Convergence Almost Everywhere of Operator Averages

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§0. Introduction

The mean ergodic theorem has been given a satisfactory operator theoretical formulation but the pointwise ergodic theorem (sometimes called the individual ergodic theorem) has not received a comparable general treatment. The first step in this direction was taken by DOOB [1] who proved the almost everywhere convergence of the averages $n^{-1} \sum_{m=0}^{n-1} T^m f$ where T is a MARKOV process and where f is a bounded function. DOOB reduced this statement to that of the ergodic theorem of G. D. BIRKHOFF and thus derived his result from BIRKHOFF's theorem. S. KAKUTANI [1] improved DOOB's result by proving the almost everywhere convergence for any function f in L_p with $p > 1$. Recently E. HOPF [1], by means of a remarkably powerful argument, extended the Doob-Kakutani result to cover the case where f is only assumed to be Lebesgue integrable and not necessarily in any space L_p with $p > 1$. Unlike the proofs of DOOB and KAKUTANI the proof of HOPF is independent of the pointwise convergence theorem of G. D. BIRKHOFF. The central lemma in HOPF's argument (our Lemma 3.2) is one which, as his proof shows, is valid for a class of positive operators which is considerably larger than the class of Markov operators. This lemma of E. HOPF provides the inspiration and starting point of the investigation presented here.

As early as 1940 K. YOSIDA observed that for quite general operators T in a Lebesgue space the pointwise ergodic theorem could be derived from the mean ergodic theorem for T by applying a general principle of BANACH provided one could show that for each integrable function f

$$(\dagger) \quad \sup_n \left| \frac{1}{n} \sum_{m=0}^{n-1} (T^m f)(s) \right| < \infty$$

for almost all points s . However, YOSIDA did not indicate any class of operators for which the above inequality could be proved. Here we shall show that $(\dagger)$ holds for every f in L_1 and every operator T in L_1 which satisfies the inequalities $(\ddagger)$ below. Regardless of whether the mean ergodic theorem holds or not, we shall show, making systematic use of BANACH's principle and the lemma of E. HOPF, that the pointwise ergodic theorem holds for any operator T in any Lebesgue space $L_1(S, \Sigma, \mu)$ provided that the inequalities

$$(\ddagger) \quad \int_S |(Tf)(s)| \mu(ds) \leq \int_S |f(s)| \mu(ds),$$

$$\operatorname{ess\,sup}_{s \in S} |(Tf)(s)| \leq \operatorname{ess\,sup}_{s \in S} |f(s)|,$$

hold for every bounded measurable function f . We also extend this result to the case of the continuous k -parameter semi-groups.

We shall state here, quite explicitly, the two principal results contained in the present communication and also introduce the notation to be used throughout this paper.

Let μ be a non-negative countably additive set function defined on the σ -field Σ of subsets of a set S and, for $1 \leq p \leq \infty$, let $\|f\|_p$ denote the norm of an element in the Lebesgue space $L_p = L_p(S, \Sigma, \mu)$. For a measurable function f on S which is not in L_p let $\|f\|_p = \infty$. If T is a linear operator with domain $D(T) \supseteq L_1 \cap L_\infty$ and range $R(T) \subseteq L_1 \cup L_\infty$, let $\|T\|_p$ denote the smallest K such that $\|Tf\|_p \leq K \|f\|_p$ for $f \in D(T)$. Then the following two theorems constitute the substance of the present paper.

1. Theorem. *Let $\{T_i(t)\}$, $i = 1, \dots, k$, be a set of strongly measurable semi-groups of operators in L_1 , such that $\|T_i(t)\|_1 \leq 1$, $\|T_i(t)\|_\infty \leq 1$, $i = 1, \dots, k$, $t > 0$. For each $\alpha_1, \dots, \alpha_k > 0$, and each $f \in L_p$ let*

$$A(\alpha_1, \dots, \alpha_k)f = (\alpha_1 \cdots \alpha_k)^{-1} \int_0^{\alpha_1} \cdots \int_0^{\alpha_k} T_1(t_1) \cdots T_k(t_k)f \, dt_1 \cdots dt_k.$$

Then, for $p > 1$ and $f \in L_p$, the limit (as $\alpha_1, \dots, \alpha_k$ approach ∞ independently) of $A(\alpha_1, \dots, \alpha_k)f$ exists almost everywhere. Moreover, there exists a function $f^ \in L_p$ dominating all the functions $A(\alpha_1, \dots, \alpha_k)f$, $\alpha_i > 0$.*

2. Theorem. *Let $\{T(t_1, \dots, t_k)\}$ be a k -parameter strongly measurable semi-group of operators in L_1 , such that $\|T(t_1, \dots, t_k)\|_1 \leq 1$ and $\|T(t_1, \dots, t_k)\|_\infty \leq 1$ for $t_i > 0$, $i = 1, \dots, k$. For each $\alpha > 0$ and each $f \in L_1$, put*

$$A(\alpha)f = \alpha^{-k} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \dots, t_k)f \, dt_1 \cdots dt_k.$$

Then, for each $f \in L_1$, $\lim_{\alpha \rightarrow \infty} A(\alpha)f$ exists almost everywhere.

The discrete analogs of both the theorems are also true and will be proved below.

The organization of the present paper is as follows. In §1 we collect a number of "mean" ergodic theorems which will be of use in developing the more complicated individual ergodic theory. In §2 we collect a number of theorems about the L_p spaces as lattices which are also useful in the sections following. In §3 we begin the individual ergodic theory proper with the simplest discrete cases of Theorem 1. Lemmas 3.1 and 3.2 give a simplified and somewhat generalized proof of the fundamental lemma of HOPF first published in HOPF [1]. Lemma 3.4 proves a lemma on operators in L_1 which enables us to eliminate the hypothesis of positivity from the theory. Theorem 3.6 gives the BIRKHOFF type of individual ergodic theorem for iterates of a single operator without assumptions of positivity and without assuming the measure space involved to be finite or even σ -finite. Theorem 3.8 gives the dominated ergodic theorem in an equally general operator-theoretic setting. On the basis of this theorem it is possible

to prove Theorem 3.9, which gives a result on convergence almost everywhere of simultaneous averages of several non-commuting operators in L_p , $p > 1$. In §4 we extend the considerations of §3 to the case of continuous-parameter semi-groups of operators. Theorem 4.1 is a mean ergodic theorem for semi-groups, for which we give a proof even though the result is well known, to emphasize the ease with which continuous parameter convergence theorems of this kind may be reduced to discrete-parameter convergence theorems. Theorem 4.5 is the continuous-parameter version of the "Birkhoff" convergence Theorem 3.6. Lemma 4.6 and Theorem 4.7 give somewhat generalized continuous parameter versions of the dominated ergodic Theorem 3.8. Theorem 4.10 is the continuous case of the "non-commuting" ergodic Theorem 3.9, and is, at the same time, the first of the theorems stated above.

The rest of §4 is devoted to a proof of the second theorem stated above, which reappears as Theorem 4.17. This theorem is reduced, via a detour through the discrete case and by the use of Lemma 3.4, to the case in which all the operators $\{T(t_1, \dots, t_k)\}$ preserve positivity. This latter case is then treated by induction on k . Thus, for instance, consideration of the two parameter semi-group $\{T(t_1, t_2)\}$ is reduced to consideration of the one parameter semi-group

$$S(x) = (4\pi)^{-1} x^2 \int_0^\infty \int_0^\infty (t_1 t_2)^{-3/2} \exp(t_1 x^{-2}) \exp(t_2 x^{-2}) T(t_1, t_2) dt_1 dt_2.$$

The discrete analog of our theorem is reduced to the continuous case. Consideration of the two parameter discrete semi-group $\{T_1^{i_1} T_2^{i_2}\}$ is, for instance, reduced to consideration of the two parameter continuous semi-group $\exp(t_1(T_1 - I) + t_2(T_2 - I))$.

§1. Operator-Theoretic Preliminaries

We shall base our approach to the theory of convergence almost everywhere on the simpler theory of convergence ("in mean") of operator averages, and on a convergence theorem of BANACH. The principal theorems of the former theory are by now fairly well known. For convenience, we collect them, together with the above mentioned theorem of BANACH, in the present section, giving references to the literature for their proofs. Thus, the present section is principally devoted to the statement of theorems guaranteeing the ("strong") convergence of the averages

$$A(n) = A(T, n) = \frac{1}{n} \sum_{m=0}^{n-1} T^m$$

of the powers T^m of a bounded linear operator T in a Banach space X . The notations introduced in the preceding sentence will be used throughout the remainder of the present paper.

For proof of a mean ergodic theorem of the same type as those stated below, but for the case of continuous averages, see DUNFORD [2], Theorem 1.

1. Theorem. *Let the averages $A(n)$ of the iterates of the operator T in the B -space X be bounded. Then the set of those points x in X for which the sequence $\{A(n)x\}$ is convergent is a closed linear manifold consisting of all vectors x for which the set $\{A(n)x\}$ is weakly sequentially compact and $T^n x/n \rightarrow 0$.*

2. Theorem. *If, under the hypotheses of Theorem 1, the strong limit $Ex = \lim_n A(n)x$ exists for each $x \in X$, E is a projection of X upon the manifold $\{x \mid Tx = x\}$ of fixed points. The complementary projection to E has the closure of $(I - T)X$ for its range.*

3. Corollary. *If the sequence $\{A(n)\}$ is bounded then it converges in the strong operator topology if and only if $T^n x/n \rightarrow 0$ for x in a fundamental set and the sequence $\{A(n)x\}$ is weakly sequentially compact for x in a fundamental set.*

4. Corollary. *If X is reflexive then the sequence $\{A(n)\}$ converges in the strong operator topology if and only if it is bounded and $\{T^n x/n\}$ converges to zero for each x in a fundamental set.*

Our next corollary pertains to the more special situation in which X is a space of Lebesgue integrable functions.

5. Corollary. *Let (S, Σ, μ) be a finite measure space and let T be a linear operator in $L_1(S, \Sigma, \mu)$ which maps essentially bounded functions into essentially bounded functions. Let the averages $A(n)$ be bounded as operators in $L_\infty(S, \Sigma, \mu)$ as well as in $L_1(S, \Sigma, \mu)$. Then these averages are strongly convergent in $L_1(S, \Sigma, \mu)$*

if and only if $T^n f/n \rightarrow 0$ in $L_1(S, \Sigma, \mu)$ for every f in some fundamental set in $L_1(S, \Sigma, \mu)$.

In stating our next two theorems it is convenient to make use of the notation $TM(S, \Sigma, \mu)$ for the space of all *totally measurable* functions defined on S ; that is, the space of all measurable functions f such that $\lim_{n \rightarrow \infty} \mu\{s \mid |f(s)| > n\} = 0$. Into this space we introduce the norm

$$\|f\| = \inf_{\alpha > 0} \arctan(\alpha + \mu\{s \mid |f(s)| > \alpha\}),$$

so that $TM(S, \Sigma, \mu)$ becomes a complete metric space in which convergence in norm and convergence in measure are equivalent. With this understood, we can state the following theorem.

Theorem (BANACH). *Let $\{T_n\}$ be a sequence of continuous linear maps from an F -space X into the space $TM(S, \Sigma, \mu)$ of all real or complex totally measurable functions on a positive measure space (S, Σ, μ) . Suppose that for each x in X we have $\sup_n |T_n(x, s)| < \infty$ for almost all s in S . Suppose also that for each x in a dense set in X , the limit $\lim_n T_n(x, s)$ exists for almost all s in S . Then for every x in X the limit $\lim_n T_n(x, s)$ exists almost everywhere on S .*

In proving several of the theorems to be given below we shall have to use the above theorem in the following generalized form.

6. Theorem (Banach Convergence Theorem). *Let (S, Σ, μ) be a positive measure space. Let $A_1 \supseteq A_2 \supseteq \cdots$ be denumerable sets and for each $a \in A_1$ let T_a be a continuous linear map from an F -space X into the space $TM(S, \Sigma, \mu)$ of real or complex valued measurable functions. Suppose that*

(i) *for each x in X*

$$\sup_{a \in A_1} |T_a(x, s)| < \infty,$$

almost everywhere on S ; and

(ii) *for each x in a set dense in X*

$$\lim_{p \rightarrow \infty} \sup_{a, b \in A_p} |T_a(x, s) - T_b(x, s)| = 0,$$

almost everywhere on S . Then the equation appearing in (ii) is valid for every x in X .

For a proof of this latter theorem, see DUNFORD & MILLER [1], Lemma 7, page 542.

§2. Lattice-Theoretic Preliminaries

We shall now consider some order properties of the space $M(S, \Sigma, \mu)$ of all measurable functions. If f is in $M(S, \Sigma, \mu)$ and if $f(s) \geq 0$ for μ -almost all $s \in S$ we say that f is *positive* and write $f \geq 0$. If f_1 and f_2 are real or complex valued functions in $M(S, \Sigma, \mu)$ and if $f_1 - f_2 \geq 0$, $f_1 \geq f_2$ or $f_2 \leq f_1$. This makes $M(S, \Sigma, \mu)$ into a partially ordered system. We note that the space $TM(S, \Sigma, \mu)$ is also partially ordered by the same ordering.

1. Lemma. *If (S, Σ, μ) is a positive measure space then the real space $M(S, \Sigma, \mu)$ is a σ -complete lattice.*

For a proof, see BIRKHOFF [1], Theorem 7.14.

In order to discuss another type of completeness for the space $M(S, \Sigma, \mu)$, it is convenient to state a general lemma on σ -complete lattices.

2. Lemma. *Let L be a σ -complete lattice in which every set of elements of L which is well-ordered under the partial ordering of L is at most countable. Then L is complete and every subset A of L has a least upper bound which is the least upper bound of a countable subset of A .*

For this result, see J. VON NEUMANN [1], Theorem A.4, p. 8.

By Lemma 2 we can easily conclude:

3. Theorem. *Let (S, Σ, μ) be a σ -finite measure space and $M(S, \Sigma, \mu)$ the space of real or complex valued μ -measurable functions with its natural partial ordering. Then if $\{f_\alpha\}$ is a subset of $M(S, \Sigma, \mu)$ which has an upper bound, it has a least upper bound. In fact, there exists a countable subset $\{f_{\alpha_n}\}$ such that*

$$\sup_{\alpha} \{f_{\alpha}\} = \sup_n \{f_{\alpha_n}\}.$$

The natural ordering used for functions in $M(S, \Sigma, \mu)$ can be employed just as well for functions in the spaces $L_p(S, \Sigma, \mu)$, $1 \leq p \leq \infty$. Since if $f \in L_p$, $g \in M$, and $f \geq g \geq 0$ we have $g \in L_p$, it follows immediately that L_p is a complete sublattice of M . Moreover, it is useful to know that the least upper bounds of a bounded subset of L_p may be taken either in the space $L_p(S, \Sigma, \mu)$ or in $M(S, \Sigma, \mu)$ without confusion. We state this result formally.

4. Corollary. *Let (S, Σ, μ) be a positive σ -finite measure space. If a subset $\{f_\alpha\}$ of $L_p(S, \Sigma, \mu)$, $1 \leq p \leq \infty$, is bounded in the natural order of this space, then $\sup_{\alpha} \{f_{\alpha}\}$ exists and is the same whether taken in $L_p(S, \Sigma, \mu)$ or in $M(S, \Sigma, \mu)$. Further, there is a countable subset $\{f_{\alpha_n}\}$ such that $\sup_{\alpha} \{f_{\alpha}\} = \sup_n \{f_{\alpha_n}\}$.*

In general if X and Y are two partially ordered normed linear spaces it is natural to order the space $B(X, Y)$ of all continuous linear operators $T: X \rightarrow Y$ by putting $T \geq 0$ if $Tx \geq 0$ for each $x \geq 0$. This yields in particular a natural

"dual" ordering of the dual space X^* of an ordered B -space X . If $T \in B(X, Y)$ is positive, and if Y^* and X^* are both given their natural dual ordering, then the adjoint operator $T^*: Y^* \rightarrow X^*$ is readily seen to be positive. If X is a partially ordered Banach space, X^* its dual with the natural dual ordering, and X^{**} the dual of X^* with the natural dual ordering again, then the natural imbedding $\kappa: X \rightarrow X^{**}$ defined by $(\kappa x)(x^*) = x^*(x)$ is not only isometric but order-preserving.

In the particular case of spaces $L_p(S, \tau, \mu)$ it is worth noting that if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, so that by a well-known theorem L_p is isometrically isomorphic to L_q^* , then the natural dual ordering of L_p (regarded as L_q^*), and its natural ordering of L_p as a subspace of $M(S, \Sigma, \mu)$ coincide. The same holds if $q = 1, p = \infty$, provided that S is σ -finite. If we wish to consider the case $q = \infty$, we must make certain modifications. The elements x^* of L_∞^* may be represented in an isometrically isomorphic way as finitely additive set functions ν by letting x^* correspond to the set function ν defined by $\nu(e) = x^*(\chi_e)$, $e \in \Sigma$, where χ_e is the characteristic function of the set e . We have $\|x^*\| = \text{total variation } \nu = v(\nu, S) = \|\nu\|$. This isometric isomorphism is order preserving if we write $\nu \geq 0$ when $\nu(e) \geq 0$ for $e \in \Sigma$.

5. Theorem. *Let (S, Σ, μ) be a σ -finite positive measure space. If κ is the natural isometric isomorphism of $L_1 = L_1(S, \Sigma, \mu)$ into L_1^{**} , and if $\{f_\alpha\}$ has an upper bound in the partially ordered space L_1 , then $\kappa(\sup\{f_\alpha\}) = \sup_\alpha\{\kappa f\}$.*

In order to make the reader more familiar with the concepts of the present section, we sketch the proof of this theorem.

