Solution of Some Problems in the Exact Theory of Visco-Elasticity

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1. Introduction. It has been shown in a previous paper (RIVLIN, 1955), that if we assume that in a visco-elastic fluid, which is isotropic in its state of rest, the stress components t_{ij} in a rectangular Cartesian coordinate system x_i at a point x_i of the fluid are expressible as polynomials in the gradients of velocity, acceleration, second acceleration, $\dots$, $(n - 1)^{\text{th}}$ acceleration at the point considered, then the stress matrix $\mathbf{T}$ ($= || t_{ij} ||$) may be expressed as a polynomial in n kinematic matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$. The coefficients in this polynomial are scalar invariants of the matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ and are expressible as polynomials in traces of products formed from these matrices. The kinematic matrices $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ are defined by the equations

$$\mathbf{A}_r = || A_{ij}^{(r)} || \quad (r = 1, 2, \dots, n), \quad (1.1)$$

$$A_{ij}^{(1)} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \quad (1.2)$$

$$A_{ij}^{(r+1)} = \frac{\partial A_{ij}^{(r)}}{\partial t} + v_l \frac{\partial A_{ij}^{(r)}}{\partial x_l} + A_{mi}^{(r)} \frac{\partial v_m}{\partial x_j} + A_{mj}^{(r)} \frac{\partial v_m}{\partial x_i} \quad (1.3)$$

$$(r = 1, 2, \dots, n - 1),$$

where v_i is the velocity of the fluid at the point x_i .

In the present paper we shall consider the forces necessary to support in such fluids certain simple types of steady-state laminar flow. These are:

- (i) rectilinear laminar flow;
- (ii) torsional flow between two parallel discs;
- (iii) helical flow in the annular region between two coaxial cylinders.

It will be assumed throughout the paper that the fluid is incompressible and it may be remarked that the types of flow considered do not violate this con-

dition. It will also be assumed that at a rigid boundary of the fluid, the velocity of the fluid and of the boundary are equal.

It will be seen that, for each of the states of flow considered in the present paper, $\mathbf{A}_r = \mathbf{0}$ for $r > 2$. Consequently in these cases the stress matrix $\mathbf{T}$ can be expressed as a polynomial in the two kinematic matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, the coefficients in this matrix polynomial being scalar invariants of $\mathbf{A}_1$ and $\mathbf{A}_2$. It follows (RIVLIN, 1955) that the stress matrix $\mathbf{T}$ must be expressible in the form

$$\begin{aligned} \mathbf{T} = & \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_1^2 + \alpha_4 \mathbf{A}_2^2 + \alpha_5 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ & + \alpha_6 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \alpha_7 (\mathbf{A}_1 \mathbf{A}_2^2 + \mathbf{A}_2^2 \mathbf{A}_1) \\ & + \alpha_8 (\mathbf{A}_1^2 \mathbf{A}_2^2 + \mathbf{A}_2^2 \mathbf{A}_1^2), \end{aligned} \quad (1.4)$$

where $\alpha_0, \alpha_1, \dots, \alpha_8$ are polynomials in the ten scalar invariants $\text{tr } \mathbf{A}_1, \text{tr } \mathbf{A}_1^2, \text{tr } \mathbf{A}_1^3, \text{tr } \mathbf{A}_2, \text{tr } \mathbf{A}_2^2, \text{tr } \mathbf{A}_2^3, \text{tr } \mathbf{A}_1 \mathbf{A}_2, \text{tr } \mathbf{A}_1^2 \mathbf{A}_2, \text{tr } \mathbf{A}_1 \mathbf{A}_2^2$ and $\text{tr } \mathbf{A}_1^2 \mathbf{A}_2^2$.

