On the Pressures and the Flux of Energy in a Gas according to Maxwell's Kinetic Theory, II

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NOTE. This memoir continues the researches reported in Part I. All prerequisites for Chapters III–V are contained in §§1–2 of Part I; all for Chapter VI, in §§7 and 14. Chapters III–VI are independent of one another; except for the prerequisites just mentioned, each is a self-contained study. The table of contents and summary for both parts are printed at the head of Part I.

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CHAPTER III. *Analytical theory of a mathematical model*

23. Analytical remarks on differential iteration. Thus far our treatment of iterative schemes has followed the tradition of the kinetic theory in calculating blindly with no thought for the analytical justice of the method. Chapter II presented various related *differential iterations*, more in fact than were enumerated since several are not mutually exclusive and may be combined with one another. From all this the reader is expected to carry away the realization that there are various simple processes which easily yield formulæ of the Maxwell-Chapman-Enskog type, that in some cases the results are similar in kind but different in detail, while in other cases they are entirely different. Returning to the most straightforward method, Maxwellian iteration as discussed in §2, we now attempt to grasp its rationale.

The equations to be solved are the infinite system of *differential equations* for the moments, (2.4) and (2.5) being the first two. All iterative processes discussed in Part I consist in algebraic and differential substitutions: nowhere is there any *integration* or any idea equivalent to it.

A differential system in the time suggests an *initial value problem*. Clearly none of our iterative methods yields the general solution to such a problem. Each iterate is a permanent formula, and the manner of obtaining it allows no use to be made of initial values of the moments. Indeed, the entire approach

which gives rise to formulæ of the type gotten in Part I is dominated by hydrodynamics and makes no effort to pose a problem which is well set in the kinetic theory itself.

For example, we have the **Theorem of steady state determinism**: *At a point where p , ρ , and u_i are steady, every Maxwellian iterate for every moment is steady.* This is so because, as proved in §13, the Maxwellian iterates are functions only of the gross condition.

If the iterative scheme converges to a solution, the foregoing theorem shows how limited must be the class of solutions so obtainable. For such a solution, steadiness of the first five moments implies steadiness of *all* moments. The theorem of steady state determinism is a counterpart of HILBERT's theorem of macroscopic causality (cf. [1950, 1, §30]).

In Maxwellian iteration the zeroth iterates are selected by a fixed agreement (§2). In general, processes of iteration are expected to improve an initial guess, whether or not it happens to be a good one, and in Chapter II we have discussed the effect of a different choice of zeroth iterates. The foregoing theorem and remarks show that this choice may have a far reaching effect on the limits, if any, which the iterates approach.

Moreover, differential iteration cannot be expected to converge or to be meaningful in general. It rests upon repeated differentiation, indeed a strange method for solving a differential system. Plainly, *if the pressure, density, and velocity are not infinitely many times differentiable with respect to space and time, Maxwellian iteration breaks down.* To require existence of more than one or two derivatives is altogether unnatural and contrary to what experience in gas dynamics leads us to expect. A proper kinetic theory should allow us to consider less orderly conditions of the gas than are described by hydrodynamics, but any result from Maxwellian iteration is restricted to extremely smooth states.¹

To see the analytic essence of differential iteration, we apply it to the single equation

$$y = \frac{dy}{dt} \quad (23.1)$$

by putting

$$y_{n+1} \equiv \frac{dy_n}{dt}, \quad \text{or} \quad y_n = \frac{d^n y_0}{dt^n}. \quad (23.2)$$

Almost anything can come out in the limit as $n \rightarrow \infty$. If y_0 is a polynomial, $y_n \rightarrow 0$; if $y_0 = ce^{kt}$, then $y_n \rightarrow 0$, ce^t , ∞ , or no limit according as $|k| < 1$, $k = +1$, $k > 1$, or $k \leq -1$; if $y_0 = \sinh t$, the sequence $y_n(t)$ has two limit points, neither of which yields a solution.

¹ Such a restriction is by no means inherent in the continuum viewpoint. See, for example, my theory of the "Stokesian fluid" [1949, 2][1950, 2], where non-linearity is introduced without requiring any additional smoothness. The shock layer problem in this theory has been solved by GILBARG & PAOLUCCI [1953, 1, §5].

The general solutions of (23.1) is $y(0)e^t$. If the zeroth iterate, y_0 , is such that in a closed interval $d^n y_0/dt^n$ exists for all n and converges uniformly, then the limit function is indeed a solution. Since the iterative scheme makes no use of the initial data $y(0)$, we are not surprised that in cases when y_n does converge to a solution it selects as its limit some particular one, somehow mysteriously determined by the choice of the zeroth iterate, y_0 .

The example (23.1) with the iterative scheme (23.2) would suffice to indicate the inadequacy of differential iteration, were a general solution our objective. But in the kinetic theory a general solution is never what is wanted. The aim of the Maxwellian kinetic theory is to find the correct values of the gross condition, the pressures, and the heat flux as functions of place and time and of their initial conditions in a gas flow. The Maxwellian theory has always led to a *methodological paradox*. This paradox consists in the superabundance of initial conditions—the initial values of the higher moments, or, essentially, the initial value of the molecular distribution function—which is the price the kinetic theory must pay for a description presumably more accurate than that of hydrodynamics. In a particular problem of gas flow, we have no means of conjecturing or measuring these initial values. Results which are to be compared with experimental data are influenced by these new quantities but necessarily must be expressed in the end entirely without reference to them. To get such results we must somehow cut a path through the cumbrous and unwanted extra data and reach the limited but contingent quantities which have physical interpretation. Now the various classical “approximate” methods of solution eliminate these extra conditions by some formal device such as a particular kind of series expansion. The resulting expressions for the pressures and the flux of energy as functions of the gross condition, if correct, may be just what is desired. By their limitation, these “solutions” cannot be expected to be both general and exact. Are they somehow approximate—approximate, perhaps, to every solution? That is, can the dependence of the general solution on the initial value of the distribution function be shown to be in some sense weak, thus justifying approximate solutions in which the initial conditions play no part at all? Indeed, the authors of these methods have sometimes claimed (*cf.* [1952, 1, §7.2]) that they yield “normal” solutions, special solutions such that after a while any other solution will quickly be approximated by one of them; in other words, that the effect of the embarrassing initial conditions will disappear within a few times of relaxation. No mathematical evidence beyond MAXWELL’s single example of relaxation in a uniform state (*cf.* Chapter VI below) has been brought forward in favor of these declarations of faith.

To eliminate the higher moments is *the central problem of the Maxwellian kinetic theory*, and it has never been touched by any correct mathematics or even by any convincing argument of plausibility.

Guided by these ideas, replace (23.1) by

$$-\frac{1}{\mu} y = \frac{dy}{dt}, \quad (23.3)$$

where μ is a positive constant. Since the general solution is

$$y = y(0)e^{-t/\mu}, \quad (23.4)$$

we see at once that

1. There is only one solution, $y = 0$, which for fixed t is bounded near $\mu = 0$. This solution is an analytic function of μ at $\mu = 0$.
2. Hold the initial condition and the time fixed. Then $y \rightarrow 0$ as $\mu \rightarrow 0+$.
3. Hold μ and the initial condition fixed. Then $y \rightarrow 0$ as $t \rightarrow \infty$.

Thus there is a particular solution, $y = 0$, which is specially distinguished, being in the first place an *unusually smooth* solution and in the second place the *common asymptotic form*, in two different senses, of all solutions. Now apply differential iteration:

$$-\frac{1}{\mu} y_{n+1} \equiv \frac{dy_n}{dt}, \quad (23.5)$$

or

$$y_n = (-\mu)^n \frac{d^n y_0}{dt^n}. \quad (23.6)$$

Hence if y_0 is a polynomial, $y_n \rightarrow$ the asymptotic solution; if all derivatives of y_0 are uniformly bounded and if $\mu < 1$, $y_n \rightarrow$ the asymptotic solution. We have shown, then, that *there is one particular solution which is especially important, and that for a certain class of initial iterates and for small enough μ the method of differential iteration converges to that particular solution.*

Here we see a new method in analysis. For a differential equation having a single specially smooth and asymptotic solution, a method based on *infinite differentiation* selects just this very solution and converges to it.

This new method is well worth analytical study, but now we turn aside from its implications for pure mathematics so as to concentrate upon the kinetic theory. The rest of this chapter develops the ideas presented in the previous paragraphs by constructing and studying a *mathematical model* having some properties in common with the equations of moments. For this model, we solve the central problem completely and thus resolve the methodological paradox.

24. A model for the equations of moments. Thus far the extreme formal complexity of the Maxwellian kinetic theory has prevented a rigorous treatment. All work on this subject, that of HILBERT, ENSKOG, and GRAD and the iterative schemes of Part I of the present paper not excepted, is purely formal. We turn now to a mathematical model in some respects similar to the equations of moments but simple enough that we can obtain a rigorous analytic theory both for the exact solution and for differential iteration.

This model is

$$-\frac{1}{\mu} p_0 = E + \frac{dp_0}{dt} + \alpha p_0 + \beta p_1, \quad (24.1)$$

$$-\frac{1}{\mu} p_m = \frac{dp_m}{dt} + \alpha p_m + \beta p_{m+1}, \quad m \geq 1. \quad (24.2)$$

The first equation, (24.1), is to be taken as analogous to the equation (2.9) for $P_{,i}$, and accordingly we shall call p_0 the *pressure deviator*. The equation (24.2) is to be taken as analogous to the equations for the third and higher moments. We subject the constants μ , E , α , β to the restrictions

$$\mu > 0, \quad \alpha \neq -\frac{1}{\mu}, \quad \beta \neq 0. \quad (24.3)$$

The first of these asserts that μ plays the role of a *viscosity*; the second, that the dependence on μ is not illusory; the third, that the linking of the members of the system is not illusory.

The reader will see to what extent the model reflects the kinetic theory and to what extent it does not. It is an infinite system with forward linking; it is of first order in the time. There are parameters μ and E which except for physical dimensions enter the system as do their counterparts μ and $E_{,i}$ in the kinetic theory, as far as the linear terms are concerned, and thus there is a *Navier-Stokes approximation*, $p_0 = -\mu E$. All non-linear effects and spatial variation are omitted. Moreover, the members of the system, apart from the first, are all alike, while in the kinetic theory the formal complications increase with n . It is our intention that the model shall contain some of the essential analytic features of the kinetic theory, yet be simple enough that an exact treatment is easy.

We note that there exist constant solutions of the type

$$p_m = \frac{(-1)^m (\alpha + 1/\mu)^m}{\beta^m} \left(p_0 + \frac{\mu E}{1 + \alpha \mu} \right), \quad m \geq 1, \quad (24.4)$$

uniquely determined by an assigned constant p_0 . These solutions we shall call the *steady solutions*. There is exactly one of these for which the higher moments vanish identically:

$$p_0 = -\frac{\mu E}{1 + \alpha \mu}, \quad p_m = 0 \quad \text{if} \quad m \geq 1. \quad (24.5)$$

If we restrict p_0 to the class of analytic functions of μ , *this special solution is the only steady solution such that all p_m are bounded functions of μ near $\mu = 0$; in fact, for this solution p_m is analytic in μ , for all m . The Navier-Stokes approximation $p_0 = -\mu E$ is the leading term in the power series for this solution.*

25. Exact general solution of the initial value problem. Set

$$y_m \equiv \begin{cases} p_0 + \frac{\mu E}{1 + \alpha \mu}, & m = 0, \\ p_m, & m \geq 1, \end{cases} \quad (25.1)$$

$$F(t, m) \equiv (-1)^m \beta^m e^{[(1/\mu) + \alpha]t} y_m(t).$$

Then

$$\frac{\partial F(t, m)}{\partial t} = F(t, m + 1). \quad (25.2)$$

This equation I studied some years ago, and from one of my results [1948, 3, Theorem 11.7], slightly strengthened, we may write down the general solution such that $F(0, m)$ is an assigned function of m for $m = 0, 1, 2, \dots$. This solution, expressed in terms of the original variables p_m , yields the following

Theorem of existence and uniqueness. *For the moment equations (24.1) (24.2) there exists at most one solution $p_m(t)$ taking on assigned values $p_m(0) = p_m^0$, $m = 0, 1, 2, \dots$, when $t = 0$. Set*

$$\frac{1}{k} \equiv e \overline{\lim} \frac{\sqrt[m]{|p_m^0|}}{m}. \quad (25.3)$$

(If $\overline{\lim} |p_{m+1}^0/(mp_m^0)|$ exists, it equals $1/k$). If $k \neq 0$, this solution for $|t| < k/|\beta|$ is the following sequence of analytic functions of t :

$$p_0(t) = -\frac{\mu E}{1 + \alpha\mu} (1 - e^{-(1/\mu) + \alpha)t}) + e^{-(1/\mu) + \alpha)t} \sum_{r=0}^{\infty} (-1)^r p_r^0 \frac{(\beta t)^r}{r!}, \quad (25.4)$$

$$p_m(t) = e^{-(1/\mu) + \alpha)t} \sum_{r=0}^{\infty} (-1)^r p_{m+r}^0 \frac{(\beta t)^r}{r!}, \quad m \geq 1.$$

By inspection we have

Corollary 1. Asymptotic solution for small viscosity. *Hold the initial conditions p_m^0 fixed, and consider any fixed time t such that $0 < t < k/|\beta|$. Then*

$$p_0 \sim -\mu E \quad \text{as} \quad \mu \rightarrow 0+. \quad (25.5)$$

In this sense, the Navier-Stokes approximation is a universal asymptotic formula, valid for sufficiently small viscosity. Moreover, the full asymptotic solution as $\mu \rightarrow 0+$ is

$$p_0 \sim -\frac{\mu E}{1 + \alpha\mu} = -\mu E \sum_{r=0}^{\infty} (-\alpha\mu)^r, \quad (25.6)$$

$$p_m \sim 0, \quad m \geq 1$$

(in the sense that $p_0 + \mu E/(1 + \alpha\mu) = o(\mu^n)$ and $p_m = o(\mu^n)$, $m > 1$, for all $n > 0$).

This corollary shows the exceptional position of the one steady solution in which $p_0 = -\mu E/(1 + \alpha\mu)$: at any fixed time such that $0 < t < k_0/|\beta|$, that solution yields the common asymptotic value of all solutions whose initial values, held fixed as $\mu \rightarrow 0+$, are such that $k \geq k_0$. For example, for all solutions

whose initial values p_m^0 satisfy $p_m^0 = o(m)$, the above stated asymptotic behavior holds at any fixed positive time. In particular, all such solutions are asymptotically steady for small μ . Henceforth we call (25.6) the *asymptotic solution*.

The meaning of the asymptotic formulæ must be borne in mind. They do not prove that the Navier-Stokes approximation, or that any power series in μ , furnishes a good approximation to any given solution, but only that *if we hold the time and the initial conditions fixed, we can find a positive μ small enough that the Navier-Stokes approximation becomes arbitrarily close*. In other words, if we insist on a formula for linear viscosity, the Navier-Stokes formula is the best we can get; if we insist on a power series, the asymptotic solution (25.6) is the best we can get. There is no reason, however, to expect such a formula at all. In fact by inspection of (25.4) we have

Corollary 2. Non-analyticity theorem. *For initial conditions p_m^0 which do not depend on μ , there is no exact solution which is an analytic function of μ , or even a bounded function of μ , at $\mu = 0$. (While the asymptotic solution is a particular exact solution, for it p_0^0 is a function of μ ; the special analytic character of this solution was described at the end of §24.)*

Now we consider the limit $t \rightarrow \infty$, and in order to obtain simple results we limit the initial conditions p_m^0 . We note first that if

$$\frac{p_{m+1}^0}{p_m^0} = o(m), \quad (25.7)$$

the solutions $p_m(t)$ are entire functions of t . We shall impose a still stronger condition, and then by inspection of (25.4) obtain

Corollary 3. Asymptotic solution for large time. *Assume*

$$p_m^0 = O(K^m) \quad (25.8)$$

for some K , and also assume

$$\frac{1}{\mu} > |\beta| K - \alpha. \quad (25.9)$$

Then

$$\begin{aligned} p_0(t) &\rightarrow -\frac{\mu E}{1 + \alpha\mu} & \text{as } t \rightarrow \infty. \\ p_m(t) &\rightarrow 0, & m \geq 1, \end{aligned} \quad (25.10)$$

This corollary asserts that for *small enough viscosity* and not too rapidly growing initial values p_m^0 , the asymptotic value of the pressure deviator for small viscosity is also its limit value as time increases. To prove the corollary, we note that (25.8) is sufficient for (25.7), and hence that (25.4) shall hold for all t ; it is also sufficient that the series in (25.4)₁ be dominated by that for $e^{|\beta|Kt}$.

A natural example, perhaps analogous to the initial values for equilibrium in the case of a gas, is

$$p_m^0 = \begin{cases} 0 & \text{if } m = 0 \\ A \frac{1 + (-1)^m}{2} p^m & \text{if } m \geq 1, \end{cases} \quad (25.11)$$

where A and p are constants. For this case (25.4)₁ yields

$$p_0(t) = -\frac{\mu E}{1 + \alpha\mu} (1 - e^{-[(1/\mu) + \alpha]t}) + 2Ae^{-[(1/\mu) + \alpha]t} \sinh^2 \frac{1}{2}\beta p t. \quad (25.12)$$

This example illustrates both corollaries and suggests the manner in which the pressure deviator may rise quickly from an initial value, in this case 0, to the asymptotic value $p_0(t) \sim -\mu E$ for small μ .

A general estimate of the accuracy of the asymptotic formula for any t is easy to obtain from (25.4)₁ subject to the assumptions (25.8) and (25.9). It is

$$\begin{aligned} \left| p_0(t) + \frac{\mu E}{1 + \alpha\mu} \right| &\leq e^{-[(1/\mu) + \alpha]t} \left[\frac{\mu |E|}{|1 + \alpha\mu|} + Ce^{|\beta|Kt} \right], \\ &\leq e^{-[(1/\mu) + \alpha - |\beta|K]t} \left[\frac{\mu |E|}{|1 + \alpha\mu|} + C \right], \end{aligned} \quad (25.13)$$

where C is the constant implied by (25.8). This is to be contrasted with the obvious

$$p_0(t) = p_0^0 - \left[E + \left(\alpha + \frac{1}{\mu} \right) p_0^0 + \beta p_1^0 \right] t + O(t^2) \quad (25.14)$$

as $t \rightarrow 0$ for fixed μ .

The result of this section, then, is to show that for a fairly broad class of solutions there is at any fixed time a *common asymptotic expansion, independent of the time and of the (fixed) initial conditions*, as $\mu \rightarrow 0+$, and for a somewhat narrower class of solutions this same power series in μ is a *common asymptotic expansion as $t \rightarrow \infty$ when μ is fixed at a small enough value*. This asymptotic expansion is in fact a *particular and uniquely smooth exact solution*.

Thus there exists a broadly valid approximation independent of the initial conditions. We now investigate whether this approximation can be calculated by differential iteration.

26. Exact theory of differential iteration. The differential iterative scheme analogous to that of §2 is

$$\begin{aligned} -\frac{1}{\mu} p_0^{(n+1)} &\equiv E + \frac{dp_0^{(n)}}{dt} + \alpha p_0^{(n)} + \beta p_1^{(n)}, \\ -\frac{1}{\mu} p_m^{(n+1)} &\equiv \frac{dp_m^{(n)}}{dt} + \alpha p_m^{(n)} + \beta p_{m+1}^{(n)}, \quad m \geq 1. \end{aligned} \quad (26.1)$$

Set

$$\begin{aligned} y_m^{(n)} &\equiv \begin{cases} p_0^{(n)} + \frac{\mu E}{1 + \alpha\mu}, & m = 0, \\ p_m^{(n)}, & m \geq 1, \end{cases} \\ z_m^{(n)} &\equiv e^{\alpha t} y_m^{(n)}. \end{aligned} \quad (26.2)$$

Then (26.1) is equivalent to

$$\begin{aligned} -\frac{1}{\mu} z_m^{(n+1)} &= \frac{dz_m^{(n)}}{dt} + \beta z_{m+1}^{(n)}, \\ &= (\mathfrak{D}_t + \beta \mathfrak{E}_m) z_m^{(n)}, \end{aligned} \quad (26.3)$$

where $\mathfrak{D}$ and $\mathfrak{E}$ are differential and displacement operators in the subscript variables, respectively. The general solution of this iterative scheme is

$$\begin{aligned} z_m^{(n)} &= (-\mu)^n (\mathfrak{D}_t + \beta \mathfrak{E}_m)^n z_m^{(0)}, \\ &= (-\mu)^n \sum_{r=0}^n \binom{n}{r} \beta^{n-r} \frac{d^r z_{m+n-r}^{(0)}}{dt^r}. \end{aligned} \quad (26.4)$$

Therefore,

$$\begin{aligned} p_0^{(n)} &= -\frac{\mu E}{1 + \alpha\mu} [1 - (-\alpha\mu)^n] + (-\mu)^n \sum_{r=0}^n \sum_{q=0}^r \binom{n}{r} \binom{r}{q} \beta^{n-r} \alpha^{r-q} \frac{d^q p_{n-r}^{(0)}}{dt^q}, \\ p_m^{(n)} &= (-\mu)^n \sum_{r=0}^n \sum_{q=0}^r \binom{n}{r} \binom{r}{q} \beta^{n-r} \alpha^{r-q} \frac{d^q p_{m+n-r}^{(0)}}{dt^q}, \quad m \geq 1. \end{aligned} \quad (26.5)$$

Hence we see that if $p_0^{(0)} = \alpha p_1^{(0)} = 0$, the first iterate to the pressure deviator is the Navier-Stokes approximation.

In the general result (26.5)₁ for the pressure deviator, the first term is independent of the zeroth iterates $p_m^{(0)}$ and, considered by itself, when $\mu < 1/|\alpha|$ has the effect of converting the Navier-Stokes approximation $p_0 = -\mu E$ into the full asymptotic solution $p_0 = -\mu E/(1 + \alpha\mu)$. The second term, however, depends on the zeroth iterates and can yield almost any kind of result as $n \rightarrow \infty$. We begin with a simple sufficient condition for convergence:

Theorem on convergence to the asymptotic solution. *If there are positive functions $A(t)$ and $B(t)$ such that*

$$\left| \frac{d^q p_m^{(0)}}{dt^q} \right| < AB^q, \quad (26.6)$$

uniformly in m and for t in some closed interval, then for

$$\mu < \frac{1}{|\alpha| + |\beta| + B} \quad (26.7)$$

differential iteration converges to the asymptotic solution.

To prove this theorem, we need only observe that the hypothesis (26.6) implies that the double sums in (26.5) are dominated by $(-\mu)^n A(|\alpha| + |\beta| + B)^n$. Hence follows also the

Theorem on approximation of the iterates in time. *If for fixed n (26.6) holds for $m \leq n$ and moreover*

$$AB^m = o(1) \text{ when } t \rightarrow \infty, \quad (26.8)$$

then as $t \rightarrow \infty$

$$\begin{aligned} p_0^{(n)}(t) &\rightarrow -\frac{\mu E}{1 + \alpha\mu} [1 - (-\alpha\mu)^n], \\ &= -\frac{\mu E}{1 + \alpha\mu} + O(\mu^{n+1}) \quad \text{as } \mu \rightarrow 0. \end{aligned} \quad (26.9)$$

This theorem shows that the iterates up to a certain n may with time settle into results near to the asymptotic solution even when the iteration itself is not convergent.