Proof. Since (S, Σ, μ) is σ -finite it follows that $L_1^{**}(S, \Sigma, \mu) = ba(S, \Sigma^*, \mu)$, the space of all bounded additive functions on the Lebesgue extension Σ^* of Σ which vanish on sets from Σ^* of μ -measure zero. If $f \in L_1$, it is readily seen that κf is the set function defined by $\kappa f(E) = \int_E f(s) \mu(ds)$, $E \in \Sigma^*$. In particular κf is countably additive and μ -continuous (*i.e.*, "absolutely continuous with respect to μ "). Now suppose that $0 \leq f_0 = \sup_\alpha \{f_\alpha\}$. Since κ preserves order it is clear that $0 \leq \lambda = \sup_\alpha \{\kappa f_\alpha\} \leq \kappa f_0$, the least upper bound being taken in the partially ordered space L_1^{**} of finitely additive set functions. It follows that the set function λ is countably additive and μ -continuous, so by the Radon-Nikodym theorem there is a function $g \in L_1(S, \Sigma^*, \mu)$ such that

$$\lambda(E) = \int_E g(s) \mu(ds), \quad E \in \Sigma^*.$$

Now it is clear that $g(s) = f_0(s)$ for s in the complement of a set $E_0 \in \Sigma^*$ of μ -measure zero, and therefore for s in the complement of a set $E_1 \in \Sigma$ of μ -measure zero. Q.E.D.

§3. Pointwise Ergodic Theorems

In proving theorems about convergence almost everywhere, we shall replace the B -space X of Section 1 by a space $L_p = L_p(S, \Sigma, \mu)$ where (S, Σ, μ) is an arbitrary positive measure space. For an operator T in L_p and a point f in L_p the symbols $(Tf)(s)$ and $T(f, s)$ will be used for the value at the place s of one of the functions in the equivalence class Tf . As before, the symbol $A(n)$ will denote the average $n^{-1} \sum_{m=0}^{n-1} T^m$ and sometimes, when it is desirable to show the dependence of $A(n)$ upon T , the symbol $A(T, n)$ will be used instead of $A(n)$. As usual the symbol $|f|_p$ will be used for the p -norm

$$|f|_p = \left\{ \int_S |f(s)|^p \mu(ds) \right\}^{1/p}, \quad 1 \leq p < \infty,$$

$$= \operatorname{ess\,sup}_{s \in S} |f(s)|, \quad p = \infty,$$

of a measurable function on S . At times it will be convenient to use the notion of the p norm of an operator T whose domain contains the characteristic functions of sets of finite measure. This norm is defined by the equation $|T|_p = \sup |Tf|_p$ where the supremum is taken over all f in the domain of T with $|f|_p \leq 1$. If $|T|_p < \infty$ and $p < \infty$, then T has a unique continuous extension to L_p and the same symbol T will be used for the extended operator. Most of the results of this section are concerned with an operator T for which $|T|_1 \leq 1$ and $|T|_\infty \leq 1$. It follows from these two inequalities, according to the Riesz convexity theorem (for this theorem, see THORIN [1], page 5), that $|T|_p \leq 1$ for $1 \leq p \leq \infty$, so that T is a continuous linear map in each space L_p , $1 \leq p < \infty$. Our principal objectives will be to demonstrate the almost everywhere convergence of $A(n)(f, s)$ and to obtain estimates on the function $\sup_n |A(n)(f, s)|$.

1. Lemma (HOFF). *Let P be a positive operator in L_∞ with $|P|_\infty \leq 1$. Let $\{e^k\}$ be a decreasing sequence of characteristic functions with e^k void for $k > n$. Then there is a sequence $\{g_k, 0 \leq k\}$ in L_∞ with*

- (i) $0 \leq g_i \leq g_i e^{i+1}, \quad 0 \leq i,$
- (ii) $0 \leq e^{i+2}(g_{i+1} - g_i), \quad 0 \leq i,$
- (iii) $e^{i+1} = e^{i+1}(g_i + Pg_{i+1} + P^2g_{i+2} + \cdots), \quad 0 \leq i.$

Proof. Let $g_i = 0$ for $i \geq n$, and, for $i < n$, define g_i by downward induction by the equation

$$(1) \quad g_i = e^{i+1}(1 - Pg_{i+1} - P^2g_{i+2} - \cdots), \quad 0 \leq i < n.$$

Then clearly

$$(2) \quad e^{i+1} = e^{i+1}(g_i + Pg_{i+1} + \cdots), \quad 0 \leq i,$$

which proves (iii). We shall next show that

$$(3) \quad g_i + Pg_{i+1} + P^2g_{i+2} + \cdots \leq 1, \quad 0 \leq i.$$

This is clear for $i \geq n$ while for $i < n$ it may be proved by downward induction as follows: since $(1 - e^{i+1})g_i = 0$ we have, by the induction hypothesis,

$$\begin{aligned} (1 - e^{i+1})(g_i + Pg_{i+1} + \cdots) &= (1 - e^{i+1})P(g_{i+1} + Pg_{i+2} + \cdots) \\ &\leq (1 - e^{i+1})P1 \leq 1 - e^{i+1}, \end{aligned}$$

which, when combined with (2), proves (3). From (1) and (3) it is seen that

$$e^{i+1}g_i = g_i = e^{i+1}[1 - P(g_{i+1} + Pg_{i+2} + \cdots)] \geq 0,$$

which proves (i). To prove (ii) we shall first prove, by induction downwards, that

$$(4) \quad (g_i + Pg_{i+1} + \cdots) - (g_{i+1} + Pg_{i+2} + \cdots) \geq 0, \quad 0 \leq i.$$

Suppose that (4) holds for $i > j$. Then, by (2) and (3),

$$\begin{aligned} (5) \quad e^{i+1}(g_i + Pg_{i+1} + \cdots - g_{i+1} - Pg_{i+2} - \cdots) \\ = e^{i+1} - e^{i+1}(g_{i+1} + Pg_{i+2} + \cdots) \geq 0. \end{aligned}$$

Also, since $(1 - e^{i+1})g_i = (1 - e^{i+1})g_{i+1} = 0$, we see from the induction hypothesis that

$$\begin{aligned} (6) \quad (1 - e^{i+1})(g_i + Pg_{i+1} + \cdots - g_{i+1} - Pg_{i+2} - \cdots) \\ = (1 - e^{i+1})P(g_{i+1} + Pg_{i+2} + \cdots - g_{i+2} - Pg_{i+3} - \cdots) \geq 0. \end{aligned}$$

Thus (4) follows by adding (5) and (6). Since $e^{i+2}e^{i+1} = e^{i+2}$ we see from (1) and (4) that

$$e^{i+2}(g_{i+1} - g_i) = e^{i+2}P(g_{i+1} + Pg_{i+2} + \cdots - g_{i+2} - Pg_{i+3} - \cdots) \geq 0,$$

which proves (ii). Q.E.D.

2. Lemma (HOPF). *Let P be a positive operator in L_1 with $\|P\|_1 \leq 1$. Then, for every real f in L_1 and every positive integer n , we have $\int_{\mathbb{R}} f(s)\mu(ds) \geq 0$ where $E = \{s \mid \sup_{1 \leq k \leq n} A(P, k)(f, s) \geq 0\}$.*

Proof. For $i > n$ let E_i be void and for $1 \leq i \leq n$ let

$$E_i = \{s \mid A(i)(f, s) \geq 0; A(j)(f, s) < 0; j < i\},$$

where $A(n) = A(P, n)$. Let $E^i = E_i \cup E_{i+1} \cup \cdots$ so that $E^1 = E$ and let e_i and e^i be the characteristic functions of E_i and E^i respectively. Then

$$e_i \sum_{j=0}^{i-1} P^j f \geq 0, \quad e_i \sum_{j=0}^{n-1} P^j f \leq 0, \quad m < i,$$

so that

$$e_i \sum_{j=m}^{i-1} P^j f \geq 0, \quad m < i,$$

Upon adding these inequalities we get

$$(1) \quad \sum_{i>m} \sum_{j=m}^{i-1} e_i P^j f = \sum_{i \geq m} e^{i+1} P^i f \geq 0, \quad 0 \leq m.$$

The preceding lemma is now applied to the adjoint operator P^* in L_∞ so that there are functions $g_i \in L_\infty$, $0 \leq i$, with the properties (i), (ii) of that lemma and also with the property

$$(\gamma)' \quad e^1 = e^1(g_0 + P^*g_1 + P^{*2}g_2 + \dots).$$

Let $g_{-1} = 0$ so that from (1) and (ii) we have

$$\begin{aligned} 0 &\leq \sum_{m \geq 0} e^{m+1}(g_m - g_{m-1}) \sum_{i \geq m} e^{i+1} P^i f = \sum_{m \geq 0} \sum_{i \geq m} e^{i+1}(g_m - g_{m-1}) P^i f \\ &= \sum_{i \geq 0} \sum_{m=0}^i (g_m - g_{m-1}) e^{i+1} P^i f = \sum_{i \geq 0} g_i e^{i+1} P^i f = \sum_{i \geq 0} g_i P^i f. \end{aligned}$$

Consequently

$$0 \leq \int_S \left(\sum_{i \geq 0} g_i P^i f \right)(s) \mu(ds) = \int_S \sum_{i \geq 0} (P^{*i} g_i)(s) f(s) \mu(ds).$$

Since $P^{*i} g_i \geq 0$ and $f = f_1$ is negative on $S - E \subseteq S - E_1$ we see therefore that

$$0 \leq \int_E \sum_{i \geq 0} (P^{*i} g_i)(s) f(s) \mu(ds).$$

Since e^1 is the characteristic function of E it follows from $(\gamma)'$ that $\sum_{i \geq 0} (P^{*i} g_i)(s) = 1$ for S in E . Thus $\int_E f(s) \mu(ds) \geq 0$. Q.E.D.

3. Lemma. Let A be a bounded subset of L_∞ with $\alpha A \subseteq A$ for $|\alpha| = 1$. Then $\sup_{f \in A} |f(\cdot)| = \sup_{f \in A} \Re f$.

Proof. The suprema are taken in the lattice ordering of L_∞ . Let $g_1 = \sup_{f \in A} |f(\cdot)|$, $g_2 = \sup_{f \in A} \Re f$. Then, since $|f(\cdot)| \geq \Re f$, we have $g_1 \geq g_2$. Hence, to prove the lemma, it will suffice to show that $g_2 \geq |f(\cdot)|$ for every f in A . If this is false there is an f in A , an $\epsilon > 0$, and a measurable set e of positive measure such that $|f(s)| \geq g_2(s) + \epsilon$ for s in e . Let N be a positive integer with $4|f|_\infty \pi < N\epsilon$. Since $0 \leq \arg f(s) < 2\pi$ there is an integer $i \leq N$ and a measurable subset e_1 of e with positive measure and such that $2\pi i/N \leq \arg f(s) < 2\pi(i+1)/N$ for s in e_1 . Since $e^{-2\pi i/N} f \in A$ we may and will assume that $0 \leq \arg f(s) < 2\pi/N$ for s in e_1 . Then

$$|\Re f(s) - f(s)| \leq |f(s) \sin \arg f(s)| \leq \frac{2\pi |f|_\infty}{N} < \frac{1}{2}\epsilon, \quad s \in e_1,$$

and hence $\Re f \geq g_2 + \frac{1}{2}\epsilon$ for s in e_1 which contradicts the definition of g_2 and proves the lemma. Q. E. D.

4. Lemma. *Let (S, Σ, μ) be a σ -finite measure space and let T be a bounded linear operator in L_1 whose L_∞ norm is also finite. Then there is a positive linear operator P in L_1 whose L_1 and L_∞ norms do not exceed those of T and which is such that*

$$|(T^n f)(\cdot)| \leq P^n(|f(\cdot)|), \quad 1 \leq n, \quad f \in L_1.$$

Proof. Using the completeness of the lattice L_∞ (Corollary 2.4), we define Pf for $0 \leq f \in L_1 \cap L_\infty$ as $\sup \Re Tg$ where g varies over all elements in L_∞ with $|g(\cdot)| \leq f$. Thus, by the preceding lemma,

$$Pf = \sup_{|g(\cdot)| \leq f} \Re Tg = \sup_{|g(\cdot)| \leq f} |(Tg)(\cdot)|, \quad 0 \leq f \in L_\infty.$$

It is clear that $Pf \geq 0$ if $0 \leq f \in L_1 \cap L_\infty$. Also, if $|g_1(\cdot)| \leq f_1 \in L_1 \cap L_\infty$ and $|g_2(\cdot)| \leq f_2 \in L_1 \cap L_\infty$, then $\Re Tg_1 + \Re Tg_2 = T(g_1 + g_2) \leq P(f_1 + f_2)$ from which it follows that $Pf_1 + Pf_2 \leq P(f_1 + f_2)$ if $0 \leq f_1, f_2 \in L_1 \cap L_\infty$. Now let $f_1, f_2, h \in L_1 \cap L_\infty$ with $f_1, f_2 \geq 0$ and with $|h(\cdot)| \leq f_1 + f_2$ and define $h_2 = h - h_1$ where $h_1(s) = 0$ if $h(s) = 0$ and

$$h_1(s) = \frac{h(s)}{|h(s)|} [|h(s)| \wedge f_1(s)],$$

otherwise. Then clearly $|h_1(\cdot)| \leq f_1, |h_2(\cdot)| \leq f_2$, so that $\Re Th = \Re Th_1 + \Re Th_2 \leq Pf_1 + Pf_2$ which proves that $P(f_1 + f_2) \leq Pf_1 + Pf_2$ and establishes the equation

$$P(f_1 + f_2) = Pf_1 + Pf_2, \quad 0 \leq f_1, \quad f_2 \in L_1 \cap L_\infty.$$

If f_1, f_2, g_1, g_2 are non-negative elements of $L_1 \cap L_\infty$ with $f_1 - f_2 = g_1 - g_2$ then $f_1 + g_2 = f_2 + g_1$ so that $Pf_1 + Pg_2 = Pf_2 + Pg_1$ and $Pf_1 - Pf_2 = Pg_1 - Pg_2$. This shows that Pf may be defined for an arbitrary real element in $L_1 \cap L_\infty$ by the equation $Pf = Pf_1 - Pf_2$ where f_1 and f_2 are non-negative functions with $f = f_1 - f_2$. It is clear that $P\alpha f = \alpha Pf$ for real scalars α and real functions f in $L_1 \cap L_\infty$. For an arbitrary f in $L_1 \cap L_\infty$ we let $Pf = P\Re f + iP\Im f$. It is evident that P is additive and that $P\alpha f = \alpha Pf$ for α real. Thus to see that P is linear it suffices to note that $iPf = Pif$, i.e., that $P\Re(if) + iP\Im(if) = i(P\Re f + iP\Im f)$ for each f in $L_1 \cap L_\infty$.

To see that $\|P\|_\infty \leq \|T\|_\infty$ we note that $P\Re f = \Re Pf$ and thus, by Lemma 3,

$$\begin{aligned} |(Pf)(\cdot)| &= \sup_{|\alpha|=1} \Re P\alpha f = \sup_{|\alpha|=1} P\Re \alpha f \\ &\leq P|f(\cdot)| = \supremum_{|g(\cdot)| \leq |f(\cdot)|} |(Tg)(\cdot)| \leq \|T\|_\infty |f|_\infty, \end{aligned}$$

for $f \in L_1 \cap L_\infty$, which proves that $\|P\|_\infty \leq \|T\|_\infty$.

To prove that $\|P\|_1 \leq \|T\|_1$ we apply the method used above to the adjoint operator T^* , which is defined in $L_1^* = L_\infty$. In this way we construct an operator P_1 in L_∞ with

$$[1] \quad \begin{aligned} P_1 &\geq 0, \quad \|P_1\|_\infty \leq \|T^*\|_\infty = \|T\|_1, \\ \|(T^*f)(\cdot)\| &\leq \|P_1\| \|f(\cdot)\| = \sup_{\|g(\cdot)\| \leq \|f(\cdot)\|} \|Tg(\cdot)\|, \quad 0 \leq f \in L_\infty. \end{aligned}$$

Let $0 \leq f \in L_\infty$, $g \in L_1$, and let κ be the natural embedding of L_1 into L_1^{**} . Then

$$\begin{aligned} \int_S f(s) (\mathcal{R}Tg)(s) \mu(ds) &= \mathcal{R} \int_S f(s) (Tg)(s) \mu(ds) \\ &\leq \left| \int_S f(s) (Tg)(s) \mu(ds) \right| = \left| \int_S (T^*f)(s) g(s) \mu(ds) \right| \\ &\leq \int_S |(T^*f)(s)| |g(s)| \mu(ds) \leq \int_S (P_1 f)(s) |g(s)| \mu(ds) = \nu f, \end{aligned}$$

where $\nu = P_1^* \kappa |g(\cdot)|$. In the preceding inequality, replace f by the characteristic function χ_e of the set $e \in \Sigma$, and write $\nu(e)$ for $\nu(\chi_e)$. Then $\nu(e)$ is a bounded non-negative additive set function defined on Σ , and we have

$$\left| \int_e (Tg)(s) \mu(ds) \right| \leq \nu(e), \quad g \in L_1, \quad e \in \Sigma.$$

It consequently follows that the variation $v(\beta, e)$ of the set function

$$\beta(e) = \int_e (Tg)(s) \mu(ds)$$

over the set e satisfies $v(\beta, e) \leq \nu(e)$. Since it is well known that

$$v(\beta, e) = \int_e |(Tg)(s)| \mu(ds),$$

we have

$$\int_e |(Tg)(s)| \mu(ds) \leq \nu(e) \quad \text{for } e \in \Sigma,$$

i.e.,

$$[2] \quad \kappa |Tg(\cdot)| \leq P_1^* \kappa |g(\cdot)|, \quad g \in L_1.$$

If h is a non-negative measurable function with $h \leq f \in L_1 \cap L_\infty$ then h is also in both L_1 and L_∞ . Thus an element defined by an expression of the form $\sup_{h \in A} \|h(\cdot)\|$ will be the same whether the supremum is taken in the complete lattice L_1 or the complete lattice L_∞ provided only that there is an element f in $L_1 \cap L_\infty$ with $\|h(\cdot)\| \leq f$ for every h in A . Similarly, by Theorem 2.5, for a set A in L_1 for which there exists a non-negative f in L_1 such that $\|h(\cdot)\| \leq f$ for h in A , we have $\kappa \sup_{h \in A} \|h(\cdot)\| = \sup_{h \in A} \kappa \|h(\cdot)\|$. Finally we observe that a

bounded set A in L_∞ and the characteristic function χ_e of a set e in Σ satisfy the relation $\sup_{h \in A} \chi_e h = \chi_e \sup_{h \in A} h$.

