Characteristic Surfaces in the Prandtl-Reuss Plasticity Theory

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1. Introduction. In a previous article in this Journal we derived conditions for the existence of characteristic surfaces in the von Mises theory of plasticity on the basis of the usual Cauchy procedure.¹ These same conditions were later obtained² by employing the concept of the singular surface which has been extensively used by HADAMARD³ in his researches on the propagation of waves. In this paper⁴ the problem of characteristic surfaces in the Prandtl-Reuss theory of plasticity is discussed and the differential equation for these surfaces is deduced. We here publish these results for the record with the understanding that this publication does not constitute an endorsement on our part of the Prandtl-Reuss equations for the treatment of plastic flow.

¹T. Y. THOMAS, *On the characteristic surfaces of the von Mises plasticity equations*, J. Rational Mech. and Anal., **1**, 1952, pp. 343-357. See also, J. L. ERICKSEN, *Singular surfaces in plasticity*, J. Math. and Physics, **34**, 1955, pp. 74-79; and J. W. CRAGGS, *Characteristic surfaces in ideal plasticity in three dimensions*, Quar. J. Mech. and Appl. Math., **7**, 1954, pp. 35-39. The characteristic conditions obtained by CRAGGS can be given the invariant formulation

$$\sigma_{\alpha\beta}^* \nu_{\alpha} \nu_{\beta} = 0; \quad 2\sigma_{\alpha\beta}^* \sigma_{\alpha\gamma}^* \nu_{\beta} \nu_{\gamma} = K$$

in the notation of the present paper. It can readily be shown by transformation of these invariant equations to canonical coordinates ($\sigma_{\alpha\beta}^* = \sigma_{\alpha}^* \delta_{\alpha\beta}$ at the origin of this system) that $\sigma_2^* = 0$ where σ_{α}^* are the principal values of the deviator σ^* with the ordering $\sigma_1^* \geq \sigma_2^* \geq \sigma_3^*$. Also the unit normal ν_{α} to the characteristic surface must have the components $\pm 1/\sqrt{2}$, 0, $\pm 1/\sqrt{2}$ relative to the canonical system and hence the characteristic surface elements lie in the planes of maximum or minimum shearing stress. These remarks are made to show that the result obtained by CRAGGS conforms to that deduced in the other two references.

²T. Y. THOMAS, *Singular surfaces and flow lines in the theory of plasticity*, J. Rational Mech. and Anal., **2**, 1953, pp. 339-381.

³See, J. HADAMARD, *Leçons sur la Propagation des Ondes et les Équations de l'Hydrodynamique*, Paris, 1903.

⁴Prepared for the Applied Mathematics Branch, Mechanics Division, Naval Research Laboratory, Washington, D. C.

In addition to the general dynamical relations

$$(1) \quad \frac{d\rho}{dt} + \rho v_{\alpha, \alpha} = 0, \quad (\text{equation of continuity}),$$

$$(2) \quad \sigma_{\alpha\beta, \beta} = \rho \frac{dv_{\alpha}}{dt}, \quad (\text{equations of motion}),$$

the theory under consideration involves the following special equations, namely

$$(3) \quad \frac{D\sigma_{\alpha\beta}^*}{Dt} = 2\mu\epsilon_{\alpha\beta} - \frac{2\mu}{K} \epsilon_{ik}\sigma_{ik}^*\sigma_{\alpha\beta}^*, \quad (\text{Prandtl-Reuss equations}),$$

$$(4) \quad \epsilon_{ii} = v_{\alpha, \alpha} = 0, \quad (\text{equation of incompressibility}),$$

$$(5) \quad \sigma_{\alpha\beta}^*\sigma_{\alpha\beta}^* = K, \quad (\text{quadratic yield condition}).$$

Here ρ is the density, the v_{α} are components of velocity, $\epsilon_{\alpha\beta}$ the components of the rate of strain tensor, $\sigma_{\alpha\beta}$ the components of the stress tensor and $\sigma_{\alpha\beta}^*$ the components of the stress deviation. The quantities μ and K are material constants. It is obvious from the form of the above equations that rectangular coordinates are employed. In these equations we distinguish between the ordinary or total time derivative which appears in (1) and (2) and the *absolute time derivative* in the left member of (3) which is introduced in order to present the Prandtl-Reuss equations as a dynamically correct system.⁵

If we sum on the indices α and β in (3) the resulting equation is satisfied identically on account of (4). Hence we can consider that there are only five independent equations in the set (3). Moreover, the equations (3) are not entirely independent of the yield condition (5). Thus if we consider the equations (3) without regard to the condition expressed by (5) we can deduce

$$(6) \quad \sigma_{\alpha\beta}^* \frac{D\sigma_{\alpha\beta}^*}{Dt} = \sigma_{\alpha\beta}^* \frac{d\sigma_{\alpha\beta}^*}{dt} = 2\mu \left(1 - \frac{\sigma_{\alpha\beta}^*\sigma_{\alpha\beta}^*}{K} \right) \epsilon_{ik}\sigma_{ik}^*.$$