However, it is possible for differential iteration to converge to *any* of the steady solutions (24.4). To see this, put $p_m^{(0)} = (-1)^m K(\alpha + 1/\mu)^m \beta^{-m}$ in (26.5); then when $\mu < 1/|\alpha|$ the iterative scheme converges to the steady solution defined by $p_0 + \mu E/(1 + \alpha\mu) = K$.

It is also possible for the iterative scheme to converge to a time-dependent solution or to diverge. This statement may be verified by considering $z_m^{(0)} = e^{Ct}$ or $p_m^{(0)} = e^{Ct}$, where C is a constant.

Finally we consider the position of the Navier-Stokes approximation relative to the higher iterates. So as to simplify the results, we make two specializations: (1) we set $\alpha = 0$, making the Navier-Stokes approximation and the asymptotic solution coincide, (2) we suppose the $p_m^{(0)}$ are uniformly bounded constants. Then (26.5) becomes

$$\begin{aligned} p_0^{(n)} &= -\mu E + (-\mu\beta)^n p_n^{(0)}, \\ p_m^{(n)} &= (-\mu\beta)^n p_{m+n}^{(0)}, \quad m \geq 1. \end{aligned} \quad (26.10)$$

If $p_1^{(0)} = 0$ and $\mu < 1/|\beta|$, the iterative process yields the Navier-Stokes approximation at the first stage, $p_0^{(1)} = -\mu E$, and converges to it ultimately: $p_0^{(n)} \rightarrow -\mu E$ as $n \rightarrow \infty$. Every higher iterate is a worse approximation than the first.

In physical works it is customary to speak of the "order" of a perturbation and to believe as an article of faith that in any approximate calculation only terms of a given fixed "order" should be retained. The naivety of this faith is apparent from (26.5)₁ under conditions such that the iteration converges. In that formula are two terms proportional to μ^n . The former is that obtained from the expansion of $-\mu E [1 - (-\alpha\mu)^n] / (1 + \alpha\mu)$ in a series of powers of μ . As $n \rightarrow \infty$, the terms so arising steadily convert the Navier-Stokes approximation $-\mu E$ into the true asymptotic solution $-\mu E / (1 + \alpha\mu)$. The latter term of "order" n is $(-\mu)^n \times$ the double sum. Since the double sum depends on the

initial iterates, while the limit function is independent of them, retention of this term of "order" n can in general have no other effect than that of slowing the convergence.

In §2 we referred to the now well known fact that for some physical phenomena solutions corresponding to the "higher" Chapman-Enskog approximations in the kinetic theory differ from the experimental data more than do those given by the Navier-Stokes equations. The foregoing example shows that for the model, at least, these higher iterates may be, as mathematical approximations to an exact solution, worse than the first. Thus the discrepancy between theory and experiment may reflect not on the Maxwellian kinetic theory itself but only on misplaced confidence in "approximate" mathematical methods.

27. Final remarks on differential iteration. The conclusions from the preceding section have been set at the head of the memoir in §1. Here we remark further that the conditions which determine the solution of the equations of moments bear no relation to those determining whether the iterative scheme converges or not. The former concern only *the growth of the initial values of the higher moments*. The latter concern only *the growth of the derivatives of all orders of all the zeroth iterates*. But this one thing both conditions have in common: behavior of the *higher* moments, either in initial value or in our first guess at them, determines everything. Physically, of course, in the kinetic theory it is the higher moments that we know nothing about.

Finally, we notice the extreme sensitivity of the outcome of differential iteration to the choice of zeroth iterates. Virtually anything can result. Within a fairly broad class of zeroth iterates, however, including the case when they are uniformly bounded constants, the scheme does converge to a result independent of the particular choice of zeroth iterates, and the limit is the asymptotic solution.

Our analysis in this chapter has concerned only a model, not the true equations of moments. It is of course possible that the classical approach to the kinetic theory yields results better than those gained by application of corresponding techniques to the model. However, in Chapter V we shall see that the reverse is true and that, at least in the special case there considered, the expectation we might form from the model is in fact unwarrantedly optimistic.

CHAPTER IV. *Integral iteration*

28. A scheme akin to Grad's method of moments. We now recognize the equations of moments as partial differential equations and endeavor to find a method of solution based on integration. We order the $Y_{2r|s}$ as in §5 and write the equations $m\mathcal{C}(Y_{2r|s}) = \mathcal{L}(Y_{2r|s})$ in the new form

$$\mathfrak{M}(Y_{2r|s}) = \mathfrak{C}(Y_{2r|s}), \quad (28.1)$$

where

$$\mathfrak{C}(Y_{2r|s}) \equiv -(\overline{\rho c_i Y_{2r|s}})_{,i}. \quad (28.2)$$

Now set

$$\mathfrak{M}^{(n+1)}(Y_{2r|s}) = \mathfrak{C}^{(n)}(Y_{2r|s}). \quad (28.3)$$

These iterative equations are partial differential equations: At each step, an integration must be performed. The zeroth iterates may be selected in various ways, such as those suggested in §2, §18, or §22.

This method differs from GRAD's in that on the one hand we never solve any of the exact equations except those of mass, momentum, and energy, while on the other hand each additional step in the iteration increases the order of moments considered by one. It may be expected that this method and GRAD's are equivalent in the limit, both being ultimately exact and general, but that in special cases the iterative equations (28.3) may be easier to solve than are GRAD's. Of course, no universal formulæ can result: Each initial and boundary value problem must be faced and solved individually.

Against this method, as against GRAD's, there are two basic objections:

1. As greater accuracy is desired, the initial values of more and more moments must be assigned. We have no physical information regarding these higher moments and no physical interest in them. That is, this method does not appear to lend itself to resolving the methodological paradox of the kinetic theory (*cf.* §23).

2. The equations themselves are so complicated that, however, appealing they may be in principle, practical results may scarcely be obtained from them.

In the following section we formulate a new method in which the latter objection is in great part removed.

29. A scheme employing only linear partial differential equations. The non-linearity of the system of §28 presents difficulties at present insuperable in most cases, but it is possible that these difficulties may be avoided.

In (28.3), interpret $\mathfrak{M}^{(n+1)}(Y_{2r|s})$ as the result of replacing in $\mathfrak{M}(Y_{2r|s})$ the function $P_{2r|s}$ itself by its $n + 1^{\text{st}}$ iterate, but all other P 's by their n^{th} iterates. The system (28.3) then is linear.

For example, we get

$$\partial_i \rho^{(n+1)} + (\rho^{(n+1)} u_i^{(n)})_{,i} = 0, \quad (29.1)$$

$$\rho^{(n)} \partial_i u_i^{(n+1)} + \rho^{(n)} u_{i,i}^{(n+1)} u_i^{(n)} + p_{,i}^{(n)} - \rho^{(n)} f_i = 0, \quad (29.2)$$

$$3\partial_i p^{(n+1)} + 3p_{,i}^{(n+1)} u_i^{(n)} + 5p^{(n+1)} E^{(n)} + 2P_{ii}^{(n+1)} E_{ii}^{(n)} + 2h_{i,i}^{(n)} = 0, \quad (29.3)$$

$$\begin{aligned} \partial_i P_{ij}^{(n+1)} + P_{ij,k}^{(n+1)} u_k^{(n)} + P_{ii}^{(n+1)} E^{(n)} + 2p^{(n)} E_{ii}^{(n)} \\ + \frac{p^{(n)}}{\mu^{(n)}} P_{ii}^{(n+1)} - \frac{2}{3} P_{kl}^{(n+1)} E_{kl}^{(n)} \delta_{ii} + 2P_{k(i}^{(n+1)} u_{j),k}^{(n)} \\ + P_{ijk,k}^{(n)} - \frac{4}{15} h_{k,k}^{(n)} \delta_{ij} + \frac{4}{5} h_{(i,i)}^{(n)} = 0, \quad \text{etc.} \end{aligned} \quad (29.4)$$

Notice that in order to obtain linear equations we have deprived the pressure, density, and velocity of the preferred role they are usually given in the kinetic theory and have placed them on a par with the other moments, to be obtained by iteration¹. Similarly, $\mu^{(n)}$ stands for $(2.3)_1$ with p/ρ replaced by $p^{(n)}/\rho^{(n)}$.

For zeroth iterates to p , ρ , and u_i it is natural to take the *hydrodynamical solution of the boundary value problem in question*. Thus our method allows use of the equations of the kinetic theory to refine results from hydrodynamics. For zeroth iterates to P_{ij} and h_i we could take the solutions from the theory of viscous fluids, although such solutions are available for only a few cases. In any event, we quickly come to the higher moments, about which we have no knowledge *a priori*. For these, and possibly also for P_{ij} and h_i , in the absence of any other information, we may take the zeroth iterates as zero. (Of course, we could use alternatively one of the methods of initiation suggested in §2, §18, or §22. Methods of successive approximation when appropriately formulated usually lead to results independent of the choice of the first approximation, within a certain class, and thus the choice of zeroth iterates may not be important here.)

Solving even the first few of the linear partial differential equations (29.1)–(29.4) is a formidable problem. In the next section we consider a particular case in which the details are extraordinarily simple.

¹ Recall that in Maxwellian iteration (Chapter I), P_{ij} and all moments of order greater than 2 are expressed in terms of the gross condition by iterates, in the process of obtaining which all the time derivatives of these same moments are replaced by certain functions of the gross condition. The Chapman-Enskog process in effect (§15) treats $(p/\rho)^*$ and $(u_{i,i})^*$ in the same way. The results of either process are to be substituted in the equations of momentum and energy, which become the differential equations to be solved. In the present method not only are *all* moments to be obtained by an iterative scheme involving linear partial differential equations, but also the result of each iterative step is an approximate solution of the flow problem as a whole.

30. An example: shearing flow. We shall give as our single example the beginning of the iteration in the case of simple shearing flow when $f_i = 0$, the first case on which it is customary to illustrate any new theory of fluids. We shall treat this problem in semi-inverse style, assuming

$$\rho = \text{const.}, \quad u_x = Ky, \quad u_y = 0, \quad u_z = 0, \quad K = \text{const.}, \quad (30.1)$$

and taking $\rho^{(n)} = \rho$, $u_i^{(n)} = u_i$. Since $p/\mu \propto \rho$ and $\rho = \text{const.}$ in time as well as space, we set $p^{(n)}/\mu^{(n)} = p/\mu$. We try to find a solution in which all the $P_{2r|s}$ are constant in space. We choose to restrict attention to p and P_{ij} , not making any study of the h_i or other higher moments (but cf. §51). To facilitate the calculation, we use both the Maxwellian *time of relaxation*,

$$\tau \equiv \mu/p, \quad (30.2)$$

and the dimensionless *truncation number*:¹

$$\mathfrak{T} \equiv K\tau = \frac{\mu K}{p}. \quad (30.3)$$

In the present case, these two quantities are constant both in space and in time.

For our special problem, the linking of the successive equations of moments disappears, and, since the equations of mass and momentum are satisfied, the entire set of iterative equations for the pressures becomes

$$\begin{aligned} \tau p^{(n)'} &= -\frac{2}{3}\mathfrak{T}P_{xy}^{(n)}, \\ \tau P_{xy}^{(n+1)'} + P_{xy}^{(n+1)} &= -\mathfrak{T}(p^{(n)} + P_{yy}^{(n+1)}), \\ \tau P_{xx}^{(n+1)'} + P_{xx}^{(n+1)} &= -\frac{4}{3}\mathfrak{T}P_{xy}^{(n+1)}, \\ \tau P_{yy}^{(n+1)'} + P_{yy}^{(n+1)} &= \tau P_{zz}^{(n+1)'} + P_{zz}^{(n)} = \frac{2}{3}\mathfrak{T}P_{xy}^{(n+1)}, \\ \tau P_{yz}^{(n+1)'} + P_{yz}^{(n+1)} &= 0, \quad \tau P_{xz}^{(n+1)'} + P_{xz}^{(n+1)} = 0, \end{aligned} \quad (30.4)$$

where a prime denotes differentiation with respect to time. The first of these equations follows from (29.3), the remainder from (29.4).

To solve this system, we set

$$p_0 \equiv p(0), \quad \varpi^{(n)} \equiv p^{(n)}/p_0, \quad (30.5)$$

$$\mathfrak{p}_{ij}^{(n)} \equiv P_{ij}^{(n)}/p_0,$$

and we demand that each iterate shall satisfy the initial conditions of the exact problem:

$$\varpi^{(n)}(0) = 1, \quad \mathfrak{p}_{ij}^{(n)}(0) = P_{ij}(0)/p_0 \equiv \mathfrak{P}_{ij}, \quad \text{say.} \quad (30.6)$$

From the last two members of (30.4) follows

$$\mathfrak{p}_{yz}^{(n)} = \mathfrak{P}_{yz}e^{-t/\tau}, \quad \mathfrak{p}_{xz}^{(n)} = \mathfrak{P}_{xz}e^{-t/\tau}. \quad (30.7)$$

¹ For references and remarks concerning these parameters, see §31 and §33.

Thus the shearing pressures on the planes perpendicular to the shearing planes relax steadily to zero. To solve the remaining equations in (30.4), we note that the first, third, fourth, and fifth may be integrated:

$$\begin{aligned}\varpi^{(n)}(t) &= 1 - \frac{2}{3}K \int_0^t \varpi_{xy}^{(n)}(\xi) d\xi, \\ -\frac{3}{4}(e^{t/\tau} p_{xx}^{(n)} - \mathfrak{P}_{xx}) &= \frac{3}{2}(e^{t/\tau} p_{yy}^{(n)} - \mathfrak{P}_{yy}) \\ &= \frac{3}{2}(e^{t/\tau} p_{zz}^{(n)} - \mathfrak{P}_{zz}) = K \int_0^t p_{xy}^{(n)}(\xi) e^{\xi/\tau} d\xi.\end{aligned}\tag{30.8}$$

Putting these formulæ into (30.4)₂ yields:

$$\tau^2(e^{t/\tau} p_{xy}^{(n+1)})'' + \frac{2}{3}\mathfrak{T}^2(e^{t/\tau} p_{xy}^{(n+1)}) = -\tau\mathfrak{T}(e^{t/\tau} \varpi^{(n)})',\tag{30.9}$$

and hence

$$\begin{aligned}p_{xy}^{(n+1)} &= -\frac{\mathfrak{T}}{1 + \frac{2}{3}\mathfrak{T}^2} \left\{ 1 - \frac{2}{3}K \int_0^t p_{xy}^{(n)}(\xi) d\xi + \frac{2}{3}K e^{-t/\tau} \int_0^t p_{xy}^{(n)}(\xi) e^{\xi/\tau} \right. \\ &\quad \cdot [\cos \sqrt{\frac{2}{3}}K(\xi - t) + \sqrt{\frac{2}{3}}\mathfrak{T} \sin \sqrt{\frac{2}{3}}K(\xi - t)] d\xi \Big\} \\ &\quad + e^{-t/\tau} \{ A \cos \sqrt{\frac{2}{3}}Kt + B \sin \sqrt{\frac{2}{3}}Kt \}.\end{aligned}\tag{30.10}$$

The constants A and B do not depend on n . By (30.6) we get

$$A = \mathfrak{P}_{xy} + \frac{\mathfrak{T}}{1 + \frac{2}{3}\mathfrak{T}^2}.\tag{30.11}$$

To express B in terms of the initial conditions, notice from (30.4)₂ that

$$\tau p_{xy}^{(n+1)'}(0) = -\mathfrak{P}_{xy} - \mathfrak{T}(1 + \mathfrak{P}_{yy}).\tag{30.12}$$

Substitution in (30.10) yields

$$\sqrt{\frac{2}{3}}B = \frac{\frac{2}{3}\mathfrak{T}^2}{1 + \frac{2}{3}\mathfrak{T}^2} - \mathfrak{P}_{yy}.\tag{30.13}$$

From (30.10) and (30.8) we see that *the iterates $p_{ii}^{(n)}$ can be calculated by quadratures.*

Let us consider the results. Clearly the essential formula is the integral-difference relation (30.10). To begin the iteration, we take the simplest zeroth iterate consistent with the initial conditions (30.6)²:

$$p_{ii}^{(0)} = \mathfrak{P}_{ii}.\tag{30.14}$$

Putting $n = 0$ in (30.10) then yields

$$p_{xy}^{(1)} = -\frac{\mathfrak{T}}{1 + \frac{2}{3}\mathfrak{T}^2} + e^{-t/\tau} [A \cos \sqrt{\frac{2}{3}}Kt + B \sin \sqrt{\frac{2}{3}}Kt],\tag{30.15}$$

² The agreement (30.6) was introduced only to prune away superfluous details in the calculation. The ultimate results appear to be independent of the choices (30.6) and (30.14).

consisting in a steady term plus a transient term. The logarithmic decrement of the transient term is one time of relaxation. For small $\mathfrak{T}$, the steady term yields $\mathfrak{p}_{xy}^{(n)} \approx -\mathfrak{T}$, agreeing with the Navier-Stokes formula ($P_{xy}/p = -\mathfrak{T}$) so long as the pressure may be considered constant. For large $\mathfrak{T}$, the steady term is arbitrarily small. While this is a very interesting result, we must not trust it, since it results only from the first step in a method of successive approximation.

To calculate the mean pressure in first approximation, we have only to put (30.15) into (30.8)₁. Thus we get

$$\varpi^{(1)} = 1 + \frac{\frac{2}{3}\mathfrak{T}}{1 + \frac{2}{3}\mathfrak{T}^2} \{Kt + e^{-t/\tau}[(B - \sqrt{\frac{2}{3}}\mathfrak{T}A) \sin \sqrt{\frac{2}{3}}Kt + (A + \sqrt{\frac{2}{3}}\mathfrak{T}B) \cos \sqrt{\frac{2}{3}}Kt]\}. \quad (30.16)$$

Thus, apart from a rapidly damped portion, $\varpi^{(1)}$ increase linearly with the time. That the pressure must increase is evident *a priori*, since such increase is equivalent to the heating which must result from the work continually done on the fluid.

Putting (30.15) into (30.8)₂ and (30.8)₃ yields formulæ for the normal pressures

$$\begin{aligned} \mathfrak{p}_{xx}^{(1)} &= \frac{\frac{4}{3}\mathfrak{T}^2}{1 + \frac{2}{3}\mathfrak{T}^2} + e^{-t/\tau} \left\{ \mathfrak{P}_{xx} + \frac{2\sqrt{2}}{\sqrt{3}} [-A \sin \sqrt{\frac{2}{3}}Kt + B(\cos \sqrt{\frac{2}{3}}Kt - 1)] \right\}, \\ \mathfrak{p}_{yy}^{(1)} &= \frac{-\frac{2}{3}\mathfrak{T}^2}{1 + \frac{2}{3}\mathfrak{T}^2} + e^{-t/\tau} \left\{ \mathfrak{P}_{yy} + \sqrt{\frac{2}{3}}[A \sin \sqrt{\frac{2}{3}}Kt + B(\cos \sqrt{\frac{2}{3}}Kt - 1)] \right\}, \\ \mathfrak{p}_{zz}^{(1)} &= -\mathfrak{p}_{xx}^{(1)} - \mathfrak{p}_{yy}^{(1)}. \end{aligned} \quad (30.17)$$

We note that the steady parts of these pressures are proportional to $\mathfrak{T}^2$ for small $\mathfrak{T}$, as is the case in results obtained by previous students of the kinetic theory but that they exhibit for large $\mathfrak{T}$ a behavior which cannot be obtained from the polynomials in $\mathfrak{T}$ which emerge from the older type of calculation.

We have seen from (30.16) that the undamped part of $\varpi^{(1)}$ increases steadily with the time. From (30.9) it follows that $\mathfrak{p}_{xy}^{(2)}$ must change in time accordingly. To calculate this change, we have only to put (30.15) into (30.10), obtaining

$$\begin{aligned} \mathfrak{p}_{xy}^{(2)} &= -\frac{\mathfrak{T}(1 + \mathfrak{A} + 2\mathfrak{A}^2)}{(1 + \mathfrak{A})^3} - \frac{\mathfrak{A}Kt}{(1 + \mathfrak{A})^2} \\ &+ e^{-t/\tau} \left\{ \frac{\frac{1}{3}\mathfrak{T}Kt}{1 + \mathfrak{A}} [(A + \sqrt{\frac{2}{3}}B\mathfrak{T}) \cos Kt + (\sqrt{\frac{2}{3}}A\mathfrak{T} - B) \sin Kt] \right. \\ &\left. + L \cos Kt + M \sin Kt \right\}, \end{aligned} \quad (30.18)$$

where

$$\begin{aligned} \mathfrak{A} &\equiv \frac{2}{3}\mathfrak{T}^2, \quad L \equiv \mathfrak{P}_{xy} + \frac{\mathfrak{T}(1 + \mathfrak{A} + 2\mathfrak{A}^2)}{(1 + \mathfrak{A})^3}, \\ M &\equiv \mathfrak{P}_{yy} + \frac{\mathfrak{A}}{1 + \mathfrak{A}} + \frac{\frac{1}{3}\mathfrak{T}(A + \sqrt{\frac{2}{3}}B\mathfrak{T})}{1 + \mathfrak{A}}. \end{aligned} \quad (30.19)$$

In this result the dependence on $\mathfrak{T}$ is considerably more complicated than for $\mathfrak{p}_{xv}^{(1)}$, and the damped portion is no longer proportional to a periodic function.

It is not without reason that here I pass over the physical implications of the work. Rather it is for shortness of writing, since in the next chapter I will obtain the *exact* solution to the problem of shearing flow.

The calculations just presented illustrate the thorough difference between the type of results which follow from our new method of integration and those obtainable by classical methods. As will appear from the analysis in the next chapter, the reason for this difference is that while the iterates following from the classical approach converge to limits closely approximating a particular and special solution, the iterates $P_{ii}^{(n)}$ obtained by the present method converge to the exact and *general* solution, for any fixed $\mathfrak{T}$, K , and t . However, the convergence is rapid only for sufficiently small values of Kt or t/τ , and on this account not all of the physical conclusions which might seem to follow from the formulæ actually worked out are true. Cf. §55.

We close this chapter with the remark that the success of our approximative method for the special case of shearing flow does not justify expectation of similar success in other applications of it. Not only is the problem a specially simple one, but also we have approached it in semi-inverse style, assuming a particular gross flow. To explore the method in more representative examples would require elaborate numerical study, inappropriate to the present work, which is aimed toward the principles.

CHAPTER V. *Exact solution for shearing flow*

31. The case when the pressures and the higher moments are constant in space. Much of the complexity of the equations of moments is resolved if all moments of order greater than one are supposed spatially homogeneous. In particular, since the $n + 1^{\text{st}}$ moments enter the equation for the n^{th} moments only through their space derivatives, the formidable forward linking of the equations disappears. However, a backward linking remains: The equation for the n^{th} moments generally involves moments of lower order. Thus the moments may be calculated in turn.