Using the observations of the preceding paragraph and formula [2] it is seen that for $0 \leq f \in L_1 \cap L_\infty$ and $e \in \Sigma$ with $\mu(e) < \infty$ we have

$$\begin{aligned} \kappa \chi_e P f &= \kappa \chi_e \sup_{|g(\cdot)| \leq f} |Tg(\cdot)| \\ &= \kappa \sup_{|g(\cdot)| \leq f} \chi_e |(Tg)(\cdot)| \\ &= \sup_{|g(\cdot)| \leq f} \kappa [\chi_e |(Tg)(\cdot)|] \\ &= \sup_{|g(\cdot)| \leq f} \kappa |(Tg)(\cdot)| \leq \sup_{|g(\cdot)| \leq f} P^* \kappa |g(\cdot)| \\ &= P^* \kappa f. \end{aligned}$$

Thus, for each set e in Σ with $\mu(e) < \infty$, we have $\int_S \chi_e(s) (Pf)(s) \mu(ds) \leq |P^* \kappa f|$. Now, from [1], it is seen that $|P^*| = |P_1|_\infty \leq |T|_1$, and so $\int_S (Pf)(s) \mu(ds) \leq |T|_1 |\kappa f| = |T|_1 |f|_1$ for every non-negative f in $L_1 \cap L_\infty$. Since, as shown earlier, $|(Pf)(\cdot)| \leq P |f(\cdot)|$, we have $\int_S (Pf)(s) \mu(ds) \leq |T|_1 |f|_1$ for all f in $L_1 \cap L_\infty$. Thus $|P|_1 \leq |T|_1$, and P has a unique extension to an operator (which we still call P) defined on all of L_1 .

By definition

$$P |f(\cdot)| = \sup_{|g(\cdot)| \leq |f(\cdot)|} |(Tg)(\cdot)| \geq |(Tf)(\cdot)|, \quad f \in L_1 \cap L_\infty.$$

Since $L_1 \cap L_\infty$ is dense in L_1 and $|P|_1$ has been proved finite, this inequality holds for all $f \in L_1$. It follows by induction that $P^{n+1} |f(\cdot)| \geq P |(T^n f)(\cdot)| \geq |(T^{n+1} f)(\cdot)|$. Q.E.D.

5. Lemma. *Let (S, Σ, μ) be a σ -finite measure space and let T be a linear operator in L_1 with $|T|_\infty \leq 1$ and $|T|_1 \leq 1$. Then for $1 \leq p < \infty$ and f in $L_p(S, \Sigma, \mu)$ we have $\lim_{1 \leq n < \infty} |A(T, n)(f, s)| < \infty$, for almost all s in S .*

Proof. We may and will assume that $f \geq 0$. Let $\tilde{A}$ be the complement of the set $A = \{s | f(s) \geq 1\}$. Since $f \in L_p$ it follows that $\mu(A) < \infty$ and thus f is the sum of a summable function $f\chi_A$ and a bounded function $f\chi_{\tilde{A}}$. Now, for a bounded function g in L_p we have, since $|T|_\infty \leq 1$, $|A(n)g|_\infty \leq |g|_\infty$. Thus in proving the lemma it may be assumed that f is in L_1 . In view of Lemma 4 it may also be assumed that T is positive. Let

$$e_\infty = \{s | \sup_{1 \leq k} A(T, k)(f, s) = \infty\}.$$

Since μ is σ -finite on S it will suffice to prove that $\mu(ee_\infty) = 0$ for every set e in Σ with $\mu(e) < \infty$. Let g be the characteristic function of ee_∞ where $\mu(e) < \infty$. If $\alpha > 0$ and Lemma 2 is applied to the function $f - \alpha g$ it is seen that

$$(*) \quad \alpha \mu(ee_\infty E) \leq \int_S f(s) \mu(ds)$$

where $E = \{s \mid \sup_{1 \leq k \leq n} A(T, k) (f - \alpha g, s) \geq 0\}$. Since $E \supseteq e_n$ where $e_n = \{s \mid \sup_{1 \leq k \leq n} A(T, k) (f, s) \geq \alpha\}$, the inequality (*) then gives

$$\alpha \mu(ee_n) \leq \int_s f(s) \mu(ds), \quad n = 1, 2, \dots$$

and since $e_n \rightarrow e_\infty$ we have $\alpha \mu(ee_\infty) \leq \int_s f(s) \mu(ds)$. Since this holds for every $\alpha > 0$ it follows that $\mu(ee_\infty) = 0$. Q.E.D.

6. Theorem. *Let (S, Σ, μ) be a positive measure space and let T be a linear operator in $L_1(S, \Sigma, \mu)$ with $\|T\|_\infty \leq 1$ and $\|T\|_1 \leq 1$. Then, for every p with $1 \leq p < \infty$ and every function f in $L_p(S, \Sigma, \mu)$ the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} (T^m f)(s)$$

exists for almost all s in S .

Proof. Since the functions $T^n f$, $n = 0, 1, \dots$, all vanish on the complement of a σ -finite set we may and shall assume that (S, Σ, μ) is σ -finite. Since $\|T\|_\infty \leq 1$ and $\|T\|_1 \leq 1$ it follows from the Riesz convexity theorem that $\|T\|_p \leq 1$. For $1 < p < \infty$, L_p is reflexive and thus, from Corollary 1.4, it follows that $A(T, n)f$ converges in L_p for every f in L_p . The decomposition given in Theorem 1.1 then shows that vectors of the form $h = f + (I - T)g$, with $f, g \in L_p$, $Tf = f$, and g bounded are dense in L_p . For such a vector h we have $A(T, n)h = f + (I - T^n)g/n$, and thus, for almost all s in S , we have

$$\|A(T, n)(h, s) - f(s)\| \leq 2 \|g\|_\infty / n \rightarrow 0.$$

This shows that $A(T, n)h$ converges almost everywhere for every h in a dense set in L_p and, by Lemma 5, $\sup_{1 \leq n} \|A(T, n)(h, s)\| < \infty$ almost everywhere for every h in L_p . Thus, by Banach's convergence Theorem 1.6, the sequence $A(T, n)h$ converges almost everywhere for every h in L_p . Since $L_p \cap L_1$ is dense in L_1 we may apply Lemma 5 and Theorem 1.6 again to see that the sequence $A(T, n)f$ converges almost everywhere for every f in L_1 . Q.E.D.

It is seen from the preceding theorem and from Corollary 1.4 that for $1 < p < \infty$ and for f in L_p the sequence $\{A(n)f\}$ of averages not only converges in L_p but also converges almost everywhere on S . It will next be shown that this sequence of averages is bounded by a function in L_p . The proof of this is based upon the next lemma, whose statement makes use of the following notation: $A(n) = A(T, n)$, $f^*(s) = \sup_{1 \leq n} \|A(n)(f, s)\|$, $e^*(\alpha) = \{s \mid f^*(s) > \alpha\}$, $e(\alpha) = \{s \mid \|f(s)\| > \alpha\}$.

7. Lemma. *Let T be an operator in L_1 with $\|T\|_\infty \leq 1$, $\|T\|_1 \leq 1$, and let $1 \leq p < \infty$. Then, for every function f in L_p and every positive number α , we have*

$$\alpha \mu(e^*(2\alpha)) \leq \int_{e(\alpha)} \|f(s)\| \mu(ds).$$

Proof. The lemma will first be proved under the assumption that $\mu(S) < \infty$. If P is the positive operator associated with T as in Lemma 4, then $e^*(\alpha) \subseteq \{s \mid \sup_{1 \leq n} A(P, n)(|f(\cdot)|, s) > \alpha\}$ from which it follows that we may, without loss of generality, assume that T is a positive operator and that f is a non-negative function in L_p . Let $g(s) = f(s)$ if $f(s) > \alpha$ and $g(s) = 0$ otherwise. Then $f \leq g + \alpha$, $f - 2\alpha \leq g - \alpha$, and, since $\|T\|_\infty \leq 1$, we have

$$A(n)f - 2\alpha \leq A(n)(f - 2\alpha) \leq A(n)(g - \alpha).$$

This shows that $e^*(2\alpha) \subseteq B \subseteq C$, where

$$B = \{s \mid \sup_{1 \leq n} A(n)(f - 2\alpha, s) \geq 0\}, \quad C = \{s \mid \sup_{1 \leq n} A(n)(g - \alpha, s) \geq 0\}.$$

By Lemma 2, $\int_C (g - \alpha)(s) \mu(ds) \geq 0$, and so

$$\alpha \mu(e^*(2\alpha)) \leq \alpha \mu(C) \leq \int_C g(s) \mu(ds).$$

If s is a point such that $g(s) \neq 0$ then $0 < g(s) - \alpha = A(1)(g - \alpha, s)$ and thus s is in C . This means that g vanishes on the complement of C and hence

$$\alpha \mu(e^*(2\alpha)) \leq \int_S g(s) \mu(ds) = \int_{e^*(\alpha)} f(s) \mu(ds).$$

This proves the lemma under the additional assumption that $\mu(S) < \infty$. We now allow (S, Σ, μ) to be an arbitrary positive measure space. Since all of the functions $A(n)f$ vanish on the complement of some σ -finite set we may and will assume that S is σ -finite. Let $\{S_n\}$ be an increasing sequence of sets whose union is S and with $\mu(S_n) < \infty$. Let $T_n = P_n T P_n$ where P_n is the operation of multiplication by the characteristic function of S_n . Let

$$e^*(m, \alpha) = \{s \mid \sup_{1 \leq k \leq m} A(k)(f, s) > \alpha\},$$

and

$$e^*(n, m, \alpha) = \{s \mid \sup_{1 \leq k \leq m} A(T_n, k)(f, s) > \alpha\}.$$

Then, by what has already been proved,

$$(*) \quad \alpha \mu(e^*(n, m, 2\alpha)) \leq \int_{S_n \cap e^*(\alpha)} f(s) \mu(ds).$$

Since $T_n f \rightarrow T f$ in the metric topology of L_p for each $f \in L_p$ the function $\sup_{1 \leq k \leq m} A(T_n, k)(f, \cdot)$ approaches $\sup_{1 \leq k \leq m} A(k)(f, \cdot)$ in the norm of L_p as $n \rightarrow \infty$. On the other hand, since $f \geq 0$ and since T_n increases with n , the sequence $\{\sup_{1 \leq k \leq m} A(T_n, k)(f, s)\}$ is increasing in n . This shows that the sequence $\{e^*(n, m, \alpha)\}$ increases with n and has the limit $e^*(m, \alpha)$. By letting $n \rightarrow \infty$ in the inequality (*) one obtains, therefore, the inequality

$$\alpha\mu(e^*(m, 2\alpha)) \leq \int_{e(\alpha)} f(s)\mu(ds),$$

from which the desired inequality follows by letting $m \rightarrow \infty$. Q.E.D.

8. Theorem. *Let T be an operator in L_1 with $\|T\|_\infty \leq 1$ and $\|T\|_1 \leq 1$. For each f in L_p let*

$$f^*(s) = \sup_{1 \leq n} \left| \frac{1}{n} \sum_{m=0}^{n-1} (T^m f)(s) \right|.$$

Then, if $1 < p < \infty$, we have

$$\int_s f^*(s)^p \mu(ds) \leq \frac{2^p p}{p-1} \int_s |f(s)|^p \mu(ds),$$

while if $p = 1$ and $\mu(S) < \infty$ we have

$$\int_s f^*(s)\mu(ds) \leq 2 \left[\mu(S) + \int_s |f(s)| \log^+ |f(s)| \mu(ds) \right].$$

Note: The symbol $\log^+ a$ is defined for $a > 0$ to be the larger of $\log a$ and 0.

Proof. In view of Lemma 4 we may and will assume that T and f are both non-negative. The case $1 < p$ will be treated first by using the preceding lemma as follows:

$$\begin{aligned} \int_s f^*(s)^p \mu(ds) &= p \int_s \left(\int_0^{f^*(s)} a^{p-1} da \right) \mu(ds) \\ &= p \int_s \int_0^\infty a^{p-1} \chi_{e^*(a)}(s) \mu(ds) da \\ &= p \int_0^\infty a^{p-1} \mu(e^*(a)) da \\ &\leq 2p \int_0^\infty a^{p-2} \left\{ \int_{e(a/2)} f(s) \mu(ds) \right\} da \\ &= 2p \int_0^\infty a^{p-2} f(s) \chi_{e(a/2)}(s) \mu(ds) da \\ &= 2p \int_s f(s) \left\{ \int_0^{2f(s)} a^{p-2} da \right\} \mu(ds) \\ &= \frac{2^p p}{p-1} \int_s f(s)^p \mu(ds). \end{aligned}$$

The case $p = 1$ is treated similarly as follows:

$$\begin{aligned}
 \int_S f^*(s) \mu(ds) &= \int_S \left\{ \int_0^{f^*(s)} da \right\} \mu(ds) \\
 &= \int_S \int_0^\infty \chi_{e^*(a)}(s) \mu(ds) da \\
 &= \int_0^\infty \mu(e^*(a)) da \leq 2\mu(S) + \int_2^\infty \mu(e^*(a)) da,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_2^\infty \mu(e^*(a)) da &\leq 2 \int_2^\infty a^{-1} \left\{ \int_{e(a/2)} f(s) \mu(ds) \right\} da \\
 &= 2 \int_1^\infty a^{-1} \left\{ \int_{e(a)} f(s) \mu(ds) \right\} da \\
 &= 2 \int_1^\infty \int_S a^{-1} f(s) \chi_{e(a)}(s) \mu(ds) da \\
 &= 2 \int_S f(s) \left\{ \int_1^{\max(1, f(s))} a^{-1} da \right\} \mu(ds) \\
 &= 2 \int_S f(s) \log^+ f(s) \mu(ds). \quad \text{Q.E.D.}
 \end{aligned}$$

9. Theorem. Let T_i , $i = 1, \dots, k$, be linear operators in L_1 with $\|T_i\|_\infty \leq 1$ and $\|T_i\|_1 \leq 1$, $i = 1, \dots, k$. Then, for every f in L_p with $p > 1$, the multiple sequence

$$(1) \quad (n_1 \dots n_k)^{-1} \sum_{m_1=0}^{n_1-1} \dots \sum_{m_k=0}^{n_k-1} (T_1^{m_1} \dots T_k^{m_k} f)(s)$$

is convergent (as $n_1, \dots, n_k \rightarrow \infty$ independently) almost everywhere on S , as well as in the norm of L_p . Furthermore, this multiple sequence is dominated by a function in L_p .

Proof. Let $V(n_1, \dots, n_k) = A(T_k, n_k) \dots A(T_1, n_1)$, so that

$$V(n_1, \dots, n_k)(f, s)$$

is the multiple sequence (1). Since $\|T_i\|_\infty \leq 1$ and $\|T_i\|_1 \leq 1$ it follows from the Riesz convexity theorem that $\|T_i\|_p \leq 1$ and thus that

$$(2) \quad \|A(T_i, n_i)\| \leq 1, \quad \|V(n_1, \dots, n_k)\| \leq 1,$$

where here, and throughout the remaining part of the proof, the norm without a subscript refers to the L_p norm. By Corollaries 1.2 and 1.4 there are projection operators E_i with

$$(3) \quad \lim_n A(T_i, n)f = E_i f, \quad f \in L_p, \quad i = 1, \dots, k.$$

From (2) and (3) it follows immediately that

$$(4) \quad \lim V(n_1, \dots, n_k)f = E_k \cdots E_1 f, \quad f \in L_p;$$

for indeed suppose this fact has been established for $k - 1$ operators $T_2, \dots, T_k$ and note that

$$\begin{aligned} & | A(T_k, n_k) \cdots A(T_1, n_1)f - E_k \cdots E_1 f | \\ & \leq | A(T_k, n_k) \cdots A(T_2, n_2) \{ A(T_1, n_1) - E_1 \} f | \\ & + | \{ A(T_k, n_k) \cdots A(T_2, n_2) - E_k \cdots E_2 \} E_1 f | \\ & \leq | \{ A(T_1, n_1) - E_1 \} f | + | \{ A(T_k, n_k) \cdots A(T_2, n_2) - E_k \cdots E_2 \} E_1 f | \end{aligned}$$

approaches zero by the induction hypothesis. By Corollary 1.2, E_i projects L_p onto the manifold $\mathfrak{N}_i$ of those f in L_p for which $T_i f = f$ and the complementary projection $E'_i = I - E_i$ projects L_p onto the closure of the manifold $(I - T_i)L_p$. Thus, if we let $\mathfrak{M}_i$ denote the set of functions of the form $(I - T_i)f$ with f bounded, we have

$$(5) \quad \mathfrak{M}_i + \mathfrak{N}_i \text{ is dense in } L_p, \quad 1 \leq i \leq k,$$

a fact which will be needed later. Now let $g = (I - T_1)f \in \mathfrak{M}_1$ with $|f(s)| \leq K$. Then $A(T_1, n)(g, s) = n^{-1} [f(s) - T_1^n(f, s)]$ and thus $|V(n_1, \dots, n_k)(g, s)| \leq 2K/n_1$ for almost all s . This shows that, for almost all s in S ,

$$(6) \quad \lim V(n_1, \dots, n_k)(g, s) = 0, \quad g \in \mathfrak{M}_1.$$

For a function f in $\mathfrak{N}_1$ we have $T_1(f, s) = f(s)$ for almost all s in S and thus $A(T_1, n)(f, s) = f(s)$ for almost all s and all $m = 1, 2, \dots$. Since the theorem is true for $k = 1$ (Theorem 6) we shall apply induction and assume that it has been proved for the case of $k - 1$ transformations $T_2, \dots, T_k$. The induction hypothesis yields then for f in $\mathfrak{N}_1$ the almost everywhere convergence of the multiple sequence

$$V(n_1, \dots, n_k)(f, s) = A(T_k, n_k) \cdots A(T_2, n_2)(f, s).$$

This fact, combined with (5) and (6) shows that the sequence

$$V(n_1, \dots, n_k)(f, s)$$

converges almost everywhere on S for every f in a set dense in L_p . It will next be shown that for every f in L_p the sequence (1) is dominated by a function

in L_p . To prove this we first use Lemma 4 to find positive operators P_i , $i = 1, \dots, k$ with $\|P_i\|_\infty \leq 1$, $\|P_i\|_1 \leq 1$ and such that

$$\|A(T_i, n)(f, \cdot)\| \leq A(P_i, n)(\|f(\cdot)\|), \quad 1 \leq i \leq k, \quad 1 \leq n.$$

If $g(s) = \|f(s)\|$ then it follows that $\|A(T_1, n_1)(f, s)\| \leq A(P_1, n_1)(g, s)$ for almost all s in S and hence that

$$\|V(n_1, \dots, n_k)(f, s)\| \leq A(P_k, n_k) \cdots A(P_1, n_1)(g, s).$$

Thus, in proving that $V(n_1, \dots, n_k)f$ is dominated by a function in L_p we may and will assume that $f \geq 0$ and that $T_i \geq 0$, $i = 1, \dots, k$. By Theorem 8 there are functions $f_1, \dots, f_k$ in L_p with

$$A(T_1, n)f \leq f_1, \quad n_1 \geq 1,$$

$$A(T_2, n_2)A(T_1, n_1)f \leq A(T_2, n_2)f_1 \leq f_2, \quad n_1, n_2 \geq 1,$$

$$A(T_k, n_k) \cdots A(T_1, n_1)f \leq f_k \quad n_1, \dots, n_k \geq 1,$$

which proves that the multiple sequence $V(n_1, \dots, n_k)f$ is dominated by a function in L_p . We may now deduce the almost everywhere convergence of this sequence from BANACH'S Convergence Theorem 1.6. In that theorem let A_p be the set of all k -tuples $a = (n_1, \dots, n_k)$ of integers with $n_i \geq p$, $i = 1, \dots, k$, and let, for $a = (n_1, \dots, n_k)$ in A_1 , the operator T_a of Theorem 1.6 be replaced by $V(n_1, \dots, n_k)$. The hypotheses of Theorem 1.6 have all been verified and by that theorem then we conclude that the multiple sequence

$$V(n_1, \dots, n_k)(f, s)$$

converges almost everywhere on S for every f in L_p . Q.E.D.