The first part of this relation is actually an identity in view of the definition of the absolute time derivative (see §3). Now write (6) in the form

$$(7) \quad \frac{dW}{dt} = -\frac{4\mu}{K} \epsilon_{ik}\sigma_{ik}^* W,$$

where

$$W = 1 - \frac{\sigma_{\alpha\beta}^*\sigma_{\alpha\beta}^*}{K},$$

⁵See, C. TRUESDELL, *The simplest rate theory of pure elasticity*, Comm. Pure and Appl. Math., 8, 1955, pp. 123-132. Also, T. Y. THOMAS, *On the structure of the stress-strain relations*, Proc. Nat. Acad. Sci., 41, 1955, pp. 716-720; and, *Kinematically preferred coordinate systems*, *ibid.*, 44, 1955, pp. 762-770.

and let Σ be a surface, fixed or moving relative to the coordinate system, which is intersected by the trajectories of all particles of the region of flow under consideration; this region can be limited to the immediate neighborhood of the surface Σ in this discussion. We assume that the quantity W vanishes at all times over Σ . From the uniqueness theorem for differential equations of the type (7) it now follows that $W = 0$ throughout the flow. In other words, we can consider the yield condition (5) to be of the nature of a *boundary condition* over the above surface Σ . With the understanding that this boundary condition is imposed we can therefore limit our attention to the set of equations (1), $\dots$, (4). There are ten independent equations (1), $\dots$, (4) and these equations involve the ten quantities ρ , v_α and $\sigma_{\alpha\beta}$ as dependent variables.

2. Selection of the coordinate system. Denote by x_1, x_2, x_3 the coordinates of an admissible rectangular system and consider a moving surface $\Sigma(t)$ defined by an equation $F(x, t) = 0$ where the function F satisfies the following conditions. *First*, $F(x, t)$ possesses continuous first partial derivatives with respect to the space variables x and the time t ; *second*, the (invariant) condition $F_{,\alpha}F_{,\alpha} > 0$ is satisfied on $\Sigma(t)$. For any assigned value of t the surface $\Sigma(t)$ will divide the region $R(t)$ under consideration into a region $R_1(t)$ and a region $R_2(t)$. It may be assumed that $F < 0$ in $R_1(t)$ and $F > 0$ in $R_2(t)$. Then if ν denotes the unit normal vector along $\Sigma(t)$, this vector being directed from the region $R_1(t)$ into the region $R_2(t)$, we shall have²

$$(8) \quad \nu_\alpha = \frac{F_{,\alpha}}{\sqrt{F_{,\beta}F_{,\beta}}} ; \quad G = -\frac{\partial F/\partial t}{\sqrt{F_{,\beta}F_{,\beta}}} ,$$

where G is the velocity of Σ in the direction ν .

We assign the functions ρ , v_α and $\sigma_{\alpha\beta}$ over the surface $\Sigma(t)$ subject to the condition (5) and obvious differentiability requirements. The moving surface $\Sigma(t)$ is said to be a *characteristic surface* when the equations (1), $\dots$, (5) and the assigned data do not suffice for the determination of all first partial derivatives of ρ , v_α and $\sigma_{\alpha\beta}$ over $\Sigma(t)$. To obtain the condition for $\Sigma(t)$ to be a characteristic surface it will be convenient to introduce a specially selected coordinate system.

Denote by y_1, y_2, y_3 the coordinates of a rectangular system which may move with uniform velocity relative to the original x system. The transformation $x \leftrightarrow y$ is permissible since this transformation will leave invariant all equations of the system (1), $\dots$, (5). Now consider an arbitrary point P on $\Sigma(t)$ at some particular time t_0 and choose the y system so that its origin coincides with P and its y_1 and y_2 axes lie in the tangent plane to Σ at the time t_0 . In addition we assume that at time t_0 the velocity G of $\Sigma(t)$ is zero at the point P relative to the y system.

We may now represent the surface $\Sigma(t)$ locally, *i.e.*, for points y neighboring P and for times t neighboring t_0 , by an equation of the form

$$(9) \quad y_3 = f(y_1, y_2, t).$$

Hence at (P, t_0) , i.e., at the point P and time t_0 , we shall have

$$(10) \quad \frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial y_2} = \frac{\partial f}{\partial t} = 0,$$

since the y_1, y_2 plane is tangent to $\Sigma(t_0)$ and $G = 0$ from the choice of the moving y system.

Retaining the above symbols ρ, v_α and $\sigma_{\alpha\beta}$ to denote components of density, velocity and stress relative to the y system, we suppose that

$$\begin{aligned} \rho(y_1, y_2, y_3, t) &= \bar{\rho}(y_1, y_2, t), \\ v_\alpha(y_1, y_2, y_3, t) &= \bar{v}_\alpha(y_1, y_2, t), \\ \sigma_{\alpha\beta}(y_1, y_2, y_3, t) &= \bar{\sigma}_{\alpha\beta}(y_1, y_2, t), \end{aligned}$$

on $\Sigma(t)$. The deviator components $\bar{\sigma}_{\alpha\beta}^*$ which are determined from the $\bar{\sigma}_{\alpha\beta}$ are subject to the condition (5). Otherwise the $\bar{\sigma}_{\alpha\beta}$ as well as the quantities $\bar{\rho}$ and $\bar{v}_\alpha$ are arbitrary differentiable functions of the variables y_1, y_2 and t . Now