The equations in question may be read off from the results of §6 and §8. Using a prime to denote a partial derivative with respect to time, we have

$$\rho' = -(\rho u_i)_{,i} , \quad (31.1)$$

$$u'_i = -u_{i,j}u_j + f_i , \quad (31.2)$$

$$3p' = -5pE - 2P_{ij}E_{ij} , \quad (31.3)$$

$$P'_{ij} = -P_{ij}E - 2P_{j(i}u_{j),k} - 2pE_{ij} + \frac{2}{3}P_{kl}E_{kl}\delta_{ij} - P_{ij}/\tau , \quad (31.4)$$

$$h'_i = -\frac{5}{3}h_iE - P_{ijk}E_{jk} - h_iu_{i,j} - \frac{4}{3}h_iE_{ij} - \frac{2}{3}h_i/\tau , \quad (31.5)$$

$$\begin{aligned} P'_{ijk} = & -P_{ijk}E - 3P_{l(i}u_{j)k,l} + \frac{6}{5}P_{lm(i}\delta_{jk)}E_{lm} \\ & - \frac{12}{5}h_{(i}E_{jk)} + \frac{24}{5}h_lE_{l(i}\delta_{jk)} - \frac{3}{2}P_{ijk}/\tau , \end{aligned} \quad (31.6)$$

etc., where the *time of relaxation*, τ , is given by

$$\tau \equiv \frac{\mu}{p} = \frac{1}{3} \sqrt{\frac{2m}{G}} \cdot \frac{m}{\rho A_2} . \quad (31.7)$$

Now the equation (31.4) for the second moments has a familiar look. More generally, consider the case when p_{ijk} is spatially homogeneous, or at least $p_{ijk,k} = \frac{2}{3}h_{i,k}\delta_{ij}$. Then (2.9) reduces to

$$\dot{P}_{ij} + 2P_{k(i}u_{j),k} + P_{ij}E - \frac{2}{3}P_{kl}E_{kl}\delta_{ij} + 2pE_{ij} + P_{ij}/\tau = 0. \quad (31.8)$$

This equation, apart from the terms containing contracted products of P_{ij} by $u_{k,l}$, is of the type occurring in the theories of “relaxation phenomena”. The concept of relaxation was introduced by MAXWELL in this very connection, but he gave only a rough sketch of a one-dimensional model to substantiate his rather vague remarks concerning a visco-elastic continuum [1867, pp. 30–31]. MAXWELL’s application of the idea of relaxation to the kinetic theory consisted in observing from (31.8) that in a gas at rest and at uniform constant temperature and pressure, any deviatoric pressures P_{ij} must fall off according to the law

$$P_{ij} = P_{ij}(0)e^{-t/\tau}, \quad (31.9)$$

thus justifying¹ the name “time of relaxation” for τ .

While an enormous literature on visco-elasticity has grown up about MAXWELL's idea or the related subsequent idea of O.-E. MEYER, this literature is restricted either explicitly or implicitly to linearizations supposedly appropriate for small motion and thus is worthless for gas flows. Until recently ZAREMBA [1903][1937] was the only student of visco-elasticity to have formulated a properly invariant theory, but, curiously enough, although he was guided by MAXWELL's work his final equations are not sufficiently general as to include (31.8). NOLL has just constructed and studied a broad and properly invariant theory of *hygrosteric materials*, containing (31.8) as a special case of what he calls a *linear fluent body* [1955, 1, §12].² However, while NOLL obtained some exact solutions of his equations, they do not happen to be appropriate to our present problems.³ In the next section, therefore, we shall obtain a class of exact solutions of the system (31.1)–(31.6). While the same method would yield corresponding results for more general continuous media, since physicists and engineers at present appear to put considerable faith in the kinetic theory we shall rest content with the special case which it affords.

Obviously the coincidence between the equations for the second moments in a Maxwellian gas and the constitutive equations for a certain type of hygrosteric continuum is restricted to problems in which $p_{ijk,k} - \frac{2}{3}h_{k,k}\delta_{ij} = 0$. It does not follow that any solution P_{ij} of (2.9) with the terms containing third moments annulled necessarily yields values of P_{ij} consistent with an exact solution for a Maxwellian gas, for the condition $p_{ijk,k} - \frac{2}{3}h_{k,k}\delta_{ij} = 0$ may not be consistent with the equations of higher moments. If, however, $u_{i,j}$, the P_{ij} and all moments of order greater than 2 are spatially homogeneous, the required consistency is immediate: The equations for the higher moments become an ordinary differential system of first order with constant coefficients. This case we shall investigate in the next section. We pause to suggest the possibility of solutions in which u_i is a polynomial of degree m in the spatial coordinates, P_{ij} a polynomial of degree $m - 1$, p_{ijk} of degree $m - 2$, etc., while the moments of order $m + 1$ and greater are spatially homogeneous. The problem of compatibility of the equations of higher moments is then not trivial. To investigate this possibility and to apply other semi-inverse methods so as to find exact solutions in a Maxwellian gas will require a memoir in itself.

¹ This idea will be pursued to its conclusion in Chapter VI.

² Specifically, to get (31.8) from NOLL's (12.2c), put $\mu_1 = 2p$, $\lambda_1 = -p/\mu$, $\mu_1 + 3\mu_2 = 0$, $\alpha_1 = \frac{2}{3}$, $\alpha_3 = -1$, $\alpha_5 = -1$.

³ Specifically, NOLL's §19b concerns relaxation of initial stress in a body at rest, yielding τ as the logarithmic decrement, just as in the special case (31.9). His treatment of shearing flow in §21b is restricted to incompressible substances subject to steady stress but nevertheless obtains results strikingly similar to what we shall find in §33. The most serious limitation in NOLL's solutions, as far as the present problem is concerned, arises from his assumption $\mu_1 = \text{const.}$, while for a Maxwellian gas $\mu_1 = 2p$.

32. The possibility of a general exact solution for steady rectilinear flow. The case when the acceleration vanishes and the velocity field is linear is not trivial. It has been characterized [1955, 3] and shown to include interesting subcases. Here we shall consider only the especially simple case when u_i is steady, $\rho = \text{const.}$, $u_{i,i} = \text{const.}$ in space and time. Then $u_i = a_{ij}x_j + b_i$, where a_{ij} is a constant matrix whose square and trace both vanish, while b_i is a constant vector orthogonal to a_{ij} . Under these assumptions $\rho' = 0$, and (31.2) is satisfied if $f_i = 0$.

By (31.7), $\tau = \text{const.}$ in space and time for this class of problems. Therefore the equations (31.3) and (31.4) become a homogeneous linear system with constant coefficients. Thus p and P_{ij} are linear combinations of exponential functions or temporally damped oscillations. Similarly, the equations (31.5) and (31.6) for the third moments become a homogeneous linear system with constant coefficients. The equations for the fourth moments also are linear equations with constant coefficients, but they are not homogeneous: In them occur forcing terms depending on the second moments. From these observations and from some of the results in §13 it follows that *for a gas whose molecules are Maxwellian, any flow whose velocity vector satisfies*

$$u_i = a_{ij}x_j + b_i, \quad a_{ij}a_{jk} = 0, \quad a_{ij}b_j = 0, \quad a_{ii} = 0, \quad (32.1)$$

where the a_{ij} and b_i are constant in space and time, is dynamically possible when the extrinsic force is zero. For such a flow, the general solution $P_{2r|s}(t)$ can be calculated by a finite number of quadratures. Though these flows themselves are steady, the $P_{2r|s}$ cannot be steady (cf. §32B). For example, (31.3) describes the continued heating which must result from the work of deformation when, as is assumed here, the flux of energy h_i are uniform in space.

The exact solutions just discovered are the first representing any fluid motion affected by viscosity ever to be gotten in the kinetic theory. We shall work out the details only for the most important special case, that of simple shearing flow.

32A. Appendix. Previously known exact solutions.

1. *Locally Maxwellian solutions.* The only distribution function known to satisfy the Maxwell-Boltzmann equation exactly is the Maxwellian. The resulting motions (e.g. [1949, 1, Appendix 2]), which are not restricted to the case of Maxwellian molecules, are precisely those gas flows which are independent of the values of the molecular constants.

2. *The uniform state.* In a Maxwellian gas when the gross condition is steady and uniform, MAXWELL showed that the pressures are given by (31.9) and found similar formulæ for the third moments. This class of solutions, which will be studied further in Chapter VI, does not represent any streaming of the gas.

GRAD [1949, 1, pp. 368-369] has obtained what might suggest an exact solution for steady one-dimensional flow of energy in the presence of a temperature gradient, but it is valid only in his thirteen moment approximation. If we seek a similar solution of the exact equations, assuming the P_{ijk} constant in space, instead of the equation numbered (5.24) in GRAD's paper we get

$$\tau h_i = -\frac{1}{3}P_{2|i i, i} - \frac{1}{3}P_{4|i, i}, \quad (32A.1)$$

as well as an infinite sequence of conditions on the P_{ijk} and the higher moments to assure their

compatibility with the assumptions made. Not only does no such result as the Newton-Fourier law follow without further hypotheses, but it seems questionable whether the implied conditions can be satisfied. GRAD's procedure is in effect to eliminate the fourth moments by (17.1), and thus his method of calculating the thermal conductivity is in essence the same as MAXWELL's (*cf.* our presentation of MAXWELL's derivation of (2.15) by using (2.12)₆).

32B. Appendix. Previous attempts to solve the problem of shearing flow.

An analysis of shearing flow according to Burnett's equations is given by SCHAMBERG [1947, §§6.1–6.2], but unfortunately the results are triply false. A steady flow in which $u_y = u_x = 0$ while u_x , p , and θ are functions of y only, is assumed; according to the result, u_x is linear and θ is quadratic in y .

1. That no steady solution is possible according to the exact equations was proved at the end of §32.

2. From the equations of momentum and energy, for any steady flow in which $u_y = 0$ and all quantities depend on y only, we get

$$p_{xy} = \text{const.}, \quad p_{yy} = \text{const.}, \quad u_x p_{xy} + h_y = \text{const.} \quad (32B.1)$$

In the case of Maxwellian molecules, either from the equations of Burnett and Chapman & Cowling or from the Navier-Stokes and Fourier formulæ we have $p_{xy} = -\mu u'_x = -\mu_0 \theta u'_x$, where $\mu_0 = \text{const.}$, and $h_y = -\alpha \mu_0 \theta \theta'$. Eliminating u_x between (32B.1)₁ and (32B.1)₃ yields

$$\frac{1}{2} \alpha (\theta^2)'' + A^2 / \theta = 0, \quad (32B.2)$$

where $A = \theta u'_x$ is a non-vanishing real constant. If we let θ_0 be the value of θ when $y = 0$, we may put the solution of (32B.2) into the form

$$(\theta_0 - \theta)(2\theta_0 + \theta)^2 = \frac{9A^2}{2\alpha} y^2. \quad (32B.3)$$

The temperature falls off from θ_0 as y^2 increases, reaching $\theta = 0$ on the planes $y = \pm \frac{2}{3}(\sqrt{2\alpha}/A)\theta_0^{\frac{3}{2}}$. The curve $\theta = \theta(y)$ is somewhat like a parabola, but with such a θ and either the Burnett formula or the Navier-Stokes formula for p_{yy} it is impossible to satisfy (32B.1)₂.

3. SCHAMBERG in fact uses not Burnett's formula but another which results from it when μ is taken as constant. Even so, (32B.1)₂ is not satisfied (*cf.* SCHAMBERG's eq. (6.15)) except "in the present approximation", since " μ^2/p^2 may be replaced by $\mu^2/\rho p^2$." Now SCHAMBERG has assumed μ constant in order to integrate the equations, and we have both $\rho = \text{const.}$ and $p = (k/m)\rho\theta$ to every order of approximation. Thus taking μ^2/p^2 as constant is equivalent to taking $\theta = \text{const.}$ immediately after having decided that it is a parabolic function.

The argument is in fact quite general. Eulerian hydrodynamics excepted, in every sort of approximation from the kinetic theory and in every sort of fluid theory the pressures do work in deforming the gas. Except for very special (perhaps non-existent) forms of flux of energy, this work will be converted in part into heat, and the temperature will rise as time goes on. If the density is steady and if there is an equation of state of form $p = p(\rho, \theta)$, $\partial p / \partial \theta > 0$, then the pressure also must increase in time. No order of approximation is involved in these simple physical facts. Only in very special cases, such as barotropic flow ($f(p, \rho) = 0$) of a fluid with linear viscosity independent of temperature, is a steady pressure possible.

T. C. LIN [1951, 4] gives a transformation by which SCHAMBERG's method is modified to take account of the dependence of μ on temperature. LIN's analysis is likewise faulty (*cf.* the remarks following LIN's eq. (4.18)).

The attempt of GRAD [1949, 1, p. 370] to solve the problem of shearing flow is admittedly abortive because he, too, insists that P_{ij} be steady.

33. The exact solution for the pressures in simple shearing flow. We consider the velocity field

$$u_x = Ky, \quad u_y = 0, \quad u_z = 0, \quad K = \text{const.} \quad (33.1)$$

So as to write our equations and solutions in dimensionally invariant form, we employ the dimensionless *truncation number*,

$$\mathfrak{T} \equiv \frac{\mu K}{p}, \quad (33.2)$$

introduced in my studies of the continuum theories of fluids [1948, 2] [1950, 2, §8]¹. Putting (33.1) into (31.3) and (31.4) yields the following system for the second moments:

$$\begin{aligned} \tau p' + \frac{2}{3}\mathfrak{T}P_{xy} &= 0, \\ \tau P'_{xy} + P_{xy} + \mathfrak{T}p + \mathfrak{T}P_{yy} &= 0, \\ \tau P'_{yy} + P_{yy} - \frac{2}{3}\mathfrak{T}P_{xy} &= 0, \\ \tau P'_{xx} + P_{xx} + \frac{4}{3}\mathfrak{T}P_{xy} &= 0, \\ \tau P'_{xz} + P_{xz} &= 0, \\ \tau P'_{yz} + P_{yz} &= 0, \end{aligned} \quad (33.3)$$

while of course $P_{zz} = -(P_{xx} + P_{yy})$. For the solution of the last two members we may write immediately

$$P_{xz}(t) = P_{xz}(0)e^{-t/\tau}, \quad P_{yz}(t) = P_{yz}(0)e^{-t/\tau}, \quad (33.4)$$

just as in relaxation of stress in a uniform state, and thus we need give no further consideration to any of the shearing pressures except P_{xy} .

The first three members of (33.3) form in themselves a homogeneous linear system of the third order. The growth modulus χ in a solution proportional to $e^{\chi t/\tau}$ must satisfy

$$\chi(\chi + 1)^2 = \mathfrak{A}, \quad (33.5)$$

where we have set

$$\mathfrak{A} \equiv \frac{2}{3}\mathfrak{T}^2. \quad (33.6)$$

¹ We avoid the unpleasant custom of calling every dimensionless parameter depending on μ but not on the heat conductivity "the Reynolds number". In the non-linear theories of fluids $\mathfrak{T}$ and the Reynolds number are independent and must both be controlled in questions of dynamical similarity. The name *truncation number* arises from the fact that in the theory of the Stokesian fluid the condition $\mathfrak{T} \ll 1$ is necessary and sufficient that the general equations can be broken off after the first term, thus yielding the usual theory of viscous fluids in first approximation.

For any positive $\mathfrak{A}$, the characteristic equation (33.5) has one positive root r and two complex conjugate roots $\Re \pm i\Im\mathfrak{T}$. The positive root r is a monotone increasing function of $\mathfrak{A}$, but $r/\mathfrak{A}$ is monotone decreasing, and r enjoys the bounds²

$$(\mathfrak{A} + 1)^{\frac{1}{3}} - 1 < r < \text{Min}(\mathfrak{A}, [\mathfrak{A} + \frac{1}{9}]^{\frac{1}{3}} - \frac{1}{3}); \quad (33.7)$$

moreover,

$$\begin{aligned} r &= \mathfrak{A}(1 - 2\mathfrak{A} + 7\mathfrak{A}^2) + O(\mathfrak{A}^4) \quad \text{as } \mathfrak{A} \rightarrow 0, \\ r &= \mathfrak{A}^{\frac{1}{3}} - \frac{2}{3} + O(\mathfrak{A}^{-\frac{1}{3}}) \quad \text{as } \mathfrak{A} \rightarrow \infty. \end{aligned} \quad (33.8)$$

In fact, by Cardano's formula we get

$$\begin{aligned} r &= \sqrt[3]{\frac{1}{2}\mathfrak{A} + \frac{1}{27} + \frac{1}{2}\sqrt{\mathfrak{A}\sqrt{\mathfrak{A} + \frac{4}{27}}}} \\ &\quad + \sqrt[3]{\frac{1}{2}\mathfrak{A} + \frac{1}{27} - \frac{1}{2}\sqrt{\mathfrak{A}\sqrt{\mathfrak{A} + \frac{4}{27}}}} - \frac{2}{3}, \end{aligned} \quad (33.9)$$

while directly from (33.5) follow simple expressions for the complex roots $\Re \pm i\Im\mathfrak{T}$:

$$\begin{aligned} \Re &= -(1 + \tfrac{1}{2}r), \\ \Im^2\mathfrak{T}^2 &= r(1 + \tfrac{3}{4}r). \end{aligned} \quad (33.10)$$

Hence

$$\begin{aligned} \Im &= \sqrt{\tfrac{2}{3}}(1 - \tfrac{5}{8}\mathfrak{A}) + O(\mathfrak{A}^2) \quad \text{as } \mathfrak{A} \rightarrow 0, \\ \Im &= \frac{1}{\sqrt{2}}\mathfrak{A}^{-1/6} + O(\mathfrak{A}^{-7/6}) \quad \text{as } \mathfrak{A} \rightarrow \infty. \end{aligned} \quad (33.11)$$

Figure 33.1 displays r and $\Im$ as functions of $\mathfrak{A}$.

We pause to determine the radius of convergence of the series for r in powers of $\mathfrak{A}$. By using a well known theorem, we find that (33.5) has a single-valued inverse in the neighborhood of every point except $\chi = -1$ and $\chi = -\frac{2}{3}$; equivalently $\mathfrak{A} = 0$ and $\mathfrak{A} = -\frac{4}{27}$. The real inverse r is regular at $\mathfrak{A} = 0$. Its singularity nearest the origin is accordingly $\mathfrak{A} = -\frac{4}{27}$, or $\mathfrak{T} = \pm i\sqrt{2}/3$. Hence *the series for r in powers of $\mathfrak{T}$ converges for $\mathfrak{T} < \frac{1}{3}\sqrt{2}$, diverges for $\mathfrak{T} > \frac{1}{3}\sqrt{2}$.*

We may now write the general solution of the first three members of (33.3) in the form

² The upper bound $\text{Min}(\mathfrak{A}, \mathfrak{A}^{\frac{1}{3}})$ is obvious from (33.5). The sharper bound asserted in (33.7) may be derived from (33.9) by replacing $\sqrt{\mathfrak{A}}$ by $\sqrt{\mathfrak{A} + \frac{4}{27}}$ in the first cube root, $\sqrt{\mathfrak{A} + \frac{4}{27}}$ by $\sqrt{\mathfrak{A}}$ in the second.

FIGURE 33.1. The amplification factor r and the frequency factor $\mathfrak{I}$ as functions of $\mathfrak{A}$.

$$\begin{aligned}
 p/p_0 &= Ae^{\mathfrak{r}t/\tau} + e^{-(1+\frac{1}{2}\mathfrak{r})t/\tau}[B \cos \mathfrak{I}Kt + C \sin \mathfrak{I}Kt], \\
 P_{xu}/p_0 &= \frac{1}{\frac{2}{3}\mathfrak{I}} \{-A\mathfrak{r}e^{\mathfrak{r}t/\tau} + e^{-(1+\frac{1}{2}\mathfrak{r})t/\tau} \\
 &\quad \cdot [(\{1 + \frac{1}{2}\mathfrak{r}\}B - \mathfrak{I}\mathfrak{I}C) \cos \mathfrak{I}Kt + (\{1 + \frac{1}{2}\mathfrak{r}\}C + \mathfrak{I}\mathfrak{I}B) \sin \mathfrak{I}Kt]\}, \\
 P_{vu}/p_0 &= -\frac{1}{\mathfrak{r} + 1} \left\{ A\mathfrak{r}e^{\mathfrak{r}t/\tau} + e^{-(1+\frac{1}{2}\mathfrak{r})t/\tau} \right. \\
 &\quad \cdot \left[\left(\left\{ \frac{3}{2} + \mathfrak{r} \right\} B + \frac{\mathfrak{I}\mathfrak{I}}{\mathfrak{r}} C \right) \cos \mathfrak{I}Kt \right. \\
 &\quad \left. \left. + \left(\left\{ \frac{3}{2} + \mathfrak{r} \right\} C - \frac{\mathfrak{I}\mathfrak{I}}{\mathfrak{r}} B \right) \sin \mathfrak{I}Kt \right] \right\}, \tag{33.12}
 \end{aligned}$$

where the dimensionless constants A , B , C , are determined by the initial conditions, as follows:

$$\begin{aligned}
1 &= A + B, \\
\frac{2}{3}\mathfrak{T}P_{xx}(0)/p_0 &= -Ar + (1 + \frac{1}{2}r)B - \mathfrak{I}C, \\
-(r + 1)P_{yy}(0)/p_0 &= Ar + (\frac{3}{2} + r)B + \frac{\mathfrak{I}}{r}C.
\end{aligned} \tag{33.13}$$

For use in §46 we record the formulæ

$$\begin{aligned}
p &= p_0 - \frac{2}{3}K \int_0^t P_{xx}(\xi) d\xi, \\
P_{yy} &= P_{yy}(0)e^{-t/\tau} + \frac{2}{3}Ke^{-t/\tau} \int_0^t e^{\xi/\tau} P_{xx}(\xi) d\xi, \\
P_{zz} &= P_{zz}(0)e^{-t/\tau} + \frac{2}{3}Ke^{-t/\tau} \int_0^t e^{\xi/\tau} P_{xx}(\xi) d\xi, \\
P_{xx} &= P_{xx}(0)e^{-t/\tau} - \frac{4}{3}Ke^{-t/\tau} \int_0^t e^{\xi/\tau} P_{xx}(\xi) d\xi,
\end{aligned} \tag{33.14}$$

which may be read off from (33.3)₁, (33.3)₃, and (33.3)₄.