For an incompressible material the stress corresponding to a specified state of flow must be indeterminate to the extent of an arbitrary hydrostatic pressure p . Consequently, we may write $\alpha_0 = -p$ where p is indeterminate for a specified state of flow, and we obtain

$$\begin{aligned} \mathbf{T} = & -p \mathbf{I} + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2 + \alpha_3 \mathbf{A}_1^2 + \alpha_4 \mathbf{A}_2^2 + \alpha_5 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) \\ & + \alpha_6 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \alpha_7 (\mathbf{A}_1 \mathbf{A}_2^2 + \mathbf{A}_2^2 \mathbf{A}_1) \\ & + \alpha_8 (\mathbf{A}_1^2 \mathbf{A}_2^2 + \mathbf{A}_2^2 \mathbf{A}_1^2). \end{aligned} \quad (1.5)$$

2. Rectilinear Laminar Flow. We consider that the fluid is contained between two infinite rigid surfaces located on the planes $x_2 = c$ and $x_2 = 0$ of a rectangular Cartesian coordinate system x_i , that the surface at $x_2 = 0$ is fixed and the surface at $x_2 = c$ moves parallel to the x_1 axis with a velocity Kc . We assume that the velocity v_i at each point of the fluid is given by

$$v_1 = Kx_2, \quad v_2 = v_3 = 0, \quad (2.1)$$

where K is constant throughout the fluid and independent of time. Substituting these expressions for v_i in equations (1.2) and (1.3), it is seen that, for the flow described by equations (2.1), the kinematic matrices take the forms

$$\mathbf{A}_1 = \begin{Bmatrix} 0, & K, & 0 \\ K, & 0, & 0 \\ 0, & 0, & 0 \end{Bmatrix}, \quad \mathbf{A}_2 = \begin{Bmatrix} 0, & 0, & 0 \\ 0, & 2K^2, & 0 \\ 0, & 0, & 0 \end{Bmatrix}$$

and

$$\mathbf{A}_r = \mathbf{0} \quad (r > 2). \quad (2.2)$$

Thus,

$$\begin{aligned} \operatorname{tr} \mathbf{A}_1 &= 0, & \operatorname{tr} \mathbf{A}_1^2 &= 2K^2, & \operatorname{tr} \mathbf{A}_1^3 &= 0, & \operatorname{tr} \mathbf{A}_2 &= 2K^2, & \operatorname{tr} \mathbf{A}_2^2 &= 4K^4, \\ \operatorname{tr} \mathbf{A}_2^3 &= 8K^6, & \operatorname{tr} \mathbf{A}_1 \mathbf{A}_2 &= 0, & \operatorname{tr} \mathbf{A}_1^2 \mathbf{A}_2 &= 2K^4, & \operatorname{tr} \mathbf{A}_1 \mathbf{A}_2^2 &= 0 \end{aligned}$$

and

$$\operatorname{tr} \mathbf{A}_1^2 \mathbf{A}_2^2 = 4K^6. \quad (2.3)$$

Substituting from (2.2) in equation (1.5), we see that the stress components t_{ij} are given by

$$\begin{aligned} t_{11} &= -p + \alpha_3 K^2, & t_{22} &= -p + K^2[(2\alpha_2 + \alpha_3) + 4(\alpha_4 + \alpha_6)K^2 + 8\alpha_8 K^4], \\ t_{12} &= K[\alpha_1 + 2\alpha_5 K^2 + 4\alpha_7 K^4], & t_{33} &= -p, & t_{31} &= t_{23} = 0. \end{aligned} \quad (2.4)$$

If no body forces act on the fluid, it is immediately apparent that the equations of equilibrium are satisfied by the expressions (2.4) for the stress components and that p is then constant throughout the fluid.

3. Torsional Flow in a Cylindrical Mass of Fluid. We consider a cylindrical mass of fluid of radius a and length l to have its curved surface force-free and to be maintained in a state of torsional flow by means of two parallel rigid discs of radius a in contact with its plane ends, one of which rotates relative to the other with a constant angular velocity Ω . Let us take as our reference system a rectangular Cartesian coordinate system x_i , the x_3 axis of which coincides with the axis of the cylinder and the origin of which lies at the center of one of the discs. We shall assume that the disc located on $x_3 = 0$ is stationary and the disc located on $x_3 = l$ rotates with an angular velocity Ω .

