*Cylindrically Symmetrical Deformations of
Incompressible Elastic Materials Reinforced
with Inextensible Cords*

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1. Introduction. The introduction of thin, flexible, inextensible cords into an elastic body imposes a constraint upon the deformation which, if there is no relative movement between each cord and the adjoining elastic material, may be expressed mathematically by formulating the condition that in any elastic deformation, the material undergoes no extension along paths taken by the cords. If the cords lie parallel to each other and close together in smooth surfaces in the undeformed body, these surfaces may be regarded as forming boundary surfaces for the elastic material on either side which are inextensible in certain directions. Employing this method of approach, the problems of torsion, shear and flexure have been investigated by ADKINS & RIVLIN (1955) for reinforced, isotropic, incompressible bodies.

A more general type of cylindrically symmetrical deformation has been considered for unreinforced, incompressible, transversely isotropic bodies by ERICKSEN & RIVLIN (1954). The same problem is here examined for reinforced materials, and arrangements of cords which permit the deformation to take place are investigated. The analysis is performed for incompressible materials exhibiting a suitable type of curvilinear æolotropy, employing the formulæ obtained by ADKINS (1955 a, b) which express the stresses, and the resultant forces and couples required to maintain the deformation, in terms of the parameters which define the problem. These results are independent of symmetries in the elastic material and apply to isotropic bodies as a special case. The changes in the analysis required for a consideration of the allied problem of a rectangular block subjected to shear and flexure are indicated.

2. Preliminary relations. We summarize here the results obtained by ADKINS (1955a, b) for cylindrically symmetrical deformations of ideally elastic materials exhibiting curvilinear æolotropy related to cylindrical polar coordinates in the undeformed body, slight changes of notation being made for convenience in the subsequent analysis.

We consider an elastic body, capable of finite deformation, which in the unstrained state is bounded in the cylindrical polar co-ordinate system (r, θ, x^3) by the curved surfaces $r = A, r = B$ ($A > B$), and the planes $\theta = \pm \Theta, x^3 = \pm L$. During deformation, a point initially at (r, θ, x^3) is displaced to (ρ, ϑ, y^3) in the same co-ordinate system where

$$\rho = \rho(r), \quad \vartheta = a\theta + bx^3, \quad y^3 = c\theta + dx^3, \quad (2.1)$$

a, b, c, d being constants. The initial and final co-ordinates of this point in the rectangular Cartesian co-ordinate system x^k are denoted by x^k, y^k respectively so that $x^1 + ix^2 = re^{i\theta}, y^1 + iy^2 = \rho e^{i\vartheta}$.

With the notation of the previous papers, the strain components $\bar{\gamma}_{(ij)}$ for the deformation described by (2.1) are given by

$$\begin{aligned}
\bar{\gamma}_{(11)} &= \frac{1}{2}(\rho_r^2 - 1), & \bar{\gamma}_{(22)} &= \frac{1}{2}\{(a^2\rho^2 + c^2)/r^2 - 1\}, \\
\bar{\gamma}_{(33)} &= \frac{1}{2}(b^2\rho^2 + d^2 - 1), \\
\bar{\gamma}_{(23)} &= \frac{1}{2}(ab\rho^2 + cd)/r, & \bar{\gamma}_{(12)} &= \bar{\gamma}_{(13)} = 0,
\end{aligned} \tag{2.2}$$

where the suffixes 1, 2, 3 refer to the directions of r , θ , x^3 respectively, and are enclosed in brackets to indicate that the quantities thus defined are not the components of a tensor.

For a material which exhibits curvilinear æolotropy in such a manner that the elastic properties are equivalent at all points when referred to the radial, tangential and longitudinal directions in the undeformed body, the strain-energy function W may be regarded as a function of the quantities $\bar{\gamma}_{(ij)}$ alone, and we may therefore write

$$W = W(\bar{\gamma}_{(ij)}). \tag{2.3}$$

Symmetries existing in the material may restrict the form of W as a function of its strain components. For the particular case of isotropic bodies we have

$$W = W(I_1, I_2, I_3), \tag{2.4}$$

where I_1, I_2, I_3 are the strain invariants given by

$$\begin{aligned}
I_1 &= 3 + 2 \sum_{i=1}^3 \bar{\gamma}_{(ii)} \\
&= \rho_r^2 + (a^2\rho^2 + c^2)/r^2 + b^2\rho^2 + d^2, \\
I_2 &= 3 + 4 \sum_{i=1}^3 \bar{\gamma}_{(ii)} + 2 \sum_{i=1}^3 \sum_{j=1}^3 \{\bar{\gamma}_{(ii)}\bar{\gamma}_{(jj)} - [\bar{\gamma}_{(ij)}]^2\} \\
&= \lambda\rho^2/r^2 + \rho_r^2\{(a^2\rho^2 + c^2)/r^2 + b^2\rho^2 + d^2\}, \\
I_3 &= |\delta_{ii} + 2\bar{\gamma}_{(ii)}| = \lambda^2\rho^2\rho_r^2/r^2, \\
\lambda &= ad - bc,
\end{aligned} \tag{2.5}$$

and δ_{ii} is the Kronecker delta. When there is transverse isotropy about the radial direction at each point, so that the elastic properties are symmetrical with respect to that direction and the plane normal to it, W takes the form

$$W = W(I_1, I_2, I_3, \bar{\gamma}_{(11)}, I_r), \tag{2.7}$$

where, remembering (2.2)

$$I_r = \sum_{i=1}^3 [\bar{\gamma}_{(1i)}]^2 = \frac{1}{4}(\rho_r^2 - 1)^2. \tag{2.8}$$