To describe this solution we shall prefer to use the entire pressure tensor p_{ij} rather than its deviator P_{ij} . We write p° and p_{ij}° for the damped parts, and we call the remaining parts $p^\mathbf{D}$ and $p_{ij}^\mathbf{D}$ the *dominant* parts:

$$p = p^\mathbf{D} + p^\circ, \quad p_{ij} = p_{ij}^\mathbf{D} + p_{ij}^\circ. \tag{33.15}$$

Also we introduce the following dimensionless ratios:

$$\begin{aligned}
\pi_{ij} &\equiv p_{ij}/p, & \pi_{ij}^\mathbf{D} &\equiv p_{ij}^\mathbf{D}/p^\mathbf{D}, \\
\varpi &\equiv p/p_0, & \varpi_{ij} &\equiv p_{ij}/p_0 = \varpi\pi_{ij}, \\
\varpi^\mathbf{D} &\equiv p^\mathbf{D}/p_0, & \varpi_{ij}^\mathbf{D} &\equiv p_{ij}^\mathbf{D}/p_0, \\
\Pi_{ij} &\equiv p_{ij}(0)/p_0 = \varpi_{ij}(0).
\end{aligned} \tag{33.16}$$

The oscillatory terms in p_{ij}° are very rapidly damped out. Their logarithmic decrement is $\tau/(1 + \frac{1}{2}r)$, always less than one time of relaxation and monotone decreasing to 0 as $\mathfrak{T}$ increases with τ fixed. Thus with virtual certainty we may neglect³ the damped parts and confine our attention henceforth almost entirely to the dominant parts. (An important exception to the foregoing statement will be considered in §§43–45.)

34. Universal relations for the pressures. If we neglect the damped parts, we can eliminate the factor $Ae^{t/\tau}$ from the several pairs of equations (33.12).

³ *I.e.*, neglect the damped parts of the *solutions* once obtained. As we shall see in §37, to neglect the damped parts *before* calculating the exact solution would lead to erroneous results such as those usually presented in physical treatments of this subject.

There results the **First main theorem on shearing flow**: *In shearing flow of a Maxwellian gas, at any fixed time the dominant parts of the pressures satisfy*

$$-\pi_{xy}^D = \frac{r}{\frac{2}{3}\mathfrak{T}} = \mathfrak{T} \frac{r}{\mathfrak{A}}; \quad (34.1)$$

$$\pi_{yy}^D = \frac{1}{r+1}; \quad (34.2)$$

$$\frac{P_{yy}^D}{P_{xy}^D} = \frac{\frac{2}{3}\mathfrak{T}}{r+1}, \quad \text{or} \quad \frac{\pi_{yy}^D}{-\pi_{xy}^D} = \frac{\frac{2}{3}\mathfrak{T}}{r(r+1)}; \quad (34.3)$$

$$p_{zz}^D = p_{yy}^D, \quad p_{xx}^D = -2p_{yy}^D + 3p^D; \quad (34.4)$$

$$\frac{\pi_{xx}^D}{-\pi_{xy}^D} = \frac{2}{3}\mathfrak{T} \frac{3r+1}{r(r+1)}. \quad (34.5)$$

Each of these formulæ is of great interest in itself, but before going into details we note the major

Corollary. *In shearing flow of a Maxwellian gas, the ratio π_{ij}^D is a function of $\mathfrak{T}$ only.*

At first glance this corollary might seem to justify both the theory of the Stokesian fluid, which begins by postulating the existence of such relations, and the classical approach (cf. §1) to the kinetic theory, since its principal objectives are just such formulæ. In the next section we shall pursue this thought further, but finally in §§37–42 we shall prove that in fact the reverse follows. In this section, however, we find out what our exact formulæ tell us about a shearing flow.

Although later we shall need finer distinctions, for the moment we shall say that relations valid in the limit $\mathfrak{T} \rightarrow 0$ are appropriate to *ordinary states*; for $\mathfrak{T} \rightarrow \infty$, to *ultimate states*.

First, from (34.1) and (33.8) we get

$$-\pi_{xy}^D = \mathfrak{T}(1 - \frac{4}{3}\mathfrak{T}^2 + \frac{2}{9}\mathfrak{T}^4) + O(\mathfrak{T}^7) \quad (34.6)$$

for ordinary states, while for ultimate states

$$-\pi_{xy}^D = (\frac{3}{2})^{2/3}\mathfrak{T}^{-1/3} - \mathfrak{T}^{-1} + O(\mathfrak{T}^{-5/3}). \quad (34.7)$$

The shearing stress $-\pi_{xy}^D$ is positive for all $\mathfrak{T}$; for ordinary states $-\pi_{xy}^D$ rises linearly with $\mathfrak{T}$ but falls away to a single maximum at $\mathfrak{T} = \sqrt{6}$, $-\pi_{xy}^D = \sqrt{\frac{3}{8}}$, decreasing thereafter to zero for ultimate states.

Next we consider the normal pressures p_{xx}^D , p_{yy}^D , and p_{zz}^D . That such normal pressures are necessary to produce a shearing flow and are of order $\mathfrak{T}^2$ as $\mathfrak{T} \rightarrow 0$ is only what is to be expected from recent studies in continuum mechanics.

The formulæ (34.2)–(34.5) (or the expressions (33.12) from which they were derived) give the first rigorous results concerning such stresses in the kinetic theory.

From (34.2) we calculate the normal pressure which must be applied to the shearing planes in order to effect the motion at constant density. *For ordinary states this pressure p_{vv}^D is substantially the thermostatic pressure p^D , but π_{vv}^D falls away from 1 monotonically with increasing $\mathfrak{T}^2$ and decreases ultimately to zero.*

The formula (34.3)₁ compares the normal pressure deviation with the shearing stress. This ratio is always positive and increases monotonically with $\mathfrak{T}$, linearly for ordinary states and with $\mathfrak{T}^{\frac{1}{2}}$ for ultimate states. By (34.3)₂, the ratio of total normal pressure to shearing pressure falls off monotonically from ∞ to 0 as $\mathfrak{T}$ increases.

Turning now to (34.4)₁ we discern a **universal relation**¹ valid not only at all times but also at all values of $\mathfrak{T}$: *Any two planes parallel to the flow suffer equal dominant normal pressures.* This central conclusion is particularly apt for trial by experiment, because it differs from its counterpart in the simpler continuum theories of fluids. For in shearing flow of a Reiner-Rivlin or Stokesian fluid we have $p_{xx} = p_{vv} \neq p_{zz}$ (cf. [1952, 2, §72A]). We have seen already that a Maxwellian gas in shearing flow behaves like a special case of Noll's linear fluent body. In such a body we have in general $p_{xx} \neq p_{vv} \neq p_{zz} \neq p_{xx}$, both the possibilities $p_{xx} = p_{vv}$ and $p_{vv} = p_{zz}$ being included as distinct special cases corresponding to special values of the material coefficients. It is of course possible that $p_{vv}^D = p_{zz}^D$ holds only for a Maxwellian gas, not for other molecular models in the kinetic theory.

The relation (34.4)₂, or $P_{xx}^D = -2P_{vv}^D$, is an immediate consequence of $p_{vv}^D = p_{zz}^D$ and $P_{ii}^D = 0$, but it is nevertheless of interest in showing that *the normal pressure on a plane normal to the flow always exceeds the thermostatic pressure*, exceeding it in fact by twice the amount by which the pressures on planes parallel to the flow fall short of it. Moreover,

$$\pi_{xx}^D = 3 - \frac{2}{r+1}. \quad (34.8)$$

From all our results concerning the normal pressures it follows that *for ordinary states the normal pressures are virtually hydrostatic; as $\mathfrak{T}$ increases, the relative magnitudes of the pressures change steadily, until for ultimate states virtually the entire thermostatic pressure consists in the pressure on the planes normal to the flow* (i.e., $p^D \sim \frac{1}{3}p_{xx}^D$ as $\mathfrak{T} \rightarrow \infty$).

¹ From (33.3)₃, (33.3)₄, and $P_{zz} = -(P_{xx} + P_{vv})$ follows

$$\begin{aligned} P_{vv}(t) - P_{zz}(t) &= [P_{vv}(0) - P_{zz}(0)]e^{-t/\tau}, \\ P_{xx}(t) + 2P_{vv}(t) &= [P_{xx}(0) + 2P_{vv}(0)]e^{-t/\tau}. \end{aligned}$$

The statements in the text, which refer to the dominant parts, thus hold also for the entire solution if and only if they hold at one instant.

The results obtained so far in this section are illustrated in Figure 34.1.

A deep difference between the exact theorems (34.1)–(34.5) and the type of formulæ which can result from the simpler continuum theories can be seen from the quadric of pressures. In the Reiner-Rivlin and Stokesian theories the pressure tensor is assumed to be an isotropic function of $u_{(i,i)}$. Consequently (*cf.* [1952, 2, §6]), p_{ii} and E_{ii} have the same principal axes. Now for shearing flow the quadric of E_{ii} is simply

$$Kxy = \text{const.}, \quad (34.9)$$

a pair of conjugate rectangular hyperbolic cylinders whose principal axes, whatever is the value of K , are the z -axis and the lines bisecting the angles between the x and y -axes. For our solution (34.1)–(34.4) the quadric of relative pressures is

$$\pi_{ii}^D x_i x_i = \frac{x^2}{\frac{r+1}{3r+1}} + \frac{y^2 + z^2}{r+1} - \frac{xy}{\frac{\mathfrak{T}}{3r}} = \text{const.} \quad (34.10)$$

FIGURE 34.1. The exact ratios π_{ii}^D , defined by $\pi_{ii}^D = p_{ii}^D / p^D$, for the dominant parts of the pressure tensor. (Note the change of scale between $\mathfrak{T} = 3$ and $\mathfrak{T} = 5$.)

That in the kinetic theory the quadric of pressures is always an ellipsoid follows from the definition of the pressure tensor. Our interest lies in the principal directions. The z -axis is a principal axis, but the bisectors of the angles between the x and y -axes are not. In fact, it is easy to see that the angle ψ between one of the principal axes of extension and one of the principal axes of dominant stress is given by

$$\begin{aligned}\psi &= \frac{1}{2} \operatorname{Arctan} \frac{2\mathfrak{T}}{\mathfrak{r} + 1}, \\ &= \mathfrak{T} - 2\mathfrak{T}^3 + O(\mathfrak{T}^5) \quad \text{as } \mathfrak{T} \rightarrow 0, \\ &= \frac{\pi}{4} \left[1 - \left(\frac{2}{3\mathfrak{T}} \right)^{\frac{1}{2}} \right] + O\left(\frac{1}{\mathfrak{T}}\right) \quad \text{as } \mathfrak{T} \rightarrow \infty.\end{aligned}\tag{34.11}$$

Thus as $\mathfrak{T}$ increases, the principal axes of dominant stress in the plane of shearing are steadily rotated away from the principal axes of extension, until finally in ultimate states the two pairs of axes bisect each other. It is interesting to note from (34.11)₁ and (34.3)₁ that

$$\psi = \frac{1}{2} \operatorname{Arctan} \left(3 \frac{P_{yy}^D}{P_{xy}^D} \right),\tag{34.12}$$

so that the growth of the deviatoric normal pressure P_{yy}^D is a measure of the rotation of the axes of stress. This rotation furnishes as clear an example as one could wish of the difference between a simple theory of fluidity and a theory general enough to include relaxation.

The intersection of the ellipsoid (34.10) with the $x = 0$ plane is a circle whose radius is proportional to $\sqrt{\mathfrak{r} + 1}$ (this is but a geometrical form of the condition $p_{xx}^D = p_{yy}^D$). For small $\mathfrak{T}$, the ellipsoid of pressures is nearly spherical; as $\mathfrak{T}$ increases, one of the axes of the ellipsoid in the plane of flow rotates toward the direction of flow and at the same time grows steadily shorter; both the others grow steadily longer; until as $\mathfrak{T} \rightarrow \infty$ the ellipsoid approaches an infinite circular disc in the plane perpendicular to the direction of flow (see Figure 34.2).

35. Comparison of the first, second, and third iterates according to various processes with the exact pressure ratios. In this section we compare the ratios π_{xy}^D and π_{yy}^D with those following from various schemes within the classical approach. The Navier-Stokes formula, the first Maxwellian iterate, the first step in the Chapman-Enskog process, and most of the alternatives discussed in Chapter II yield

$$-p_{xy}^{(1)} = \mu K = p\mathfrak{T}, \quad \text{or} \quad -\pi_{xy}^{(1)} = \mathfrak{T}.\tag{35.1}$$

Thus by (34.1) follows

$$\frac{\pi_{xy}^D}{\pi_{xy}^{(1)}} = \frac{\mathfrak{r}}{\mathfrak{A}},\tag{35.2}$$

FIGURE 34.2. The ellipsoid $\pi_{ij}^D x_i x_j = 1$, for three values of Ξ . The dashed line is a principal axis. The direction of gas flow is parallel to the x -axis.

a function of $\mathfrak{A}$ which is 1 for $\mathfrak{A} = 0$ and is monotone decreasing: *The Navier-Stokes shearing stress $p_{xy}^{(1)}$ is always too large a function of the gross condition, but for ordinary states it is asymptotically correct.*¹

At the second step the various processes within the classical approach begin to differ from one another. From Maxwellian iteration we get, from (2.18),

$$-\pi_{xy}^{(2)} = \mathfrak{T}(1 - \tau p'/p); \quad (35.3)$$

from atemporal Maxwellian iteration (§15) we get

$$-\pi_{xy}^{(2)} = \mathfrak{T}(1 - \mathfrak{A}); \quad (35.4)$$

from Burnett's equations² (§15) or from iteration by powers³ (equations (19.24), (19.20), and (19.21)) we get

$$\pi_{xy}^{[2]} = \pi_{xy}^{[1]} = \pi_{xy}^{(1)} = -\mathfrak{T}. \quad (35.5)$$

Discussion of (35.3) we postpone until §43. By (34.6), the term $-\mathfrak{A}$ in (35.4) is not the correct second term in a power series expansion of the exact solution.

For the second iterates to the normal pressures all the above-mentioned methods agree in yielding

$$\pi_{yy}^{(2)} = \pi_{yy}^{[2]} = \pi_{yy}^{[1]} = 1 - \mathfrak{A}. \quad (35.6)$$

Thus

$$\frac{\pi_{yy}^D}{\pi_{yy}^{(2)}} = \frac{1}{(1 - \mathfrak{A})(1 + \mathfrak{r})} = 1 + O(\mathfrak{T}^4). \quad (35.7)$$

Hence these methods within the classical approach yield at the second step a total normal pressure on the shear planes which is always too small a fraction of the total pressure, but it is correct for ordinary states. The first iterate yields $p_{yy}^{(1)} = p$ and is correct only to order 1 in $\mathfrak{T}$. However, while the second iterate is correct to order 3, in a sense it is worse than the first, since for $\mathfrak{A} > 1$ it yields negative total normal pressure, a result in contradiction with the very definition of the pressure tensor in the kinetic theory. Such anomalies are characteristic of effects of second order and occur in this instance because the expansion for p_{yy}^D in

¹ Recall that the term "ordinary states" refers to the *limit process* $\mathfrak{T} \rightarrow 0$, not to the value at the limit $\mathfrak{T} = 0$. Our theorem above calculates an indeterminate form 0/0. While the dominant shearing pressure p_{xy}^D itself tends to zero as $\mathfrak{T} \rightarrow 0$ at any fixed time, nevertheless if $\mathfrak{T}$ is a positive quantity, however small, eventually $p_{xy}^D \gg p_{yy}^0$. This must be borne in mind: The smaller is $\mathfrak{T}$, the longer, in general, must we expect to wait until p_{xy}^D becomes numerically the important part of the solution, and hence until ratios such as (35.2) become significant measures of the phenomenon. At the *limit* $\mathfrak{T} = 0$, of course, we have $p_{xy}^D = 0$, $p_{xy} = p_{xy}(0)e^{-t/\tau}$, and hence if $p_{xy}(0) \neq 0$ it follows that $(p_{xy}/p_0)/\pi_{xy}^{(1)} = \infty$.

² The term in Burnett's equations proportional to $u_{k,(j)u_i),k}$, which I criticized some years ago (§16), in the present special case happens to be zero.

³ For uniformity of notation we set

$$\pi_{ij}^{[n]} \equiv \delta_{ij} + \sum_{k=1}^n P_{ij}^{[k]}/p.$$

powers of $\mathfrak{A}$ diverges at $\mathfrak{A} = \frac{4}{27}$, so that in fact the range of even qualitative correctness of the second order approximation is very small.

At the third step, the process which we have indicated in §15 to be equivalent to the Chapman-Enskog process agrees with the method of iteration by powers in yielding

$$-\pi_{xy}^{[3]} = \mathfrak{T}(1 - 2\mathfrak{A}). \quad (35.8)$$

Thus

$$\frac{\pi_{xy}^D}{\pi_{xy}^{[3]}} = \frac{\mathfrak{r}}{\mathfrak{A}(1 - 2\mathfrak{A})} = 1 + O(\mathfrak{T}^4). \quad (35.9)$$

If we compare (35.4) with (35.8), we see that atemporal Maxwellian iteration gives at the second stage half of the correction to the shearing pressure which is necessary to get a correct polynomial approximation of degree 3 in $\mathfrak{T}$, while the Chapman-Enskog process yields all of this correction at the third stage. These results are presented graphically in Figure 35.1, which makes entirely plain the inadequacy of all these iterative methods when $\mathfrak{T}$ is large.

FIGURE 35.1. Comparison of $-\pi_{xy}^D$, $-\pi_{xy}^{(1)}$, $-\pi_{xy}^{(2)}$, and $-\pi_{xy}^{[3]}$, for small values of $\mathfrak{T}$.

By the theorem on polynomial approximations, there exists a range of small values of $\mathfrak{T}$ for which the third Chapman-Enskog iterate is more accurate than the second Maxwellian iterate. Despite this fact, a glance at Figure 35.1 shows that the third Chapman-Enskog iterate is little better than the Navier-Stokes approximation, and that the second Maxwellian iterate is fairly close to the

exact solution for at least twice the range of corresponding accuracy of the third Chapman-Enskog iterate.

Approximations of a very different type emerge from the scheme of §20, for by (20.2) we get

$$\begin{aligned} -\pi_{xy}^{((1))} &= \frac{\mathfrak{T}}{1 + \mathfrak{A}}, & \pi_{yy}^{((1))} &= \frac{1}{1 + \mathfrak{A}}, \\ -\pi_{xy}^{((2))} &= \frac{\mathfrak{T}}{1 + \mathfrak{A}} \left[1 + \frac{\mathfrak{A}(\mathfrak{A} - 1)}{(1 + \mathfrak{A})^2} \right], & \pi_{yy}^{((2))} &= \frac{1}{1 + \mathfrak{A}} \left[1 + \frac{\frac{5}{2}\mathfrak{A}^2}{(1 + \mathfrak{A})^2} \right], \end{aligned} \quad (35.10)$$

where so as to get results which can be compared with π_{xy}^D and π_{yy}^D we have used the method of §15 to eliminate time derivatives in calculating the second iterates.

FIGURE 35.2. Comparison of the first and second iterates by the scheme of §20 with the exact values, over a very large range of values of $\mathfrak{T}$.

Inspection of Figure 35.2 show that the *first* iterates by this process, which altogether neglects “order”, furnish a far better overall approximation to the exact pressure ratios than could the results at *any* stage of the Chapman-Enskog

process or any other process yielding polynomial approximations. While (35.10) does not give the correct order at $\mathfrak{T} = \infty$, the general form of the curves is right. Note that the range of values of $\mathfrak{T}$ shown in Figure 35.2 is 100 times that shown in Figure 35.1. At $\mathfrak{T} = 1$, when the *third* Chapman-Enskog iterate yields a shearing pressure of the wrong sign, the error of the *first* iterate by the method of §20 is less than 10%. This should not be surprising: Despite its popularity, approximation by polynomials is not always the best possible.⁴

36. Proof that atemporal Maxwellian iteration converges to the correct pressure ratios. The proof of convergence which follows now probably could be adapted to any of the schemes mentioned in §35 and to all those discussed in Chapter II except that of §22. We select atemporal Maxwellian iteration partly because for it the proof is easiest and partly because, as remarked in §35, that scheme fails to satisfy the psychological criterion of “order” which occurs in some works in the kinetic theory.

The argument is broken into two parts. First, we overturn a common prejudice. Since the classical approach obviously cannot yield solutions satisfying the full set of initial conditions which seem appropriate to the Maxwell-Boltzmann equation, it has often been argued that the resulting series in powers of μ are not convergent, but only asymptotic. In the case of shearing flow, at least, this cannot be. For by (34.1)–(34.5) and the analysis preceding these formulæ (especially the paragraph after (33.11)), the ratios π_{ij}^D possess expansions in powers of $\mathfrak{T}$ which are *convergent* for $\mathfrak{T} < \frac{1}{3}\sqrt{2}$. Thus we have the **alternative**: *The polynomials in $\mathfrak{T}$ which result from most schemes within the classical approach are either formally wrong or else convergent to the dominant parts of the exact solution.* That we are able to assert this rather astonishing alternative follows from the existence of the universal formulæ of §34.

Now we turn to the scheme of Maxwellian iteration. For simple shearing flow, (2.10) reduces to

$$\begin{aligned} -P_{xy}^{(n+1)} &= p\mathfrak{T} + \mathfrak{T}P_{yy}^{(n)} + \tau P_{xy}^{(n)'} , \\ P_{yy}^{(n+1)} &= \frac{2}{3}\mathfrak{T}P_{xy}^{(n)} - \tau P_{yy}^{(n)'} . \end{aligned} \quad (36.1)$$

For atemporal Maxwellian iteration we write $\{n\}$ for (n) , and whenever p' occurs we replace it by $p'^{\{n\}}$, where

$$\tau p'^{\{n\}} + \frac{2}{3}\mathfrak{T}P_{xy}^{\{n\}} = 0. \quad (36.2)$$

The solution of the system (36.1), (36.2) is of the type

$$p_{ij}^{\{n\}} = p\pi_{ij}^{\{n\}}. \quad (36.3)$$

⁴ The most familiar example of the superiority of expansions in rational fractions as compared with power series is furnished by EULER's formulæ for sec, csc, etc. Apparently this superiority has recently been noticed by physicists, since MORSE & FESHBACH [1953, 4, §9.1] assert that in a certain problem in quantum mechanics an expansion in rational fractions has a larger “radius” of convergence than does the conventional perturbation series.