If the velocity v_i throughout the fluid is given by

$$v_1 = -\psi x_2 x_3, \quad v_2 = \psi x_1 x_3 \quad \text{and} \quad v_3 = 0, \quad (3.1)$$

where

$$\psi = \Omega/l, \quad (3.2)$$

it can readily be seen that the boundary conditions on the two discs are satisfied. Substituting from (3.1) in equations (1.2) and (1.3), we obtain

$$\mathbf{A}_1 = \begin{vmatrix} 0, & 0, & -\psi x_2 \\ 0, & 0, & \psi x_1 \\ -\psi x_2, & \psi x_1, & 0 \end{vmatrix}, \quad \mathbf{A}_2 = \begin{vmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 2\psi^2 r^2 \end{vmatrix} \quad (3.3)$$

and

$$\mathbf{A}_r = \mathbf{0} \quad (r > 2), \quad (3.4)$$

where $r^2 = x_1^2 + x_2^2$. If the direction of the x_1 axis is chosen so that $x_2 = 0$ for the point of the fluid considered, we have $x_1 = r$ and hence from (3.3)

$$\mathbf{A}_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & \psi r \\ 0 & \psi r & 0 \end{vmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\psi^2 r^2 \end{vmatrix}. \quad (3.5)$$

Introducing the expressions (3.5) into (1.5), we obtain

$$\begin{aligned} t_{11} &= -p, & t_{22} &= -p + \alpha_3 \psi^2 r^2, \\ t_{33} &= -p + (2\alpha_2 + \alpha_3) \psi^2 r^2 + 4(\alpha_4 + \alpha_6) \psi^4 r^4 + 8\alpha_8 \psi^6 r^6, \\ t_{23} &= \psi r (\alpha_1 + 2\alpha_5 \psi^2 r^2 + 4\alpha_7 \psi^4 r^4) \end{aligned}$$

and

$$t_{31} = t_{12} = 0. \quad (3.6)$$

The quantities t_{ij} given by equations (3.6) are the stress components in a cylindrical polar coordinate system (r, θ, z) the origin of which is at the origin of the rectangular Cartesian coordinate system x_i and the z -axis of which coincides with the x_3 axis of the system x_i . The 1, 2 and 3 directions are radial, azimuthal and longitudinal respectively. The α 's are all expressible as polynomials in the quantities $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$, $\text{tr } \mathbf{A}_1^3$, $\text{tr } \mathbf{A}_2$, $\text{tr } \mathbf{A}_2^2$, $\text{tr } \mathbf{A}_2^3$, $\text{tr } \mathbf{A}_1 \mathbf{A}_2$, $\text{tr } \mathbf{A}_1^2 \mathbf{A}_2$, $\text{tr } \mathbf{A}_1 \mathbf{A}_2^2$, $\text{tr } \mathbf{A}_1^2 \mathbf{A}_2^2$ which, from (3.5), are obtainable from the expressions (2.3) for these quantities by substituting ψr for K .

If no body forces are applied to the fluid and centrifugal forces are neglected, one of the equations of equilibrium takes the form

$$\frac{\partial t_{11}}{\partial r} + \frac{t_{11} - t_{22}}{r} = 0, \quad (3.7)$$

the remaining equations being automatically satisfied. From (3.7) and (3.6), we obtain

$$\frac{\partial p}{\partial r} = -\alpha_3 \psi^2 r. \quad (3.8)$$

Whence,

$$p = -\int_a^r \alpha_3 \psi^2 r \, dr, \quad (3.9)$$

if we assume that the curved surface of the cylindrical mass of fluid is force-free (*i.e.* $t_{11} = 0$ when $r = a$). Introducing the expression (3.9) for p into the expression (3.6) for t_{33} , we obtain

$$t_{33} = \int_a^r \alpha_3 \psi^2 r \, dr + (2\alpha_2 + \alpha_3) \psi^2 r^2 + 4(\alpha_4 + \alpha_6) \psi^4 r^4 + 8\alpha_8 \psi^6 r^6. \quad (3.10)$$