The forms of W appropriate to materials which are transversely isotropic about the tangential and longitudinal directions are obtained from (2.7) by replacing $\bar{\gamma}_{(11)}, I_r$ by $\bar{\gamma}_{(22)}, I_\theta$ and $\bar{\gamma}_{(33)}, I_l$ respectively where

$$\begin{aligned}
I_\theta &= \sum_{i=1}^3 [\tilde{\gamma}_{(2i)}]^2 \\
&= \frac{1}{4} \left\{ \left(\frac{a^2 \rho^2 + c^2}{r^2} - 1 \right)^2 + \frac{(ab\rho^2 + cd)^2}{r^2} \right\}, \\
I_t &= \sum_{i=1}^3 [\tilde{\gamma}_{(3i)}]^2 \\
&= \frac{1}{4} \left\{ (b^2 \rho^2 + d^2 - 1)^2 + \frac{(ab\rho^2 + cd)^2}{r^2} \right\}.
\end{aligned} \tag{2.9}$$

For orthotropic bodies, where there is symmetry about planes normal to the directions of r , θ , x^3 at each point we may write

$$W = W(\tilde{\gamma}_{(ii)}, \Gamma_i, I_3),$$

where

$$\Gamma_1 = [\tilde{\gamma}_{(23)}]^2 = (ab\rho^2 + cd)^2/(4r^2), \tag{2.10}$$

$$\Gamma_2 = [\tilde{\gamma}_{(13)}]^2 = 0, \quad \Gamma_3 = [\tilde{\gamma}_{(12)}]^2 = 0.$$

The corresponding results for incompressible materials are obtained by putting $I_3 = 1$ in the foregoing relations, and (2.5) then yields

$$\lambda \rho \rho_r = , \quad \lambda \rho^2 = r^2 + K, \tag{2.11}$$

K being a constant of integration.

If the physical components of stress, referred to the radial, tangential and longitudinal directions at the point (ρ, ϑ, y^3) in the deformed body, are denoted by $t_{\rho\rho}$, $t_{\vartheta\vartheta}$, t_{33} , $t_{\rho\vartheta}$, $t_{\vartheta 3}$, $t_{3\rho}$, it may be shown that for materials of the type considered, these quantities are functions of the radial co-ordinate ρ alone, and do not involve ϑ or y^3 . For the general type of material defined by (2.3), we shall assume the form of W to be restricted so that $\partial W/\partial \tilde{\gamma}_{(12)} = \partial W/\partial \tilde{\gamma}_{(13)} = 0$ if $\tilde{\gamma}_{(12)} = \tilde{\gamma}_{(13)} = 0$, and that if in addition $\tilde{\gamma}_{(23)} = 0$, the corresponding derivative $\partial W/\partial \tilde{\gamma}_{(23)}$ is also zero. This limitation upon the form of W ensures that the stress components $t_{\rho\vartheta}$ and $t_{3\rho}$ are zero for the present problem, and that $t_{\vartheta 3}$ also vanishes if $\tilde{\gamma}_{(23)} = 0$. The restriction implies for example, that W is not linear in any of the quantities $\tilde{\gamma}_{(12)}$, $\tilde{\gamma}_{(13)}$, $\tilde{\gamma}_{(23)}$ alone, and this condition is satisfied automatically for the more special types of material defined by (2.4) and the subsequent relations. The equations of equilibrium now yield

$$\frac{dt_{\rho\rho}}{d\rho} + \frac{t_{\rho\rho} - t_{\vartheta\vartheta}}{\rho} + R = 0, \tag{2.12}$$

where to preserve generality we have assumed the existence of a radial body force such as would be produced by the uniform rotation of the system about the x^3 -axis; R is thus a function only of ρ (or r).

