-1} \binom{2k}{j} + \frac{1}{2} \binom{2k}{k} \right\} \\ = 2^{2k} a_{01} \{a_{10}d|\sin \alpha|\}^{2k}, \end{aligned} \quad (5.11)$$

with a similar expression for $\sum_{\mathbf{r}} |\mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{2k+1}|$.

We now include viscosity. Following the iteration procedure (4.7), (4.8), we see that the solutions obtained by successive approximations will differ from those in the inviscid case by the inclusion of the factors $F\{\dots, r_{*1j}\}_i$ as defined by (5.1), (5.2), (5.3). The solution obtained from the n^{th} approximation is

$$\begin{aligned} \mathbf{A}_n(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}) \exp \{-\nu r^2 t\} + \mathfrak{F}\{\mathbf{a}_{10} * \mathbf{a}_{01}\} F\{r_{*1j}\} \\ + \mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01})\}_2 F\{\dots, r_{*1j}\}_2 + \dots \\ + \mathfrak{F}\{\mathbf{a}_{10} * \mathfrak{F}[\mathbf{a}_{10} * \dots * \mathfrak{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \dots]\}_{n-1} F\{\dots, r_{*1j}\}_{n-1}. \end{aligned} \quad (5.12)$$

From (5.7), (5.11), (5.12) we have, therefore,

$$\begin{aligned}
 & \sum_{\mathbf{r}} |\mathbf{A}_{2k+1}(\mathbf{r}, t) - \mathbf{A}_{2k}(\mathbf{r}, t)| \\
 &= \sum_{\mathbf{r}} |\mathcal{F}\{\mathbf{a}_{10} * \mathcal{F}[\mathbf{a}_{10} * \cdots * \mathcal{F}(\mathbf{a}_{10} * \mathbf{a}_{01}) \cdots]\}_{2k} \cdot F\{\cdots, r_{\mathbf{a}1j}\}_{2k}| \\
 &\leq \frac{a_{01}}{(2k)!} \left\{ \frac{2a_{10}d \sin \alpha}{\nu r_{10}^2} \right\}^{2k} \{1 - \exp(-\nu r_{10}^2 t)\}^{2k} \exp(-\nu r_m^2 t) \\
 &\leq \frac{a_{01}}{(2k)!} \left\{ \frac{2a_{10}d \sin \alpha}{\nu r_{10}^2} \right\}^{2k} \tag{5.13}
 \end{aligned}$$

for all $t, t \geq 0$. A similar expression can be written for $\sum_{\mathbf{r}} |\mathbf{A}_{2k}(\mathbf{r}, t) - \mathbf{A}_{2k-1}(\mathbf{r}, t)|$. The series $\mathbf{A}_1(\mathbf{r}, t) + \sum_{n=1}^{\infty} |\mathbf{A}_{n+1}(\mathbf{r}, t) - \mathbf{A}_n(\mathbf{r}, t)|$ therefore converges absolutely and uniformly for all $t, t \geq 0$. Each term of the series is continuous in t , so that the limit of $\mathbf{A}_n(\mathbf{r}, t)$ as $n \rightarrow \infty$ exists and is continuous in t . The existence of the solution now follows in the usual manner.

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