Thus, in order to maintain in the cylindrical mass of fluid the motion described by equations (3.1), we have to apply to the plane ends of the fluid mass azimuthal surface tractions of magnitude t_{23} per unit area and normal surface tractions of magnitude t_{33} per unit area, where t_{23} is given by the fourth of equations (3.6) and t_{33} is given by equation (3.10). The azimuthal surface tractions t_{23} are, of course, statically equivalent to a couple of magnitude

$$\int_0^a 2\pi r^2 t_{23} dr = 2\pi\psi \int_0^a r^3(\alpha_1 + 2\alpha_5\psi^2 r^2 + 4\alpha_7\psi^4 r^4) dr. \quad (3.11)$$

4. Helical Flow in an Annular Mass of Fluid. We consider an annular mass of fluid to be contained between two infinite coaxial cylinders of radii a_1 and a_2 respectively ($a_1 > a_2$). We consider also that the outer and inner cylinders rotate about their common axis with angular velocity Ω_1 and Ω_2 respectively and that a constant pressure gradient P , parallel to the axis of the cylinders, exists in the fluid.

We assume that each particle of the fluid moves in a helical path about the axis of the cylinders with an angular velocity ω about the axis and a velocity u parallel to it, ω and u being functions of r only, the radial distance of the particle from the axis of the cylinders. In a rectangular coordinate system x_i , the x_3 axis of which coincides with the axis of the cylinders, the velocity v_i of a particle of the fluid at x_i is given by

$$v_1 = -\omega x_2, \quad v_2 = \omega x_1 \quad \text{and} \quad v_3 = u. \quad (4.1)$$

Substituting from (4.1) in equations (1.2) and (1.3), we obtain

$$\mathbf{A}_1 = \begin{vmatrix} -2 \frac{x_1 x_2}{r} \omega', & \frac{x_1^2 - x_2^2}{r} \omega', & \frac{x_1}{r} u' \\ \frac{x_1^2 - x_2^2}{r} \omega', & 2 \frac{x_1 x_2}{r} \omega', & \frac{x_2}{r} u' \\ \frac{x_1}{r} u', & \frac{x_2}{r} u', & 0 \end{vmatrix},$$

$$\mathbf{A}_2 = \begin{vmatrix} 2x_1^2(\omega'^2 + u'^2/r^2), & 2x_1 x_2(\omega'^2 + u'^2/r^2), & 0 \\ 2x_1 x_2(\omega'^2 + u'^2/r^2), & 2x_2^2(\omega'^2 + u'^2/r^2), & 0 \\ 0, & 0, & 0 \end{vmatrix}, \quad (4.2)$$

and

$$\mathbf{A}_2 = \mathbf{0} \quad \text{for} \quad r > 2,$$

where $\omega' = d\omega/dr$ and $u' = du/dr$. If the direction of the x_1 axis is chosen so that $x_2 = 0$ for the point of the fluid considered, we have $x_1 = r$ and equations (4.2) yield

$$\mathbf{A}_1 = \begin{vmatrix} 0, & r\omega', & u' \\ r\omega', & 0, & 0 \\ u', & 0, & 0 \end{vmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{vmatrix} 2(r^2\omega'^2 + u'^2), & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{vmatrix}. \quad (4.3)$$