The solution of the problem defined by (2.1) evidently assumes a variety of

forms depending upon the expression for W as a function of the strain components. We may, however, regard ρ , and hence the strain components $\bar{\gamma}_{(ii)}$ and the strain-energy function W , as functions of the radial co-ordinate r and the parameters a, b, c, d defining the deformation. Employing this approach, ADKINS (1955b) has obtained a number of results which are independent of symmetries in the elastic material, and which are therefore applicable for all of the forms of W considered above. We are here concerned only with incompressible materials, for which the stresses may be written in the forms

$$\begin{aligned} t_{\rho\rho} &= \int_{\rho_0}^{\rho} \left\{ \frac{2}{\rho} \left(a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} \right) - R \right\} d\rho + [t_{\rho\rho}]_{\rho=\rho_0}, \\ t_{\vartheta\vartheta} &= t_{\rho\rho} + 2 \left(a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} \right), \\ t_{33} &= t_{\rho\rho} + a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} + c \frac{\partial W}{\partial c} + d \frac{\partial W}{\partial d}, \\ t_{\vartheta 3} &= \frac{1}{\rho} \left(c \frac{\partial W}{\partial a} + d \frac{\partial W}{\partial b} \right) = \rho \left(a \frac{\partial W}{\partial c} + b \frac{\partial W}{\partial d} \right), \\ t_{\rho\vartheta} &= t_{3\rho} = 0, \end{aligned} \quad (2.13)$$

ρ_0 being a constant. In addition we have the identity

$$a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} = \frac{\lambda \rho^2}{2K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right), \quad (2.14)$$

which may be used to modify the formulae for the principal stress components.

Resultant forces and couples acting across planes in the deformed body are obtained from (2.13) by integration. Thus if N_ϑ , S_ϑ are the resultant forces acting in the directions of ϑ and y^3 respectively across a plane $\vartheta = \text{constant}$, and measured per unit length of y^3 -axis in the deformed body, we have

$$\begin{aligned} N_\vartheta &= [\rho t_{\rho\rho}]_{B'}^{A'} + \int_{B'}^{A'} \rho R \, d\rho, \quad [A' = \rho(A), B' = \rho(B)], \\ S_\vartheta &= \frac{1}{\lambda} \int_B^A \left(a \frac{\partial W}{\partial c} + b \frac{\partial W}{\partial d} \right) r \, dr, \end{aligned} \quad (2.15)$$

and if M_ϑ is the moment about the x^3 -axis of the forces on this surface, similarly measured, we have

$$M_\vartheta = \frac{1}{2\lambda} \int_B^A \left[2 \left(a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} \right) + \rho R \right] r \, dr + \frac{1}{2} [\rho^2 t_{\rho\rho}]_{B'}^{A'}. \quad (2.16)$$

Similarly the force acting in the longitudinal direction on the part of the plane $y^3 = \text{constant}$ in the deformed body which lies between the planes $\vartheta = \vartheta_1$, $\vartheta = \vartheta_2$, ($\vartheta_1 > \vartheta_2$) is

$$N_3 = \frac{1}{2} (\vartheta_1 - \vartheta_2) \left\{ \frac{1}{\lambda} \int_B^A \left[2 \left(c \frac{\partial W}{\partial c} + d \frac{\partial W}{\partial d} \right) + \rho R \right] r dr + [\rho^2 t_{\rho\rho}]_B^A \right\}, \quad (2.17)$$

and the moment about the y^3 -axis of the forces on this surface is

$$M_3 = \frac{\vartheta_1 - \vartheta_2}{\lambda} \int_B^A \left(c \frac{\partial W}{\partial a} + d \frac{\partial W}{\partial b} \right) r dr. \quad (2.18)$$

It should be emphasized that in the derivation of these results r , a , b , c , d and K are regarded as independent parameters. The corresponding relations which exhibit the form of W as a function of its strain components are obtained by introducing the appropriate expression for W in terms of these quantities, and making use of (2.2) or the relevant formulae for the combinations of $\bar{\gamma}_{(ij)}$ which are involved.

3. Constraints imposed by sets of cords. We now suppose the elastic body considered in the previous section to be reinforced by means of sets of thin flexible, inextensible cords which lie in the helical paths

$$r = r_i, \quad x^3 = r_i \theta \cot \alpha_i + \text{constant} \quad (i = 1, 2, \dots, n), \quad (3.1)$$

in the undeformed body. Each cord in the layer $r = r_i$ therefore cuts generators of this surface at a constant angle α_i . Two sets of cords i and j will be described as independent if $r_i \neq r_j$, or if $r_i = r_j$, and $\alpha_i \neq n\pi + \alpha_j$ where n is an integer. After deformation we assume that the surfaces $r = r_i$ become $\rho = \rho_i$, and that the angle α_i is changed to β_i .