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{\partial \rho}{\partial t} + \rho_{,3} \frac{\partial f}{\partial t}; \quad \bar{\rho}_{,\gamma} = \rho_{,\gamma} + \rho_{,3} f_{,\gamma}$$

for $\gamma = 1, 2$. Hence

$$(11) \quad \frac{\partial \bar{\rho}}{\partial t} = \frac{\partial \rho}{\partial t}; \quad \rho_{,\gamma} = \bar{\rho}_{,\gamma} \quad (\gamma = 1, 2), \quad \text{at } (P, t_0),$$

on account of (10). Similar equations hold for the quantities v_α and $\sigma_{\alpha\beta}$. Hence all derivatives of ρ, v_α and $\sigma_{\alpha\beta}$ are determined at (P, t_0) from the assigned data on $\Sigma(t)$ with the exception of the derivatives of these quantities with respect to the coordinate y_3 . The determination of these latter derivatives by means of the equations (1), $\dots$, (5) will be considered in the following section.

3. The determinantal condition. Let us suppose that the assignment of data on $\Sigma(t)$ is such that the normal velocity of the flow relative to the velocity of Σ does not vanish on this surface. The yield condition (5) can then be disregarded and our attention confined to the system (1), $\dots$, (4) as shown in §1. This is the general case and the resulting characteristic condition will be referred to as the *general characteristic condition*. The special case for which the velocity G of $\Sigma(t)$ is equal to the velocity of the flow normal to $\Sigma(t)$ will be treated separately in §6.

Since (1) becomes $d\rho/dt = 0$ on account of (4) we find that (1), (2) and (4) give the conditions

$$(12) \quad v_3 \rho_{,3} = \dots; \quad \sigma_{\alpha 3,3} - \rho v_3 v_{\alpha,3} = \dots; \quad v_{3,3} = \dots, \quad \text{at } (P, t_0),$$

where the dots are used to denote terms not involving differentiation with respect to y_3 . To obtain conditions corresponding to (12) from the equations

(3) we shall need the formula for the absolute time derivative of the tensor σ^* which is given by

$$\frac{D\sigma_{\alpha\beta}^*}{Dt} = \frac{d\sigma_{\alpha\beta}^*}{dt} + \sigma_{\alpha\mu}^* \phi_{\mu\beta} + \sigma_{\beta\mu}^* \phi_{\mu\alpha} ,$$

where

$$\phi_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} - v_{\beta,\alpha}) .$$

Making this substitution for $\phi_{\alpha\beta}$ and also making the substitutions

$$\frac{d\sigma_{\alpha\beta}^*}{dt} = \frac{\partial\sigma_{\alpha\beta}^*}{\partial t} + \sigma_{\alpha\beta,\gamma}^* v_\gamma ,$$

$$\sigma_{\alpha\beta}^* = \sigma_{\alpha\beta} - \frac{1}{3}\sigma_{ii}\delta_{\alpha\beta} ,$$

we find that the equations (3) give the following conditions on derivatives with respect to y_3 at (P, t_0) , namely

$$(13) \quad \begin{aligned} & v_3 \sigma_{\alpha\beta,3} - \frac{1}{3} v_3 \sigma_{ii,3} \delta_{\alpha\beta} + \frac{1}{2} \sigma_{\alpha\mu}^* v_{\mu,\beta} + \frac{1}{2} \sigma_{\beta\mu}^* v_{\mu,\alpha} \\ & - \frac{1}{2} \sigma_{\alpha 3}^* v_{\beta,3} - \frac{1}{2} \sigma_{\beta 3}^* v_{\alpha,3} - \mu(v_{\alpha,\beta} + v_{\beta,\alpha}) + \frac{2\mu}{K} \sigma_{\alpha\beta}^* \sigma_{i3}^* v_{i,3} = \dots . \end{aligned}$$

The equation $\alpha = \beta = 3$ will be omitted from the set (13) in view of the remark on the dependence of these equations in §1.

The determinant

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma & \frac{2}{3}v_3 & 0 & 0 & -\frac{1}{3}v_3 & 0 & -\frac{1}{3}v_3 & 0 \\ \delta & \epsilon & \eta & 0 & v_3 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & v_3 & 0 & 0 & 0 & 0 \\ d & e & f & -\frac{1}{3}v_3 & 0 & 0 & \frac{2}{3}v_3 & 0 & -\frac{1}{3}v_3 & 0 \\ p & q & r & 0 & 0 & 0 & 0 & v_3 & 0 & 0 \\ -\rho v_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\rho v_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\rho v_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3 \end{vmatrix}$$

is constructed from the coefficients of the quantities

$$v_{1,3} ; v_{2,3} ; v_{3,3} ; \sigma_{11,3} ; \sigma_{12,3} ; \sigma_{13,3} ; \sigma_{22,3} ; \sigma_{23,3} ; \sigma_{33,3} ; \rho_{,3}$$

in equations (12) and (13). For brevity in writing certain of the elements in this determinant we have put