Introducing the expressions (4.3) into (1.5), we obtain

$$\begin{aligned} t_{11} &= -p + (2\alpha_2 + \alpha_3)(r^2\omega'^2 + u'^2) + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 + 8\alpha_8(r^2\omega'^2 + u'^2)^3, \\ t_{22} &= -p + \alpha_3 r^2\omega'^2, \quad t_{33} = -p + \alpha_3 u'^2, \\ t_{23} &= \alpha_3 r\omega' u', \quad t_{31} = u'[\alpha_1 + 2\alpha_5(r^2\omega'^2 + u'^2) + 4\alpha_7(r^2\omega'^2 + u'^2)^2] \end{aligned}$$

and

$$t_{12} = r\omega'[\alpha_1 + 2\alpha_5(r^2\omega'^2 + u'^2) + 4\alpha_7(r^2\omega'^2 + u'^2)^2]. \quad (4.4)$$

From the symmetry of the flow field it is apparent that p is a function of r and z only.

The quantities t_{ij} given by equation (4.4) are again the stress components in a cylindrical polar coordinate system (r, θ, z) , the origin of which is at the origin of the rectangular Cartesian coordinate system x_i and the z -axis of which coincides with the x_3 axis of the system x_i . The α 's are expressible as polynomials in the quantities $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$, $\text{tr } \mathbf{A}_1^3$, $\text{tr } \mathbf{A}_2$, $\text{tr } \mathbf{A}_2^2$, $\text{tr } \mathbf{A}_2^3$, $\text{tr } \mathbf{A}_1\mathbf{A}_2$, $\text{tr } \mathbf{A}_1^2\mathbf{A}_2$, $\text{tr } \mathbf{A}_1\mathbf{A}_2^2$, $\text{tr } \mathbf{A}_1^2\mathbf{A}_2^2$ which, from (4.3) are obtainable from the expressions (2.3) for these quantities by substituting $(r^2\omega'^2 + u'^2)$ for K^2 .

If the body forces applied to the fluid are zero, the equations of motion take the form

$$\begin{aligned} \frac{\partial t_{11}}{\partial r} + \frac{t_{11} - t_{22}}{r} &= -\rho r\omega^2, \\ \frac{\partial t_{12}}{\partial r} + \frac{2}{r} t_{12} &= 0 \quad \text{and} \quad \frac{\partial t_{31}}{\partial r} + \frac{1}{r} t_{31} + \frac{\partial t_{33}}{\partial z} = 0 \end{aligned} \quad (4.5)$$

where ρ is the density of the fluid.

Since a constant pressure gradient P exists in the fluid, we have

$$\frac{\partial t_{33}}{\partial z} = -P. \quad (4.6)$$

From (4.6) and the third of equations (4.5), we obtain

$$t_{31} = \frac{1}{2}Pr + \frac{A}{r}, \quad (4.7)$$

where A is a constant of integration. From the second of equations (4.5), we obtain

$$t_{12} = B/r^2, \quad (4.8)$$

where B is a constant of integration. Employing the expressions (4.4) for t_{31} and t_{12} in equation (4.7) and (4.8), we obtain

$$u'[\alpha_1 + 2\alpha_5(r^2\omega'^2 + u'^2) + 4\alpha_7(r^2\omega'^2 + u'^2)^2] = \frac{1}{2}Pr + \frac{A}{r}$$

and

$$\omega'[\alpha_1 + 2\alpha_5(r^2\omega'^2 + u'^2) + 4\alpha_7(r^2\omega'^2 + u'^2)^2] = B/r^3. \quad (4.9)$$

Equations (4.9) are a pair of simultaneous, first-order (generally non-linear) differential equations for the determination of u and ω subject to the conditions that $u = 0$ when $r = a_1$ and $r = a_2$ and $\omega = \Omega_1$ when $r = a_1$ and $\omega = \Omega_2$ when $r = a_2$.