The preceding theorem is not true if $p = 1$, but there is a k -parameter analogue of Lemma 5. This will be found at the end of the next section.

§4. The Ergodic Theory of Continuous Flows

It is known that the ergodic hypothesis in classical statistical mechanics has led to questions involving averages of a particular one parameter semi-group of unitary operators, rather than discrete averages of powers of a single operator. In this section we shall study such questions and show how they may be resolved by a reduction to the discrete case. It will not be necessary to restrict the discussion to the type of semi-group arising in classical statistical mechanics as the results to be described are valid in a situation having the same degree of generality enjoyed by the discrete semi-group $\{T^n, 0 \leq n\}$ discussed in the preceding section.

The basic notion underlying the present discussion is that of a *one parameter semi-group of bounded linear operators in a real or complex B-space*. We recall that such a semi-group is a set $\{T(t), 0 \leq t\}$ of bounded linear operators in a B-space X for which

$$(i) \quad T(0) = I, \quad T(t+u) = T(t)T(u), \quad 0 \leq t, u.$$

In order to be able to assign a meaning to the averages

$$(ii) \quad A(\alpha) = \frac{1}{\alpha} \int_0^\alpha T(t) dt, \quad 0 \leq \alpha,$$

which are to be our basic concern in this section, it will be desirable to supplement the algebraic condition (i) by a condition concerning the analytical dependence of the semi-group upon the parameter t . We recall that the semi-group $\{T(t), 0 \leq t\}$ is said to be *strongly continuous* if its dependence upon t is continuous in the strong operator topology, *i.e.*, if $\lim_{t \rightarrow u} \|T(t)x - T(u)x\| = 0$ for each x in X and each $u \geq 0$. We also recall that the semi-group is said to be *strongly measurable* if, for each x in X , the function $T(\cdot)x$ is measurable, with respect to Lebesgue measure, on the infinite interval $0 \leq t$. A related concept is that of strong integrability. The semi-group is said to be *strongly integrable over every finite interval* if, for each x in X the function $T(\cdot)x$ is integrable with respect to Lebesgue measure on every finite interval $0 \leq t \leq \alpha$. If the semi-group is strongly integrable on every finite interval we may, for each $\alpha \geq 0$, define a bounded linear operator $A(\alpha)$ called *the average of $\{T(t), 0 \leq t\}$ on the interval $0 \leq t \leq \alpha$* . The operator $A(\alpha)$ is defined for $\alpha = 0$ as $A(0) = I$ and for $\alpha > 0$ by the equation $A(\alpha)x = \alpha^{-1} \int_0^\alpha T(t)x dt$. To see that $A(\alpha)$ is a bounded operator we note first that the map $x \rightarrow T(\cdot)x$ of X into the space L of X valued Lebesgue integrable functions on $[0, \alpha]$ is closed: indeed if $x_n \rightarrow x$ and $T(\cdot)x_n \rightarrow f$ in L then, for some subsequence $\{x_{n_i}\}$, $T(t)x_{n_i} \rightarrow f(t)$ for almost all t in $[0, \alpha]$. Since $T(t)x_n \rightarrow T(t)x$ for every t we have $T(t)x = f(t)$ for almost all t which proves that the map $x \rightarrow T(\cdot)x$ is closed and thus continuous by the closed graph theorem (HILLE [1], page 30). It follows immediately that $A(\alpha)$ is a continuous

linear operator in the space X . It is in this sense that the integral appearing in equation (ii) is defined.

This notion of the average $A(\alpha)$ is sufficient for the statement of the first few results for they are concerned with limits (in mean) of the type $\lim_{\alpha \rightarrow \infty} A(\alpha)x$. In most of the section however the B -space X is a space $L_p = L_p(S, \Sigma, \mu)$ where (S, Σ, μ) is a positive measure space, and we shall be concerned with the almost everywhere convergence and the almost everywhere boundedness of the averages $A(\alpha)f$ for a particular function f in L_p . Since the point $f_\alpha = A(\alpha)f$ in L_p is a class of equivalent functions and two functions $f_\alpha(\cdot)$ and $g_\alpha(\cdot)$ represent the same point if $f_\alpha(s) = g_\alpha(s)$ except on a μ -null set E_α which may vary with α , it is imperative to define more explicitly what is to be meant by the averages $A(\alpha)(f, s)$ of f at a point s in S . To make such a definition we suppose that the semi-group $\{T(t), 0 \leq t\}$ is strongly integrable on every finite interval and that it operates in L_p . Then, by a theorem of HILLE (HILLE [1], page 183) $T(t)f$ is continuous in t for $t > 0$, so that for each f in L_p the set of all vectors $T(t)f$, $t \geq 0$, is a separable subset of L_p and hence, by Lemma III.8.5, these functions all vanish on the complement of a σ -finite subset of S . Thus when defining the average $A(\alpha)(f, s)$ we may and will assume that the measure space (S, Σ, μ) is σ -finite. Thus by a theorem of DUNFORD (DUNFORD [1], Theorems 3.1, 3.2, 3.3, 3.4) there is, for each f in L_p , a scalar representation $T(t)(f, s)$ of $T(t)f$ which is measurable in (t, s) (i.e., measurable with respect to the product of Lebesgue measure and μ) and such a measurable representation is unique except on a set of points (t, s) whose product measure is zero. Moreover, there is a null set $E(f)$ which may depend upon f but which is independent of t and is such that $T(\cdot)(f, s)$ is integrable on every finite t -interval for every s not in $E(f)$. Furthermore for each $\alpha > 0$ the integral

$$(iii) \quad A(\alpha)(f, s) = \frac{1}{\alpha} \int_0^\alpha T(t)(f, s) dt, \quad 0 \leq \alpha, \quad s \notin E(f),$$

is a scalar representation of the vector $A(\alpha)f$ in L_p . If we had started with another (t, s) -measurable representation $K(t, s)$ of $T(t)f$ then, since for almost all s , $K(t, s) = T(t)(f, s)$ for almost all t , it is seen that for almost all s , $\alpha^{-1} \int_0^\alpha K(t, s) dt = A(\alpha)(f, s)$ for all α . Thus for almost all s in S the average $A(\alpha)(f, s)$ as given in equation (iii) is uniquely defined for all positive values of α and furthermore furnishes an (α, s) -measurable scalar representation of the vector $A(\alpha)f$ as defined in equation (ii).

This uniquely defined average $A(\alpha)(f, s)$ is not only a measurable function of (α, s) but for every s not in the null set $E(f)$ it is continuous in α on the interval $0 \leq \alpha < \infty$. This continuity in α shows that, for almost all s in S , the number $f^*(s)$ defined by the equation

$$(iv) \quad f^*(s) = \sup_{0 < \alpha < \infty} |A(\alpha)(f, s)|$$

is the same as $\sup_{\alpha \in R} |A(\alpha)(f, s)|$ where R is the denumerable set of positive

rational numbers. In view of Lemma 2.4, it follows that f^* is also the least upper bound of the set $\{ \|A(\alpha)(f, \cdot)\|, 0 < \alpha \}$ when it is regarded as a set of elements in the lattice of real μ -measurable functions. Thus if $\{ \|A(\alpha)(f, \cdot)\|, 0 < \alpha \}$ is a bounded set in this lattice its supremum

$$(v) \quad f^* = \sup_{0 < \alpha} \|A(\alpha)(f, \cdot)\|$$

taken in the lattice ordering is represented by the scalar function (iv) so that no confusion concerning the interpretation of least upper bounds can arise. Similar remarks are valid in the situation discussed in Theorem 7 where averages of the product of k different semi-groups are considered.

All the above remarks will be used below freely without explicit references being given in each case.

We shall continue, when convenient, to use the notation $\|U\|_p$, which was introduced at the beginning of the last section, for the L_p norm of an operator U . We recall that $\|U\|_p = \sup \|Uf\|_p$ where f varies over all elements in the domain of U with $\|f\|_p \leq 1$. We shall begin by a discussion of the strong convergence of the averages $A(\alpha)$ as $\alpha \rightarrow \infty$, and for this it will not be necessary to restrict the B -space upon which the semi-group acts.

1. Theorem. *For $\alpha \geq 0$ let $A(\alpha)$ be the average on the interval $[0, \alpha]$ of the one parameter semi-group $\{T(t), 0 \leq t\}$ which is assumed to be strongly integrable on every finite interval. Suppose also that*

$$(1) \quad \lim_{n \rightarrow \infty} \frac{T(n)}{n} x = 0, \quad x \in X;$$

$$(2) \quad \|A(\alpha)\| \leq K, \quad 0 \leq \alpha;$$

(3) *for each x in a fundamental set in X the set $\{A(\alpha)x, 0 < \alpha\}$ is weakly sequentially compact.*

Then the averages $A(\alpha)$ converge, as $\alpha \rightarrow \infty$, in the strong operator topology.

Proof. The proof will be based upon the following identity, which will serve to reduce the present theorem to the discrete case already discussed. Let $n = [\alpha]$ so that $\alpha = n + r$ with $0 \leq r < 1$. Then, for $\alpha \geq 1$,

$$\begin{aligned} A(\alpha) &= \frac{1}{\alpha} \int_0^n T(t) dt + \frac{1}{\alpha} \int_n^{n+r} T(t) dt \\ &= \frac{1}{\alpha} \sum_{m=0}^{n-1} \int_m^{m+1} T(t) dt + \frac{1}{\alpha} \int_n^{n+r} T(t) dt \\ &= \frac{1}{\alpha} \sum_{m=0}^{n-1} \int_0^1 T(t+n) dt + \frac{1}{\alpha} \int_0^r T(t+n) dt, \end{aligned}$$

and so

$$(vi) \quad A(\alpha) = \frac{n}{\alpha} \left[\frac{1}{n} \sum_{m=0}^{n-1} T(m)A(1) + \frac{T(n)}{n} rA(r) \right].$$

For a fixed x , $rA(r)x$ is continuous in r and thus the set of all vectors $rA(r)x$ with $0 \leq r \leq 1$ is compact. It follows readily from hypothesis (1) that

$$\lim_{n \rightarrow \infty} \frac{T(n)}{n} rA(r)x = 0,$$

uniformly on $0 \leq r \leq 1$, and thus hypothesis (3) and the identity (vi) show that the sequence $\{n^{-1} \sum_{m=0}^{n-1} T(m)x\}$ is weakly sequentially compact for each x in a set fundamental in the closure of the range of $A(1)$. Since $T(m) = \overline{T^m(1)}$, Theorem 1.1 may now be applied to the operator $T(1)$ on the space $A(1)X$ to give the strong convergence of $A(\alpha)$. Q.E.D.

2. Corollary. *Under the hypothesis of the theorem the limit $Ex = \lim_{\alpha} A(\alpha)x$ projects X onto the manifold $\{x \mid T(t)x = x, 0 \leq t\}$ of fixed points of the semi-group and the complementary projection $E' = I - E$ projects X onto the closed linear manifold determined by the ranges of all of the operators $I - T(t)$ with $t \geq 0$.*

Proof. Clearly $Ex = x$ if x is a fixed point. Now the identity

$$T(u)A(\alpha) = \frac{1}{\alpha} \int_0^{\alpha} T(t+u) dt = \frac{1}{\alpha} \left[\int_0^{\alpha} T(t) dt - \int_0^u T(t) dt + \int_{\alpha}^{u+\alpha} T(t) dt \right]$$

shows that $\| (T(u) - I)A(\alpha) \| \leq 2u\alpha^{-1}$, from which it follows that $T(u)E = E$. Thus every x in the range space of E is a fixed point of the semi-group. Since $E'(T(u) - I) = T(u) - I$, the range of E' contains the union of the ranges of all the operators $T(u) - I$. If x^* is a functional vanishing on the ranges of all the operators $T(u) - I$ then $x^* = x^*T(u) = x^*A(u) = x^*E$ and thus x^* vanishes on the range of E' . The desired conclusion follows from the Hahn-Banach theorem. Q.E.D.

3. Corollary. *In a reflexive space the averages $A(\alpha)$ are strongly convergent if they are bounded and if $T(n)/n \rightarrow 0$ strongly.*

4. Corollary. *Let (S, Σ, μ) be a finite measure space and $\{T(t), 0 \leq t\}$ a semi-group of operators in $L_1(S, \Sigma, \mu)$ which is strongly integrable on every finite interval. It is assumed that for some constant K*

$$\| A(\alpha) \|_1 \leq K, \quad \| T(t) \|_{\infty} \leq K, \quad 0 < \alpha, t,$$

and that $T(n)/n \rightarrow 0$ strongly in $L_1(S, \Sigma, \mu)$. Then the averages converge strongly in $L_1(S, \Sigma, \mu)$.

The remaining part of the section will be concerned exclusively with the case where the semi-group operates in a space $L_p = L_p(S, \Sigma, \mu)$. The basic assumption in the theorems to follow are that the semi-group $\{T(t), 0 \leq t\}$ is a strongly

measurable semi-group in L_1 for which $\|T(t)\|_1 \leq 1$ and $\|T(t)\|_\infty \leq 1$. There are two immediate conclusions that may be drawn from these assumptions and which will be used repeatedly without further explicit mention of them. The first of these is that $\|T(t)\|_p \leq 1$ for all $p \geq 1$. This follows from the Riesz convexity theorem. Thus for each $p \geq 1$ the semi-group $\{T(t), 0 \leq t\}$ is a semi-group of bounded linear operators in $L_p = L_p(S, \Sigma, \mu)$. The second consequence of assumption (vii) which we wish to mention here is that, as a semi-group of operators in L_p with $1 \leq p < \infty$, $\{T(t), 0 \leq t\}$ is strongly integrable on every finite interval and thus the averages $A(\alpha)$ are defined for $0 \leq \alpha < \infty$ as linear operators in L_p and have norms $\|A(\alpha)\|_p \leq 1$. To see this let $f \in L_1 \cap L_p$ so that $T(t)(f, s)$ is (t, s) -measurable and $\|T(t)f\|_p \leq \|f\|_p$. It follows from the above mentioned result of DUNFORD that the L_p -valued function $T(\cdot)f$ is measurable. Since $1 \leq p < \infty$, $L_1 \cap L_p$ is dense in L_p , and thus the semi-group is strongly measurable when regarded as operating in L_p . Since $\|T(t)\|_p \leq 1$ it is also strongly integrable over every finite interval and therefore $A(\alpha)$ is a bounded operator in L_p with $\|A(\alpha)\|_p \leq 1$. In what follows the symbol L_p will always be used for the space $L_p(S, \Sigma, \mu)$.

5. Theorem. *If $\{T(t), 0 \leq t\}$ is a strongly measurable semi-group in L_1 , with $\|T(t)\|_1 \leq 1$ and $\|T(t)\|_\infty \leq 1$ then, for every f in L_p with $1 \leq p < \infty$, the averages $A(\alpha)(f, s)$ converge almost everywhere as $\alpha \rightarrow \infty$.*

Proof. Let $1 \leq p < \infty$, $f \in L_p$, and let $g(s) = \int_0^1 \|T(t)(f, s)\|_p dt$. As indicated at the beginning of this section, the number $g(s)$ is finite for every s not in a null set $E(f)$ and for such s we have $\|rA(r)(f, s)\|_p \leq g(s)$ for every r in the interval $0 \leq r \leq 1$. Furthermore, since

$$\begin{aligned} \|g\|_p^p &= \int_S ds \left[\int_0^1 \|T(t)(f, s)\|_p dt \right]^p \\ &\leq \int_S ds \int_0^1 \|T(t)(f, s)\|_p^p dt = \int_0^1 \|T(t)f\|_p^p dt \leq \|f\|_p^p, \end{aligned}$$

it is seen that g is in L_p . By Lemma 3.3 there is a positive operator P in L_∞ with $\|P\|_1 \leq 1$, $\|P\|_\infty \leq 1$, and $\|T^n(f, \cdot)\|_p \leq P^n(\|f, \cdot\|_p)$, $n = 1, 2, \dots$. By the Riesz convexity theorem we have also $\|P\|_p \leq 1$ and, since $P^n = \sum_{m=0}^{n-1} P^m - \sum_{m=0}^{n-1} P^m$, it is seen from Theorem 3.5 that

$$\left| \frac{T^n}{n} rA(r)(f, s) \right| \leq \frac{P^n}{n}(g, s) \rightarrow 0$$

almost everywhere on S and uniformly on the interval $0 \leq r \leq 1$. The desired conclusion now follows from the identity (vi) and Theorem 3.5. Q.E.D.