Substitution of (36.2) and (36.3) into (36.1) yields

$$\begin{aligned} -\pi_{xy}^{(n+1)} &= \mathfrak{T}\pi_{yy}^{(n)} - \frac{2}{3}\mathfrak{T}(\pi_{xy}^{(n)})^2, \\ \pi_{yy}^{(n+1)} &= 1 + \frac{2}{3}\mathfrak{T}\pi_{xy}^{(n)}\pi_{yy}^{(n)}. \end{aligned} \quad (36.4)$$

According to the initiation agreement of §2, $\pi_{ii}^{(0)} = \delta_{ii}$; hence, obviously, $\pi_{ii}^{(n)}$ is a uniquely determined polynomial in $\mathfrak{T}$. Now let $n \rightarrow \infty$ formally in (36.4):

$$\begin{aligned} -\pi_{xy}^{(\infty)} &= \mathfrak{T}\pi_{yy}^{(\infty)} - \frac{2}{3}\mathfrak{T}(\pi_{xy}^{(\infty)})^2, \\ \pi_{yy}^{(\infty)} &= 1 + \frac{2}{3}\mathfrak{T}\pi_{xy}^{(\infty)}\pi_{yy}^{(\infty)}. \end{aligned} \quad (36.5)$$

If there is any solution of (36.5) which is a convergent power series in $\mathfrak{T}$, $\pi_{ii}^{(n)}$ must converge to it for small enough $\mathfrak{T}$. By substitution we may verify that such a solution is

$$\pi_{xy}^{(\infty)} = -\mathfrak{T} \frac{r}{\mathfrak{A}}, \quad \pi_{yy}^{(\infty)} = \frac{1}{r+1}. \quad (36.6)$$

This solution agrees exactly with (34.1) and (34.2) and has a series expansion convergent when $\mathfrak{T} < \frac{1}{3}\sqrt{2}$. Collecting the foregoing argument, we have the

Convergence theorem. *For shearing flow of a Maxwellian gas, atemporal Maxwellian iteration converges when $\mathfrak{T} < \frac{1}{3}\sqrt{2}$ and diverges when $\mathfrak{T} > \frac{1}{3}\sqrt{2}$. The results are correct in the following sense:*

$$\pi_{ii}^{(n)} \rightarrow \pi_{ii}^D. \quad (36.7)$$

That is, the limit functions are the same functions of the gross condition as are the dominant parts of any exact solution.

The results in this section bring the classical approach to its high water mark.¹ With the next section, the ebb tide sets in quickly.

37. Proof that the classical approach yields false absolute pressures. Having analyzed those properties of the solution (33.12) which are independent of the initial conditions, we turn now to the constant A , which determines the numerical values of the dominant pressures p^D and p_{ij}^D as functions of time. It is obvious

¹ We must comment on the scheme of §20. When applied to the problem of shearing flow, its iterative equations may be obtained by replacing $P_{yy}^{(n)}$ by $P_{yy}^{(n+1)}$ in (34.1)₁, $P_{xy}^{(n)}$ by $P_{xy}^{(n+1)}$ in (34.1)₂; when time derivatives are eliminated by the method of §15, the iterative scheme is obtained by replacing $\pi_{yy}^{(n)}$ by $\pi_{yy}^{(n+1)}$ in (36.4)₂. Hence this latter scheme converges when $\mathfrak{T} < \frac{1}{3}\sqrt{2}$. However, there is no indication that it diverges for greater values of $\mathfrak{T}$. On the contrary, in §35 we saw reason to expect that it gives a uniform approximation in every finite interval of $\mathfrak{T}$. This I have not been able to prove. If true, it would show greater possibility in the classical approach than can be realized by such special cases as the Chapman-Enskog method or atemporal Maxwellian iteration but would not prevent the negative conclusion of §37.

that A depends on the initial conditions. In fact, by eliminating B and C from (33.13) we get

$$A = \frac{(1 + \mathfrak{r})(1 + \mathfrak{r}\Pi_{\nu\nu}) - \frac{2}{3}\mathfrak{T}\Pi_{xx}}{1 + 3\mathfrak{r}}, \quad (37.1)$$

There is no escaping this formula. It yields the

Second main theorem on shearing flow. *In shearing flow of a Maxwellian gas, the entire system of dominant pressures depends for all time on the values of the pressures to which the fluid is subject initially.* From the differential equations, this is only to be expected. In fact, what is surprising is that so much faith has been put in the contrary expectation up to now.

Recall the claim (§23) that there are certain “normal” solutions, depending only on the gross condition, an appropriate one of which is approached in time by an arbitrary solution. The present example, being a very special one, however conformable to this claim could hardly serve as a substantiation of it. However, it can and does serve as a counter-example to show that the claim is false. There is indeed a part of the solution which is rapidly damped out, but the remaining, dominant part p_{ij}^D depends on the constant A , a function not only of $\mathfrak{T}$ but also of the initial conditions. The second main theorem may be rephrased as follows: *For shearing flow of a Maxwellian gas, no “normal” solution exists.*

Let us now see whether the classical approach, which endeavors to express p_{ij} in terms of the gross condition by universal formulæ, deserves any status as a method of approximation. Putting (33.8)₁ into (37.1) yields

$$A = 1 - \frac{2}{3}\mathfrak{T}\Pi_{xx} + \frac{2}{3}\mathfrak{T}^2(\Pi_{\nu\nu} - 2) + O(\mathfrak{T}^3) \quad (37.2)$$

as $\mathfrak{T} \rightarrow 0$. Hence follows the

Third main theorem on shearing flow. *For normal states the dominant pressures are independent of the initial conditions, to the first order in $\mathfrak{T}$. To any higher order, they are not.*

In the limit process $\mathfrak{T} \rightarrow 0$, the initial conditions Π_{xx} and $\Pi_{\nu\nu}$ are held fixed. The third main theorem asserts that if we compare the flows corresponding to the *same* initial pressures but different values of $\mathfrak{T}$, as $\mathfrak{T} \rightarrow 0$ these solutions have an asymptotic form independent of the initial pressures.

From this result we see that the leading terms in the series expansions of the dominant parts of the solutions (33.12) are independent of the initial conditions. Thus ϖ^D is independent of initial conditions to the zeroth order in $\mathfrak{T}$; ϖ_{xx}^D , to the first order; and ϖ_{xx}^D , $\varpi_{\nu\nu}^D$, and ϖ_{zz}^D to the second order. *None are independent of the initial conditions to any higher order.* While our result concerns but a special flow of a special gas, it serves as a counter-example to the usual methods of “integration” in the kinetic theory, since all these express p_{ij} “approximately” in terms of the gross condition, without respect to initial values. Most of these methods eliminate time derivatives in terms of space derivatives (*cf.* §15). If,

for the moment, we limit the term “classical approach” to this major subclass, we obtain our

Principal theorem. *In general, the formulæ which the classical approach yields for the pressures as functions of space and time can be correct only asymptotically as $\mu \rightarrow 0+$, and at that only to the first non-vanishing term. The “higher approximations” obtained by these methods in general fail to yield correct additional terms in the asymptotic expansions.*

In our treatment of the model for the equations of moments (§25), we found that a broad class of solutions possessed a common asymptotic expansion, independent of the initial conditions, as $\mu \rightarrow 0+$. We then were able to find a fairly general class of zeroth iterates such that the process of differential iteration converges to this common asymptotic solutions (§26). Our result above shows that in the kinetic theory there is no such possibility: There can be at most (as in fact there is in our example of shearing flow) a common *leading asymptotic term*, but no more than that. Thus methods which take no account of initial conditions can be correct as far as the first term *at most*. Whether in fact the classical approach yields results correct as far as this term is the matter to which we turn our next efforts.

38. The general nature of the solution for the pressures which results from most schemes within the classical approach. Before going into details, we pause to succor such readers as may be confused by the negative results of §37 after the positive ones of §36. We have shown that the classical approach (in one specific case) yields approximations whose limits $\pi_{ij}^{(\infty)} \equiv p_{ij}^{(\infty)}/p$ are precisely equal to the corresponding ratios of *dominant parts* of the exact solution: $\pi_{ij}^{(\infty)} = \pi_{ij}^D \equiv p_{ij}^D/p^D$. In other words, the classical approach yields *pressure ratios* π_{ij} which as functions of gross condition are, after a certain number of times of relaxation, arbitrarily good approximations to the exact values. Nevertheless, the absolute pressures $\varpi_{ij} \equiv p_{ij}/p_0$, *functions of time*, as obtained by the classical approach are necessarily false. How can this be?

Now most of the schemes¹ within the classical approach when applied to shearing flow yield limits such as $\pi_{ij}^{(\infty)}$ which are functions of $\mathfrak{T}$ only; in particular,

$$p_{xv}^{(\infty)}/p = \pi_{xv}^{(\infty)} = -f(\mathfrak{T}), \quad \text{say.} \quad (38.1)$$

Suppose we now use this formula in the spirit of the classical approach and calculate the corresponding *solution* ${}^{(\infty)}\varpi_{xv} = {}^{(\infty)}p_{xv}/p_0$ as a function of *time*. The differential equations to be satisfied are those of mass, momentum, and energy. The first two of these reduce to $0 = 0$, while the last, (2.8), becomes

$$3\tau p' = 2\mathfrak{T}f(\mathfrak{T})p, \quad (38.2)$$

¹ Specifically, all those which eliminate time derivatives (cf. §15). As shown by (35.3), Maxwellian iteration itself furnishes an exception to (38.1). It will be discussed in §43.

whence follows, since τ and $\mathfrak{T}$ are constants,

$${}^{(\infty)}\varpi = \exp [\tfrac{2}{3}\mathfrak{T}f(\mathfrak{T})t/\tau], \quad (38.3)$$

where we have written ${}^{(\infty)}\varpi$ for the solution p/p_0 of (38.2) so as to indicate that this solution corresponds to the limit of the classical approach. From (38.1) now follows $p_{xy} = -pf(\mathfrak{T})$, where p is given by (38.3). Denoting the resulting ratio p_{xy}/p_0 by ${}^{(\infty)}\varpi_{xy}$, we have

$${}^{(\infty)}\varpi_{xy} = -f(\mathfrak{T}) \exp [\tfrac{2}{3}\mathfrak{T}f(\mathfrak{T})t/\tau] \quad (38.4)$$

as the solution corresponding to (38.1). Similar formulæ hold for the other components ${}^{(\infty)}\varpi_{ij}$.

The result (38.3) from the classical approach is very different from the true solution (33.12), : The two agree if and only if

$$A = 1, \quad r = \tfrac{2}{3}\mathfrak{T}f. \quad (38.5)$$

By (37.1), the former is equivalent to

$$1 = \frac{(1+r)(1+r\Pi_{yy}) - \tfrac{2}{3}\mathfrak{T}\Pi_{xy}}{1+3r}. \quad (38.6)$$

For arbitrary initial conditions, the requirement (38.6) obviously is not satisfied. *This is true whatever the form of f :* That is, whether or not a particular scheme within the classical approach yields correct pressure ratios $\pi_{ij} \equiv p_{ij}/p$, in general it *cannot* yield correct absolute pressures $\varpi_{ij} \equiv p_{ij}/p_0$. However, for fixed Π_{yy} and Π_{xy} the condition (38.6) is satisfied asymptotically as $\mathfrak{T} \rightarrow 0$, as is easy to see also from (37.2).

This should not be surprising. It is what is called a “boundary layer” or “edge” effect in other parts of mechanics.² A theory essentially hydrodynamical (for such, at last, we see is the outcome of the classical approach) leads to a differential system of *lower order* than that for a theory of relaxation. In this case, the hydrodynamical approach leads to the single equation (38.2) instead of the (essentially) third order relaxation system (33.3). This depression of order renders it impossible to satisfy all initial conditions of the problem. Nevertheless the result is an asymptotic solution for small $\mathfrak{T}$.

From yet another viewpoint we repeat the explanation. That the classical approach yields $\pi_{ij}^{(\infty)} = \pi_{ij}^D$ is equivalent to $\pi_{ij}^{(\infty)} \approx \pi_{ij}$ with error which in every case approaches zero very rapidly in time. Now the equation of energy is an exact equation. If we had *precisely* $\pi_{ij}^{(\infty)} = \pi_{ij}$, by solving the exact equation of energy we should get necessarily ${}^{(\infty)}\varpi_{ij} = \varpi_{ij}$ also. It is *neglect of the small damped part* π_{xy}^0 when we substitute π_{xy} in the energy equation which renders

² Persons to whom examples from the mechanics of mass points are more immediate than those from continuum mechanics may prefer the elementary remarks given in [1949, Ch. I, §5, ¶1-2].

the solution ${}^{(1)}\varpi_{ij}$ generally incorrect. This may be added to several examples from recent studies in non-linear continuum mechanics showing that in problems of physical interest it is quite possible for a very small term in a differential equation to have a very great effect on the solution.

Having seen that the classical approach can yield absolute pressures correct at most to the first order, we turn now to a search for what asymptotic agreement in fact there is.

39. Definition of certain limit processes. We have referred to *normal states* in connection with $\mathfrak{T} \rightarrow 0$, *ultimate states* for $\mathfrak{T} \rightarrow \infty$. These two limit processes suffice for discussion of quantities which depend on $\mathfrak{T}$ only, but for a full grasp of the exact solution (33.12) we need to consider τ as well as $\mathfrak{T}$. Recall that

$$\mathfrak{T} = \tau K, \quad \tau \equiv \frac{\mu}{p} = \frac{1}{3} \sqrt{\frac{2}{mG}} \cdot \frac{1}{n\mathfrak{U}_2}. \quad (39.1)$$

The limit $\mu \rightarrow 0+$, other conditions being fixed, yields $\tau \rightarrow 0$ as well as $\mathfrak{T} \rightarrow 0$. There are two simple ways of visualizing this limit: mG , the repulsion exerted by one molecule upon another of unit mass at unit distance, becomes infinite, or $n \rightarrow \infty$, indicating dense packing. The limit of *dense states* shall be understood to refer to either of these circumstances. Similarly, *rare states*¹ shall refer to the limit $\tau \rightarrow \infty$. Finally, relations valid in the limits $K \rightarrow 0$ and $K \rightarrow \infty$ for fixed τ shall be said to be associated with *slow shearing* and *fast shearing*, respectively.

The relations of these terms to those used before are as follows:

Ordinary States ($\mathfrak{T} \rightarrow 0$)	$\left\{ \begin{array}{l} \text{Dense States } (K \text{ fixed, } \tau \rightarrow 0) \\ \text{Slow Shearing } (\tau \text{ fixed, } K \rightarrow 0) \end{array} \right.$
Ultimate States ($\mathfrak{T} \rightarrow \infty$)	$\left\{ \begin{array}{l} \text{Rare States } (K \text{ fixed, } \tau \rightarrow \infty) \\ \text{Fast Shearing } (\tau \text{ fixed, } K \rightarrow \infty) \end{array} \right.$

40. Status of the Navier-Stokes solution. For comparison of the solution according to the Navier-Stokes equations with the exact solution for a Maxwellian gas, we assume $\mu \propto \theta$; in the problem of shearing at constant density, this is equivalent to $\mathfrak{T} = \text{const.}$, so that we may use (38.3) and (38.4) in combination with (35.1). The resulting *Navier-Stokes solution* is:

$$\begin{aligned} {}^{(1)}\varpi &= e^{\mathfrak{U}/\tau}, \\ {}^{(1)}\varpi_{xy} &= -\mathfrak{T}e^{\mathfrak{U}/\tau}, \quad \text{all other } {}^{(1)}\varpi_{ij} = 0. \end{aligned} \quad (40.1)$$

These results we now compare² with the exact dominant parts, given by (33.12).

¹ The terms *dense* and *rare* as used in this memoir refer only to these limit processes within the Maxwellian theory. From the physical standpoint the Maxwellian theory itself seems to be inadequate both for very dense gases and for very rare gases.

² Henceforth we assume $A > 0$. This is not a limitation, since as shown in §50 only initial conditions such that $A > 0$ are possible.

We get

$$\begin{aligned}\frac{{}^{(1)}p}{p^D} &= \frac{1}{A} e^{(\mathfrak{A}-r)t/\tau}, \\ \frac{{}^{(1)}p_{xy}}{p_{xy}^D} &= \frac{\mathfrak{A}}{Ar} e^{(\mathfrak{A}-r)t/\tau}.\end{aligned}\tag{40.2}$$

Since $r < \mathfrak{A}$, these formulæ yield

Theorem 1. *In every case the Navier-Stokes solution becomes infinitely larger than the dominant parts of the exact solution, exponentially with time.*

Corollary. *If at any one time the Navier-Stokes solution gives the correct pressures, from that time onward it becomes steadily a worse and worse approximation.*

Next we develop (40.2) in a series of powers of $\mathfrak{T}$ when t/τ , Π_{xy} , and Π_{yy} are kept constant, by (37.2) and (33.8)₁ obtaining

$$\begin{aligned}\frac{{}^{(1)}p}{p^D} &= \frac{1}{A} [1 + \frac{8}{9}\mathfrak{T}^4 t/\tau + O(\mathfrak{T}^6)], \\ &= \frac{1 + O(\mathfrak{T}^4)}{A}, \\ &= 1 + \frac{2}{3}\mathfrak{T}\Pi_{xy} + O(\mathfrak{T}^2),\end{aligned}\tag{40.3}$$

$$\begin{aligned}\frac{{}^{(1)}p_{xy}}{p_{xy}^D} &= \frac{\mathfrak{A}}{Ar} [1 + \frac{8}{9}\mathfrak{T}^4 t/\tau + O(\mathfrak{T}^6)], \\ &= \frac{\mathfrak{A}}{Ar} [1 + O(\mathfrak{T}^4)], \\ &= 1 + \frac{2}{3}\mathfrak{T}\Pi_{xy} + O(\mathfrak{T}^2).\end{aligned}\tag{40.4}$$

If we hold Kt rather than t/τ fixed, the order of the first time-dependent term in (40.3)₁ and (40.4)₁ is lowered from 4 to 3. Inspection of the foregoing formulæ yields

Theorem 2. *At a fixed time, consider the solutions corresponding to a given set of initial conditions Π_{xy} , Π_{yy} but different values of $\mathfrak{T}$. For $\left\{ \begin{smallmatrix} \text{dense states} \\ \text{slow shearing} \end{smallmatrix} \right\}$, the Navier-Stokes solution stands in asymptotically steady relation to the dominant part of the exact solution, to order $\left\{ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right\}$ in $\mathfrak{T}$. The ratios ${}^{(1)}p/p^D$ and ${}^{(1)}p_{xy}/p_{xy}^D$ are asymptotically equal to each other to order 3. To order 0, the Navier-Stokes solution is asymptotic to the dominant part of the exact solution.*

41. Status of the Burnett solution. If we use (35.5) we get $\pi_{xy}^{[2]} = \pi_{xy}^{[1]}$, and hence the corresponding solution for ϖ and ϖ_{xy} is given by (40.1): ${}^{[2]}\varpi = {}^{(1)}\varpi$,

$^{[2]}\varpi_{xy} = ^{(1)}\varpi_{xy}$. By (35.6) follows a non-constant value for $^{[2]}p_{yy}$, which we shall call the *Burnett solution*:

$$\frac{^{[2]}p_{yy}}{p_{yy}^D} = \frac{(r+1)(1-\mathfrak{A})}{A} e^{(\mathfrak{A}-r)t/\tau}. \quad (41.1)$$

From the grounds mentioned in §35, there is no point in considering this ratio except when $\mathfrak{A} < 1$. Reasoning and developments as in §40 then yield

Theorem 3. *In every case the absolute pressure ratios according to the Burnett solution become infinitely larger than the corresponding dominant part of any exact solution, exponentially with time.*

(This is particularly interesting in view of the statement following (35.7): *The relative pressure ratio $\pi_{yy}^{[2]}$ is always too small, and the more so the larger is $\mathfrak{T}$, while the absolute pressure ratio $^{[2]}\varpi_{yy}$ is ultimately far too large, the more so the larger is t/τ . There is thus the possibility of cancellation of the two errors. For initial conditions such that $A > (r+1)(1-\mathfrak{A})$, then for times near to*

$$\frac{t}{\tau} = \frac{1}{\mathfrak{A}-r} \log \frac{A}{(r+1)(1-\mathfrak{A})} \quad (41.2)$$

the Burnett solution for the normal pressures is nearly correct.)

From (41.1) follows

$$\begin{aligned} \frac{^{[2]}p_{yy}}{p_{yy}^D} &= \frac{(r+1)(1-\mathfrak{A})}{A} [1 + \frac{8}{9}\mathfrak{T}^4 t/\tau + O(\mathfrak{T}^6)], \\ &= \frac{1-\mathfrak{A}^2}{A} [1 + O(\mathfrak{T}^4)], \\ &= 1 + \frac{2}{3}\mathfrak{T}\Pi_{xy} + O(\mathfrak{T}^2). \end{aligned} \quad (41.3)$$

Hence we read off

Theorem 4. *The pressures given by the Burnett solution are asymptotically correct for ordinary states. The conclusions of Theorem 2 of §40 can be extended to all the ratios $^{[2]}p_{ii}/p_{ii}^D$.*

Our Principal Theorem in §37 asserts that the *most* that can be expected from the classical approach is the correct leading terms in asymptotic expansions as $\mathfrak{T} \rightarrow 0$ with initial conditions fixed. In the present section we have shown that in the case of shearing flow the Burnett solution is in fact correct to this lowest order, provided we hold t/τ or Kt fixed also.

42. Status of the “higher approximations” according to the classical approach. Consider the solution $^{[3]}\varpi$ and $^{[3]}\varpi_{ij}$ which corresponds to (35.8). By (38.3) and (38.4) we get the *tertiary solution*:

$$\begin{aligned}
{}^{[3]}\varpi &= \exp [\mathfrak{A}(1 - 2\mathfrak{A})t/\tau], \\
{}^{[3]}\varpi_{xy} &= -\mathfrak{T}(1 - 2\mathfrak{A}) \exp [\mathfrak{A}(1 - 2\mathfrak{A})t/\tau], \\
{}^{[3]}\varpi_{yy} &= (1 - \mathfrak{A}) \exp [\mathfrak{A}(1 - 2\mathfrak{A})t/\tau].
\end{aligned} \tag{42.1}$$

Hence

$$\begin{aligned}
\frac{{}^{[3]}p}{{p}^D} &= \frac{1}{A} \exp [(\mathfrak{A} - 2\mathfrak{A}^2 - \mathfrak{r})t/\tau], \\
\frac{{}^{[3]}p_{xy}}{{p}_{xy}^D} &= \frac{\mathfrak{A}(1 - 2\mathfrak{A})}{A\mathfrak{r}} \exp [(\mathfrak{A} - 2\mathfrak{A}^2 - \mathfrak{r})t/\tau], \\
\frac{{}^{[3]}p_{yy}}{{p}_{yy}^D} &= \frac{(\mathfrak{r} + 1)(1 - \mathfrak{A})}{A} \exp [(\mathfrak{A} - 2\mathfrak{A}^2 - \mathfrak{r})t/\tau].
\end{aligned} \tag{42.2}$$

As we have seen already, since A is independent of the initial conditions only to the zeroth order in $\mathfrak{T}$, the functions on the right-hand sides cannot (except for special initial conditions) be asymptotically 1 to any order beyond 0, and thus the tertiary solution cannot be asymptotically better than the secondary one.