The restriction upon the deformation (2.1) due to the inextensibility of the cords readily follows from geometrical considerations. Thus, if ds is an element of length lying along a path taken by one of the cords in the layer at $r = r_i$, we have before deformation

$$ds \cos \alpha_i = dx^3, \quad ds \sin \alpha_i = r_i d\theta,$$

and after deformation

$$\begin{aligned} ds \cos \beta_i &= dy^3 = c d\theta + d dx^3, \\ ds \sin \beta_i &= \rho_i d\vartheta = \rho_i (a d\theta + b dx^3). \end{aligned}$$

Combining these relations we obtain

$$\cos \beta_i = (c/r_i) \sin \alpha_i + d \cos \alpha_i, \quad \sin \beta_i = \rho_i \{ (a/r_i) \sin \alpha_i + b \cos \alpha_i \}, \quad (3.2)$$

and

$$\rho_i^2 (a \sin \alpha_i + b r_i \cos \alpha_i)^2 + (c \sin \alpha_i + d r_i \cos \alpha_i)^2 = r_i^2. \quad (3.3)$$

For incompressible materials, we have from (2.11)

$$\lambda \rho_i^2 = r_i^2 + K, \quad (3.4)$$

and (3.3) may then be regarded as a relationship between the constants a , b , c , d and K defining the deformation. Five independent sets of cords would imply five such independent relationships which could, in principle be solved for the unknown constants. We may therefore infer that a continuously varying deformation of the type (2.1) is usually only possible if there are not more than four independent sets of cords.

Intuitively we should expect that not more than two of these sets can lie in any one layer, and this conclusion is readily confirmed by means of (3.3). For if we write this relation in the form

$$\begin{aligned} \{(b^2\rho_i^2 + d^2)r_i^2 - (a^2\rho_i^2 + c^2)\} \cos 2\alpha_i + 2(ab\rho_i^2 + cd)r_i \sin 2\alpha_i \\ = 2r_i^2 - (a^2\rho_i^2 + c^2) - (b^2\rho_i^2 + d^2)r_i^2, \end{aligned} \quad (3.5)$$

and subtract from it a similar equation, with α_i replaced by α'_i , we obtain

$$\begin{aligned} \{(b^2\rho_i^2 + d^2)r_i^2 - (a^2\rho_i^2 + c^2)\} \sin(\alpha_i + \alpha'_i) \sin(\alpha_i - \alpha'_i) \\ = 2(ab\rho_i^2 + cd)r_i \cos(\alpha_i + \alpha'_i) \sin(\alpha_i - \alpha'_i). \end{aligned} \quad (3.6)$$

Thus if two sets making angles α_i , α'_i with the x^3 -direction in the undeformed layer are independent, so that $\alpha'_i \neq n\pi + \alpha_i$, (3.6) yields

$$\begin{aligned} \tan(\alpha_i + \alpha'_i) &= 2(ab\rho_i^2 + cd)r_i / \{(b^2\rho_i^2 + d^2)r_i^2 - (a^2\rho_i^2 + c^2)\} \\ &= Q \quad (\text{say}). \end{aligned} \quad (3.7)$$

Similarly, for a third set making an angle $\alpha''_i \neq n\pi + \alpha_i$ with the x^3 -direction we should obtain $\tan(\alpha_i + \alpha''_i) = Q$ and this could only be satisfied simultaneously with (3.7) if $\alpha''_i = n\pi + \alpha'_i$, which implies that the second and third sets of cords are not independent.

In general, each set of cords reduces by unity the number of degrees of freedom available to the deformation (2.1), and conversely each limitation upon this deformation obtained by restricting one of the constants a , b , c , d , K to have its initial value, diminishes by unity the number of independent sets of cords permissible. An exception occurs when $b = c = 0$ so that (3.3) is unchanged by changing the sign of α_i , and each layer may then contain two independent sets of cords making equal angles with generators of the surface in which they lie. For simple flexure, two such double layers are admissible; in the case of a cylindrical tube subjected to combined inflation and extension, the additional restriction $a = 1$ permits only one double layer.

4. Stress resultants in a layer of cords. If the cords are thin and sufficiently close together, each layer may be regarded as a thin continuous flexible surface which is inextensible along the paths containing the cords. Owing to the tensions in the individual cords, we may define, for the deformed layer at $\rho = \rho_i$, the following system of stress resultants:

forces T_{1i} , S_i acting in the longitudinal and tangential directions respectively across lines $y^3 = \text{constant}$, and measured per unit length of these lines in the deformed state, and

forces T_{2i} , S_i similarly measured, acting in the tangential and longitudinal directions respectively across the lines $\vartheta = \text{constant}$. The equality in the values of S_i can readily be demonstrated.

We assume the layer to contain only one set of cords lying initially in the paths (3.1), and denote by δ_i the distance between adjacent cords in the undeformed layer, measured along a curve orthogonal to them. The tension in each cord after deformation is denoted by τ_i . The spacing of the cords need not be uniform throughout the layer, but in the problems considered the quotient $\sigma_i = \tau_i/\delta_i$ is constant.