$$\begin{aligned}
 \alpha &= \frac{2\mu}{K} \sigma_{11}^* \sigma_{13}^* - \sigma_{13}^* ; & \beta &= \frac{2\mu}{K} \sigma_{11}^* \sigma_{23}^* ; & \gamma &= \frac{2\mu}{K} \sigma_{11}^* \sigma_{33}^* , \\
 \delta &= \frac{2\mu}{K} \sigma_{12}^* \sigma_{13}^* - \frac{1}{2} \sigma_{23}^* ; & \epsilon &= \frac{2\mu}{K} \sigma_{12}^* \sigma_{23}^* - \frac{1}{2} \sigma_{13}^* ; & \eta &= \frac{2\mu}{K} \sigma_{12}^* \sigma_{33}^* , \\
 a &= \frac{2\mu}{K} \sigma_{13}^* \sigma_{13}^* + \frac{1}{2} (\sigma_{11}^* - \sigma_{33}^*) - \mu ; & b &= \frac{2\mu}{K} \sigma_{13}^* \sigma_{23}^* + \frac{1}{2} \sigma_{12}^* ; & c &= \frac{2\mu}{K} \sigma_{13}^* \sigma_{33}^* , \\
 d &= \frac{2\mu}{K} \sigma_{22}^* \sigma_{13}^* ; & e &= \frac{2\mu}{K} \sigma_{22}^* \sigma_{23}^* - \sigma_{23}^* ; & f &= \frac{2\mu}{K} \sigma_{22}^* \sigma_{33}^* , \\
 p &= \frac{2\mu}{K} \sigma_{23}^* \sigma_{13}^* + \frac{1}{2} \sigma_{12}^* ; & q &= \frac{2\mu}{K} \sigma_{23}^* \sigma_{23}^* + \frac{1}{2} (\sigma_{22}^* - \sigma_{33}^*) - \mu ; & r &= \frac{2\mu}{K} \sigma_{23}^* \sigma_{33}^* .
 \end{aligned}$$

It is easily shown that the determinant Δ can be reduced to the following second order determinant

$$(14) \quad \begin{vmatrix} \frac{1}{2} \sigma_{11}^* - \frac{1}{2} \sigma_{33}^* - \mu + \frac{2\mu}{K} \sigma_{13}^* \sigma_{13}^* + \rho v_3^2 & \frac{1}{2} \sigma_{12}^* + \frac{2\mu}{K} \sigma_{13}^* \sigma_{23}^* \\ \frac{1}{2} \sigma_{12}^* + \frac{2\mu}{K} \sigma_{13}^* \sigma_{23}^* & \frac{1}{2} \sigma_{22}^* - \frac{1}{2} \sigma_{33}^* - \mu + \frac{2\mu}{K} \sigma_{23}^* \sigma_{23}^* + \rho v_3^2 \end{vmatrix}$$

apart from a factor involving the normal component of velocity v_3 which, by assumption, does not vanish. The general characteristic condition will therefore be given by the vanishing of the determinant (14). Expanding (14) this condition can be written

$$\begin{aligned}
 (15) \quad & -\frac{2\mu^2}{K} \sigma_{\alpha 3}^* \sigma_{\alpha 3}^* + \frac{2\mu^2}{K} (\sigma_{33}^*)^2 + \mu^2 + \mu \sigma_{33}^* - 2\mu \rho v_3^2 - \frac{3}{2} \rho v_3^2 \sigma_{33}^* \\
 & + \frac{1}{2} (\sigma_{33}^*)^2 + (\rho v_3^2)^2 + \frac{1}{4} (\sigma_{11}^* \sigma_{22}^* - \sigma_{12}^* \sigma_{12}^*) + \frac{2\mu}{K} \rho v_3^2 \sigma_{\alpha 3}^* \sigma_{\alpha 3}^* \\
 & - \frac{2\mu}{K} \rho v_3^2 (\sigma_{33}^*)^2 - \frac{\mu}{K} \sigma_{33}^* \sigma_{\alpha 3}^* \sigma_{\alpha 3}^* + \frac{\mu}{K} (\sigma_{33}^*)^3 \\
 & + \frac{\mu}{K} (\sigma_{11}^* \sigma_{23}^* \sigma_{23}^* + \sigma_{13}^* \sigma_{13}^* \sigma_{22}^* - 2\sigma_{12}^* \sigma_{13}^* \sigma_{23}^*) = 0 .
 \end{aligned}$$

In the next section we shall deduce the invariant formulation of the characteristic condition (15).

4. Invariant formulation of the characteristic condition. The above characteristic condition (15) is valid at (P, t_0) relative to the special y coordinate system.

But with reference to the y system we have $\nu_\alpha = (0, 0, 1)$ and $G = 0$ at (P, t_0) . Hence the following substitutions can be made in (15), namely

$$(16) \quad v_3 = v_\alpha \nu_\alpha - G; \quad \sigma_{\alpha 3}^* \sigma_{\alpha 3}^* = \sigma_{\alpha \beta}^* \sigma_{\alpha \gamma}^* \nu_\beta \nu_\gamma; \quad \sigma_{33}^* = \sigma_{\alpha \beta}^* \nu_\alpha \nu_\beta.$$

Making these substitutions in (15) the resulting terms will be invariant under allowable dynamical transformations of the rectangular coordinate system, *i.e.*, orthogonal coordinate transformations and uniform translations. We now seek to replace the remaining terms in (15) by expressions of such invariant character.