Often in physical conversation and sometimes in print [1936] [1948, 4, 6] we encounter the contention that *because* the Navier-Stokes equations emerge only as the first approximation by the Chapman-Enskog process, *therefore* they cannot be correct in circumstances when the magnitudes of their differences from the so-called higher approximations become relatively large. I have remarked earlier [1952, 4] [1953, 3, §5A] that such a claim confuses necessity with sufficiency and rests upon acceptance of the unproved Chapman-Enskog process. My *caveat* is now reinforced by fact: *For the problem of shearing flow, at least, the Navier-Stokes solution for the shearing pressure has a definite asymptotic status as $\mathfrak{T} \rightarrow 0$, while the so-called higher approximations introduce arbitrary additional terms having no general asymptotic status.*

However, for fixed t/τ we have by (33.8)₁

$$\exp [(\mathfrak{A} - 2\mathfrak{A}^2 - \mathfrak{r})t/\tau] = 1 + O(\mathfrak{T}^6), \tag{42.3}$$

in respect to independence from time an improvement over (40.3)₂, (40.4)₂, and (41.3)₂. That is, *the ratio of the tertiary solution to the dominant part of any exact solution is asymptotically steady to a higher order than is the corresponding ratio for the Burnett solution.*

It is suggested by (33.8)₁ and is in fact not difficult to prove that $\mathfrak{r} > \mathfrak{A} - 2\mathfrak{A}^2$. Hence from (42.2) follows ${}^{[3]}p_{ii}/p_{ii}^D \rightarrow 0$ as $t \rightarrow \infty$. More generally, (33.9) shows that $\mathfrak{r}$ is not a polynomial in $\mathfrak{T}$, while from the iterative scheme (36.4) and the solution (38.3) we see that

$${}^{[n]}\varpi_{ii} \propto e^{\mathfrak{r}_n t/\tau}, \tag{42.4}$$

where r_n is a polynomial in $\mathfrak{T}$. The same conclusion holds for iteration by powers (§19) and for the Chapman-Enskog process. Thus follows

Theorem 5. *For every n , as $t \rightarrow \infty$*

$$\frac{{}^{[n]}p_{ij}}{p_{ij}^D} \rightarrow \text{either } \infty \text{ or } 0, \quad \text{exponentially.} \quad (42.5)$$

This theorem completely overturns the popular idea of a “normal” solution (§23, *cf.* also §37). Giving $\mathfrak{T}$ *any* non-zero value, large or small, we select *any* n , however large, and compare the solution ${}^{[n]}p_{ij}$ by the Chapman-Enskog process (or any method of the classical type yielding a series in powers of $\mathfrak{T}$) with the exact dominant part corresponding to *any* initial conditions. The larger is n , other things being equal, the longer is the time during which this ratio is approximately constant. Ultimately, however, it tends to ∞ or to 0, exponentially with time. Thus, ultimately, *the result at any finite stage of the Chapman-Enskog process becomes a steadily worse and worse approximation as time increases*. In general, we have no indication that the Chapman-Enskog process ever yields accurate results. What we have shown is that a *necessary* condition for it to be accurate is that *the time be neither small nor large compared to τ* , for when t/τ is small, we are not generally justified in neglecting the damped parts, while when t/τ is large we have the error indicated by (42.5).

We now consider the solution ${}^{(\infty)}\varpi$ and ${}^{(\infty)}\varpi_{ij}$ corresponding to the limit functions $\pi_{ij}^{(\infty)}$ given by (36.6). Substitution in (38.3) and (38.4) yields

$$\begin{aligned} {}^{(\infty)}\varpi &= e^{rt/\tau}, \\ {}^{(\infty)}\varpi_{xy} &= -\mathfrak{T} \frac{r}{\mathfrak{A}} e^{rt/\tau}, \\ {}^{(\infty)}\varpi_{yy} &= \frac{1}{r+1} e^{rt/\tau}. \end{aligned} \quad (42.6)$$

By (33.12)₁, (34.1), and (34.2) follows

Theorem 6. *The absolute pressures according to the limit of atemporal Maxwellian iteration have the correct dependence on time and stand in a common constant ratio to the corresponding dominant parts of any exact solution:*

$$\frac{{}^{(\infty)}p_{ij}}{p_{ij}^D} = \frac{1}{A}. \quad (42.7)$$

This ratio is asymptotically 1 for ordinary states.

This theorem squeezes out the ultimate possible use for the classical approach. In the limit it can possibly yield solutions proportional to an exact solution. It *cannot* in general yield the correct magnitude of the solution, except asymptotically as $\mathfrak{T} \rightarrow 0$. The result (42.7) might be used in this way: If the proportion

of error in the limit solution from the classical approach can be evaluated at some one time, perhaps by experiment, then this same proportion of error holds at all later times, so that the solution by the classical approach can be corrected.

We may now face the last question posed by the classical approach: Does it yield solutions at all? That is, are there any particular initial conditions yielding a solution p_{ij} such that $p_{ij}^D = {}^{(\infty)}p_{ij}$? Now by (42.7), ${}^{(\infty)}p_{ij}$ if once a solution is always a solution. Thus it is both necessary and sufficient that the initial conditions for the solution p_{ij} agree with those for ${}^{(\infty)}p_{ij}$. By (42.6) then, the required conditions are

$$\Pi_{xy} = -\frac{3}{2}r\mathfrak{T}, \quad \Pi_{yy} = \frac{1}{r+1}. \quad (42.8)$$

Substitution in (33.13) yields $B = 0$, $C = 0$. Hence follows the

Final theorem on the classical approach. *For shearing flow of a Maxwellian gas, the absolute pressures which correspond to the pressure ratios obtained by the limit of atemporal Maxwellian iteration as $n \rightarrow \infty$ constitute a particular exact solution. For this solution, the initial conditions are given by (42.8). Equivalently, this solution is the only one for which the damped part is exactly zero.*

We might say that in an essentially unsteady problem, atemporal Maxwellian iteration gives us the steadiest possible solution.

An evaluation of the classical approach, based on the analysis of §§35–42, has been set at the head of the memoir in §1.

43. Remark on the elimination of time derivatives. Apart from writing down (35.3), we have confined most of our discussion of the classical approach in §§35–42 to the results of processes in which time derivatives are eliminated (*cf.* §15). Such elimination, while usual, is not a necessary feature of the classical approach and is not used in Maxwellian iteration (§2). Consider (35.3). It is not included as a special case of (38.1). Substitution in (2.8) yields

$$3\tau p' = 2\mathfrak{T}^2 p(1 - \tau p'/p). \quad (43.1)$$

The general solution ${}^{(2)}\varpi \equiv p/p_0$ is

$${}^{(2)}\varpi = \exp \left[\frac{\mathfrak{U}}{1 + \mathfrak{U}} t/\tau \right]. \quad (43.2)$$

Hence by (35.3) follows

$${}^{(2)}\varpi_{xy} = -\frac{\mathfrak{T}}{1 + \mathfrak{U}} \exp \left[\frac{\mathfrak{U}}{1 + \mathfrak{U}} t/\tau \right]. \quad (43.3)$$

If we expand (43.3) in powers of $\mathfrak{U}$, we see that the decay modulus $\mathfrak{U}/(1 + \mathfrak{U}) = \mathfrak{U}(1 - \mathfrak{U}) + \dots$ is not correct to the second order. However, the result $\mathfrak{U}/(1 + \mathfrak{U})$ as it stands gives a far better approximation to the true decay modulus r than could any polynomial, such as those resulting from most of the

schemes discussed in §§35–42. The series approximated by the polynomials diverges at $\mathfrak{A} = 4/27$, and any polynomial approximation gives ridiculous results for large $\mathfrak{A}$. The factor $\mathfrak{A}/(1 + \mathfrak{A})$, however, while not giving the correct behavior at $\mathfrak{A} = \infty$, at least falls off steadily from $\mathfrak{A}$ to the value 1 and thus represents a first step toward the right answer over a very large range.

The foregoing remarks suggest that it is at least possible that Maxwellian iteration converges for all values of $\mathfrak{A}$. However, the example given does not fully distinguish the effect of elimination of time derivatives, since, as follows from (35.10), the method of §20 also yields (43.3).

Finally, methods which do not eliminate time derivatives retain the possibility of satisfying some at least of the initial conditions of the problem and thus of avoiding the negative conclusion of the Principal Theorem in §37. In Maxwellian iteration this possibility first occurs at the third step.

44. Fast shearing. For ultimate states ($\mathfrak{T} \rightarrow \infty$) we get from (37.1)

$$\frac{Ar}{\frac{2}{3}\mathfrak{T}} = \frac{1}{\sqrt{6}} \mathfrak{A}^{\frac{1}{3}} \Pi_{vv} - \frac{1}{3} \Pi_{xx} + O(\mathfrak{A}^{-\frac{1}{3}}). \quad (44.1)$$

Thus A grows with $\mathfrak{T}$, but only as $\mathfrak{T}^{\frac{1}{3}}$.

To consider fast shearing, we hold t/τ fixed as $\mathfrak{T} \rightarrow \infty$. Since $\mathfrak{r} \rightarrow \infty$, the damped parts of (33.12) vanish in the limit, and we obtain

$$\begin{aligned} \varpi &\sim \frac{1}{3} \mathfrak{A}^{\frac{1}{3}} \Pi_{vv} \exp [(\mathfrak{A}^{\frac{1}{3}} - \frac{2}{3})t/\tau], \\ \varpi_{xx} &\sim -\frac{1}{\sqrt{6}} \mathfrak{A}^{\frac{1}{3}} \Pi_{vv} \exp [(\mathfrak{A}^{\frac{1}{3}} - \frac{2}{3})t/\tau], \\ \varpi_{vv} &\sim \frac{1}{3} \Pi_{vv} \exp [(\mathfrak{A}^{\frac{1}{3}} - \frac{2}{3})t/\tau], \\ p_{xx} &\sim 3p. \end{aligned} \quad (44.2)$$

Since the damped parts are asymptotically zero, the asymptotic ratios of the pressures are consequences of (34.1)–(34.5), which are more general than our present results in that they are not limit formulæ. In particular, that the normal pressure p_{xx} in the direction of flow ultimately dominates the entire pressure system we saw earlier in §34. What the present formulæ add is information on the asymptotic magnitudes. For this purpose any one of relations in (42.2) is sufficient, since by the results of §34 it implies all the rest. We have

Theorem 7. *For fast shearing, every pressure increases to infinity at least as fast as $\exp(\xi \mathfrak{T}^{\frac{1}{3}})$, where ξ is a numerical multiple of t/τ . Each pressure is ultimately proportional to the initial normal pressure on the shearing planes.*

The formulæ (44.2) should not be taken as very significant. They result from holding the ratios Π_{xv} and Π_{vv} fixed as $\mathfrak{T} \rightarrow \infty$. A different sequence of solutions for comparison—for example, $\Pi_{xv} = -\mathfrak{T}$, $\Pi_{vv} = 0$ —would have led to quite different asymptotic expressions.

45. The damped parts. Before considering rare states we return once more to the oscillatory terms in (33.12). Their circular frequency is

$$\omega = \Im K = \Im \mathfrak{T} / \tau. \quad (45.1)$$

For ordinary states (33.11)₁ implies

$$\omega \sim \sqrt{\frac{2}{3}}(1 - \frac{5}{12}\mathfrak{T}^2)K = \sqrt{\frac{2}{3}}\mathfrak{T}(1 - \frac{5}{12}\mathfrak{T}^2)/\tau. \quad (45.2)$$

Hence by holding K or τ fixed as $\mathfrak{T} \rightarrow 0$ we conclude

Theorem 8. *For dense states the frequency of the oscillatory terms is a constant multiple of the rate of shearing. For slow shearing, it is zero.*

For ultimate states (33.11)₂ implies

$$\omega \sim 3^{\frac{1}{2}}2^{-\frac{1}{2}}\mathfrak{T}^{-\frac{1}{2}}K = 3^{\frac{1}{2}}2^{-\frac{1}{2}}\mathfrak{T}^{\frac{1}{2}}/\tau. \quad (45.3)$$

By holding K or τ fixed as $\mathfrak{T} \rightarrow \infty$ we conclude

Theorem 9. *For rare states, the frequency is zero. For fast shearing, it is infinite.*

The exact frequency and its four limit values are presented in Figure 45.1, which shows both $\omega/K = \Im$ and $\omega\tau = \Im\mathfrak{T}$ as functions of $\mathfrak{T}$.

FIGURE 45.1. Comparison of the exact values of $\Im\mathfrak{T}$ and $\Im$ with their respective asymptotic forms for ordinary states and ultimate states.

Both for slow shearing and for rare states the oscillatory terms of the solution are asymptotically constant. This raises the question of whether these steady contributions need to be considered. Recall that the damping factor in (33.12) is $e^{-(1+\frac{1}{2}r)t/\tau}$. For slow shearing $r \rightarrow 0$ while τ is held constant; thus the damping factor is asymptotically the same as for relaxation in a gas at rest with uniform thermostatic state, and the oscillatory functions which it multiplies may justly be neglected, as they were in our previous discussion.

For rare states it is an entirely different matter. The damping factor may be written $e^{-(1+\frac{1}{2}r)Kt/\mathfrak{T}}$ and by (33.8)₂ is asymptotic to $\exp(-6^{-\frac{1}{2}}\mathfrak{T}^{-\frac{1}{2}}Kt)$, which approaches 1. Therefore the damped part is asymptotically constant and contributes to the solution on an equal basis with the dominant part. In the next section we calculate the asymptotic form of the entire solution for rare states.

46. Rare states. In the limit of rare states we hold Kt fixed as $\mathfrak{T} \rightarrow \infty$. In the treatment of fast shearing in §44 it would have made no difference had we considered only the dominant parts, since the damped parts are ultimately zero as $\mathfrak{T} \rightarrow \infty$ with t/τ held fixed. For rare states, we have seen in the foregoing section that the damped parts do not vanish in the limit. Thus we need to consider the entire solution (33.12). Moreover, since $rt/\tau = (r/\mathfrak{T})Kt$, by (33.8)₂ we get $rt/\tau \rightarrow 0$; also by (33.11)₂ follows $\Im Kt \rightarrow 0$. Thus all temporal arguments in (33.12) vanish. We might expect that the limits could be obtained by setting $t = 0$ in (33.12), thus yielding the initial values of p and P_{ij} as asymptotic values, but this is not true, because A , B , and C all become infinite as $\mathfrak{T} \rightarrow \infty$. The limiting values in (33.12) are thus to be obtained as indeterminate forms.

From (33.13)₂ and (44.1) we get

$$\begin{aligned} \frac{(1 + \frac{1}{2}r)B - \Im \mathfrak{T} C}{\frac{2}{3}\mathfrak{T}} &= A + \Pi_{xx}, \\ &= \frac{1}{\sqrt{6}} \mathfrak{A}^{\frac{1}{2}} \Pi_{xx} + \frac{2}{3} \Pi_{xx} + O(\mathfrak{A}^{-\frac{1}{2}}). \end{aligned} \quad (46.1)$$

From (33.13)₃, (33.8)₂, and (44.1) we get

$$\begin{aligned} \Im \mathfrak{T} \frac{(1 + \frac{1}{2}r)C + \Im \mathfrak{T} B}{\frac{2}{3}\mathfrak{T}} &= -\frac{\mathfrak{T}}{r+1} (\Pi_{xx} - 1) - \frac{1}{2}r \left(3 \frac{Ar}{\frac{2}{3}\mathfrak{T}} + \Pi_{xx} \right), \\ &= -\frac{1}{2} \sqrt{6} \mathfrak{A}^{\frac{1}{2}} \Pi_{xx} + O(\mathfrak{A}^{\frac{1}{2}}), \end{aligned} \quad (46.2)$$

whence by (33.11)₂ follows

$$\frac{(1 + \frac{1}{2}r)C + \Im \mathfrak{T} B}{\frac{2}{3}\mathfrak{T}} = -\sqrt{2} \mathfrak{A}^{\frac{1}{2}} \Pi_{xx} + O(\mathfrak{A}^{-\frac{1}{2}}). \quad (46.3)$$

Also

$$e^{-(1+\frac{1}{2}r)t/\tau} = 1 - \frac{1}{\sqrt{6}} \frac{Kt}{\mathfrak{A}^{\frac{1}{2}}} + o\left(\frac{1}{\mathfrak{A}^{\frac{1}{2}}}\right),$$

$$\begin{aligned}
e^{\mathfrak{r}t/\tau} &= 1 + \sqrt{\frac{2}{3}} \frac{Kt}{\mathfrak{A}^{\frac{1}{3}}} + O\left(\frac{1}{\mathfrak{A}^{\frac{1}{3}}}\right), \\
\cos \Im Kt &= 1 + O\left(\frac{1}{\mathfrak{A}^{\frac{1}{3}}}\right), \\
\sin \Im Kt &= \frac{1}{\sqrt{2}} \frac{Kt}{\mathfrak{A}^{\frac{1}{3}}} + O\left(\frac{1}{\mathfrak{A}^{\frac{1}{3}}}\right).
\end{aligned} \tag{46.4}$$

Putting the foregoing estimates into (33.12)₂ yields

$$-\varpi_{xy} = -\Pi_{xy} + \frac{3}{2}Kt\Pi_{yy} + O(\mathfrak{A}^{-\frac{1}{3}}). \tag{46.5}$$

By substituting this result into (33.14) we get

$$\begin{aligned}
\varpi &= 1 - \frac{2}{3}\Pi_{xy}Kt + \frac{1}{2}\Pi_{yy}K^2t^2 + O(\mathfrak{A}^{-\frac{1}{3}}), \\
\varpi_{yy} &= \Pi_{yy} + O(\mathfrak{A}^{-\frac{1}{3}}), \\
\varpi_{xx} &= \Pi_{xx} - 2\Pi_{xy}Kt + \frac{3}{2}\Pi_{yy}K^2t^2 + O(\mathfrak{A}^{-\frac{1}{3}}), \\
P_{xx} + 2P_{yy} &= P_{xx}(0) + 2P_{yy}(0) + O(\mathfrak{A}^{-\frac{1}{3}}).
\end{aligned} \tag{46.6}$$

(The exponentials occurring in (33.14) may be neglected to order $\mathfrak{A}^{-\frac{1}{3}}$ and thus do not affect (46.6).) From (46.6) we read off

Theorem 10. *For rare states, the pressures approach finite limits. Only the normal pressures on the planes parallel to the flow are asymptotically steady. All other limiting pressures are linear or quadratic functions of Kt .*

The result asserted by the theorem is mainly negative: Rare states do not correspond to free molecular flow except for very small values of Kt .

47. Free molecular flow. According to the *kinetic view* of the gaseous state, in an ultimately rarefied gas the molecules pursue their several ways unhindered by impacts except with boundaries. It has always been a primary objective of students of the formal Maxwellian *kinetic theory* to prove such a result, but (as far as rigorous work is concerned) in vain. In the first place, the only such solutions of the Maxwell-Boltzmann equation are included among the relatively trivial locally Maxwellian ones (*cf. e.g.*, [1949, 1, Appendix 2]), which do not include any flows of hydrodynamical interest.

Second, it might be thought that free molecular solutions could emerge asymptotically. Such solutions appropriate to an unbounded volume of gas

would yield as the limiting value of ϖ_{ij} its initial value Π_{ij} . Indeed, if we let $\tau \rightarrow \infty$ for any fixed t in the solution (31.9) for relaxation of the pressures in a uniform gas at rest, this result follows. However, the work of §46 shows that in a shearing flow *such a result cannot hold in general for rare states*.

We now formulate a limit process which, however unnatural it may seem, does yield formulæ appropriate to free molecular flow. In this process t/τ , rt/τ , and $\Im Kt$ all approach zero while the initial conditions are held fixed. Since $\mathfrak{T} = K\tau = (Kt)(\tau/t)$, $\mathfrak{T}$ may behave in a variety of ways. One possibility is $\tau \rightarrow \infty$, $K \rightarrow 0$, $\mathfrak{T}$ fixed. Inspection (33.12) then yields

Theorem 11. *At a fixed time, consider the solutions corresponding to fixed initial conditions and fixed $\mathfrak{T}$. In these, let $\tau \rightarrow \infty$. (This implies $K \rightarrow 0$.) Then, whatever is t , the pressures ϖ_{ij} approach their initial values Π_{ij} .*

48. Other limiting processes. In §§37–47 we have considered a variety of limiting processes in which the initial conditions were held fixed. There is no reason that these limiting processes, or, as far as I can see, that any other limiting process should be physically appropriate. By different choices of limiting processes we can get quite different results.

For example, suppose we take

$$\Pi_{xy} = K_{xy}\mathfrak{T}, \quad \Pi_{yy} = 1 + K_{yy}\mathfrak{T}^2 \quad (48.1)$$

and hold K_{xy} and K_{yy} fixed as $\mathfrak{T} \rightarrow 0$. That is, we consider a family of initial conditions specially selected so as to have the same dependence on $\mathfrak{T}$ as do the Burnett pressures. Then from (37.1) it is plain that

$$A = 1 + O(\mathfrak{T}^2) \quad (48.2)$$

and hence that for the class of initial values (48.1) the Navier-Stokes and Burnett solutions in the limit as $\mathfrak{T} \rightarrow 0$ are correct to an order one greater than for the class obtained by holding Π_{xy} and Π_{yy} fixed.

In fact, for a class of initial values depending on $\mathfrak{T}$ in a way specially chosen for the purpose, we can render arbitrarily high the asymptotic accuracy of the solutions obtained by the classical approach. Pursuing this idea to its natural conclusion leads to the initial conditions (42.8) and thence to the one particular exact solution which the classical approach yields in the limit.

49. Numerical values in special cases.

Note added in proof. Long after this memoir was standing in type I was notified that the daily expected figures for this section, which were to have been supplied by an automatic computer, exceeded the capacity of the machine. It seems advisable to suppress the explanation of the missing figures. Since the exact general solution is given explicitly by (33.12), (37.1), (46.1)₁, and (46.2)₁, the reader may form for himself a good idea of the curves corresponding to various initial conditions. In particular, it should be plain that the special solution characterized by the Final Theorem in §42 is neither typical nor representative.

50. The dissipation of energy; proof that the H-theorem does not always imply the entropy principle.

Restrictions on the constants of integration. In all the foregoing analysis of shearing flow we have placed no restriction on the initial values of the pressure field, thus leaving the constants A , B , and C in the solution (33.12) entirely arbitrary. In fact some restrictions are necessary. While these do not modify the conclusions of the earlier parts of this chapter, they are of interest in themselves and will now be ascertained.