Substituting from (4.4) in the first of equations (4.5), we obtain

$$\begin{aligned} \frac{\partial p}{\partial r} = & \rho r \omega^2 + \frac{\partial}{\partial r} [(2\alpha_2 + \alpha_3)(r^2\omega'^2 + u'^2) \\ & + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 + 8\alpha_8(r^2\omega'^2 + u'^2)^3] \\ & + \frac{1}{r} [2\alpha_2(r^2\omega'^2 + u'^2) + \alpha_3 u'^2 + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 \\ & + 8\alpha_8(r^2\omega'^2 + u'^2)^3]. \end{aligned} \quad (4.10)$$

Whence, with (4.6),

$$\begin{aligned} p = & (2\alpha_2 + \alpha_3)(r^2\omega'^2 + u'^2) + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 \\ & + 8\alpha_8(r^2\omega'^2 + u'^2)^3 + \int^r \rho r \omega^2 dr \\ & + \int^r \frac{1}{r} [2\alpha_2(r^2\omega'^2 + u'^2) + \alpha_3 u'^2 + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 \\ & + 8\alpha_8(r^2\omega'^2 + u'^2)^3] dr + Pz + C, \end{aligned} \quad (4.11)$$

where C is an integration constant.

Introducing this expression for p into the expressions (4.4) for t_{11} , t_{22} and t_{33} , we obtain

$$\begin{aligned} t_{11} = & -\int^r \rho r \omega^2 dr - \int^r \frac{1}{r} [2\alpha_2(r^2\omega'^2 + u'^2) + \alpha_3 u'^2 \\ & + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 + 8\alpha_8(r^2\omega'^2 + u'^2)^3] dr \\ & - Pz - C, \end{aligned}$$

$$\begin{aligned} t_{22} = & -\alpha_3 u'^2 - 2\alpha_2(r^2\omega'^2 + u'^2) - 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 \\ & - 8\alpha_8(r^2\omega'^2 + u'^2)^3 - \int^r \rho r \omega^2 dr - \int^r \frac{1}{r} [2\alpha_2(r^2\omega'^2 + u'^2) \\ & + \alpha_3 u'^2 + 4(\alpha_4 + \alpha_6)(r^2\omega'^2 + u'^2)^2 + 8\alpha_8(r^2\omega'^2 + u'^2)^3] dr \\ & - Pz - C \end{aligned}$$

and

$$\begin{aligned}
 t_{33} = & -\alpha_3 r^2 \omega'^2 - 2\alpha_2 (r^2 \omega'^2 + u'^2) - 4(\alpha_4 + \alpha_6)(r^2 \omega'^2 + u'^2)^2 \\
 & - 8\alpha_8 (r^2 \omega'^2 + u'^2)^3 - \int^r \rho r \omega^2 dr - \int^r \frac{1}{r} [2\alpha_2 (r^2 \omega'^2 + u'^2) \\
 & + \alpha_3 u'^2 + 4(\alpha_4 + \alpha_6)(r^2 \omega'^2 + u'^2)^2 + 8\alpha_8 (r^2 \omega'^2 + u'^2)^3] dr \\
 & - P_z - C.
 \end{aligned} \tag{4.12}$$

We see that if the lengths of the concentric cylinders are l and we neglect end effects, then the "pressure head" which must be applied in order to maintain the state of flow considered is $[t_{33}]_{x_3=0} - [t_{33}]_{x_3=l} = Pl$. Here we have neglected gravitational forces on the fluid. If these are taken into account and we assume that the positive direction of x_3 is vertically upwards, we must of course replace P by $P - \rho g$ where ρ is the density of the fluid and g is the gravitational acceleration. The rate at which the fluid flows through the annular region in the longitudinal direction is given by $\int_{a_2}^{a_1} 2\pi r u dr$.