Let the lines $y^3 = \text{constant}$, $\vartheta = \text{constant}$ in the deformed layer at $\rho = \rho_i$ be intersected by $n_{\vartheta i}$, n_{3i} cords per unit length respectively. To evaluate these quantities we observe that the helix

$$r = r_i, \quad x^3 = r_i \theta \cot \alpha_i + k$$

is carried by the deformation (2.1) into the curve

$$\rho = \rho_i, \quad \vartheta = (a + br_i \cot \alpha_i)\theta + kb, \quad y^3 = (c + dr_i \cot \alpha_i)\theta + kd,$$

or, when θ is eliminated

$$\rho = \rho_i, \quad y^3 = \frac{(c + dr_i \cot \alpha_i)\vartheta + k\lambda}{a + br_i \cot \alpha_i}.$$

In passing to a point on a neighbouring curve, the increments Δy^3 , $\Delta \vartheta$ and Δk of y^3 , ϑ and k must therefore be connected by the relation

$$\Delta y^3 = \frac{(c + dr_i \cot \alpha_i) \Delta \vartheta + \lambda \Delta k}{a + br_i \cot \alpha_i}.$$

By putting Δy^3 , $\Delta \vartheta$ successively equal to zero in this equation, and observing from geometrical considerations that $\Delta k = \delta_i \operatorname{cosec} \alpha_i$, we may infer that

$$\begin{aligned} n_{\vartheta i} &= (c \sin \alpha_i + dr_i \cos \alpha_i)/(\lambda \rho_i \delta_i), \\ n_{3i} &= (a \sin \alpha_i + br_i \cos \alpha_i)/(\lambda \delta_i). \end{aligned} \tag{4.1}$$

Remembering (3.2), a simple resolution of forces in the layer initially at $r = r_i$ now yields

$$\begin{aligned} T_{1i} &= n_{\vartheta i} \tau_i \cos \beta_i = \sigma_i (c \sin \alpha_i + dr_i \cos \alpha_i)^2 / (\lambda r_i \rho_i), \\ T_{2i} &= n_{3i} \tau_i \sin \beta_i = \sigma_i \rho_i (a \sin \alpha_i + br_i \cos \alpha_i)^2 / (\lambda r_i), \\ S_i &= n_{\vartheta i} \tau_i \sin \beta_i = n_{3i} \tau_i \cos \beta_i \\ &= \sigma_i (a \sin \alpha_i + br_i \cos \alpha_i)(c \sin \alpha_i + dr_i \cos \alpha_i) / (\lambda r_i), \end{aligned} \tag{4.2}$$

and from these, with (3.3), we have

$$\begin{aligned}\lambda\rho_i(T_{1i} + T_{2i}) &= r_i\sigma_i, \\ T_{1i}T_{2i} &= S_i^2.\end{aligned}\tag{4.3}$$

The stress resultants for a layer containing two sets of cords may be obtained by adding expressions of the type (4.2) for each set of cords.

5. The forces on the composite body. In considering the composite body, we may apply the formulæ of §2 separately to each section of the elastic body contained between adjacent layers of cords, these being regarded as boundary surfaces, and consider similarly the equilibrium of each layer of cords under the influence of the forces exerted by the elastic material on either side. We shall confine our attention to incompressible materials, for which the formulæ (2.13) to (2.18) apply, and suppose the elastic body described in §2 to be reinforced with four layers, each containing one independent set of cords of the type described in §§3 and 4. In (3.1) we therefore assume $n = 4$, $r_i \geq r_{i+1}$.

The normal components of the surface forces acting on the boundaries of the elastic material adjoining the layer of cords initially at $r = r_i$ are denoted by Π_{iA} , Π_{iB} . These components are directed outwards from the elastic material on which they act, and are measured per unit area of deformed surface; the forces Π_{iA} , Π_{iB} act on the sections of the elastic material which lie in the undeformed body nearer to the surfaces $r = A$, $r = B$ respectively. The normal components of the surface traction on the latter surfaces are denoted by R_A , R_B respectively. It follows from (2.13) that the tangential components of the surface forces on the cylindrical boundary surfaces are zero.

Applying the first of equations (2.13) to each section of the tube in turn and remembering (2.14) we now have

$$\begin{aligned}R_A &= \int_{\rho_1}^{A'} \left\{ \frac{\lambda\rho}{K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right) - R \right\} d\rho + \Pi_{1A}, \\ \Pi_{iB} &= \int_{\rho_{i+1}}^{\rho_i} \left\{ \frac{\lambda\rho}{K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right) - R \right\} d\rho + \Pi_{i+1,A}, \\ &\hspace{25em} (i = 1, 2, 3),\end{aligned}\tag{5.1}$$

$$\Pi_{4B} = \int_{B'}^{\rho_4} \left\{ \frac{\lambda\rho}{K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right) - R \right\} d\rho + R_B.$$