It can be shown that

$$(17) \quad \sigma_{11}^* \sigma_{22}^* - \sigma_{12}^* \sigma_{12}^* = \sigma_{\alpha 3}^* \sigma_{\alpha 3}^* - \frac{1}{2} \sigma_{\alpha \beta}^* \sigma_{\alpha \beta}^*.$$

In fact if we expand the right member of (17) and make use of the identity $\sigma_{ii}^* = 0$ the relation is readily verified. On account of the condition (5) we can therefore substitute

$$(18) \quad \sigma_{11}^* \sigma_{22}^* - \sigma_{12}^* \sigma_{12}^* = \sigma_{\alpha \beta}^* \sigma_{\alpha \gamma}^* \nu_\beta \nu_\gamma - \frac{1}{2} K$$

for the above parenthesis expression in (15). To obtain the corresponding substitution for the last parenthesis expression in (15), which we now denote by J for brevity, we first observe that

$$\sigma_{11}^* \sigma_{23}^* \sigma_{23}^* = -\sigma_{22}^* \sigma_{23}^* \sigma_{23}^* - \sigma_{33}^* \sigma_{23}^* \sigma_{23}^* = -\sigma_{\alpha 2}^* \sigma_{\alpha 3}^* \sigma_{23}^* + \sigma_{12}^* \sigma_{13}^* \sigma_{23}^*,$$

and,

$$\sigma_{22}^* \sigma_{13}^* \sigma_{13}^* = -\sigma_{11}^* \sigma_{13}^* \sigma_{13}^* - \sigma_{33}^* \sigma_{13}^* \sigma_{13}^* = -\sigma_{\alpha 1}^* \sigma_{\alpha 3}^* \sigma_{13}^* + \sigma_{21}^* \sigma_{23}^* \sigma_{13}^*.$$

Hence we can write

$$J = -\sigma_{\alpha 2}^* \sigma_{\alpha 3}^* \sigma_{23}^* - \sigma_{\alpha 1}^* \sigma_{\alpha 3}^* \sigma_{13}^* = -\sigma_{\alpha \beta}^* \sigma_{\beta 3}^* \sigma_{\alpha 3}^* + \sigma_{\alpha 3}^* \sigma_{\alpha 3}^* \sigma_{33}^*,$$

and this leads to the desired expression for J , namely

$$(19) \quad J = -\sigma_{\alpha \beta}^* \sigma_{\beta \gamma}^* \sigma_{\alpha \delta}^* \nu_\gamma \nu_\delta + \sigma_{\alpha \beta}^* \sigma_{\alpha \gamma}^* \sigma_{\delta \epsilon}^* \nu_\beta \nu_\gamma \nu_\delta \nu_\epsilon.$$

When we make the substitutions (16), (18), and (19) in the relation (15) we obtain the following invariant characteristic condition

$$(20) \quad \begin{aligned} & \rho^2 (v_n - G)^4 - 2\mu \rho (v_n - G)^2 + \mu^2 - \frac{1}{8} K + [\mu - \frac{3}{2} \rho (v_n - G)^2] \sigma_{\alpha \beta}^* \nu_\alpha \nu_\beta \\ & + \left[\frac{2\mu}{K} \rho (v_n - G)^2 + \frac{1}{4} - \frac{2\mu^2}{K} \right] \sigma_{\alpha \beta}^* \sigma_{\alpha \gamma}^* \nu_\beta \nu_\gamma - \frac{\mu}{K} \sigma_{\alpha \beta}^* \sigma_{\beta \gamma}^* \sigma_{\alpha \delta}^* \nu_\gamma \nu_\delta \\ & + \left[\frac{1}{2} - \frac{2\mu}{K} \rho (v_n - G)^2 \right] (\sigma_{\alpha \beta}^* \nu_\alpha \nu_\beta)^2 + \frac{\mu}{K} (\sigma_{\alpha \beta}^* \nu_\alpha \nu_\beta)^3 = 0, \end{aligned}$$

where we have put $v_n = v_\alpha \nu_\alpha$ for the component of velocity normal to the characteristic surface $\Sigma(t)$. The relation (20) can be regarded as giving the condition on the unit normals ν to the surface elements of characteristic surfaces

$\Sigma(t)$ in the plastic flow. However, these surface elements will depend not only on the flow quantities ρ , v_α and $\sigma_{\alpha\beta}$ but also on the velocity G of the surface $\Sigma(t)$. If we make the substitutions (8) in (20) this equation becomes the general differential equation for the determination of the function $F(x, t)$ defining the characteristic surfaces.⁶ We shall not consider the question of the existence of solutions F of this differential equation in this paper.

5. Characteristics as singular or wave surfaces. Let $Z_1(x, t), \dots, Z_n(x, t)$ be a set of functions of x_1, x_2, x_3 and t which we assume to be continuous in the region $R(t)$ under consideration. We suppose moreover that the first derivatives of the functions $Z(x, t)$ with respect to the x, t exist and are continuous in the regions $R_1(t) + \Sigma(t)$ and $R_2(t) + \Sigma(t)$ separately but that not all of the derivatives of the $Z(x, t)$ with respect to the space variables x_α are continuous at points of $\Sigma(t)$. Such a surface $\Sigma(t)$ will be called a wave surface or more particularly a singular surface of order one relative to the system $Z(x, t)$. This concept of the wave or singular surface is of especial significance from the physical standpoint. We shall now show that the above characteristic surfaces can be identified with wave or singular surfaces of order one relative to the quantities ρ , v_α and $\sigma_{\alpha\beta}$ which determine the plastic flow.