If we were to take the standpoint of continuum mechanics, we should impose two conditions. First, a gas cannot rest in equilibrium except under non-negative pressure, and since the thermal equation of state is assumed to hold in all flows, we must have

$$p \geq 0. \quad (50.1)$$

(In the present problem, this is immediate from $p = (k/m)\rho\theta$ and the definition of absolute temperature.) Second, the motion must be consistent with the entropy principle; by (15.2), in the present problem this principle is equivalent to the assertion that in undergoing deformation the gas must consume work and cannot give it out:

$$-\mathfrak{T}p_{xv} \geq 0. \quad (50.2)$$

In our treatment of the Maxwellian kinetic theory so far, we have used only the equations of transfer. Beyond these, there is a single further fundamental assumption: F , the distribution function, is non-negative. The only consequent restriction on the second moments is that the normal pressures are non-negative. For the problem of shearing flow, the second moments are unaffected by the differential equations for the moments of higher order, and hence the solution given in the foregoing sections is an exact solution for the kinetic theory provided it satisfies $p_{xx} \geq 0$, $p_{vv} \geq 0$, $p_{zz} \geq 0$, for all time.¹ In particular, we have not only (50.1) but also

$$p_{vv} \geq 0. \quad (50.3)$$

We assume $p_0 > 0$. Then from the general solution (33.12) follow as equivalent to (50.1), (50.2), and (50.3) the conditions

$$A + e^{-(1+\frac{3}{2}\tau)t/\tau} [B \cos \mathfrak{K}t + C \sin \mathfrak{K}t] \geq 0, \quad (50.4)$$

¹ It is necessary to mention one possibility of additional restriction. We must have also $p_{xxxx} \geq 0$, $p_{xyyy} \geq 0$, $p_{xxxxx} \geq 0$, etc. Now the differential equation for the fourth moments contains terms involving the second moments. If the most general non-negative solution for the second moments is inserted in this equation, there is a conceivable possibility that there exist no initial values for the fourth moments such that p_{xxxx} , p_{xyyy} , etc. are non-negative for all t and $\mathfrak{T}$. A similar question arises for the moments of every even order. It seems very unlikely that a difficulty of this kind really materializes, but I have been unable to find a rigorous argument disposing of it.

$$\begin{aligned} rA - e^{-(1+\frac{1}{2}r)t/\tau}[(\{1 + \tfrac{1}{2}r\}B - \Im C) \cos \Im Kt \\ + (\{1 + \tfrac{1}{2}r\}C + \Im B) \sin \Im Kt] \geq 0, \end{aligned} \quad (50.5)$$

$$\begin{aligned} A + e^{-(1+\frac{1}{2}r)t/\tau} \left[-\left(\tfrac{1}{2}B + \frac{\Im}{r} C\right) \cos \Im Kt \right. \\ \left. + \left(-\tfrac{1}{2}C + \frac{\Im}{r} B\right) \sin \Im Kt \right] \geq 0, \end{aligned} \quad (50.6)$$

respectively. These are to hold for all $t \geq 0$. Henceforth we bear in mind that (50.4) applies equally to the kinetic and to the continuum views, while (50.5) and (50.6) are alternatives.

Letting $t \rightarrow \infty$, we get for both theories the necessary condition

$$A > 0. \quad (50.7)$$

If we confine attention to the dominant parts in (33.12), we get (50.7) in the weaker form $A \geq 0$ as both necessary and sufficient for every one of our three conditions. By (37.1), our result is equivalent to the statement that if the initial conditions satisfy

$$\frac{2}{3}\Im\Pi_{xy} < (1 + r)(1 + r\Pi_{yy}), \quad (50.8)$$

then all three restrictions (50.1), (50.2), and (50.3) will be satisfied for sufficiently large t . Sufficient for (50.8) are $-\Im\Pi_{xy} \geq 0$, $\Pi_{yy} > 0$, *i.e.*, that the motion satisfies both (50.2) and (50.3) at the initial instant.

Since we have assumed $p_0 > 0$, (50.4) is satisfied at $t = 0$. From (50.5) and (50.6) follow the further necessary conditions

$$\Im C \geq 1 - A + \tfrac{1}{2}r(1 - 3A), \quad (50.9)$$

$$\Im C \geq \tfrac{1}{2}r(1 - 3A), \quad (50.10)$$

where B has been eliminated by (33.13). Of these, the former implies or is implied by the latter according as $A \leq 1$ or $A \geq 1$.

All three conditions (50.4), (50.5), and (50.6) are of the type

$$f(t) \geq 0, \quad (50.11)$$

$$f(t) \equiv A + e^{-\beta t}g(t), \quad g(t) \equiv \gamma \cos \Im Kt + \delta \sin \Im Kt.$$

The infimum of $f(t)$ for $t \geq 0$ occurs either at $t = 0$ or at the first minimum of the quantity $e^{-\beta t}g(t)$. Thus the value of f at its infimum can be determined by elementary means. Necessary and sufficient conditions, expressed in terms of A and C only, are then $f_{\text{inf}} \geq 0$.

We shall not work out these conditions relating A and C . It is enough to remark that for any positive A there exists a range of values of C such that all

three are satisfied. To see this, recall that for any initial conditions whatever, after a sufficient time t_0 the damped parts of the solution become and remain less in absolute value than the dominant parts. Given a positive A , the values of the pressures at any time subsequent to t_0 may be taken as new initial values. For these initial values, all three requirements (50.1), (50.2), and (50.3) are satisfied.

Entropy and the H-theorem. The H-theorem expresses a certain irreversibility formally akin to the entropy principle of thermodynamics. In the case of a finite vessel with adiabatic walls, the two are identical, but for more general circumstances the H-theorem contains a surface integral which has never been satisfactorily identified with the surface integral of the flux of energy divided by temperature occurring in the general statement of the entropy principle. We shall now prove that the two are indeed different.

The H-theorem in its general form holds for all solutions of the Boltzmann equation. Therefore, to show that it is not the same as the entropy principle of thermodynamics, we need only exhibit in the kinetic theory a solution which violates the entropy principle. We shall now find a class of particular solutions in which (50.2) is not satisfied and hence, for a short time, the gas gives out work rather than using it up during the deformation.

To this end, choose

$$A = 1, \quad B = 0, \quad C = -\frac{2r}{\sqrt{r(r + \frac{3}{4})}} = -\frac{2r}{3\mathfrak{T}}. \quad (50.12)$$

For small enough $\mathfrak{T}$, it is evident that (50.4) is satisfied and hence (50.1) holds. A simple argument shows that for sufficiently small values of $\mathfrak{T}$ the infimum of the left-hand side of (50.6) cannot occur until $3Kt > \frac{1}{2}\pi$, while for all such values of $\mathfrak{T}$ a lower bound for the left-hand side is

$$1 - 2(2 - \frac{1}{2}C)e^{-\pi/(23\mathfrak{T})}; \quad (50.13)$$

for sufficiently small $\mathfrak{T}$, this quantity is positive. Hence (50.6) is satisfied. Accordingly, the selection of constants (50.12) yields a solution admissible for the kinetic theory. By (33.13), this set of constants corresponds to the initial conditions

$$-\frac{2}{3}\mathfrak{T}\Pi_{xx} = -r, \quad \Pi_{yy} = 3/(1 + r). \quad (50.14)$$

Hence (50.2) is not satisfied at the initial instant and for some time thereafter.

In mechanical terms, this result shows that according to Maxwell's kinetic theory it is possible to start a shearing flow in such a way that in the first few moments the tangential pressure will assist the shearing rather than oppose it and the process of deformation will create work rather than dissipating it.

While such a result may at first seem shocking, in fact it is not. According to the viewpoint of statistical mechanics, mean dissipation is highly probable but

not certain. It is consistent with this view that in specially ordered situations the improbable may occur. In Maxwell's theory, a particular (if to this day rather mysterious) dissipative mechanism is introduced in the hypotheses embodied in the general equation of transfer. The best known consequence of this dissipative mechanism is the H-theorem. According to this theorem, for a volume V over whose bounding surface a certain flux integral vanishes, the quantity $\int_V (-\log \bar{F}) d\mathbf{x}$ increases unless F is Maxwellian. Another theorem asserts that in equilibrium

$$\eta_K = \eta + \text{const.}, \quad (50.15)$$

where, by definition,

$$\eta_K \equiv -\frac{k}{m} \overline{\log \bar{F}} + \text{const.}, \quad \eta \equiv \frac{3}{2} \frac{k}{m} \log p \rho^{-5/3} + \text{const.}, \quad (50.16)$$

η_K and η being the specific entropies of the kinetic theory and of thermostatics, respectively. In all states of flow the energy equation (15.2), or, if we prefer,

$$\rho \theta \dot{\eta} = -P_{ij} E_{ij} - h_{i,i}, \quad (50.17)$$

is an exact consequence of Maxwell's theory. The η that occurs in this equation is defined by (50.16)₂ and has never been proved, for general conditions of flow, to bear any simple relation to the kinetic entropy, η_K , that is defined by (50.16)₁. Equation (50.17) shows $\dot{\eta}$ to be a measure of the rate of working and the rate of energy addition.

In the case of shearing flow the flux integral in the H-theorem does not necessarily vanish, so we have no indication that the total kinetic entropy $\int_V \rho \eta_K d\mathbf{x}$ increases. Our calculation pursues this idea and shows that in fact it is possible to start the motion in such a way that the thermostatic entropy η , which is uniform in space, at first decreases. After a short time, however, it will begin to increase, and dissipation will result thenceforth. The work given out initially is finite, while that eventually used up is always infinite. That this is so is a consequence of (50.7). In fact, for *any* solution satisfying the requirements of either the kinetic theory or continuum mechanics, the rate of dissipation of energy per unit volume tends to ∞ as time increases.

As was remarked in §2, the differential equations of continuum mechanics are exact consequences of Maxwell's theory. The foregoing example, since it proves the existence of solutions not compatible with the second law of thermodynamics for irreversible processes, is the first rigorous argument yielding a distinction between continuum mechanics and the kinetic theory.

51. The third moments; status of Grad's method. Substitution of the velocity field (33.1) into the equations of third moments (*cf.* (6.5), (6.6), (8.5), and (8.6)) yields the following system:

$$\tau h'_x + \frac{2}{3}h_x + \frac{7}{3}h_y\mathfrak{T} + P_{xxv}\mathfrak{T} = 0, \quad (51.1)$$

$$\tau h'_y + \frac{2}{3}h_y + \frac{2}{3}h_x\mathfrak{T} + P_{yyx}\mathfrak{T} = 0, \quad (51.2)$$

$$\tau h'_z + \frac{2}{3}h_z + P_{xyz}\mathfrak{T} = 0, \quad (51.3)$$

$$\tau P'_{xyz} + \frac{3}{2}P_{xyz} + P_{yyz}\mathfrak{T} + \frac{2}{3}h_z\mathfrak{T} = 0, \quad (51.4)$$

$$\tau P'_{yyz} + \frac{3}{2}P_{yyz} - \frac{2}{3}P_{xyz}\mathfrak{T} = 0, \quad (51.5)$$

$$\tau P'_{xxz} + \frac{3}{2}P_{xxz} + \frac{8}{5}P_{xyz}\mathfrak{T} = 0, \quad (51.6)$$

$$\tau P'_{zzz} + \frac{3}{2}P_{zzz} - \frac{6}{5}P_{xyz}\mathfrak{T} = 0, \quad (51.7)$$

$$\tau P'_{zzv} + \frac{3}{2}P_{zzv} - \frac{2}{3}P_{yyx}\mathfrak{T} - \frac{4}{25}h_x\mathfrak{T} = 0, \quad (51.8)$$

$$\tau P'_{xxv} + \frac{3}{2}P_{xxv} + \frac{8}{5}P_{yyx}\mathfrak{T} + \frac{1}{2}\frac{8}{5}h_x\mathfrak{T} = 0, \quad (51.9)$$

$$\tau P'_{vvv} + \frac{3}{2}P_{vvv} - \frac{6}{5}P_{yyx}\mathfrak{T} - \frac{1}{2}\frac{2}{5}h_x\mathfrak{T} = 0, \quad (51.10)$$

$$\tau P'_{vvx} + \frac{3}{2}P_{vvx} - P_{zzv}\mathfrak{T} - \frac{7}{5}P_{xxv}\mathfrak{T} + \frac{1}{2}\frac{6}{5}h_y\mathfrak{T} = 0, \quad (51.11)$$

$$\tau P'_{zzx} + \frac{3}{2}P_{zzx} + P_{zzv}\mathfrak{T} - \frac{2}{5}P_{xxv}\mathfrak{T} - \frac{4}{25}h_y\mathfrak{T} = 0, \quad (51.12)$$

$$\tau P'_{xxx} + \frac{3}{2}P_{xxx} + \frac{9}{5}P_{xxv}\mathfrak{T} - \frac{1}{2}\frac{2}{5}h_y\mathfrak{T} = 0. \quad (51.13)$$

This system is of tenth order, since three of the equations (*e.g.* (51.7), (51.10), and (51.13)) may be replaced by $P_{iii} = 0$.

There are various means of splitting this system into smaller ones. We begin with the observation that (51.3)–(51.5) form a determinate system for h_z , P_{xyz} , and P_{yyz} . These components, since they have z for one index, might be expected to be of no importance in the phenomenon, but we shall find a surprising result. The characteristic equation for solutions proportional to $e^{\chi t/\tau}$ is

$$(\chi + \frac{2}{3})^2(\chi + \frac{2}{3}) = \frac{1}{3}\mathfrak{T}^2. \quad (51.14)$$

For $\mathfrak{T} = 0$, this equation yields the times of relaxation $\frac{3}{2}\tau$ and $\frac{2}{3}\tau$ for h_z and for P_{xyz} and P_{yyz} , as in a uniform state (*cf.* §52). When $\mathfrak{T} \neq 0$, however, the solution is entirely different, since then (51.14) always has one real root and two complex conjugate roots. The real root $\tilde{\tau}$ is easily shown to be a monotone increasing convex function of $\mathfrak{T}$, increasing from the value $-\frac{2}{3}$ for $\mathfrak{T} = 0$, with

$$\tilde{\tau} = -\frac{2}{3} + \frac{1}{2}\frac{2}{5}\mathfrak{T}^2 + O(\mathfrak{T}^4) \quad \text{as } \mathfrak{T} \rightarrow 0, \quad (51.15)$$

to the asymptotic value $3^{-1}\mathfrak{T}^{\frac{1}{2}}$ as $\mathfrak{T} \rightarrow \infty$. When $\mathfrak{T} = 3/\sqrt{2}$, $\tilde{\tau} = 0$, and for greater values of $\mathfrak{T}$ we have $\tilde{\tau} > 0$. *That is, for $\mathfrak{T} < 3/\sqrt{2}$ the three components h_z , P_{yyz} , and P_{xyz} fall off to zero as time increases, but the more slowly the larger is $\mathfrak{T}$; if, however, $\mathfrak{T} > 3/\sqrt{2}$, these components ultimately increase exponentially with time.*

The phenomenon just revealed is a type of instability. For sufficiently fast shearing or sufficiently rarefied gases, an initial flux of energy h_z parallel to the shear planes and normal to the flow is amplified rather than damped with time. It is difficult to find a physical explanation for this fact. Some critics may choose to regard its existence as indicating some fault in the kinetic theory itself (cf. §2). Others may suggest that the assumption of a linear velocity field and spatially homogeneous pressures and moments of higher order may become physically unacceptable at or before the value $\mathfrak{T} = 3/\sqrt{2}$, with turbulence or some other type of physical instability occurring in practice.

Returning to the solution, we find that the complex roots $\tilde{\mathfrak{R}} \pm i\tilde{\mathfrak{I}}$ satisfy

$$\begin{aligned}\tilde{\mathfrak{R}} &= -\frac{1}{6} - \frac{1}{2}\tilde{\mathfrak{r}}, \\ \tilde{\mathfrak{I}}^2 &= 3(\tilde{\mathfrak{R}} + \frac{3}{2})(\tilde{\mathfrak{R}} + \frac{1}{18}) = \frac{3}{4}(\tilde{\mathfrak{r}} + \frac{2}{3})(\tilde{\mathfrak{r}} + \frac{1}{9}).\end{aligned}\tag{51.16}$$

It follows that $\tilde{\mathfrak{R}}$ is monotone decreasing from the value $-\frac{3}{2}$ at $\mathfrak{T} = 0$ to $-\infty$ as $\mathfrak{T} \rightarrow \infty$. Thus the oscillatory part of the solution is damped more rapidly than in a uniform state, and in the range of $\mathfrak{T}$ for which the monotone part is damped, the oscillatory part is damped much more rapidly than is the monotone part. Consequently we are justified in referring to the monotone part as *dominant*, and we have

$$\begin{aligned}h_z^D &= Ee^{\tilde{\mathfrak{r}}t/\tau}, \\ -\mathfrak{T}P_{xyz}^D &= (\frac{2}{3} + \tilde{\mathfrak{r}})Ee^{\tilde{\mathfrak{r}}t/\tau}, \\ -P_{vuz}^D &= \frac{2}{5} \cdot \frac{\frac{2}{3} + \tilde{\mathfrak{r}}}{\frac{3}{2} + \tilde{\mathfrak{r}}} Ee^{\tilde{\mathfrak{r}}t/\tau},\end{aligned}\tag{51.17}$$

where E is a constant of integration.

From (51.5) and (51.6) we read off

$$P_{vuz} = -\frac{1}{4}P_{xxz} + Me^{-\frac{3}{2}t/\tau},\tag{51.18}$$

where M is a constant of integration. Since P_{vuz} has already been determined, (51.18) determines P_{xxz} , and $P_{zzz} = -P_{xxz} - P_{vuz}$. If we prefer, from (51.6) and (51.7) follows a similar result, which by reason of $P_{iiz} = 0$ assumes the form

$$P_{zzz} = -\frac{3}{4}P_{xxz} - Me^{-\frac{3}{2}t/\tau}.\tag{51.19}$$

Similarly (51.8), (51.9), and (51.10) yield

$$P_{zzy} = -\frac{1}{4}P_{xzy} + Ne^{-\frac{3}{2}t/\tau},\tag{51.20}$$

$$P_{vuy} = -\frac{3}{4}P_{xzy} - Ne^{-\frac{3}{2}t/\tau}.\tag{51.21}$$

These formulæ and the original system (51.1)–(51.13) show that there is a particular solution in which h_x , h_y , and all the P_{ijk} except P_{xyz} are proportional to $e^{-\frac{3}{2}t/\tau}$, as are all the P_{ijk} in a uniform state (§52).

Putting (51.20) into (51.11) yields

$$\tau P'_{vux} + \frac{3}{2}P_{vux} - \frac{2}{3}\delta P_{xxv}\mathfrak{T} + \frac{1}{2}\frac{6}{5}h_v\mathfrak{T} = N\mathfrak{T}e^{-\frac{3}{2}t/\tau}. \quad (51.22)$$

This equation, (51.1), (51.2), and (51.9) form a system of fourth order for h_x , h_v , P_{xxv} , and P_{vux} . The characteristic equation of this system is

$$\begin{aligned} (\chi + \frac{2}{3} + \frac{2}{5}\mathfrak{T})[(\chi + \frac{3}{2})^2(\chi + \frac{2}{3}) + \frac{1}{25}\mathfrak{T}^2 \{46(\chi + \frac{2}{3})^2 - 16(\chi + \frac{3}{2})\}] \\ - \frac{1}{25}\mathfrak{T}^2(\chi + \frac{2}{3})(\chi + \frac{3}{2}) + \frac{3}{25}\mathfrak{T}^4 = 0. \end{aligned} \quad (51.23)$$

For sufficiently small $\mathfrak{T}$ there are two real roots given by

$$\begin{aligned} r_1 &= -\frac{2}{3} - \frac{2}{5}\mathfrak{T} + O(\mathfrak{T}^2), \\ r_2 &= -\frac{3}{2} + \sqrt{\frac{2}{15}}\mathfrak{T} + O(\mathfrak{T}^2). \end{aligned} \quad (51.24)$$

Thus one of the real roots at once begins to increase with $\mathfrak{T}$. It is not unlikely that we have here the inception of another phenomenon of instability such as that already discerned for h_x , P_{xvz} , and P_{vux} .

While it would be possible to calculate the entire solution by elementary means, the work would be wearisome and probably would not reveal much more than what we have already learned, which is enough for comparison of the results from various approximative methods with the exact solution.

The Newton-Fourier law of heat conduction yields $h_i = 0$, and this same result follows from the equations of CHAPMAN & COWLING, from (11.1), and from (12.2). In fact, it seems that $h_i^{(n)} = 0$ for this flow. We have seen that while h_x is damped to zero for $\mathfrak{T} < 3/\sqrt{2}$, for $\mathfrak{T} > 3/\sqrt{2}$ it is amplified. Thus the results from the methods cited are in a certain rude sense correct, as far as h_x is concerned, so long as $\mathfrak{T} < 3/\sqrt{2}$, while for $\mathfrak{T} > 3/\sqrt{2}$ they are entirely wrong.

The method of GRAD is different in principle (§17). Since GRAD's scheme for truncation affects only moments of order greater than two, for the present problem the equations for the second moments in all his approximations are the exact equations. Thus the pressures in a shearing flow may be obtained exactly by GRAD's method. Not so for the third moments. For his thirteen moment approximation, GRAD sets $P_{ijk} = 0$. In the present problem, from (51.4), (51.9), and (51.11) we see that such an assumption is possible only if $\mathfrak{T}h_i = 0$; *i.e.*, if the flux of energy and/or $\mathfrak{T}$ are "small". Overlooking this fact, we may read off GRAD's equations for h_i from (51.1), (51.2), and (51.3). Their general solution is

$$\begin{aligned} h_x &= h_x(0)e^{-\frac{3}{2}t/\tau}, \\ h_v &= h_v(0)e^{-(\frac{3}{2}+2\mathfrak{T}/5)t/\tau}, \\ h_x &= [h_x(0) - \frac{7}{2}h_v(0)]e^{-\frac{3}{2}t/\tau} + \frac{7}{2}h_v(0)e^{-(\frac{3}{2}+2\mathfrak{T}/5)t/\tau}. \end{aligned} \quad (51.25)$$

The oscillatory part of h_z is missing, but since this part is rapidly damped, little is lost. The important defect in the solution (51.25) is its failure to predict the instability of h_z for $\mathfrak{T} = 3/\sqrt{2}$. In fact, if we compare the exponents in (51.25) with (51.15) and (51.24)₂, we find agreement only to terms linear in $\mathfrak{T}$. Thus GRAD's thirteen moment approximation when applied to shearing flow yields a flux of energy which is approximately correct only as far as terms linear in $\mathfrak{T}$, and at that the root r_2 (cf. (51.24)₂) does not appear at all. Since from the linear approximation (51.25) one would conclude only that any initial flux of energy is damped out even more rapidly than for a uniform state, in this problem the most important property of the third moments is not reflected in any way in GRAD's thirteen moment approximation.

If we turn to GRAD's twenty moment approximation, only the equations for the moments of order 4 and greater differ from the exact equations, so the resulting third moments as well as the second are exact.

To summarize the status of GRAD's method for the present problem, we may say that *insofar as GRAD's equations are the exact equations, their solution is the exact solution, but to this same extent GRAD's process fails to simplify the original problem in any way; insofar as GRAD's equations are not the same as the exact equations, they fail to yield correct results except in first approximation for small $\mathfrak{T}$.*

CHAPTER VI. *The trend toward equilibrium*

52. Proof of the trend toward equilibrium in the spatially homogeneous case.

BOLTZMANN'S H-theorem indicates a trend toward equilibrium in the following restricted sense: In any volume over whose boundary a certain flux integral vanishes, the kinetic entropy steadily increases so long as the distribution is not Maxwellian. It has remained a major open problem of the kinetic theory to prove the conjecture that in these circumstances the distribution function itself approaches Maxwellian form as $t \rightarrow \infty$. In a celebrated analysis, CARLEMAN [1933] has achieved the proof in the special case of rigid spherical molecules and for a distribution function assumed to depend only on the speed (rather than on the velocity vector) and to satisfy initially certain artificial assumptions of uniformity. WILD [1951, 3, §6] has treated the case of a spatially homogeneous distribution of Maxwellian molecules, but his hypothesis concerning the initial value of the distribution function is an elaborate inequality having no evident significance. Using powerful analytic instruments, MORGENSTERN [1955, 4, Part III] has given a proof valid for a certain class of scattering factors, but the only molecular model this class has been shown to include is the Maxwellian in a space of two dimensions.