Also, we may regard the layer of cords at $\rho = \rho_i$ as a thin flexible sheet in equilibrium under the action of a uniform radial force $\Pi_{iA} - \Pi_{iB}$ and the tensions at its boundaries, and write

$$T_{2i} = \rho_i(\Pi_{iA} - \Pi_{iB}), \quad (i = 1 \text{ to } 4).\tag{5.2}$$

Combining (5.1) and (5.2) we obtain

$$\begin{aligned}
 R_A - R_B &= \int_{B'}^{A'} \left\{ \frac{\lambda \rho}{K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right) - R \right\} d\rho + \sum_{i=1}^4 \frac{T_{2i}}{\rho_i}, \\
 R_A - \Pi_{iA} &= \int_{\rho_i}^{A'} \left\{ \frac{\lambda \rho}{K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right) - R \right\} d\rho + \sum_{s=1}^{i-1} \frac{T_{2s}}{\rho_s} \\
 &\quad (i = 2, 3, 4), \\
 \Pi_{iB} - R_B &= \int_{B'}^{\rho_i} \left\{ \frac{\lambda \rho}{K} \left(b \frac{\partial W}{\partial b} - c \frac{\partial W}{\partial c} - r \frac{\partial W}{\partial r} \right) - R \right\} d\rho + \sum_{s=i+1}^4 \frac{T_{2s}}{\rho_s} \\
 &\quad (i = 1, 2, 3).
 \end{aligned} \tag{5.3}$$

Since the cords are thin, and the parameters a, b, c, d have constant values throughout the elastic body, it follows from (2.11) that K has the same value in each section of the material, and ρ may be regarded as a single continuous function of r irrespective of the presence of the cords. The sum of several integrals of the type contained in (2.13), (2.15) to (2.18) and (5.1) for adjacent sections of the body may therefore be written as a single integral evaluated between appropriate limits, and this procedure has been adopted in deriving (5.3).

The resultant forces and couples acting across planes $\vartheta = \text{constant}$ and $y^3 = \text{constant}$ in the deformed body may be obtained by applying the formulæ of §2 to each section of the elastic material, and adding the contributions arising from the cords. Thus if we employ the notation for the forces and couples adopted in §2 with the addition of a prime to indicate the resultant quantities for the composite body, then remembering the values which $t_{\rho\rho}$ must assume at each interface between the cords and the elastic material, we have from (2.15) and (5.2)

$$\begin{aligned}
 N'_\vartheta &= A'R_A - \sum_{i=1}^4 \rho_i(\Pi_{iA} - \Pi_{iB}) - B'R_B + \int_{B'}^{A'} \rho R d\rho + \sum_{i=1}^4 T_{2i} \\
 &= A'R_A - B'R_B + \int_{B'}^{A'} \rho R d\rho.
 \end{aligned} \tag{5.4}$$

Similarly, from (2.15) to (2.18), (4.2) and (5.2) we have

$$\begin{aligned}
 M'_\vartheta &= \frac{1}{2\lambda} \left\{ \int_B^A \left[2 \left(a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} \right) + \rho R \right] r dr + \lambda (A'^2 R_A - B'^2 R_B) \right. \\
 &\quad \left. + \sum_{i=1}^4 \frac{\sigma_i \rho_i^2}{r_i} (a \sin \alpha_i + b r_i \cos \alpha_i)^2 \right\},
 \end{aligned}$$

$$\begin{aligned}
S'_j &= \frac{1}{\lambda} \left\{ \int_B^A \left(a \frac{\partial W}{\partial c} + b \frac{\partial W}{\partial d} \right) r \, dr \right. \\
&\quad \left. + \sum_{i=1}^4 \frac{\sigma_i}{r_i} (a \sin \alpha_i + b r_i \cos \alpha_i) (c \sin \alpha_i + d r_i \cos \alpha_i) \right\}, \\
N'_3 &= \frac{\vartheta_1 - \vartheta_2}{2\lambda} \left\{ \int_B^A \left[2 \left(c \frac{\partial W}{\partial c} + d \frac{\partial W}{\partial d} \right) + \rho R \right] r \, dr + \lambda (A'^2 R_A - B'^2 R_B) \right. \\
&\quad \left. + \sum_{i=1}^4 \frac{\sigma_i}{r_i} [3(c \sin \alpha_i + d r_i \cos \alpha_i)^2 - r_i^2] \right\}, \\
M'_3 &= \frac{\vartheta_1 - \vartheta_2}{\lambda} \left\{ \int_B^A \left(c \frac{\partial W}{\partial a} + d \frac{\partial W}{\partial b} \right) r \, dr \right. \\
&\quad \left. + \sum_{i=1}^4 \frac{\sigma_i \rho_i^2}{r_i} (a \sin \alpha_i + b r_i \cos \alpha_i) (c \sin \alpha_i + d r_i \cos \alpha_i) \right\}.
\end{aligned} \tag{5.5}$$

An alternative expression for M'_3 may be obtained by introducing the identity (2.14) into the integral term in the first of these equations. We may observe that if there are no body forces or surface forces on the curved surfaces $\rho = A'$, $\rho = B'$, N'_3 is zero. This may be regarded as a generalization of the results obtained in similar cases by RIVLIN (1949) and others.

Expressions for the stresses may be obtained from (2.13) by giving the term $[t_{\rho\rho}]_{\rho=\rho_0}$, and the limits of integration in the first of these equations, values appropriate to the section of the material under consideration, and making use of (5.3). Remembering (3.4) and (4.2), equations (3.3), (5.5) and the first of (5.3) may be regarded as nine relations for the determination of the nine unknown quantities σ_i , a , b , c , d and K when the forces applied to the elastic body are known.