The bracket [] will be used to denote the difference in the values of a quantity on the two sides of the surface $\Sigma(t)$, e.g., $[Z_{n,\beta}]$ stands for the difference $Z_{2n,\beta} - Z_{1n,\beta}$ where $Z_{1n,\beta}$ and $Z_{2n,\beta}$ are the derivatives of the function Z_n with respect to x_β on the sides of $\Sigma(t)$ neighboring the regions $R_1(t)$ and $R_2(t)$ respectively. It can be shown that for a singular surface $\Sigma(t)$ of order one we have

$$(21) \quad [Z_{n,\beta}] = \lambda_n \nu_\beta ; \quad \left[\frac{\partial Z}{\partial t} \right] = -G \lambda_n ,$$

where the λ_n are certain proportionality factors defined over $\Sigma(t)$ while ν and G denote the unit normal and velocity of the surface $\Sigma(t)$ as previously defined.

Taking the above functions $Z(x, t)$ to be the quantities ρ , v_α and $\sigma_{\alpha\beta}$ in turn it now follows from (21) that

$$(22) \quad [\rho, \alpha] = \zeta \nu_\alpha ; \quad \left[\frac{\partial \rho}{\partial t} \right] = -G \zeta ,$$

$$(23) \quad [v_{\alpha,\beta}] = \lambda_\alpha \nu_\beta ; \quad \left[\frac{\partial v_\alpha}{\partial t} \right] = -G \lambda_\alpha ,$$

$$(24) \quad [\sigma_{\alpha\beta,\gamma}] = \xi_{\alpha\beta} \nu_\gamma ; \quad \left[\frac{\partial \sigma_{\alpha\beta}}{\partial t} \right] = -G \xi_{\alpha\beta} .$$

⁶It is required only that this differential equation hold on the surface $F(x, t) = 0$. Without this restriction the equation $F(x, t) = \text{const.}$ will give a congruence of characteristic or wave surfaces $\Sigma(t)$.

These equations give the *kinematical conditions of compatibility* for the problem under consideration.

The *dynamical conditions* of the problem are obtained from the equations of the set (1), $\dots$, (5). From (1) and (4) we have

$$\left[\frac{d\rho}{dt} \right] + \rho[v_{\alpha,\alpha}] = \left[\frac{\partial \rho}{\partial t} \right] + v_\alpha[\rho_{,\alpha}] + \rho[v_{\alpha,\alpha}] = 0; \quad [v_{\alpha,\alpha}] = 0.$$

Hence from (22) and (23) it follows that

$$(25) \quad (v_n - G)\zeta = 0; \quad \lambda_\alpha \nu_\alpha = 0.$$

The second of these relations expresses the condition that the vector λ is tangent to the surface Σ .

From (3) we obtain

$$(26) \quad \left[\frac{D\sigma_{\alpha\beta}^*}{Dt} \right] = 2\mu[\epsilon_{\alpha\beta}] - \frac{2\mu}{K} [\epsilon_{ik}] \sigma_{ik}^* \sigma_{\alpha\beta}^*.$$

But,

$$[\epsilon_{\alpha\beta}] = \frac{1}{2}([v_{\alpha,\beta}] + [v_{\beta,\alpha}]) = \frac{1}{2}(\lambda_\alpha \nu_\beta + \lambda_\beta \nu_\alpha).$$

Hence (26) becomes

$$(27) \quad \left[\frac{D\sigma_{\alpha\beta}^*}{Dt} \right] = \mu(\lambda_\alpha \nu_\beta + \lambda_\beta \nu_\alpha) - \frac{2\mu}{K} \sigma_{ik}^* \lambda_i \nu_k \sigma_{\alpha\beta}^*.$$

To deduce the expression for the quantity in the left member of (27) we first consider the relations

$$\sigma_{\alpha\beta,\gamma}^* = \sigma_{\alpha\beta,\gamma} - \frac{1}{3}\sigma_{ii,\gamma}\delta_{\alpha\beta}; \quad \frac{\partial \sigma_{\alpha\beta}^*}{\partial t} = \frac{\partial \sigma_{\alpha\beta}}{\partial t} - \frac{1}{3} \frac{\partial \sigma_{ii}}{\partial t} \delta_{\alpha\beta}$$

from which it follows that

$$[\sigma_{\alpha\beta,\gamma}^*] = [\sigma_{\alpha\beta,\gamma}] - \frac{1}{3}[\sigma_{ii,\gamma}]\delta_{\alpha\beta}; \quad \left[\frac{\partial \sigma_{\alpha\beta}^*}{\partial t} \right] = \left[\frac{\partial \sigma_{\alpha\beta}}{\partial t} \right] - \frac{1}{3} \left[\frac{\partial \sigma_{ii}}{\partial t} \right] \delta_{\alpha\beta}.$$

Application of (24) now permits us to write

$$(28) \quad [\sigma_{\alpha\beta,\gamma}^*] = \xi_{\alpha\beta}\nu_\gamma - \frac{1}{3}\xi_{ii}\delta_{\alpha\beta}\nu_\gamma,$$

$$(29) \quad \left[\frac{\partial \sigma_{\alpha\beta}^*}{\partial t} \right] = -G\xi_{\alpha\beta} + \frac{1}{3}G\xi_{ii}\delta_{\alpha\beta}.$$