6. Lemma. *Let (S, Σ, μ) be a positive measure space and for $t > 0$ let $T(t)$ be an operator in L_1 with $\|T(t)\|_1 \leq 1$ and $\|T(t)\|_\infty \leq 1$. It is assumed that either $\{T(t), 0 < t\}$ is a strongly measurable semi-group in L_1 or that the operators $T(t)$*

are positive, strongly continuous in t and satisfy the inequality $T(t)T(v) \geq T(t+v)$. Let $1 \leq p < \infty$ and for each u in a set U let f_u be an element of L_p such that the lattice supremum $f = \sup_{u \in U} |f_u(\cdot)|$ is also in L_p . For $\alpha > 0$ let

$$e(\alpha) = \{s \mid f(s) > \alpha\} \text{ and } e^*(\alpha) = \{s \mid f^*(s) > \alpha\},$$

where

$$f^* = \sup_{u \in U} \sup_{0 < \alpha} |A(\alpha)(f_u, \cdot)|,$$

and $A(\alpha) = \alpha^{-1} \int_0^\alpha T(t)dt$. Then

$$\alpha \mu(e^*(2\alpha)) \leq \int_{e(\alpha)} f(s) \mu(ds), \quad \alpha > 0.$$

Proof. There is, for each u in U , a null set $E(u)$ for which the averages $A(\alpha)(f_u, s)$ exist for all non-negative α , all u in U , and all s not in the null set $E(u)$. The operator $T(t)$ is strongly continuous in t at every point $t > 0$ and thus, since $\|T(t)\|_p \leq 1$, it follows that, for each α in the set R of non-negative rational numbers,

$$A(\alpha)f_u = \lim_{n \rightarrow \infty} \frac{1}{\alpha n!} \sum_{m=0}^{\alpha n!-1} T\left(\frac{m}{n!}\right)f_u, \quad u \in U, \quad \alpha \in R.$$

Using the Cantor diagonal process we may find, for each u in U , a sequence $\{n_i\}$ depending upon u and such that

$$A(\alpha)(f_u, s) = \lim_{i \rightarrow \infty} \frac{1}{\alpha n_i!} \sum_{m=0}^{\alpha n_i!-1} T\left(\frac{m}{n_i!}\right)(f_u, s)$$

for every α in R and every s not in a null set $E_1(u) \supseteq E(u)$. For u in U and $s \notin E_1(u)$ let

$$f_n^*(u, s) = \sup_{0 < k < \infty} \left| \frac{1}{k} \sum_{m=0}^{k-1} T\left(\frac{m}{n!}\right)(f_u, s) \right|$$

so that for $u \in U$, $\alpha \in R$, $s \notin E_1(u)$, and $\epsilon > 0$ there is an integer $N(u, \alpha, s, \epsilon)$ with

$$f_n^*(u, s) \geq |A(\alpha)(f_u, s)| - \epsilon, \quad n \geq N(u, \alpha, s, \epsilon),$$

and therefore

$$\liminf_{n \rightarrow \infty} f_n^*(u, s) \geq |A(\alpha)(f_u, s)|, \quad u \in U, \quad \alpha \in R, \quad s \notin E_1(u).$$

Now if $s \notin E_1(u)$ then $s \notin E(u)$ and since, for such s , $A(\alpha)(f_u, s)$ is continuous in α on the interval $0 < \alpha < \infty$, we have

$$\sup_{0 < \alpha < \infty} |A(\alpha)(f_u, s)| = \sup_{\alpha \in R} |A(\alpha)(f_u, s)|$$

and thus

$$\liminf_{n \rightarrow \infty} f_n^*(u, s) \geq \sup_{0 < \alpha < \infty} |A(\alpha)(f_u, s)|, \quad u \in U, \quad s \notin E_1(u).$$

Let

$$f^*(u, s) = \sup_{0 < \alpha < \infty} |A(\alpha)(f_u, s)|,$$

so that

$$f^* = \sup_u f^*(u, \cdot),$$

and for every u in U

$$\liminf_{n \rightarrow \infty} f_n^*(u, s) \geq f^*(u, s),$$

for almost all s in S . Now let $P(n)$ be the positive operator associated with $T(1/n!)$ as in Lemma 3.3 so that

$$\left| \frac{1}{k} \sum_{m=0}^{k-1} T\left(\frac{m}{n!}\right)(f_u, \cdot) \right| \leq \frac{1}{k} \sum_{m=0}^{k-1} P(n)^m f, \quad u \in U, \quad 1 \leq k.$$

Let

$$f_n^* = \sup_k \frac{1}{k} \sum_{m=0}^{k-1} P(n)^m f$$

so that $f_n^* \geq f_n^*(u, \cdot)$ for every u in U and thus

$$\liminf_{n \rightarrow \infty} f_n^* \geq f^*(u, \cdot), \quad u \in U.$$

It follows that

$$(*) \quad \liminf_{n \rightarrow \infty} f_n^* \geq \sup_u f^*(u, \cdot) = f^*.$$

Thus, if $e_n^*(\alpha) = \{s \mid f_n^*(s) > \alpha\}$, we have $e^*(\alpha) \subseteq \liminf_n e_n^*(\alpha)$. Lemma 3.7 and Fatou's lemma show that

$$\alpha \mu(e^*(2\alpha)) \leq \liminf_{n \rightarrow \infty} \alpha \mu(e_n^*(2\alpha)) \leq \int_{e^*(\alpha)} f(s) \mu(ds). \quad \text{Q.E.D.}$$

Similarly the inequality $\liminf_n f_n^* \geq f^*$, together with Fatou's lemma and Theorem 3.8 show that

$$\begin{aligned} \int_S f^*(s)^p \mu(ds) &\leq \int_S \liminf_{n \rightarrow \infty} f_n^*(s)^p \mu(ds) \\ &\leq \liminf_{n \rightarrow \infty} \int_S f_n^*(s)^p \mu(ds) \\ &\leq \frac{2^p p}{p-1} \int_S f(s)^p \mu(ds), \quad 1 < p < \infty, \\ &\leq 2 \left[\mu(S) + \int_S f(s) \log^+ f(s) \mu(ds) \right], \quad p = 1. \end{aligned}$$

This proves the following theorem.

7. Theorem. Let $\{T(t), 0 \leq t\}$ be a strongly measurable semi-group in L_1 with $\|T(t)\|_1 \leq 1$ and $\|T(t)\|_\infty \leq 1$. For each u in a set U let f_u be an element of L_p such that the lattice supremum $f = \sup_{u \in U} |f_u(\cdot)|$ is also in L_p . If $p = 1$ it is also assumed that $\mu(S) < \infty$ and that $\int_S f(s) \log^+ f(s) \mu(ds) < \infty$. Then the function

$$f^* = \sup_{u \in U} \sup_{0 < \alpha < \infty} \|A(\alpha)(f_u, \cdot)\|$$

is in L_p and

$$\begin{aligned} \|f^*\|_p &\leq 2 \left(\frac{p}{p-1} \right)^{1/p} \|f\|_p, \quad 1 < p < \infty \\ &\leq 2 \left[\mu(S) + \int_S f(s) \log^+ f(s) \mu(ds) \right], \quad p = 1. \end{aligned}$$

The final two theorems in the present section are concerned with generalizations of Theorems 5 and 7 to the case of k -parameter semi-groups. We shall first define the basic concepts associated with such semi-groups and their averages.

8. Definition. A strongly measurable k -parameter semi-group of operators in a real or complex B -space X is a set $\{T(t_1, \dots, t_k), t_1, \dots, t_k > 0\}$ of bounded linear operators in X with the properties

- (a) $T(t_1, \dots, t_k)T(u_1, \dots, u_k) = T(t_1 + u_1, \dots, t_k + u_k)$,
- (b) $T(t_1, \dots, t_k)x$ is Lebesgue measurable in $(t_1, \dots, t_k)$ for each x in X .

We shall need to use the k -parameter case of the theorem of HILLE mentioned above. For convenience, we state this as follows.

9. Lemma. If $T(t_1, \dots, t_k)$ is a strongly measurable k -parameter semi-group in a B -space X then, for each x in X , $T(t_1, \dots, t_k)x$ is continuous on the domain $t_1, \dots, t_k > 0$.

If, besides being strongly measurable, the semi-group is strongly integrable on every set determined by inequalities of the form $0 < t_1, \dots, t_k \leq \alpha$ then the averages

$$A(\alpha) = \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha T(t_1, \dots, t_k) dt_1 \dots dt_k$$

are defined for $\alpha > 0$ in a manner similar to that used in the one parameter case. In particular, $A(\alpha)$ is a bounded linear operator in X . Also, as in the one parameter case, we see that if X is the space $L_p(S, \Sigma, \mu)$ and if

$$T(t_1, \dots, t_k)(f, s)$$

is a scalar representation of the vector $T(t_1, \dots, t_k)f$ which is measurable as a function of $t_1, \dots, t_k$, and s then

$$A(\alpha)(f, s) = \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha T(t_1, \dots, t_k)(f, s) dt_1 \dots dt_k$$

is a scalar representation of the vector $A(\alpha)f$ and is, for almost all s in S , continuous in α . This observation shows that the function

$$f^*(s) = \sup_{\alpha < 0} |A(\alpha)(f, s)|$$

is the same as the function $\sup_{\alpha \in R} |A(\alpha)(f, s)|$, where R is the set of positive rational numbers. Thus f^* is μ -measurable and indeed, is the same as the supremum of the set $\{|A(\alpha)(f, \cdot)|, 0 < \alpha\}$ in the lattice of μ -measurable functions on S .

Actually, in the following theorem, which is a generalization of Theorem 7, the situation discussed is somewhat more general than that of a k -parameter semi-group, for here we are dealing with the averages of the form

$$\alpha_1^{-1} \cdots \alpha_k^{-1} \int_1^{\alpha_1} \cdots \int_0^{\alpha_k} T_k(t_k) \cdots T_1(t_1) dt_1 \cdots dt_k$$

where $T_1(t_1), \dots, T_k(t_k)$ are one parameter semi-groups. The interest here lies in the fact that these one parameter semi-groups need not commute for if they do commute then $T(t_1, \dots, t_k) = T_k(t_k) \cdots T_1(t_1)$ is a k -parameter semi-group.

10. Theorem. *Let the semi-groups $\{T_i(t), 0 \leq t\}$, $i = 1, \dots, k$, be strongly measurable semi-groups in L_1 with $\|T_i(t)\|_1 \leq 1$ and $\|T_i(t)\|_\infty \leq 1$. Then, for every f in L_p with $1 < p < \infty$, the functions*

$$(1) \quad \frac{1}{\alpha_1 \cdots \alpha_k} \int_0^{\alpha_k} \cdots \int_0^{\alpha_1} T_k(t_k) \cdots T_1(t_1)(f, s) dt_1 \cdots dt_k$$

approach a limit almost everywhere on S as $\alpha_1 \rightarrow \infty, \dots, \alpha_k \rightarrow \infty$ independently. The limit also exists in the norm of L_p and the functions (1) are, for $\alpha_1, \dots, \alpha_k > 0$, all dominated by a function in L_p .

Proof. If $A_i(\alpha)$ is the average of the semi-group $\{T_i(t), 0 \leq t\}$ on the interval $(0, \alpha)$ then the multiple sequence (1) may be written as $V(\alpha_1, \dots, \alpha_k)(f, s)$ where

$$(2) \quad V(\alpha_1, \dots, \alpha_k) = A_k(\alpha_k) \cdots A_1(\alpha_1).$$

Since $\|T_i(t)\|_\infty \leq 1$ and $\|T(t)\|_1 \leq 1$ it follows from the Riesz convexity theorem that $\|T_i(t)\|_p \leq 1$ and thus that

$$(3) \quad \|A_i(\alpha)\| \leq 1, \quad \|V(\alpha_1, \dots, \alpha_k)\| \leq 1, \quad 1 \leq i \leq k,$$

where here, and throughout the proof, the norm without a subscript refers to the L_p norm.

According to Theorem 1 the limits

$$(4) \quad E_i f = \lim_{\alpha \rightarrow \infty} A_i(\alpha)f, \quad f \in L_p, \quad 1 \leq i \leq k,$$

exist in the strong L_p topology. From (3) and (4) it follows that

$$(5) \quad E_k \cdots E_1 f = \lim V(\alpha_1, \dots, \alpha_k)f, \quad f \in L_p$$

in the norm of L_p . To see this we use (4) inductively and suppose that (5) has been proved for the product of $k - 1$ factors $A_k(\alpha_k), \dots, A_2(\alpha_2)$. Then

$$\begin{aligned} |V(\alpha_1, \dots, \alpha_k)f - E_k \cdots E_1 f| &\leq |A_k(\alpha_k) \cdots A_2(\alpha_2)\{A_1(\alpha_1) - E_1\}f| \\ &\quad + |\{A_k(\alpha_k) \cdots A_2(\alpha_2) - E_k \cdots E_2\}E_1 f| \\ &\leq |\{A_1(\alpha_1) - E_1\}f| + |\{A_k(\alpha_k) \cdots A_2(\alpha_2) - E_k \cdots E_2\}E_1 f| \end{aligned}$$

approaches zero by the induction hypothesis and establishes (5). It will next be shown that the functions (1) are dominated by a function in L_p which is independent of the parameters $\alpha_1, \dots, \alpha_k$. The preceding theorem shows that this is the case if $k = 1$. Let us suppose that it has now been established in the case of $k - 1$ semi-groups $T_1, \dots, T_{k-1}$ so that the function

$$g = \sup_{0 < \alpha_1, \dots, \alpha_{k-1} < \infty} |A_{k-1}(\alpha_{k-1}) \cdots A_1(\alpha_1)(f, \cdot)|$$

is in L_p . In the preceding theorem let $u = (\alpha_1, \dots, \alpha_{k-1})$ be a point in $k - 1$ dimensional Euclidean space and let $f_u = A_{k-1}(\alpha_{k-1}) \cdots A_1(\alpha_1)f$. It follows then, from Theorem 7, that

$$\sup_u \sup_{0 < \alpha_k < \infty} |A(\alpha_k)(f_u, \cdot)|,$$

which dominates all the functions (1), is in L_p .

Finally, it will be proved that the functions (1) converge almost everywhere on S as $\alpha_1, \dots, \alpha_k \rightarrow \infty$ independently. If $k = 1$ this follows from Theorem 5 and we suppose that it has been established in the case of $k - 1$ semi-groups $T_2, \dots, T_k$. By Corollary 2, functions of the form

$$h = f + \sum_{i=1}^n [I - T_1(t_i)]g_i$$

with $T_1(t)f = f$, $0 \leq t$, and with $g_i \in L_\infty \cap L_p$ are dense in L_p . Thus, since

$$A_1(\alpha)(I - T_1(u)) = \frac{1}{\alpha} \left[\int_0^u T_1(t) dt - \int_\alpha^{u+\alpha} T(t) dt \right],$$

we have, for such a function h ,

$$V(\alpha_1, \dots, \alpha_k)h = A_k(\alpha_k) \cdots A_2(\alpha_2)f + \sum_{i=1}^n h_i$$

where $|h_i|_\infty \leq 2t_i |g_i|_\infty / \alpha_1$. The induction hypothesis then shows that $Vh \rightarrow E_k \cdots E_2 f$ almost everywhere on S . To complete the proof we apply BANACH'S Convergence Theorem 1.6. We shall let A_p be the set of all elements $a = (\alpha_1, \dots, \alpha_k)$ with α_i rational and $\alpha_i \geq p$, $i = 1, \dots, k$, and let $T_a = V(a) = V(\alpha_1, \dots, \alpha_k)$. We note that since, for almost all s , $V(\alpha_1, \dots, \alpha_k)(t, s)$ is continuous for $0 < \alpha_1, \dots, \alpha_k$ we have

$$\sup_{a, b \in A_p} |V(a)(f, s) - V(b)(f, s)| = \sup_{\alpha_i, \beta_i \geq p} |V(a)(f, s) - V(b)(f, s)|,$$

where in the second supremum $\alpha_1, \dots, \beta_k$ vary over all real numbers greater than or equal to p . Thus, by Theorem 1.6

$$\lim_{p \rightarrow \infty} \sup_{\alpha_i, \beta_i \geq p} |V(a)(f, s) - V(b)(f, s)| = 0,$$

so that $V(a)(f, s)$ converges for almost all s in S as $\alpha_1, \dots, \alpha_k \rightarrow \infty$ independently. Q.E.D.

If $p = 1$ the preceding theorem is no longer true because the averages (even if $k = 1$) need not be bounded by a summable function. However the averages of a k -parameter strongly measurable semi-group $\{T(t_1, \dots, t_k), t_1, \dots, t_k > 0\}$ in $L_1(S, \Sigma, \mu)$ with $\|T(t_1, \dots, t_k)\|_1 \leq 1, \|T(t_1, \dots, t_k)\|_\infty \leq 1$ do, when operating on a function in $L_1(S, \Sigma, \mu)$, converge almost everywhere. This result, which will be proved by induction on k , is based upon the following key lemma which generalizes Lemma 6 to the case of a k -parameter semi-group.

11. Lemma. *Let (S, Σ, μ) be a positive measure space and let $\{T(t_1, \dots, t_k), t_1, \dots, t_k > 0\}$ be a strongly measurable semi-group of operators in $L_1(S, \Sigma, \mu)$ with $\|T(t_1, \dots, t_k)\|_1 \leq 1, \|T(t_1, \dots, t_k)\|_\infty \leq 1$. Let $1 \leq p < \infty, f \in L_p$, and $f^*(s) = \sup_{0 < \alpha < \infty} \|A(\alpha)(f, s)\|$ where*

$$A(\alpha) = \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha T(t_1, \dots, t_k) dt_1 \dots dt_k.$$

Then there is an absolute constant c_k , independent of the semi-group and independent of f , such that

$$\mu(e^*(\beta)) \leq \frac{1}{c_k \beta} \int_{e(c_k \beta)} \|f(s)\| \mu(ds), \quad \beta > 0,$$

where $e^(\beta) = \{s \mid f^*(s) > \beta\}$ and $e(\beta) = \{s \mid \|f(s)\| > \beta\}$.*

The proof of Lemma 11 will be an inductive one based upon its special case $k = 1$ which has been established in Lemma 6. Since the proof is a rather circuitous one based upon three auxiliary lemmas it will perhaps be of some help if we make a schematic display of the major logical implications involved in proving Lemma 11 from Lemma 6. To make such a diagram we shall use the symbol C_k for Lemma 11 and D_k for its discrete analogue (D_k is stated explicitly in Lemma 16). Then C_1 is Lemma 6 and D_1 is Lemma 3.7. The symbol CP_k will be used for C_k with the additional hypothesis that the operators in the semi-group are positive and DP_k will stand for the discrete analogue of CP_k . The lemmas designated by CP_k and DP_k are stated explicitly in Lemmas 13 and 14 respectively. In terms of these symbols the logical structure of the proof of Lemma 11 is as follows: $C_1 \Rightarrow CP_k \Rightarrow DP_k \Rightarrow D_k \Rightarrow C_k$. The first implication in this sequence, i.e., the reduction of CP_k to C_1 is obtained by a transformation which reduces a $2k$ -parameter semi-group to a k -parameter semi-group. The device is based upon the observation that if a family $\{\phi_u, 0 < u\}$ of functions form a semi-group under convolution, i.e., $\phi_u * \phi_v = \phi_{u+v}$, and if $\phi_u(x) = 0$ for $x \leq 0$, then the transformation

$$S(x_1, \dots, x_k) = \int_0^\infty \cdots \int_0^\infty \phi_{x_1}(t_1) \phi_{x_1}(t_2) \phi_{x_2}(t_3) \phi_{x_2}(t_4) \cdots \phi_{x_k}(t_{2k-1}) \phi_{x_k}(t_{2k}) T(t_1, \dots, t_{2k}) dt_1 \cdots dt_{2k}$$

reduces the $2k$ -parameter semi-group $T(t_1, \dots, t_{2k})$ to the k -parameter semi-group $S(x_1, \dots, x_k)$. To be sure that the semi-group $S(x_1, \dots, x_k)$ has the desired properties the functions ϕ_u must be chosen with care. The following lemma defines these functions and states the properties we shall use.