The preceding analysis may evidently be adapted to a body containing fewer sets of cords by omitting the unwanted terms.

6. Inflation and extension of a cylindrical tube. Particular cases may be examined by suitably choosing, in the results of §§3 to 5, the parameters defining the deformation and the constants which specify the configuration of the cords.

Thus to consider the simple extension and inflation of a cylindrical tube containing two sets of cords symmetrically placed in a single layer we may write

$$\begin{aligned}
\Theta &= \pi, & r_1 &= r_2 = r_0, & \rho_1 &= \rho_2 = \rho_0, \\
\alpha_1 &= -\alpha_2 = \alpha_0, & \beta_1 &= -\beta_2 = \beta_0, \\
a &= 1, & b &= c = 0, & d &= \lambda, & R &= 0,
\end{aligned} \tag{6.1}$$

and omit the terms from the preceding analysis which arise from the layers of cords at $r = r_3$, $r = r_4$. We shall also assume that $\sigma_1 = \sigma_2 = \sigma$. Denoting the stress resultants in the double layer by T_1 , T_2 , S , we have from (4.2)

$$T_1 = \frac{2\sigma\lambda r_0}{\rho_0} \cos^2 \alpha_0, \quad T_2 = \frac{2\sigma\rho_0}{\lambda r_0} \sin^2 \alpha_0, \quad S = 0. \quad (6.2)$$

The first of (5.3), with (2.11) and (6.2) yields

$$R_A - R_B = -\frac{1}{K} \int_B^A r^2 \frac{\partial W}{\partial r} dr + \frac{2\sigma}{\lambda r_0} \sin^2 \alpha_0, \quad (6.3)$$

and from (5.5) the resultant longitudinal force on the tube is

$$N'_3 = 2\pi \left\{ \int_B^A r \frac{\partial W}{\partial \lambda} dr + \frac{1}{2} (A'^2 R_A - B'^2 R_B) + \frac{\sigma r_0}{\lambda} (3\lambda^2 \cos^2 \alpha_0 - 1) \right\}. \quad (6.4)$$

If the deformation is produced by means of a simple tensile force N'_3 alone, we may put $R_A = R_B = 0$ in (6.3) and (6.4), and make use of (3.3) to obtain

$$\begin{aligned} \sigma &= \frac{\lambda r_0}{2K \sin^2 \alpha_0} \int_B^A r^2 \frac{\partial W}{\partial r} dr, \\ N'_3 &= \frac{\pi}{K\lambda} \int_B^A \left\{ 2K\lambda \frac{\partial W}{\partial \lambda} + (\chi r_0^2 - K)r \frac{\partial W}{\partial r} \right\} r dr, \end{aligned} \quad (6.5)$$

where

$$\chi = 2\lambda^3 \cot^2 \alpha_0 - 1.$$

Similarly, if the deformation is produced by means of a uniform inflating pressure P , we have $R_A = N'_3 = 0$, $R_B = -P$ and

$$\begin{aligned} \sigma &= \frac{\lambda r_0 \int_B^A \left\{ (B^2 + K)r \frac{\partial W}{\partial r} - 2K\lambda \frac{\partial W}{\partial \lambda} \right\} r dr}{2K(\chi r_0^2 + B^2) \sin^2 \alpha_0}, \\ P &= -\frac{\int_B^A \left\{ 2K\lambda \frac{\partial W}{\partial \lambda} + (\chi r_0^2 - K)r \frac{\partial W}{\partial r} \right\} r dr}{K(\chi \sigma_0^2 + B^2)}. \end{aligned} \quad (6.6)$$

It has been shown by ADKINS & RIVLIN (1955), that for an isotropic incompressible tube in simple extension, the sign of σ depends only upon the magnitude of λ and the angle $2\alpha_0$ between the cords in the undeformed layer. A simple result of this nature does not however follow from (6.6) for the case of inflation.

7. Flexure and extension of a curved block. By putting

$$\begin{aligned} r_1 &= r_2 = r_0, & \rho_1 &= \rho_2 = \rho_0, \\ \alpha_1 &= -\alpha_2 = \alpha_0, & \beta_1 &= -\beta_2 = \beta_0, \\ b &= c = 0, & ad &= \lambda, & R &= 0 \end{aligned} \quad (7.1)$$

in the results of §§3 to 5, and omitting the terms which arise from the layers of cords at $r = r_3$, $r = r_4$, we may examine similarly the simple flexure and extension of a curved block reinforced with one double layer of cords lying in

the surface $r = r_0$, the cords being symmetrically placed with respect to the generators of this surface. We again assume that $\sigma_1 = \sigma_2 = \sigma$.