Similarly from the definition of the quantities $\phi_{\alpha\beta}$ in §3 and the relations (23) we obtain

$$(30) \quad [\phi_{\alpha\beta}] = \frac{1}{2}([v_{\alpha,\beta}] - [v_{\beta,\alpha}]) = \frac{1}{2}(\lambda_\alpha \nu_\beta - \lambda_\beta \nu_\alpha).$$

But from the definition of the absolute time derivative (§3) we have

$$\left[\frac{D\sigma_{\alpha\beta}^*}{Dt} \right] = \left[\frac{\partial \sigma_{\alpha\beta}^*}{\partial t} \right] + [\sigma_{\alpha\beta,\gamma}^*]\nu_\gamma + \sigma_{\alpha\mu}^*[\phi_{\mu\beta}] + \sigma_{\mu\beta}^*[\phi_{\mu\alpha}].$$

Hence, making the substitutions (28), (29), and (30), it follows that

$$(31) \quad \left[\frac{D\sigma_{\alpha\beta}^*}{Dt} \right] = (v_n - G)(\xi_{\alpha\beta} - \tfrac{1}{3}\xi_{ii}\delta_{\alpha\beta}) + \tfrac{1}{2}(\sigma_{\alpha\mu}^*\nu_\beta + \sigma_{\beta\mu}^*\nu_\alpha)\lambda_\mu - \tfrac{1}{2}(\sigma_{\alpha\mu}^*\lambda_\beta + \sigma_{\beta\mu}^*\lambda_\alpha)\nu_\mu.$$

Finally elimination of the left members of (27) and (31) from these equations yields the following relations

$$(32) \quad (v_n - G)(\xi_{\alpha\beta} - \tfrac{1}{3}\xi_{ii}\delta_{\alpha\beta}) + \tfrac{1}{2}(\sigma_{\alpha\mu}^*\nu_\beta + \sigma_{\beta\mu}^*\nu_\alpha)\lambda_\mu - \tfrac{1}{2}(\sigma_{\alpha\mu}^*\lambda_\beta + \sigma_{\beta\mu}^*\lambda_\alpha)\nu_\mu - \mu(\lambda_\alpha\nu_\beta + \lambda_\beta\nu_\alpha) + \frac{2\mu}{K}\sigma_{ik}^*\lambda_i\nu_k\sigma_{\alpha\beta}^* = 0.$$

There remain the equations (2) and (5). From the latter no conditions are obtained in conformity with the fact, observed in §1, that (5) is merely of the nature of a boundary condition and can be excluded from the system of differential equations. However, from (2) we have

$$[\sigma_{\alpha\beta,\beta}] = \rho \left[\frac{dv_\alpha}{dt} \right] = \rho \left(\left[\frac{\partial v_\alpha}{\partial t} \right] + v_\beta [v_{\alpha,\beta}] \right).$$

Hence it follows from (23) and (24) that

$$(33) \quad \xi_{\alpha\beta}\nu_\beta = \rho(v_n - G)\lambda_\alpha.$$

Equations (25), (32), and (33) give the dynamical conditions of compatibility for this problem.

Assume $v_n - G \neq 0$ on $\Sigma(t)$. Then if all $\lambda_\alpha = 0$ we have

$$(34) \quad \zeta = 0; \quad \xi_{\alpha\beta} - \tfrac{1}{3}\xi_{ii}\delta_{\alpha\beta} = 0; \quad \xi_{\alpha\beta}\nu_\beta = 0$$

from (25), (32), and (33). Hence

$$\xi_{\alpha\beta}\nu_\beta = \tfrac{1}{3}\xi_{ii}\delta_{\alpha\beta}\nu_\beta = \tfrac{1}{3}\xi_{ii}\nu_\alpha = 0.$$

It follows that $\xi_{ii} = 0$ and hence $\xi_{\alpha\beta} = 0$ from the second relation (34). Since $\zeta = 0$ also from (34) we have a contradiction with the assumption that $\Sigma(t)$ is singular as postulated. Hence not all $\lambda_\alpha = 0$ when $(v_n - G) \neq 0$ on $\Sigma(t)$.

Let us now multiply (32) by ν_β and eliminate the expression $\xi_{\alpha\beta}\nu_\beta$ from the resulting equations by the substitution (33). We thus obtain an equation which can be written in the form

$$(35) \quad A\lambda_\alpha + B_{\alpha\beta}\lambda_\beta + \tfrac{1}{3}(v_n - G)\xi_{ii}\nu_\alpha = 0,$$

where,

$$A = \mu + \tfrac{1}{2}\sigma_{\alpha\beta}^*\nu_\alpha\nu_\beta - \rho(v_n - G)^2,$$

$$B_{\alpha\beta} = -\tfrac{1}{2}(\sigma_{\alpha\beta}^* + \sigma_{\alpha\mu}^*\nu_\beta\nu_\mu + \sigma_{\beta\mu}^*\nu_\alpha\nu_\mu) - \frac{2\mu}{K}\sigma_{\alpha\gamma}^*\sigma_{\beta\delta}^*\nu_\gamma\nu_\delta.$$