12. Lemma. For $u > 0$ let $\phi_u(x) = u^{-2} \phi(xu^{-2})$ where $\phi(x) = 1/(2\pi^{1/2} x^{3/2} e^{1/4x})$ for $x > 0$ and $\phi(x) = 0$ for $x \leq 0$. Then $\phi_u * \phi_v = \phi_{u+v}$ and $\int_{-\infty}^\infty \phi(x) dx = 1$.

Proof. We have

$$\begin{aligned} \int_{-\infty}^\infty \phi(x) dx &= 2^{-1} \pi^{-1/2} \int_0^\infty x^{-3/2} e^{-1/4x} dx \\ &= 2^{-1} \pi^{-1/2} \int_0^\infty y^{-1/2} e^{-y/4} dy = \pi^{-1/2} \int_0^\infty x^{-1/2} e^{-x} dx \\ &= \pi^{-1/2} \Gamma(1/2) = 1. \end{aligned}$$

To prove that $\phi_u * \phi_v = \phi_{u+v}$ it will suffice to show that

$$(*) \quad \int_{-\infty}^\infty \phi(x) e^{-xt} dx = e^{-\sqrt{t}}, \quad t > 0.$$

For, if (*) is known, then

$$\int_{-\infty}^\infty \phi_u(x) e^{-xt} dx = \int_{-\infty}^\infty \phi(x) e^{-xu^2t} dx = e^{-u\sqrt{t}}, \quad t > 0,$$

and thus we have

$$\int_{-\infty}^\infty (\phi_u * \phi_v)(x) e^{-xt} dx = e^{-(u+v)\sqrt{t}} = \int_{-\infty}^\infty \phi_{u+v}(x) e^{-xt} dx, \quad t > 0,$$

which, in view of the known fact that two functions can have the same Laplace transform only if they are identical, proves that $\phi_u * \phi_v = \phi_{u+v}$. To establish the equation (*) let $A = \int_0^\infty e^{-\left(\frac{\alpha}{u}-u\right)^2} du$ for $\alpha > 0$. Then, by placing $u = 1/v$ we find that

$$A = \int_0^\infty \frac{\alpha}{u^2} e^{-\left(\frac{\alpha}{u}-u\right)^2} du.$$

Upon adding the two formulæ for A it is seen that

$$A = \frac{1}{2} \int_0^\infty \left(1 + \frac{\alpha}{u^2}\right) e^{-\left(\frac{\alpha}{u}-u\right)^2} du,$$

and thus, by putting $\frac{\alpha}{u} - u = v$, that

$$A = \frac{1}{2} \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{1}{2} \sqrt{\pi}.$$

Therefore

$$\begin{aligned} \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-1/4x}}{x^{3/2}} e^{-xt} dx &= \frac{e^{-\sqrt{t}}}{2\sqrt{\pi}} \int_0^{\infty} x^{-3/2} e^{-(\frac{1}{2\sqrt{x}} - \sqrt{x})^2} dx \\ &= \frac{2e^{-\sqrt{t}}}{\sqrt{\pi}} \int_0^{\infty} \frac{\sqrt{t}}{2y^2} e^{-(\frac{\sqrt{t}}{2y} - y)^2} dy \\ &= \frac{2e^{-\sqrt{t}}}{\sqrt{\pi}} A = e^{-\sqrt{t}}, \end{aligned}$$

which proves (*) and completes the proof of the lemma. Q.E.D.

We shall now state and prove the lemma referred to as CP_k . For technical reasons occurring later the following lemma is stated for what might be called a positive sub-semi-group rather than for a positive semi-group. The proof of this lemma is the most involved of all the steps in the proof of Lemma 11 as outlined in the diagram: $C_1 \Rightarrow CP_k \Rightarrow DP_k \Rightarrow D_k \Rightarrow C_k$. The proof of the implication $CP_k \Rightarrow DP_k$, which is the proof of Lemma 14, is very similar to the proof of the following lemma but considerably easier in details, and thus the reader may prefer to read the proof of Lemma 14 before that of Lemma 13.

13. Lemma. *Let (S, Σ, μ) be a positive measure space and for $t_1, \dots, t_k > 0$ let $P(t_1, \dots, t_k)$ be a positive operator in $L_1(S, \Sigma, \mu)$ with $|P(t_1, \dots, t_k)|_1 \leq 1$, $|P(t_1, \dots, t_k)|_{\infty} \leq 1$ and*

$$P(t_1, \dots, t_k)P(u_1, \dots, u_k) \geq P(t_1 + u_1, \dots, t_k + u_k).$$

It is assumed that the operator valued function P is strongly continuous on the domain $t_1, \dots, t_k > 0$. Let $1 \leq p < \infty$, $f \in L_p$, and $f^ = \sup_{0 < \alpha} A(\alpha) |f(\cdot)|$ where*

$$A(\alpha) = \frac{1}{\alpha^k} \int_0^{\alpha} \dots \int_0^{\alpha} P(t_1, \dots, t_k) dt_1 \dots dt_k.$$

Then there is an absolute constant c_k , independent of the operators $P(t_1, \dots, t_k)$ and independent of f , such that

$$\mu(e^*(\beta)) \leq \frac{1}{c_k \beta} \int_{e^*(c_k \beta)} |f(s)| \mu(ds), \quad \beta > 0,$$

where $e^(\beta) = \{s \mid f^*(s) > \beta\}$ and $e(\beta) = \{s \mid |f(s)| > \beta\}$.*

Proof. We observe that if $1 \leq m \leq k$ and if $T(t_1, \dots, t_m) = P(t_1, \dots, t_k)$ then

$$\begin{aligned} \frac{1}{\alpha^m} \int_0^\alpha \cdots \int_0^\alpha T(t_1, \dots, t_m) dt_1 \cdots dt_m \\ = \frac{1}{\alpha^k} \int_0^\alpha \cdots \int_0^\alpha P(t_1, \dots, t_k) dt_1 \cdots dt_k. \end{aligned}$$

This shows that if the lemma is known to be true for an integer k it is also true for any integer $m \leq k$. Thus, to prove the lemma, it will suffice to show that it is true if k is even. If $k = 1$ the lemma has already been proved (Lemma 6). We shall thus suppose that the lemma has been proved for the integer n and shall conclude its validity for the integer $k = 2n$. Let ϕ_u, ϕ be the functions defined in Lemma 12, and let

$$\begin{aligned} S(x_1, \dots, x_n) = \int_0^\infty \cdots \int_0^\infty \phi_{x_1}(t_1) \phi_{x_1}(t_2) \phi_{x_2}(t_3) \phi_{x_2}(t_4) \\ \cdots \phi_{x_n}(t_{k-1}) \phi_{x_n}(t_k) P(t_1, \dots, t_k) dt_1 \cdots dt_k. \end{aligned}$$

By Lemma 12 we have

$$\int_0^\infty \cdots \int_0^\infty \phi_{x_1}(t_1) \phi_{x_1}(t_2) \cdots \phi_{x_n}(t_{k-1}) \phi_{x_n}(t_k) dt_1 \cdots dt_k = \left\{ \int_0^\infty \phi(t) dt \right\}^k = 1,$$

and, since $\phi(t) > 0$, it follows that $S(x_1, \dots, x_n)$ is a positive operator with $|S(x_1, \dots, x_n)|_1 \leq 1$ and $|S(x_1, \dots, x_n)|_\infty \leq 1$. Moreover, since $\phi(x) = 0$ for $x < 0$, we have by Lemma 12, for $0 \leq f \in L_1(S, \Sigma, \mu)$,

$$\begin{aligned} S(x_1, \dots, x_n) S(y_1, \dots, y_n) f \\ = \int_0^\infty \cdots \int_0^\infty \phi_{x_1}(t_1) \cdots \phi_{x_n}(t_k) \phi_{y_1}(u_1) \\ \cdots \phi_{y_n}(u_k) P(t_1, \dots, t_k) P(u_1, \dots, u_k) f dt_1 \cdots du_k \\ \geq \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \phi_{x_1}(t_1) \cdots \phi_{x_n}(t_k) \phi_{y_1}(u_1) \\ \cdots \phi_{y_n}(u_k) P(t_1 + u_1, \dots, t_k + u_k) f dt_1 \cdots du_k \\ = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \phi_{x_1}(v_1 - w_1) \phi_{y_1}(w_1) \\ \cdots \phi_{x_n}(v_k - w_k) \phi_{y_n}(w_k) P(v_1, \dots, v_k) f dv_1 \cdots dw_k \\ = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \phi_{x_1+y_1}(v_1) \cdots \phi_{x_n+y_n}(v_k) P(v_1, \dots, v_k) f dv_1 \cdots dv_k \\ = S(x_1 + y_1, \dots, x_n + y_n) f. \end{aligned}$$

Thus $S(x_1, \dots, x_n)$ satisfies the hypothesis of Lemma 13, and so by our inductive hypothesis there is a constant c_n such that

$$\mu\{s \mid f^{**}(s) > \beta\} \leq \frac{1}{c_n \beta} \int_{s(c_n \beta)} |f(s)| \mu(ds), \quad \beta > 0,$$

where

$$f^{**} = \sup_{0 < \alpha} \frac{1}{\alpha^n} \int_0^\alpha \cdots \int_0^\alpha S(x_1, \dots, x_n) |f(\cdot)| dx_1 \cdots dx_n.$$

To complete the proof of the lemma it is therefore sufficient to establish the existence of an absolute constant d_k such that $f^{**} \geq d_k f^*$. For this it will suffice to show that for each $\alpha > 0$ there is an $\alpha_1 = \alpha_1(\alpha) > 0$ such that

$$\begin{aligned} [*] \quad A(S, \alpha_1) f &= \frac{1}{\alpha_1^n} \int_0^{\alpha_1} \cdots \int_0^{\alpha_1} S(x_1, \dots, x_n) f dx_1 \cdots dx_n \\ &\geq d_k \frac{1}{\alpha^k} \int_0^\alpha \cdots \int_0^\alpha P(t_1, \dots, t_k) f dt_1 \cdots dt_k, \end{aligned}$$

for every non-negative function f in L_1 . Since

$$\begin{aligned} A(S, \alpha_1) f &= 2^{-k} \pi^{-k/2} \int_0^\infty \cdots \int_0^\infty h(\alpha_1, t_1, t_2) h(\alpha_1, t_3, t_4) \\ &\quad \cdots h(\alpha_1, t_{k-1}, t_k) P(t_1, \dots, t_k) f dt_1 \cdots dt_k, \end{aligned}$$

where

$$\begin{aligned} h(\alpha_1, t, v) &= \frac{1}{\alpha_1} \int_0^{\alpha_1} \phi_u(t) \phi_u(v) du \\ &= \frac{1}{\alpha_1} (tv)^{-3/2} \int_0^{\alpha_1} u^2 e^{-u^2(\frac{1}{4t} + \frac{1}{4v})} du, \end{aligned}$$

it is clearly sufficient to show that $h(\alpha_1, t, v) \geq \delta/\alpha^2$ for all $t, v \leq \alpha$; for then [*] will follow with $d_k = (\delta/4\pi)^{k/2}$. This will be verified for the function $\alpha_1 = \alpha^{1/2}$, i.e., we shall show that

$$(t_1 t_2)^{-3/2} \int_0^{\alpha_1} u^2 e^{-u^2(\frac{1}{4t_1} + \frac{1}{4t_2})} du > \delta, \quad t_1, t_2 \leq \alpha_1^2.$$

By placing $u = \alpha_1 v$, $t_1 = \alpha_1^2 s_1$, $t_2 = \alpha_1^2 s_2$ this inequality transforms into

$$(s_1 s_2)^{-3/2} \int_0^1 v^2 e^{-v^2(\frac{1}{4s_1} + \frac{1}{4s_2})} dv > \delta, \quad 0 < s_1, s_2 \leq 1.$$

Let $G(a) = \int_0^1 v^2 e^{-v^2 a} dv$, so that we wish to show

$$[**] \quad (s_1 s_2)^{-3/2} G\left(\frac{1}{4s_1} + \frac{1}{4s_2}\right) > \delta, \quad 0 < s_1, s_2 \leq 1.$$

Since G is positive and continuous it suffices to prove [**] in the case where either s_1 or s_2 is near zero. We have

$$\begin{aligned} G(a) &= \int_0^1 v^2 e^{-v^2 a} dv = \int_0^1 u^{1/2} e^{-ua} du \\ &= \frac{1}{2} a^{-3/2} \int_0^a v^{1/2} e^{-v} dv, \end{aligned}$$

and so, for some positive k , $G(a) \geq ka^{-3/2}$ for all sufficiently large values of a . Thus

$$(s_1 s_2)^{-3/2} G\left(\frac{1}{4s_1} + \frac{1}{4s_2}\right) \geq 8k(s_1 + s_2)^{-3/2}.$$

if either s_1 or s_2 is near zero. This establishes [**] and completes the proof of the lemma. Q.E.D.

14. Lemma. Let (S, Σ, μ) be a positive measure space and for every set of k non-negative integers $i_1, \dots, i_k$ let $P(i_1, \dots, i_k)$ be a positive operator in $L_1(S, \Sigma, \mu)$ with $|P(i_1, \dots, i_k)|_1 \leq 1$, $|P(i_1, \dots, i_k)|_\infty \leq 1$, and

$$P(i_1, \dots, i_k)P(j_1, \dots, j_k) \geq P(i_1 + j_1, \dots, i_k + j_k).$$

For f in L_p let $f^* = \sup_n A(n) |f(\cdot)|$ where

$$A(n) = \frac{1}{n^k} \sum_{i_k=0}^{n-1} \cdots \sum_{i_1=0}^{n-1} P(i_1, \dots, i_k).$$

Then there is a constant c_k , independent of the operators $P(i_1, \dots, i_k)$ and the function f , such that

$$\mu(\{s \mid f^*(s) > \beta\}) \leq \frac{1}{c_k \beta} \int_{e(c_k \beta)} |f(s)| \mu(ds), \quad \beta > 0.$$

Proof. For $x_1, \dots, x_k > 0$ define the operator

$$S(x_1, \dots, x_k) = e^{-(x_1 + \cdots + x_k)} \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{x_1^{i_1} \cdots x_k^{i_k}}{i_1! \cdots i_k!} P(i_1, \dots, i_k).$$

Since the sum of the coefficients in this series is

$$e^{-(x_1 + \cdots + x_k)} \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \frac{x_1^{i_1} \cdots x_k^{i_k}}{i_1! \cdots i_k!} = 1$$

we see that $|S(x_1, \dots, x_k)|_1 \leq 1$, $|S(x_1, \dots, x_k)|_\infty \leq 1$, and thus, by the Riesz convexity theorem, that $|S(x_1, \dots, x_k)|_p \leq 1$. It is clear also that $S(x_1, \dots, x_k)$ is a positive operator in L_p . Also

$$\begin{aligned}
& S(x_1, \dots, x_k) S(y_1, \dots, y_k) \\
&= e^{-(x_1+y_1+\dots+x_k+y_k)} \sum \frac{x_1^{i_1} \dots x_k^{i_k} y_1^{j_1} \dots y_k^{j_k}}{i_1! \dots i_k! \dots j_1! \dots j_k!} P(i_1, \dots, i_k) P(j_1, \dots, j_k) \\
&\geq e^{-(x_1+y_1+\dots+x_k+y_k)} \sum \frac{x_1^{i_1} \dots y_k^{j_k}}{i_1! \dots j_k!} P(i_1 + j_1, \dots, i_k + j_k),
\end{aligned}$$

where Σ is taken over all non-negative integers $i_1, \dots, i_k, j_1, \dots, j_k$. Thus

$$\begin{aligned}
& S(x_1, \dots, x_k) S(y_1, \dots, y_k) \\
&\geq e^{-(x_1+y_1+\dots+x_k+y_k)} \sum_{m_1=0}^{\infty} \dots \sum_{m_k=0}^{\infty} \frac{P(m_1, \dots, m_k)}{m_1! \dots m_k!} (x_1 + y_1)^{m_1} \dots (x_k + y_k)^{m_k} \\
&= S(x_1 + y_1, \dots, x_k + y_k),
\end{aligned}$$

and so it is seen that the operators $S(x_1, \dots, x_k)$ satisfy the requirements of the preceding lemma. Consequently if

$$f^{**} = \sup_{0 < \alpha} \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha S(x_1, \dots, x_k) |f(\cdot)| dx_1 \dots dx_k,$$

where $f \in L_p$ then

$$\mu(\{s \mid f^{**}(s) > \beta\}) \leq \frac{1}{c_k \beta} \int_{e(c_k \beta)} |f(s)| \mu(ds), \quad \beta > 0.$$

Thus the present lemma will be proved if we establish the existence of an absolute constant d_k with $f^{**} \geq d_k f^*$. This inequality will follow from the statement that for $0 \leq f \in L_p$ and for $n \geq 0$ there is an $\alpha = \alpha(n, f)$ such that

$$\begin{aligned}
[*] \quad & \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha S(x_1, \dots, x_k) f dx_1 \dots dx_k \\
& \geq \frac{d_k}{n^k} \sum_{i_1=0}^{n-1} \dots \sum_{i_k=0}^{n-1} P(i_1, \dots, i_k) f.
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha S(x_1, \dots, x_k) dx_1 \dots dx_k \\
&= \sum_{i_1=0}^{\infty} \dots \sum_{i_k=0}^{\infty} c_{i_1}(\alpha) \dots c_{i_k}(\alpha) P(i_1, \dots, i_k),
\end{aligned}$$

where $c_m(\alpha) = \alpha^{-1} \int_0^\alpha e^{-x} x^m dx / m!$, to prove [*] it suffices to show that there exists a $\delta > 0$ and, for each n , an $\alpha(n) > 0$ such that $c_m(\alpha(n)) \geq \delta/n$ for $m < n$; for then [*] will hold with $d_k = \delta^k$. This last statement will be verified for $\alpha(n) = n$, i.e., we shall show that, for some $\delta > 0$, $\int_0^n e^{-x} x^m dx / m! \geq \delta$ for all $m < n$. It thus suffices to show that $\liminf_{m \rightarrow \infty} \int_0^m f_m(x) dx > 0$ where $f_m(x) =$

$e^{-x}x^m/m!$. Since $f'_m(x) \geq 0$ and $f''_m(x) \leq 0$ on the interval $m - \sqrt{m} \leq x \leq m$, we have

$$\int_0^m f_m(x) dx \geq \int_{m-\sqrt{m}}^m f_m(x) dx \geq \frac{1}{2} \sqrt{m} f_m(m) = \frac{1}{2} \frac{e^{-m} m^{m+1} m^{-1/2}}{m!} \rightarrow \frac{e}{\sqrt{8\pi}},$$

by Stirling's formula. Q.E.D.