It is also seen that the fluid exerts a normal thrust on the cylindrical containing walls of magnitude t_{11} per unit area. The difference between these thrusts for the outer and inner walls at the same position along the tube is given by

$$\begin{aligned}
 [t_{11}]_{r=a_1} - [t_{11}]_{r=a_2} = & - \int_{a_2}^{a_1} \rho r \omega^2 dr \\
 & - \int_{a_2}^{a_1} \frac{1}{r} [2\alpha_2 (r^2 \omega'^2 + u'^2) + \alpha_3 u'^2 + 4(\alpha_4 + \alpha_6)(r^2 \omega'^2 + u'^2)^2 \\
 & + 8\alpha_8 (r^2 \omega'^2 + u'^2)^3] dr.
 \end{aligned}$$

The azimuthal surface tractions which must be applied to the walls of the inner and outer cylinders are $[t_{12}]_{r=a_2}$ and $[t_{12}]_{r=a_1}$ respectively and these are statically equivalent to equal and opposite couples of magnitude

$$2\pi a_2^2 [t_{12}]_{r=a_2} = 2\pi a_1^2 [t_{12}]_{r=a_1}$$

per unit length of the cylindrical boundaries.

It will be noted that the solutions to the problem of Couette flow and of longitudinal flow in the annular region between the cylinders can be obtained from the above analysis by taking $u = 0$ and $\omega = 0$ respectively.

5. Discussion. Problems analogous to those discussed in §§2 and 3 have been previously considered for the case when the stress components are assumed to be expressible as polynomials in the velocity gradients only. In this case the stress matrix $\mathbf{T}$ is given in terms of the velocity-strain matrix $\mathbf{A}_1$ only by the expression

$$\mathbf{T} = -p\mathbf{I} + \alpha_1 \mathbf{A}_1 + \alpha_3 \mathbf{A}_1^2, \tag{5.1}$$

where α_1 and α_3 are expressible as polynomials in $\text{tr } \mathbf{A}_1$, $\text{tr } \mathbf{A}_1^2$ and $\text{tr } \mathbf{A}_1^3$. The expression (5.1) may, of course, be obtained from (1.5) by taking $\alpha_i = 0$ ($i \neq 1, 3$) and the solutions to the problems previously obtained are similarly obtainable from the results of §§2 and 3 by taking $\alpha_i = 0$ ($i \neq 1, 3$). The cases of Couette and Poiseuille flow, in a fluid for which the stress matrix is given by (5.1), have also been previously considered, with results which can again be obtained by taking $\alpha_i = 0$ ($i \neq 1, 3$) in the appropriate results of §4.

Many visco-elastic fluids, such as high-polymer solutions, exhibit normal stress effects which result from the non-vanishing of the normal coefficients of the stress for steady-state laminar flow of the fluid and considerable interest has been shown in recent years in such effects. Attention was first drawn to them by GARNER & NISSAN (1946) and by WEISSENBERG (1947).

The expression (5.1) for the stress matrix* was advanced by RIVLIN (1948) as a basis for the discussion of such effects and was used by GREENSMITH & RIVLIN (1953) in the interpretation of experimental measurements of the normal stresses associated with torsional flow of polyisobutylene solutions. The normal stress effects then arise from the non-vanishing of the coefficient α_3 in equation (5.1).

This expression for the stress matrix is based on the assumptions that the stress components at any point of a fluid are polynomials in the velocity gradients only, that the fluid is incompressible and that it is isotropic in its state of rest. Its application to visco-elastic fluids was justified by the consideration that, for steady state laminar flow, any terms in the constitutive equation for the fluid which are "elastic" in character will vanish. This consideration was validly criticized by OLDROYD (1950). That OLDROYD's criticism is correct may be readily seen from the results given in the previous sections, in which the acceleration-strain matrix $\mathbf{A}_2$ is seen not to vanish for the steady states of laminar flow considered. It is seen also, in accord with the contention of OLDROYD, that terms involving $\mathbf{A}_2$ in the constitutive equation for the fluid may also give rise to normal stress effects. It is seen further that they may make contributions to the shearing forces associated with steady states of laminar flow.

* An analogous expression for the stress matrix for a compressible fluid was earlier advanced by REINER (1945) as a basis for the discussion of dilatant materials.

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