We observe here that the second term in the expression for $B_{\alpha\beta}$ vanishes when multiplied by λ_β on account of (25); this term has been added in order to make the quantities $B_{\alpha\beta}$ symmetric. Now the equations (35) and the second relation (25) constitute a set of four linear and homogeneous equations for the determination of the variables $\lambda_1, \lambda_2, \lambda_3$ and ξ_{ii} . These equations must have a non-trivial solution since we have shown that not all the λ_α can vanish under the above assumption that $v_n - G \neq 0$ on $\Sigma(t)$. Hence we must have

$$(36) \quad \begin{vmatrix} A + B_{11} & B_{12} & B_{13} & (v_n - G)\nu_1 \\ B_{21} & A + B_{22} & B_{23} & (v_n - G)\nu_2 \\ B_{31} & B_{32} & A + B_{33} & (v_n - G)\nu_3 \\ \nu_1 & \nu_2 & \nu_3 & 0 \end{vmatrix} = 0.$$

Corresponding to any non-trivial solution $\lambda_1, \lambda_2, \lambda_3$ and ξ_{ii} of the equations (35) and the second equation (25) we can determine the $\xi_{\alpha\beta}$ by (32). The values of the $\xi_{\alpha\beta}$ so determined will satisfy (33) in which the λ_α are the above solution functions. For if we multiply (32) by ν_β and combine the resulting equations with (35) the relations (33) are obtained. Since $\zeta = 0$ from (25) we have thus found a solution $\zeta, \lambda_\alpha, \xi_{\alpha\beta}$ of the system consisting of (25), (32), and (33). The condition for the existence of this solution is of course the vanishing of the above determinant. Hence (36) gives the general condition for the existence of the singular surface $\Sigma(t)$. The special case for which $v_n - G = 0$ on $\Sigma(t)$ will be discussed in the following section.

Since the condition (36) is invariant under a rotation of the coordinate system we can choose a system such that $\nu_\alpha = (0, 0, 1)$ at an arbitrary point P on $\Sigma(t)$. Then it is immediately seen that the condition (36) reduces at P to

$$\begin{vmatrix} \mu + \frac{1}{2}\sigma_{33}^* - \rho(v_n - G)^2 - \frac{1}{2}\sigma_{11}^* - \frac{2\mu}{K}\sigma_{13}^*\sigma_{13}^* & -\frac{1}{2}\sigma_{12}^* - \frac{2\mu}{K}\sigma_{13}^*\sigma_{23}^* \\ -\frac{1}{2}\sigma_{12}^* - \frac{2\mu}{K}\sigma_{13}^*\sigma_{23}^* & \mu + \frac{1}{2}\sigma_{33}^* - \rho(v_n - G)^2 - \frac{1}{2}\sigma_{22}^* - \frac{2\mu}{K}\sigma_{23}^*\sigma_{23}^* \end{vmatrix} = 0,$$

which is equivalent to the condition expressed by the vanishing of the determinant (14) since v_3 in (14) can be replaced by the quantity $v_n - G$ in this consideration. Hence the procedure in §4 applies and leads to the relation (20) as the condition for $\Sigma(t)$ to be a singular surface. The general differential equation for the singular surface $\Sigma(t)$ is of course obtained from (20) by the substitution (8) and hence these singular or wave surfaces can be identified with the characteristic surfaces discussed in §2 and §3.

Remark. From the first equation (25) we have $\zeta = 0$ since $v_n - G \neq 0$ on $\Sigma(t)$ by assumption. Hence the density ρ is continuous across the general singular surface $\Sigma(t)$.

6. The special case. Let us adopt the viewpoint of the wave or singular surface in commenting on the case for which $v_n - G = 0$ on $\Sigma(t)$ since the necessary relations for the treatment of this special case appear in the preceding section. The condition $v_n - G = 0$ leads immediately to the differential equation of these wave surfaces, namely⁶

$$(37) \quad \frac{\partial F}{\partial t} + v_\alpha F_{,\alpha} = 0,$$

when use is made of the equations (8).

No condition is imposed on the quantity ζ by the first equation (25). Hence the derivatives of the density can be discontinuous across the surface $\Sigma(t)$. Discontinuities can also occur in the derivatives of the stress components $\sigma_{\alpha\beta}$ since the conditions (33) reduce to $\xi_{\alpha\beta}\nu_\beta = 0$ and these equations can obviously be satisfied by non-zero values of the $\xi_{\alpha\beta}$. However, the second equation (25) and the equations (32) which now reduce to

$$(\sigma_{\alpha\mu}^*\nu_\beta + \sigma_{\beta\mu}^*\nu_\alpha)\lambda_\mu - (\sigma_{\alpha\mu}^*\lambda_\beta + \sigma_{\beta\mu}^*\lambda_\alpha)\nu_\mu - 2\mu(\lambda_\alpha\nu_\beta + \lambda_\beta\nu_\alpha) + \frac{4\mu}{K}\sigma_{ik}^*\lambda_i\nu_k\sigma_{\alpha\beta}^* = 0$$

will have, in general, only the trivial solution $\lambda_\alpha = 0$ in consequence of which the derivatives of the velocity will be continuous across this wave surface.

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