Formulae similar to (6.2) and (6.3) are readily obtained by introducing (7.1) into (4.2) and (5.3), and if the curved surfaces of the deformed body are free from applied force, so that $R_A = R_B = 0$, we have

$$\sigma = \frac{dr_0}{2Ka \sin^2 \alpha_0} \int_B^A r^2 \frac{\partial W}{\partial r} dr, \quad (7.2)$$

which corresponds to the first of (6.5). From (5.4) $N'_\vartheta = 0$, and the forces on either of the planes initially at $\theta = \pm\Theta$ then reduce to a couple M given by

$$\begin{aligned} M &= 2LdM'_\vartheta \\ &= -\frac{L}{Ka} \int_B^A (r^2 - r_0^2) r^2 \frac{\partial W}{\partial r} dr, \end{aligned} \quad (7.3)$$

whilst the resultant longitudinal force N on either of the planes initially at $x^3 = \pm L$ is

$$N = \frac{\Theta}{Ka} \int_B^A \left\{ 2Ka \frac{\partial W}{\partial d} + (2d^2 r_0^2 \cot^2 \alpha_0 - a^2 \rho_0^2) r \frac{\partial W}{\partial r} \right\} r dr. \quad (7.4)$$

8. Shear and flexure of a cuboid. Consideration of the problem analogous to that defined in §§2 to 5 in which the undeformed body is a cuboid presents no difficulty. We assume that the initial boundary surfaces are the planes $x^1 = A$, $x^1 = B$, $x^2 = \pm C$, $x^3 = \pm L$, and that the cords lie in the families of parallel straight lines

$$x^1 = x^1_i, \quad x^3 = x^2 \cot \alpha_i + \text{constant}, \quad (i = 1 \text{ to } 4, x^1_i \geq x^1_{i+1}), \quad (8.1)$$

the elastic material now being homogeneous and rectilinearly æolotropic. The deformation is described by

$$\rho = \rho(x^1), \quad \vartheta = ax^2 + bx^3, \quad y^3 = cx^2 + dx^3, \quad (8.2)$$

the planes $x^1 = A$, $x^1 = B$ again being deformed into the curved surfaces $\rho = A'$, $\rho = B'$.

It has been shown by ADKINS (1955 a, b) that the formulæ of §2 continue to apply provided we replace λ/r , a/r , c/r by λ , a , c respectively, $\partial/\partial r$, dr by $\partial/\partial x^1$, dx^1 respectively, the incompressibility condition (2.11) by

$$\lambda \rho^2 = 2x^1 + K, \quad (8.3)$$

and the identity (2.14) by

$$2 \left(a \frac{\partial W}{\partial a} + b \frac{\partial W}{\partial b} \right) = \lambda \rho^2 \frac{\partial W}{\partial x^1} = (2x^1 + K) \frac{\partial W}{\partial x^1}. \quad (8.4)$$

Terms of the type $a\partial W/\partial a$, $a\partial W/\partial c$ in (2.13) and (2.15) to (2.18) are unaltered. The relations of §§3 and 4 may evidently be similarly modified by replacing

a/r_i , c/r_i , by a and c respectively, and using the appropriate form (8.3) for the incompressibility condition in place of (3.4). Equations (5.2) and (5.4) are unchanged in form, and the first of (5.1) is now replaced by

$$\begin{aligned} R_A &= \int_{\rho_1}^{A'} \left(\lambda \rho \frac{\partial W}{\partial x^1} - R \right) d\rho + \Pi_{1A} \\ &= W(A') - W(\rho_1) - \int_{\rho_1}^{A'} R d\rho + \Pi_{1A}, \end{aligned} \quad (8.5)$$

the subsequent formulæ of (5.1) and (5.3) being modified accordingly. In particular, remembering (5.2) and the modified forms of (4.2), we may replace the first of (5.3) by

$$R_A - R_B = W(A') - W(B') - \int_{B'}^{A'} R d\rho + \frac{1}{\lambda} \sum_{i=1}^4 \sigma_i (a \sin \alpha_i + b \cos \alpha_i)^2. \quad (8.6)$$

The changes in the formulæ (5.5) for the resultant forces and couples acting across planes in the deformed body follow the lines already indicated. The identity (8.4), however, enables the formula for M'_δ to be written

$$\begin{aligned} M'_\delta &= \frac{1}{2} \left\{ A'^2 [R_A + W(A')] - B'^2 [R_B + W(B')] + \int_{B'}^{A'} \rho^2 R d\rho \right. \\ &\quad \left. - \frac{2}{\lambda} \int_B^A W(x^1) dx^1 + \frac{1}{\lambda} \sum_{i=1}^4 \sigma_i \rho_i^2 (a \sin \alpha_i + b \cos \alpha_i)^2 \right\}. \end{aligned} \quad (8.7)$$

If there is a single layer at $x^1 = x_0^1$ containing either one or two sets of cords, and $R_A = R_B = R = 0$, the formulæ (8.6) and (8.7) may be combined to yield

$$M'_\delta = \frac{1}{\lambda} \left\{ (A - x_0^1) W(A) - (B - x_0^1) W(B) - \int_B^A W(x^1) dx^1 \right\}. \quad (8.8)$$

The simple flexure problem may again be considered separately by the method of §7.

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