In proving the implication $DP_k \Rightarrow D_k$ we shall need the following lemma.

15. Lemma. *For every bounded operator T in L_1 , for which $\|T_\infty\| < \infty$ let $P(T)$ be the positive operator associated with T as in Lemma 3.4. Then $P(T_1)P(T_2) \geq P(T_1 T_2)$.*

Proof. We recall that, for $0 \leq f \in L_1 \cap L_\infty$

$$P(T)f = \sup_{|g(\cdot)| \leq f} |T(g, \cdot)|.$$

Let $|g(\cdot)| \leq f$ so that, by Lemma 3.4, $|T_1 T_2(g, \cdot)| \leq P(T_1) |T_2(g, \cdot)| \leq P(T_1)P(T_2) |g(\cdot)| \leq P(T_1)P(T_2)f$. Thus $P(T_1 T_2)f \leq P(T_1)P(T_2)f$ for every non-negative function f in $L_1 \cap L_\infty$. A continuity argument shows that this same inequality is valid for all positive f in L_1 . Q.E.D.

16. Lemma. *Let $T_1, \dots, T_k$ be commuting operators in L_1 with $\|T_i\|_1, \|T_i\|_\infty \leq 1$ for $i = 1, \dots, k$. For $f \in L_p$ let $f^* = \sup_{n \geq 1} |A(n)(f, \cdot)|$, where*

$$A(n) = \frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T_1^{i_1} \cdots T_k^{i_k}.$$

Then there is a constant c_k , independent of the operators $T_1, \dots, T_k$ and the function f , such that

$$\mu(\{s \mid f^*(s) > \beta\}) \leq \frac{1}{c_k \beta} \int_{e(c_k \beta)} |f(s)| \mu(ds).$$

Proof. Using the notation $P(T)$ introduced in Lemma 15 we shall define, for each set $i_1, \dots, i_k$ of non-negative integers, the operators $P(i_1, \dots, i_k) = P(T_1^{i_1} \cdots T_k^{i_k})$. Then, by Lemmas 3.4 and 15, the operators $P(i_1, \dots, i_k)$ satisfy the hypothesis of Lemma 14. Hence, for some constant c_k

$$[*] \quad \mu(\{f^{**}(s) > \beta\}) \leq \frac{1}{c_k \beta} \int_{e(c_k \beta)} |f(s)| \mu(ds), \quad \beta > 0,$$

where

$$f^{**} = \sup_{n \geq 1} \frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} P(i_1, \dots, i_k) |f(\cdot)|.$$

But, by Lemma 3.4,

$$f^{**} \geq \sup_{n \geq 1} \left| \frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T_1^{i_1} \cdots T_k^{i_k}(f, \cdot) \right| = f^*.$$

Hence $\{s \mid f^{**}(s) > \beta\} \supseteq \{s \mid f^*(s) > \beta\}$ which, in view of [*], proves the lemma. Q.E.D.

We are now prepared to give the proof of Lemma 11. This lemma will first be proved under the slight additional assumption that the semi-group $T(t_1, \dots, t_k)$ is defined for all $t_i \geq 0$. To simplify the notation the proof will be given for the case $k = 2$ but it will be clear that the method is satisfactory for an arbitrary positive integer k . There is a μ -null set E for which the averages

$$A(\alpha)(f, s) = \frac{1}{\alpha^2} \int_0^\alpha \int_0^\alpha T(t_1, t_2)(f, s) dt_1 dt_2$$

exist for all s not in E . Thus, for $s \notin E$, $A(\alpha)(f, s)$ is continuous for $\alpha > 0$. By Lemma 9 the semi-group $\{T(t_1, t_2), t_1, t_2 \geq 0\}$ is strongly continuous at every point $t_1 > 0, t_2 > 0$; and thus, since $\|T(t_1, t_2)\|_p \leq 1$, it follows that for each α in the set R of non-negative rational numbers,

$$A(\alpha)f = \lim_{n \rightarrow \infty} \frac{1}{(\alpha n!)^2} \sum_{i_1=0}^{\alpha n!-1} \sum_{i_2=0}^{\alpha n!-1} S_1(n)^{i_1} S_2(n)^{i_2} f,$$

where $S_1(n) = T\left(\frac{1}{n!}, 0\right)$, $S_2(n) = T\left(0, \frac{1}{n!}\right)$. If a sequence of functions converges in the mean it has a subsequence converging almost everywhere. Thus, using the Cantor diagonal process, we may find a sequence $\{n_i\}$ such that

$$A(\alpha)(f, s) = \lim_{i \rightarrow \infty} \frac{1}{(\alpha n_i!)^2} \sum_{i_1=0}^{\alpha n_i!-1} \sum_{i_2=0}^{\alpha n_i!-1} S_1(n_i)^{i_1} S_2(n_i)^{i_2}(f, s)$$

for each α in R and each s not in a null set $E_1 \supseteq E$. For $s \notin E_1$ let

$$f_n^*(s) = \sup_{1 \leq k} \left| \frac{1}{k^2} \sum_{m_1=0}^{k-1} \sum_{m_2=0}^{k-1} S_1(n)^{m_1} S_2(n)^{m_2}(f, s) \right|,$$

so that for $\alpha \in R, s \notin E_1$, and $\epsilon > 0$, there is an integer $N(\alpha, s, \epsilon)$ with

$$|f_n^*(s)| \geq |A(\alpha)(f, s)| - \epsilon, \quad n \geq N(\alpha, s, \epsilon).$$

Therefore

$$\liminf_{n \rightarrow \infty} f_n^*(s) \geq |A(\alpha)(f, s)|, \quad \alpha \in R, \quad s \notin E_1.$$

Now if $s \notin E_1$ then $s \notin E$ and hence for $s \notin E_1$ the average $A(\alpha)(f, s)$ is continuous in α on the interval $0 < \alpha < \infty$. Thus

$$\liminf_{n \rightarrow \infty} f_n^*(s) \geq f^*(s) = \sup_{0 < \alpha} |A(\alpha)(f, s)|, \quad s \notin E_1.$$

Thus for $\beta > 0$ we have $\lim_n \inf \chi_n \geq \chi$ where χ_n and χ are the characteristic functions of $\{s \mid f_n^*(s) > \beta\}$ and $\{s \mid f^*(s) > \beta\}$ respectively. Consequently, by Fatou's theorem and Lemma 16, we have

$$\begin{aligned}
\mu(\{s \mid f^*(s) > \beta\}) &= \int_S \chi(s) \mu(ds) \\
&\leq \liminf_n \int_S \chi_n(s) \mu(ds) \\
&= \liminf_{n \rightarrow \infty} \mu\{s \mid f_n^*(s) > \beta\} \\
&\leq \liminf_{n \rightarrow \infty} \frac{1}{c_k \beta} \int_{s(c_k \beta)} |f(s)| \mu(ds) \\
&= \frac{1}{c_k \beta} \int_{s(c_k \beta)} |f(s)| \mu(ds).
\end{aligned}$$

This proves Lemma 11 under the additional assumption that the semi-group is defined for all $t_1, \dots, t_k \geq 0$. Now suppose that

$$\{S(t_1, \dots, t_k), t_1, \dots, t_k > 0\}$$

is a semi-group which satisfies the hypothesis of Lemma 11 and, for $\epsilon > 0$, let $T(\epsilon; t_1, \dots, t_k) = I$ if $t_1 = t_2 = \dots = t_k = 0$ and otherwise let $T(\epsilon; t_1, \dots, t_k) = S(t_1 + \epsilon u_1, \dots, t_k + \epsilon u_k)$, where $u_i = t_1 + \dots + t_{i-1} + t_{i+1} + \dots + t_k$. Then we may apply the result just proved to the semi-group $\{T(\epsilon; t_1, \dots, t_k), t_1, \dots, t_k \geq 0\}$. For f in L_p we let

$$f_\epsilon^* = \sup_{0 < \alpha} |A(\epsilon, \alpha)(f, \cdot)|, \quad f^* = \sup_{0 < \alpha} |A(\alpha)(f, \cdot)|,$$

where

$$A(\epsilon, \alpha) = \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha T(\epsilon; t_1, \dots, t_k) dt_1 \dots dt_k$$

and

$$A(\alpha) = \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha S(t_1, \dots, t_k) dt_1 \dots dt_k.$$

Thus it has been proved that

$$[*] \quad \mu(\{s \mid f_\epsilon^*(s) > \beta\}) \leq \frac{1}{c_k \beta} \int_{s(c_k \beta)} |f(s)| \mu(ds), \quad \epsilon, \beta > 0.$$

It is seen from the Lebesgue dominated convergence theorem and from Lemma 9 that for each f in L_p we have, for $\alpha > 0$, $\lim_{\epsilon \rightarrow 0} A(\epsilon, \alpha)f = A(\alpha)f$ in the norm of L_p . Thus, $A(\epsilon, \alpha)f \rightarrow A(\alpha)f$ in measure and hence, by the Cantor diagonal process, we can find a sequence $\epsilon_i \rightarrow 0$ such that $A(\epsilon_i, \alpha)(f, s) \rightarrow A(\alpha)(f, s)$ for μ -almost all s in S and for all α in the set R of positive rational numbers. It follows that $\liminf_{i \rightarrow \infty} f_{\epsilon_i}^*(s) \geq |A(\alpha)(f, s)|$ for almost all s in S and all α in R . Since, for almost all s in S , $A(\alpha)(f, s)$ is continuous for $0 < \alpha$ we have

$$f^*(s) = \sup_{\alpha \in \mathbb{R}} |A(\alpha)(f, s)| \leq \liminf_{i \rightarrow \infty} f_{i^*}^*(s)$$

almost everywhere on S . If $e^*(\beta) = \{s \mid f^*(s) > \beta\}$ and $e_{i^*}^*(\beta) = \{s \mid f_{i^*}^*(s) > \beta\}$ then, modulo a null set, $\liminf_{i \rightarrow \infty} e_{i^*}^*(\beta) \supseteq e^*(\beta)$ and so $[\ast]$ and Fatou's theorem yield

$$\mu(e^*(\beta)) \leq \frac{1}{c_k \beta} \int_{e(c_k \beta)} |f(s)| \mu(ds).$$

This completes the proof of Lemma 11 and prepares us for the following basic result on the almost everywhere convergence of the averages of a k -parameter semi-group in $L_1(S, \Sigma, \mu)$.

17. Theorem. *Let (S, Σ, μ) be a positive measure space and let $\{T(t_1, \dots, t_k), t_1, \dots, t_k \geq 0\}$ be a strongly measurable k -parameter semi-group of operators in $L_1(S, \Sigma, \mu)$ with $\|T(t_1, \dots, t_k)\|_1 \leq 1$ and $\|T(t_1, \dots, t_k)\|_\infty \leq 1$. Then for every f in $L_1(S, \Sigma, \mu)$, the limit*

$$\lim_{\alpha \rightarrow \infty} \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha T(t_1, \dots, t_k)(f, s) dt_1 \dots dt_k$$

exists for almost all s in S .

Proof. Let

$$A(\alpha) = \frac{1}{\alpha^k} \int_0^\alpha \dots \int_0^\alpha T(t_1, \dots, t_k) dt_1 \dots dt_k.$$

For $p > 1$ the set $L_p \cap L_1$ is dense in L_1 and for f in $L_p \cap L_1$ the limit $\lim_{\alpha \rightarrow \infty} A(\alpha)(f, s)$ exists for almost all s in S . In view of Lemma 11 we may apply BANACH'S Convergence Theorem 1.6 with $A_p =$ the set of all rational numbers $\alpha \geq p$, to conclude that for every f in L_1

$$\lim_p \sup_{\alpha, \beta \in A_p} |A(\alpha)(f, s) - A(\beta)(f, s)| = 0$$

for almost all s in S . But since, for almost all s , $A(\alpha)(f, s)$ is continuous in α this means that the limit $\lim_{\alpha \rightarrow \infty} A(\alpha)(f, s)$ exists almost everywhere on S . Q.E.D.

§5. Applications

Many of the general theorems of §4 can readily be applied to give more familiar analytical theorems on convergence almost everywhere. Thus, for instance, let $\{T_t\}$ be a strongly measurable semi-group of operators in L_1 such that $\|T(t)\|_1 \leq 1$, $\|T(t)\|_\infty \leq 1$, $t > 0$. Let $\beta(t)$ be a positive decreasing integrable function defined for $t > 0$. Then we have

$$\int_0^\alpha T_t f \beta(t) dt = \alpha \beta(\alpha) A(\alpha) f - \int_0^\alpha t A(t) f d\beta(t).$$

Thus, if $f^*(\cdot) = \sup_{0 < t < \infty} \|A(t)f(\cdot)\|$, we have

$$\begin{aligned} \left| \int_0^\alpha (T_t f)(s) \beta(t) dt \right| &\leq (\alpha \beta(\alpha) - \int_0^\alpha t d\beta(t)) f^*(s) \\ &= f^*(s) \int_0^\alpha \beta(t) dt \end{aligned}$$

for all α and almost all s , so that there exists a measurable function f^* which is finite almost everywhere such that

$$\left| \int_0^\infty (T_t f)(s) \beta(t) dt \right| \leq f^*(s) \int_0^\infty \beta(t) dt.$$

Moreover, if $f \in L_p$, $p > 1$, we can take $f^* \in L_p$.

We can prove the corresponding k -parameter theorem in the same way. This states: Let $\{T(t_1, \dots, t_k)\}$ be a strongly measurable k -parameter semi-group of operators in L_1 such that $\|T(t_1, \dots, t_k)\|_1 \leq 1$, $\|T(t_1, \dots, t_k)\|_\infty \leq 1$, $t_i > 0$. For each $f \in L_p$, $p \geq 1$, there exists a measurable function f^* which is finite almost everywhere such that

$$\begin{aligned} \left| \int_0^\infty \dots \int_0^\infty (T(t_1, \dots, t_k) f)(s) \beta(t_1, \dots, t_k) dt_1 \dots dt_k \right| \\ \leq f^*(s) \int_0^\infty \dots \int_0^\infty \beta(t_1, \dots, t_k) dt_1 \dots dt_k \end{aligned}$$

for each β which is positive and has the form $\beta(t_1, \dots, t_k) = \gamma(t_1^2 + \dots + t_k^2)$, γ decreasing.

Application of this principle and the Banach convergence Theorem 1.6 to the group of translations of the line gives the following result on singular integrals of convolution type, which we state in its one- and its k -parameter form.

Let $\beta(x)$ be a positive even integrable function of the real variable x , decreasing for $x > 0$; let $\beta(x) \geq |\gamma(x)|$, and let γ be measurable and $\int_{-\infty}^{+\infty} \gamma(x) dx = 1$. Then for almost all x we have

$$\lim_{n \rightarrow \infty} t \int_{-\infty}^{+\infty} \beta(t(x-y))f(y) dy = f(x)$$

for $f \in L_1$.

Let $\beta(x_1, \dots, x_n)$ be a positive integrable function of the real variable x which has the form $\beta(x_1, \dots, x_n) = \beta_1(x_1^2 + \dots + x_n^2)$, where β_1 is decreasing. Let $\beta(x_1, \dots, x_n) \geq |\gamma(x_1, \dots, x_n)|$, and let γ be measurable and have $\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \gamma(x_1, \dots, x_n) dx_1 \dots dx_n = 1$. Then for almost all x we have

$$\begin{aligned} \lim_{t \rightarrow \infty} t^n \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \gamma(t(x_1 - y_1), \dots, t(x_n - y_n)) f(y_1, \dots, y_n) dy_1 \dots dy_n \\ = f(x_1, \dots, x_n) \end{aligned}$$

for $f \in L_p$.

The familiar almost-everywhere limit relations

$$(a) \quad \lim_{t \rightarrow \infty} \sqrt{t/\pi} \int_{-\infty}^{+\infty} e^{-t(x-y)^2} f(y) dy = f(x)$$

$$(b) \quad \lim_{t \rightarrow \infty} t \int_x^\infty e^{+t(x-y)} f(y) dy = f(x)$$

$$(c) \quad \lim_{t \rightarrow 0} (\pi t)^{-1} \int_{-\infty}^{+\infty} \left(\frac{\sin t(x-y)}{x-y} \right)^2 f(y) dy = f(x)$$

$$(d) \quad \lim_{\epsilon \rightarrow 0} \epsilon/\pi \int_{-\infty}^{+\infty} \frac{1}{\epsilon^2 + (x-y)^2} f(y) dy = f(x)$$

holding for $f \in L_p$, $1 \leq p \leq \infty$, are all special cases of the 1-parameter singular integral theorem mentioned above. The fact that $f(x_1, \dots, x_n)$ is, for almost all $x_1, \dots, x_n$, equal to

$$(a) \quad \lim_{t \rightarrow \infty} \left(\frac{t}{\pi} \right)^{n/2} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{-t[(x_1-y_1)^2 + \dots + (x_n-y_n)^2]} f(y_1, \dots, y_n) dy_1 \dots dy_n$$

$$(b) \quad \lim_{t \rightarrow \infty} \frac{1}{(\pi t)^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left\{ \frac{\sin t(x_1 - y_1) \dots \sin t(x_n - y_n)}{(x_1 - y_1) \dots (x_n - y_n)} \right\}^2 \\ \cdot f(y_1, \dots, y_n) dy_1 \dots dy_n$$

for each $f \in L_p$, $1 \leq p \leq \infty$ are special cases of the k -parameter singular integral theorem mentioned above. In the same way, the 1-parameter theorems listed above suffice to prove the following theorems of FATOU and HARDY & LITTLEWOOD.

Theorem. (HARDY & LITTLEWOOD) Let h be a harmonic function defined in the circle $x^2 + y^2 < 1$, and let $\infty > p > 1$. Suppose that $\int_0^{2\pi} |h(re^{i\theta})|^p d\theta \leq K$,

$0 < r < 1$. Let $h^*(\theta) = \max_{0 < r < 1} h(re^{i\theta})$. Then there exists an absolute constant C such that $\int_0^{2\pi} |h^*(\theta)|^p d\theta \leq CK$.

Theorem. Let h be a harmonic function as in the previous theorem. Then for almost all θ , $\lim_{z_i \rightarrow e^{i\theta}} h(z_i)$ exists, provided that $|\arg(z_i - e^{i\theta})| \leq K < \pi/2$.

An easily obtained n -dimensional generalisation states:

Theorem. Let h be a harmonic function defined in the sphere $\sum_{i=1}^n x_i^2 < 1$ in Euclidean n -space. Let $\infty > p > 1$, $0 < K < \infty$. Suppose that for each r , $0 < r < 1$, the integral of the function $|h(rx)|^p$ over the surface of the unit sphere is $\leq K$. Putting $h^*(x) = \sup_{0 < r < 1} |h(rx)|$ for $|x| = 1$, there exists an absolute constant C_n such that the integral of $h^*(x)^p$ over the surface of the unit sphere is $\leq C_n K$. Moreover, $\lim_{r \rightarrow 1} h(rx)$ exists for almost all x on the surface of the unit sphere.

It is interesting to note that our method gives a proof of the n -dimensional Lebesgue theorem on differentiability of multiple integrals which is independent of the Vitali covering theorem. In fact, by applying Lemma 4.6 directly to the semi-group $\{S_t\}$ defined by

$$(S_t f)(x_1, \dots, x_n) = t^{n/2} \pi^{-n/2} \int_{-\infty}^{+\infty} e^{-t\{(x_1-y_1)^2 + \dots + (x_n-y_n)^2\}} f(y_1, \dots, y_n) dy_1 \dots dy_n$$

an inequality strong enough to yield the second result (a) of the listed above can be obtained. By the method of the latter half of §4 this then yields the Lebesgue theorem:

$$\lim_{n \rightarrow 0} h^{-n} \int_{x_1}^{x_1+h} \dots \int_{x_n}^{x_n+h} f(y_1, \dots, y_n) dy_1 \dots dy_n = f(x_1, \dots, x_n) \quad \text{a.e.}$$

By applying the dominated ergodic theorem corresponding to Theorem 10, rather than Lemma 11, to concrete analytic situations, we obtain Zygmund-Marcinkiewicz type "strong differentiability" theorems. Samples are:

Theorem. Let f be a function of n real variables which is in L_p , and suppose $p > 1$. In this case $f(x_1, \dots, x_n)$ is, for almost all $x_1, \dots, x_n$ equal to

$$(a) \lim \left(\frac{t_1 \dots t_n}{\pi^n} \right)^{1/2} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \left(\sum_{i=1}^n t_i (x_i - y_i)^2 \right) f(y_1 \dots y_n) dy_1 \dots dy_n$$

$$(b) \lim \frac{1}{(t_1 \dots t_n) \pi^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\prod_{i=1}^n \frac{\sin t_i (x_i - y_i)}{x_i - y_i} \right)^2 \cdot f(y_1, \dots, y_n) dy_1 \dots dy_n$$

$$(c) \lim (h_1 \dots h_n)^{-1} \int_{x_1}^{x_1+h_1} \dots \int_{x_n}^{x_n+h_n} f(y_1, \dots, y_n) dy_1 \dots dy_n,$$

where the limits are taken with respect to $t_1, \dots, t_n \rightarrow \infty$ and $h_1, \dots, h_n \rightarrow 0$.

Theorem. Let h be an infinitely differentiable function of the $2n$ real variables $x_1, \dots, x_n, y_1, \dots, y_n$ defined in the region $x_i^2 + y_i^2 < 1, i = 1, \dots, n$. Suppose that for each $i = 1, \dots, n$ we have $\partial^2 h / \partial x_i^2 + \partial^2 h / \partial y_i^2 = 0$. Suppose that $p > 1$, and that for $0 < r < 1$

$$\int_0^{2\pi} \cdots \int_0^{2\pi} |h(r \cos \theta_1, r \cos \theta_2, \dots, r \sin \theta_n)|^p d\theta_1 \cdots d\theta_n \leq K.$$

Then $\lim_{r_1 \rightarrow 1, \dots, r_n \rightarrow 1} h(r_1 \cos \theta_1, \dots, r_n \sin \theta_n)$ exists for almost all $\theta_1, \dots, \theta_n$.

The slight gap remaining in our discussion of the Lebesgue theorem on differentiation, which is the gap between

“every absolutely continuous function can be differentiated almost everywhere”

and

“every function of bounded variation can be differentiated almost everywhere”

can be filled in with the aid of the following general convergence principle.

Theorem. Let S be a separable metric space. Let $\mathfrak{B}$ be the field of Borel sets of S , and μ a σ -finite measure defined on $\mathfrak{B}$, every open subset of S having positive μ -measure. Let $\{T_n\}$ be a sequence of bounded operators in $L_1(S, \mathfrak{B}, \mu)$ of the form

$$(T_n f)(s) = \int_S K_n(s, t) f(t) \mu(dt).$$

Suppose that

(i) $K_n(s, t)$ is a bounded continuous function of s and t for each $n \geq 1$; $\lim_{n \rightarrow \infty} K_n(s, t) = K_n(s_0, t)$ uniformly for $t \in S$, for each $n \geq 1$.

(ii) If U is an arbitrary neighborhood of s , $K_n(s, t)$ converges, as $n \rightarrow \infty$, uniformly on $S - U$.

(iii) $\lim_{n \rightarrow \infty} (T_n f)(s)$ exists μ -almost everywhere for each $f \in L_1(S, \mathfrak{B}, \mu)$.

Then for each finite countably additive measure ν defined on $\mathfrak{B}$,

$$\lim_{n \rightarrow \infty} (T_n \nu)(s) = \lim_{n \rightarrow \infty} \int_S K_n(s, t) \nu(dt)$$

exists μ -almost everywhere.

Proof. The proof is divided into a number of steps, which we shall first outline and then establish individually.

(i) First we assert that if A is a subset of S of finite μ -measure, and ϵ a positive real number, then there exists a positive real number $\delta = \delta(A, \epsilon)$ such that if $f \in L_1(S, \mathfrak{B}, \mu)$ and $\|f\|_1 < \delta$, then, putting

$$(T^*f)(s) = \sup_{1 \leq n < \infty} |T_n f(s)|, \quad f \in L_1(S, \Sigma, \mu), \quad s \in S,$$

we have

$$\mu\{s \in A \mid (T^*f)(s) > \epsilon\} < \epsilon.$$

(ii) Next, using (i), we show that there exists a positive real number $\delta = \delta_1(A, \epsilon)$ such that for each bounded measure ν defined on Σ with total variation less than δ , we have

$$\mu\{s \in A \mid (T^*\nu)(s) > \epsilon\} < \epsilon;$$

where here we have put

$$(T^*\nu)(s) = \sup_{1 \leq n < \infty} |(T_n\nu)(s)|.$$

(iii) Next we show that if F is a closed subset of S , and ν a measure such that $\nu(S - F) = 0$, then

$$\lim_{n \rightarrow \infty} (T_n\nu)(s)$$

exists for all $s \in S - F$.

(iv) Finally, we show that (ii) and (iii) imply the conclusion of the theorem.

Now let us get down to details. We begin by giving the proof that (ii) *and* (iii) *imply the conclusion of our theorem*. Since by hypothesis

$$\lim_{n \rightarrow \infty} (T_n\nu)(s)$$

exists μ -almost everywhere for each μ -continuous (*i.e.*, μ -absolutely continuous) bounded measure ν , it is enough to show that

$$\lim_{n \rightarrow \infty} (T_n\nu)(s)$$

exists μ -almost everywhere for each μ -singular measure ν . Since ν may be decomposed into a linear combination of four positive singular measures, it is clear that we may assume without loss of generality that ν is positive. Consequently, let ν be a positive μ -singular measure, so that there exists a Borel subset E_0 such that $\mu(E_0) = 0$ and $\nu(S - E_0) = 0$. Since S is a metric space, every bounded Borel measure defined on S is regular. Consequently, if $\delta > 0$, there exists a closed subset F of E_0 such that $\nu(S - F) < \delta$. Put $\nu_1(E) = \nu(EF)$ and $\nu_2(E) = \nu(E - F)$ for each Borel subset E of S . Then $\nu = \nu_1 + \nu_2$, and the total variation of ν_2 is less than δ . Consequently, by (ii),

$$\mu\{s \in A \mid \limsup_{m, n \rightarrow \infty} |(T_m\nu_2)(s) - (T_n\nu_2)(s)| > \epsilon\} < \epsilon.$$

By (iii),

$$\lim_{n \rightarrow \infty} (T_n\nu_1)(s)$$

exists for μ -almost all $s \in S - F$. It follows from these two equations that

$$\mu\{s \in A \mid \limsup_{m, n \rightarrow \infty} |(T_m \nu)(s) - (T_n \nu)(s)| > \epsilon\} < \epsilon.$$

Since ϵ is an arbitrary positive quantity, it follows that

$$\lim_{n \rightarrow \infty} (T_n \nu)(s)$$

exists for μ -almost all $s \in A$. Since (S, Σ, μ) is a σ -finite measure space, and A is an arbitrary subset of S of finite μ -measure, it follows that

$$\lim_{n \rightarrow \infty} (T_n \nu)(s)$$

exists for μ -almost all $s \in S$. Thus, (ii) and (iii) imply the conclusion of the theorem.

The *proof of (iii)* is most trivial. Since $s \in S - F$,

$$\lim_{n \rightarrow \infty} K_n(s, t)$$

exists uniformly for $t \in F$ by hypothesis. Thus

$$\lim_{n \rightarrow \infty} \int_S K_n(s, t) \nu(dt) = \lim_{n \rightarrow \infty} \int_F K_n(s, t) \nu(dt)$$

exists since ν is a finite measure.

Next we give the *proof that (i) implies (ii)*. If a continuous bounded function f satisfies

$$\left| \int_S f(s) h(s) \mu(ds) \right| \leq 1$$

for each $h \in L_1(S, \mathcal{B}, \mu)$ such that $\|h\|_1 \leq 1$, then, since $L_\infty \cong L_1^*$, μ -ess sup $|f(s)| \leq 1$. Since f is continuous, and since every open set is assumed to have positive μ -measure, it follows that $|f(s)| \leq 1$ for every $s \in S$. Consequently,

$$\left| \int_S f(s) \nu(ds) \right| \leq 1$$

for every Borel measure ν such that the total variation $\nu(\nu, S)$ of ν is not greater than one.

Let $f_1, \dots, f_n$ be any finite set of continuous bounded functions defined on S . Then the sets of n -tuples $K_1 = \{[\alpha_1, \dots, \alpha_n] \mid \alpha_i = \int_S f_i(s) h(s) \mu(ds), \|h\|_1 \leq 1\}$, $K_2 = \{[\alpha_1, \dots, \alpha_n] \mid \alpha_i = \int_S f_i(s) \nu(ds), \nu(\nu, S) \leq 1\}$ are clearly convex, and clearly $K_1 \subseteq K_2$. If $[\beta_1, \dots, \beta_n]$ is an n -tuple such that

$$\Re \left\{ \sum_{i=1}^n \beta_i \alpha_i \right\} \leq 1$$

for $[\alpha_1, \dots, \alpha_n] \in K_1$, then, putting

$$\sum_{i=1}^n \beta_i f_i(s) = f(s),$$

we have

$$\Re \left\{ \int_S f(s) h(s) \mu(ds) \right\} \leq 1$$

for all $h \in L_1$ such that $\|h\|_1 \leq 1$. Putting $h = e^{i\theta} h_1$ and choosing θ suitably, it follows that

$$\left| \int_S f(s) h(s) \mu(ds) \right| \leq 1, \quad h \in L_1(S, \mathfrak{B}, \mu), \quad \|h\|_1 \leq 1.$$

Thus, by what has been proved above,

$$\left| \int_S f(s) \nu(ds) \right| \leq 1, \quad \nu(\nu, S) \leq 1.$$

In particular,

$$\Re \int_S f(s) \nu(ds) \leq 1, \quad \nu(\nu, S) \leq 1;$$

i.e.,

$$\Re \left\{ \sum_{i=1}^n \beta_i \alpha_i \right\} \leq 1$$

for $[\alpha_1, \dots, \alpha_n] \in K_2$. By the well-known separating hyperplane theorem for convex sets, it follows that K_1 is dense in K_2 . From this, the following assertion may be deduced immediately by an application of the Cantor diagonal argument:

Let ν be a finite countably additive measure defined on $\mathfrak{B}$. Then, if $\mathfrak{F}$ is any separable subspace of the metric space $C(S)$ of bounded continuous functions defined on S (with norm $\|f\| = \sup_{s \in S} |f(s)|$), there exists a sequence $\{d_m\}$ of functions in $L_1(S, \mathfrak{B}, \mu)$ such that

$$\int_S |d_m(s)| \mu(ds) \leq \nu(\nu, S), \quad m \geq 1,$$

and

$$\lim_{m \rightarrow \infty} \int_S f(s) d_m(s) \mu(ds) = \int_S f(s) \nu(ds), \quad f \in \mathfrak{F}.$$

Since S is separable and

$$\lim_{s \rightarrow s_0} K_n(s, t) = K_n(s_0, t)$$

uniformly in t for each n , this statement may be applied, in particular, to the collection of functions $K_n(s, \cdot)$, $n \geq 1$, $s \in S$. It follows that there exists a sequence $\{d_m\}$ of functions in $L_1(S, \mathcal{B}, \mu)$ such that

$$\int_S |d_m(s)| \mu(ds) \leq v(\mu, S), \quad m \geq 1,$$

and

$$\begin{aligned} \lim_{m \rightarrow \infty} (T_n g_m)(s) &= \lim_{m \rightarrow \infty} \int_S K_n(s, t) d_m(t) \mu(dt) \\ &= \int_S K_n(s, t) \nu(dt) = (T_n \nu)(s), \quad n \geq 1, \quad s \in S. \end{aligned}$$

Consequently,

$$|(T_n \nu)(s)| \leq \liminf_{m \rightarrow \infty} \sup_{n \geq 1} |(T_n g_m)(s)| = \liminf_{m \rightarrow \infty} (T^* g_m)(s),$$

so that

$$(T^* \nu)(s) \leq \liminf_{m \rightarrow \infty} (T^* g_m)(s).$$

If we put $\phi(\alpha) = 1$ if $|\alpha| > \epsilon$, $\phi(\alpha) = 0$ if $|\alpha| < \epsilon$, it follows by FATOU'S lemma and (i) that if $v(\nu, S) < \delta(A, \epsilon)$

$$\begin{aligned} \mu\{s \in A \mid (T^* \nu)(s) > \epsilon\} &= \int_A \phi\{(T^* \nu)(s)\} \mu(ds) \\ &\leq \liminf_{m \rightarrow \infty} \int_A \phi\{(T^* g_m)(s)\} \mu(ds) \\ &= \liminf_{m \rightarrow \infty} \mu\{s \in A \mid (T^* g_m)(s) > \epsilon\} \\ &\leq \liminf_{m \rightarrow \infty} \epsilon = \epsilon. \end{aligned}$$

This shows that (i) implies (ii).

Thus to complete the proof of the present theorem, it suffices to establish (i). This may be established just as in the proof of BANACH'S Theorem 5. For details we refer the reader to DUNFORD & MILLER [1], Lemma 7, page 542. Q.E.D.

The following corollary follows readily from the above theorem.

Corollary. *If the function β is continuous, integrable, and positive, decreasing for $x > 0$ and increasing for $x < 0$, if $t\beta(t) \rightarrow 0$ as $t \rightarrow \pm\infty$, and if γ is Borel measurable, and $|\gamma(\cdot)| \leq \beta(\cdot)$, then*

$$\lim_{t \rightarrow \infty} t \int_{-\infty}^{+\infty} \gamma(t(x - y)) \nu(dy)$$

exists (Lebesgue) almost everywhere for each finite Borel measure ν .

As a final example indicating the generality of the results obtained in the preceding theorem, we mention the following theorem, which has applications to parabolic partial differential operators of general type.

Theorem. *Let (S, Σ, μ) be a measure space. Let A be a closed unbounded operator with dense domain in $L_1(S, \Sigma, \mu)$, such that the resolvent $R(\lambda, A)$ exists for $\lambda > 0$, and $\|R(\lambda, A)\|_1 \leq 1$, $\|R(\lambda, A)\|_\infty \leq 1$ for $\lambda > 0$. Then*

$$\lim_{\lambda \rightarrow \infty} (\lambda R(\lambda, A)f)(s)$$

exists almost everywhere for $f \in L_p$, $1 \leq p < \infty$. Moreover, if S is a topological space, and $R(\lambda, A)f$ is continuous if f is continuous, $\lambda > 0$, then

$$\lim_{\lambda \rightarrow 0} (\lambda R(\lambda, A)f)(s)$$

exists almost everywhere for $f \in L_p$, $1 \leq p < \infty$.

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