Since apart from the entropy it is only the moments of the distribution function that have physical significance, the result sought is unnecessarily strong. Ample would be the following: *Each spherical moment $P_{2r|s}$ tends with time toward the corresponding spherical moment $P_{2r|s}^{(0)}$ of an appropriate Maxwellian distribution.* For Maxwellian molecules in the case when p , ρ , u_i , and $P_{2r|s}$ are spatially homogeneous, the above **Theorem of the trend toward equilibrium** is easy to prove¹ by simple calculations based on the fundamental theorem asserted in §7 and proved in §13. It is an obvious corollary of the following

Theorem (general solution for the spatially homogeneous case). *If p , ρ , u_i , and $P_{2r|s}$ are spatially homogeneous, then the general solution of the equations of transfer is of the form*

$$P_{2r|s} = \sum_{i=1}^N A_{2r|s|i}(t) e^{-\lambda_i^{2r|s} t} + P_{2r|s}^{(0)}, \quad (52.1)$$

where $\lambda_i^{2r|s} > 0$; $A_{2r|s|i}(t)$ is a polynomial in t with coefficients which are functions of p , ρ , u_i , r , and s ; $N = N(r, s)$; and $P_{2r|s}^{(0)}$ is given by (5.11).

The proof is not difficult, but it will be easier to follow if we make some preliminary remarks and work out the explicit form of (52.1) for the first few moments. First, it follows from the equations of mass, momentum, and energy ((8.17), (15.6), and (2.8)) that ρ , u_i , and p are constant in time. For P_{ii} , (52.1) is the classical result (31.9) of MAXWELL [1867, eq. (130)]:

¹ The problem is a special case of that considered in Chapter V, but there we had little interest in the higher moments and made no attempt to find them, while here it is the higher moments that are our objective.

$$P_{ii} = P_{ii}(0)e^{-t/\tau}. \quad (52.2)$$

For the third moments, from the equations that follow by equating (6.5) to (8.5) and (6.6) to (8.6) we read off another result due to MAXWELL [1879, eqq. (33)–(35) and the end of the addition following §5] (*cf.* also [1949, 1, eqq. (5.21)₂, (5.21)₃]):

$$h_i = h_i(0)e^{-\frac{2}{3}t/\tau}, \quad P_{iik} = P_{iik}(0)e^{-\frac{1}{3}t/\tau}. \quad (52.3)$$

It is at the fourth moments that interesting complications begin. Here MAXWELL tells us only “... I found that, to the order of approximation required, all harmonics of the fourth and sixth orders may be neglected, so that the rate of decay of harmonics of the second order, the time-modulus of which is $\mu \div p$, determines the rate of decay of all functions of less than six dimensions” [1879, Note added after §15]. What is meant by “determines” is not clear, and the problem is not one properly involving any “approximation”. The exact equations of fourth moments are obtained by equating (8.7), (8.8), and (8.9), respectively, to (6.7), (6.8), and (6.9). In the spatially homogeneous case, the first of these is

$$\begin{aligned} \tau \dot{P}_{4i} + \frac{2}{3}P_{4i} &= \frac{2}{3\rho} (15p^2 - P_{ii}P_{ii}), \\ &= \frac{2}{3}P_{4i}^{(0)} - \frac{2}{3\rho} P_{ii}(0)P_{ii}(0)e^{-2t/\tau}, \end{aligned} \quad (52.4)$$

where the second step follows by (5.11) and (52.2). The general solution is

$$\begin{aligned} P_{4i} &= \left[P_{4i}(0) - P_{4i}^{(0)} - \frac{1}{2\rho} P_{ii}(0)P_{ii}(0) \right] e^{-\frac{1}{3}t/\tau} \\ &\quad + \frac{1}{2\rho} P_{ii}(0)P_{ii}(0)e^{-2t/\tau} + P_{4i}^{(0)}. \end{aligned} \quad (52.5)$$

Similarly, the general solutions for the remaining spherical moments of fourth order are

$$\begin{aligned} P_{21ii} &= \left[P_{21ii}(0) - \frac{7p}{\rho} P_{ii}(0) - \frac{4}{5\rho} \{P_{ik}(0)P_{ki}(0) - \frac{1}{3}P_{kl}(0)P_{kl}(0)\delta_{ii}\} \right] e^{-(7/5)t/\tau} \\ &\quad + \frac{7p}{\rho} P_{ii}(0)e^{-t/\tau} + \frac{4}{5\rho} [P_{ik}(0)P_{ki}(0) - \frac{1}{3}P_{kl}(0)P_{kl}(0)\delta_{ii}]e^{-2t/\tau}, \end{aligned} \quad (52.6)$$

$$\begin{aligned} P_{ijkl} &= \left[P_{ijkl}(0) + \frac{3}{\rho} (P_{(ij}(0)P_{kl)}(0) \right. \\ &\quad \left. - \frac{4}{7}P_{m(i}(0)\delta_{jk}P_{l)m}(0) + \frac{2}{35}P_{mn}(0)P_{mn}(0)\delta_{(ij}\delta_{kl)}) \right] e^{-\beta_{4l}t/\tau} \\ &\quad - \frac{3}{\rho} (P_{(ij}(0)P_{kl)}(0) - \frac{4}{7}P_{m(i}(0)\delta_{jk}P_{l)m}(0) + \frac{2}{35}P_{mn}(0)P_{mn}(0)\delta_{(ij}\delta_{kl)})e^{-2t/\tau}, \end{aligned} \quad (52.7)$$

where (cf. (7.22))

$$\beta_4 \equiv \frac{3}{2} \left(1 + \frac{\mathfrak{B}_4}{6\mathfrak{B}_2} \right) = \tau\mathfrak{C}_4 = 2.096 \dots \quad (52.8)$$

From the basic formula (7.2), it is clear that the process just given is entirely general. However, since the theorem asserted above settles a famous open problem, it seems advisable to write out a detailed formal proof.

Proof (by induction). The fact that $\rho = \text{const.}$, $u_i = \text{const.}$, $P_{2r} = 3p = \text{const.}$, and the formulæ (52.2), (52.3), (52.5)–(52.7) verify (52.1) for all spherical moments of orders 0, 1, 2, 3. Henceforth we fix both r and s , and we assume that (52.1) holds for all spherical moments of all orders less than $2r + s$. By (4.1), (4.2), and (7.2), to calculate $P_{2r|s}$ we must integrate

$$\dot{P}_{2r|s} = -\mathfrak{C}_{2r|s} P_{2r|s} + Q_{2r|s} . \quad (52.9)$$

In the formula (7.3) for $Q_{2r|s}$, by the hypothesis of induction we may replace $P_{2r_1|s_1}$ and $P_{2r_2|s_2}$ by expressions of the form (52.1), thus obtaining

$$\begin{aligned} Q_{2r|s} &= \sum_{r_1, r_2, s_1, s_2} A_{r_1 r_2 s_1 s_2} \left[\sum_{j=1}^{N_1} A_{2r_1|s_1|j} e^{-\lambda_j^{2r_1|s_1} t} + P_{2r_1|s_1}^{(0)} \right] \\ &\quad \cdot \left[\sum_{k=1}^{N_2} A_{2r_2|s_2|k} e^{-\lambda_k^{2r_2|s_2} t} + P_{2r_2|s_2}^{(0)} \right], \quad (52.10) \\ &= \sum_{i=1}^R B_{i|s} e^{-\mu_i t} + Q_{2r|s}^{(0)}, \end{aligned}$$

where $\mu_i > 0$; $B_{i|s}(t)$ is a polynomial; M , N_1 , N_2 , and R are certain non-negative integers; and $Q_{2r|s}^{(0)}$ is the form assumed by $Q_{2r|s}$ for a Maxwellian distribution. It is known that the only solution of the Maxwell-Boltzmann equation which is both steady and spatially homogeneous is the Maxwellian equilibrium solution; this solution, moreover, is uniquely determined by the corresponding constant values of ρ , u_i , and p . Putting $\dot{P}_{2r|s} = 0$ in (52.9) therefore yields

$$Q_{2r|s}^{(0)} = \mathfrak{C}_{2r|s} P_{2r|s}^{(0)} . \quad (52.11)$$

The general solution of (52.9) is

$$P_{2r|s} = e^{-\mathfrak{C}_{2r|s} t} \left[C_s + \int e^{\mathfrak{C}_{2r|s} t} Q_{2r|s} dt \right]. \quad (52.12)$$

By (52.10) and (52.11) follows

$$\begin{aligned} P_{2r|s} &= C_s e^{-\mathfrak{C}_{2r|s} t} + \sum_{i=1}^R e^{-\mathfrak{C}_{2r|s} t} \int B_{i|s}(t) e^{(\mathfrak{C}_{2r|s} - \mu_i) t} dt + P_{2r|s}^{(0)}, \quad (52.13) \\ &= C_s e^{-\mathfrak{C}_{2r|s} t} + \sum_{i=1}^R C_{i|s}(t) e^{-\mu_i t} + P_{2r|s}^{(0)}, \end{aligned}$$

where $C_{i|s}(t)$ is a polynomial and C_s is a constant of integration. Just before the Inductive Lemma in §13 it was proved that $\mathfrak{C}_{2r|s} > 0$ if $2r + s \geq 3$. Hence (52.13) is of the form (52.1). Q.E.D.

N.B. I conjecture that the $A_{2r|s|i}(t)$ are in fact constants. The induction just given would establish this if supplemented by a proof that no μ_i in (52.10) equals $\mathfrak{C}_{2r|s}$. In view of (52.10)₁, it would be sufficient to prove that no non-trivial equation of the form

$$\mathfrak{C}_{2r|s} = \sum_i \mathfrak{C}_{2r_i|s_i}, \quad \sum_i (2r_i + s_i) = 2r + s, \quad (52.14)$$

can hold.

Corollary of the general solution (52.1): *The only spatially homogeneous solution satisfying $P_{2r|s}(0) = P_{2r|s}^{(0)}$ is the constant solution $P_{2r|s}(t) = P_{2r|s}^{(0)}$. This is also a corollary of the uniqueness theorem of MORGENSTERN [1954, 3].*

53. Interpretation of $(\mathfrak{C}_{2r|s})^{-1}$ as a time of relaxation. From (52.7) and (52.8) we see that the moments P_4 fall off ultimately as $e^{-2t/\tau}$, while $\mathfrak{C}_4 > 2/\tau$. This illustrates the fact that while indeed there is a term proportional to $e^{-\mathfrak{C}_{2r|s}t}$ in the solution for $P_{2r|s}$, this term in general does not give the asymptotic form of $P_{2r|s}$ as $t \rightarrow \infty$. In the case of the P_4 , for example, the dominant term is proportional to $e^{-2\mathfrak{C}_2t}$. We defer until the next section the problem of calculating this dominant term. The importance of the coefficients $\mathfrak{C}_{2r|s}$ in all the foregoing theory impels us first to find circumstances in which they have a simple mechanical significance. Thus we are led to the special problem solved by the following

Theorem. *Suppose that all moments of order less than q are constant both in space and in time. Then the moments of order q satisfy*

$$P_{2r|s} - P_{2r|s}^{(0)} = [P_{2r|s}(0) - P_{2r|s}^{(0)}]e^{-\mathfrak{C}_{2r|s}t}. \quad (53.1)$$

(In general, no such formula holds for the moments of order greater than q).

Proof. By modifying the argument of §52, or by a fresh induction up to order $q - 1$, it is easy to prove that the moments of order less than q must satisfy $P_{2r|s} = P_{2r|s}^{(0)}$. From (52.10) and the remarks following it we get $Q_{2r|s} = Q_{2r|s}^{(0)} = \mathfrak{C}_{2r|s} P_{2r|s}^{(0)}$; the theorem now follows from (52.12). Q.E.D.

In §14 it was shown that $\mathfrak{C}_{2r|s} \rightarrow \infty$ as $2r + s \rightarrow \infty$. In view of the theorem just proved, this result implies that the greater the orders of the moments initially having their equilibrium values, the faster the moments of still higher orders tend toward their equilibrium values.

54. Estimation of the rate of approach to equilibrium. From the theorem of §52 it is plain that $P_{2r|s} - P_{2r|s}^{(0)}$ is bounded both above and below by quantities which tend to zero monotonically and exponentially. We consider the possibility

that the moments of high order approach their equilibrium values relatively slowly, since if such were the case the trend toward equilibrium might be so slow as to lack physical significance. In fact, we shall prove that this doubt is unfounded: No moment approaches its equilibrium value more slowly than $e^{-\frac{2}{3}t/\tau}$.

From (52.1) we see that as $t \rightarrow \infty$

$$P_{2r|s} - P_{2r|s}^{(0)} = O(e^{-\lambda^{2r|s}t}), \quad (54.1)$$

where the O -symbol is understood as possibly including a fixed power of t , and where

$$\lambda^{2r|s} \equiv \text{Min}_i \lambda_i^{2r|s}. \quad (54.2)$$

We shall prove by induction that

$$\tau \lambda^{2r|s} \geq \frac{2}{3} \quad \text{if} \quad 2r + s \geq 3. \quad (54.3)$$

For by (52.2), (52.3), and (52.5)–(52.7), we get

$$\begin{aligned} \tau \lambda^{2|1} &= \frac{2}{3}, & \tau \lambda^3 &= \frac{3}{2}, \\ \tau \lambda^{4|1} &= \frac{2}{3}, & \tau \lambda^{2|2} &= 1, & \tau \lambda^4 &= 2, \end{aligned} \quad (54.4)$$

so that (54.3) holds for the spherical moments of orders 3 and 4. Fix r and s , and assume (54.3) holds for all orders $\leq 2r + s - 2$. By (52.13), one of the $\lambda_i^{2r|s}$ is $\mathfrak{C}_{2r|s}$; the expansion of the product in (52.10) shows that the remaining $\lambda_i^{2r|s}$ are either λ 's for certain $P_{2r|s}$ of orders $2r + s - 2$, $2r + s - 3$, $\dots$, or the sum of two such λ 's. Hence, by the hypothesis of induction,

$$\lambda^{2r|s} \geq \text{Min} \left[\mathfrak{C}_{2r|s}, \frac{2}{3\tau} \right]. \quad (54.5)$$

The assertion now follows by (14.4). Q.E.D.

In words, (54.3) may be put as follows:

Theorem. *No spherical moment relaxes more slowly than the flux of energy.*

As far as physical interpretation, this theorem disposes of any possible difficulty. There remain, however, some mathematical questions. Very likely it can be shown that F itself approaches Maxwellian form, but there is little interest in this refinement. From (54.4) and by inspection of (8.10) we see that the minimum exponent, $\frac{2}{3}/\tau$, is actually achieved for $P_{3|1}$, $P_{4|1}$, and $P_{4|1}$. In §14 it was shown that $\mathfrak{C}_{2r|s} \rightarrow \infty$ as $2r + s \rightarrow \infty$, and this may support a guess that the $\mathfrak{C}_{2r|s}$ for large $2r + s$ exceed at least some of the $\lambda_i^{2r|s}$ contributed by the lower moments. I am thus led to formulate

Conjecture 1. *The spherical moments $P_{2r|1}$ and $P_{2r|i}$, $r = 2, 3, \dots$, relax at the same rate as the flux of energy. (I.e., $\lambda^{2r|1} = \lambda^{2r|i} = \frac{2}{3}/\tau$.)*

Looking back at (52.10), we see that to prove the first assertion it would be sufficient to show that in Q_{2r+1} the coefficient of the term proportional to $P_4 P_{2r-4}$ does not vanish for $r \geq 3$. A similar result concerning Q_{2r+1} would establish the second assertion. In principle, these assertions could be tested by straightforward but apparently formidable calculation along the lines of §13.

The splitting of the moments into spherical moments, while mightily easing all the calculations, does not appear to reflect any physical distinction. For the moments themselves, Conjecture 1 or a weaker result would lead to

Conjecture 2. *For $n \geq 3$, every moment $p_{i_1 i_2 \dots i_n}$ relaxes at the same rate as the flux of energy.*

CHAPTER VII. *Concluding perspectives*

55. Relative merits of various methods. Precise theorems and estimates concerning various methods of iterative or approximate solution have been given at various points in the memoir, and our summary has already been presented in §1. We now stand off from the work as a whole and try to lay out some guide lines through it.

First, little toward solving problems of flow can be expected from the Chapman-Enskog method, which in the example of simple shearing flow has turned out to be the least accurate of all considered in this memoir. The Navier-Stokes solution has a definite asymptotic status within the kinetic theory, and hence so does the viscosity (and perhaps also the heat conductivity, though there is room for doubt here). The formula which the Chapman-Enskog method produces for the viscosity in terms of the molecular model may be correct, but perhaps that is as far as we dare go.

We recall the failure of the higher Chapman-Enskog approximations to yield results in better agreement with experimental data on shock thickness and ultrasonic absorption and dispersion than are those from the Navier-Stokes equations. We have not studied these two special problems and thus have not explained the discrepancy. However, for other problems we have seen several examples of possible error or demonstrated error in results of the Chapman-Enskog type, as follows:

1. The accumulation of assumptions, explained in footnote 1 of §18.
 2. The unnecessary elimination of time derivatives, especially the entropy rate (§15).
 3. The irregularity of differential iteration as a method of calculating a solution (§26).
 4. The slow convergence of polynomial approximations (§35).
 5. The limit solution's arbitrary determination of initial conditions (which may not be appropriate to the experiments involved) (§42, cf. also §37 and §41).
- In my opinion, these are listed in order of increasing likelihood, and I conjecture that No. 5 gives the true answer, with No. 4 adding to the difficulties.

Such criticisms extend to results obtained by the classical approach as a whole, but in lesser degree. If, after all this probing of principle and dissection of a special case, we might dare to put forward a conjecture, it would be as follows: *For small enough values of such parameters as $\mathfrak{T} \equiv \mu(E_{ii}E_{ii})^{\frac{1}{2}}/p$, the formulæ given by at least some of the methods within the classical approach converge slowly; the limit functions are numerically correct approximations for large time; but results which depend on differentiating these functions (e.g., substituting them in the equations of motion) are in general false.* That is to say, the results are correct as they stand but may not safely be used to solve flow problems.

Moreover, our work has indicated that within the classical approach the

results of those methods which neglect the least converge the most rapidly. Such a dictum would be unnecessary in many branches of mechanics, but in the kinetic theory it stands in contradiction to a generally received view based ultimately on beliefs relating to the mean free path. In particular, we found the best overall results in the early iterates from methods whose results are not restricted to power series: Maxwellian iteration (Chapter I) and the scheme of §20.

For more details regarding our results on the classical approach and on differential iteration, see §1.

Only two methods falling outside the classical approach have been considered here. The first of these is GRAD's method of moments. For it, the problem of shearing flow is a particularly favorable example, since the pressures follow from a system of equations which is the same for GRAD's method as for the exact solution, so that GRAD's method necessarily yields the correct pressures. Not so for the flux of energy. From the "thirteen moment approximation" follow formulæ for h_i which not only are wrong in quantity but moreover fail to reflect even the broadest qualitative features of the exact solution—indeed, except for very small $\mathfrak{T}$ their prediction is diametrically opposed to the truth (§51). In this example, then, predictions for whose derivation any use is made of the equations affected by GRAD's truncation process are entirely wrong.

The second method falling outside the classical approach is the method of integral iteration (§29). For the example of shearing flow, it is easy to see that the scheme (30.10) converges to the exact and general solution (33.12)₂. I do not present the proof because the case and even the approach within it are too special for this convergence to be of interest, aside from the bare fact that of all iterative methods considered in this memoir this is the only one which can be fully trusted even in this simplest of cases.¹ What is more enlightening is a comparison of the second iterate (30.17) with the exact solution. As is usual with approximate solutions when an exact solution is known, this one is far more complicated than the exact solution being approximated. It is not surprising that a series in powers of Kt fails to reveal the correct behavior as $t \rightarrow \infty$ for fixed $\mathfrak{T}$. What is more unfortunate is that for fixed t (30.17) gives $p_{xy}^{(2)} \rightarrow 0$ as $\mathfrak{T} \rightarrow \infty$, and this is not the behavior of the exact solution. Also the periodic nature of the oscillatory part and the correct law for the decay factor would hardly be inferred from (30.17), but this is less important. As a matter of fact, if we calculate $p_{xy}^{(2)}$ we find that it approaches ∞ with $\mathfrak{T}$, as it should. Although the method is correct, its convergence is slow. When we recall the labor required to push the in essence far simpler method of Maxwellian iteration to a general third iterate, we see that it may be very laborious to reach any reliable information by our new method.

¹ No qualification is needed here in respect to any given stage in GRAD's method. To see that the remark in the text above remains correct in respect to the thirteen moment approximation, however, we need to set up the equations of integral iteration for the third moments. Similar difficulties with the fourth moments may be expected with the twenty moment approximation, *etc.*

Lastly, our attack on the methodological paradox (§23) has not led to its elucidation. The counter-example of §37 shows that to determine a solution according to the kinetic theory requires in general more initial data than does a hydrodynamical solution, and that this initial data *cannot* be fully eliminated by the asymptotic process which was successful in the case of the model (§25). However, this special example in itself much reduces the initial data required to determine the pressures. The possibility remains, for example, that the paradox could ultimately be resolved by asymptotic elimination of an infinite number of initial conditions—say, the initial values of all moments except the pressures and the flux of energy themselves.

It is fitting that this appraisal should end in tribute to the magnificent genius of MAXWELL. Such ideas in the present memoir as are not entirely new derive from reading and rereading MAXWELL's two great analytical researches on the kinetic theory [1867, 1879].² Indeed, in spirit this memoir passes over all developments in the kinetic theory since 1879 and goes back for its source and inspiration to what MAXWELL left us.

56. Extension to molecular models other than the Maxwellian. The analysis in this memoir has been limited to the case of molecules repelling each other as the inverse fifth power of the distance. To see to what extent the results can be extended to other models, we must begin with the equations of transfer. These are again of the form (4.1), where the differential operator $\mathfrak{L}(Q)$ is still given by (4.2), the integral operator $m\mathfrak{C}(Q)$ by (3.2). The fundamental theorem (7.2) must be modified, however. In fact, the general form of $m\mathfrak{C}(Y_{2r|s})$ may be determined from the remarks following (13.16) (*cf.* also [1949, 1, §4 and Appendix 3]). The expression (7.3) retains its validity, except that the sum is now infinite, running over all values of the indices such that $2r_1 + s_1 > 0$, $2r_2 + s_2 > 0$. In particular, $P_{2r|s}$ itself is not excluded.

Plainly there is no simple way of generalizing the exact solutions of Chapters V and VI to the general case, and the model in Chapter III similarly fails to represent the added complications.

All iterative methods developed in Chapters I, II, and IV may be extended in a variety of ways. To illustrate the ideas it suffices to consider Maxwellian iteration. The most straightforward is to interpret $m\mathfrak{C}^{(n+1)}(Q)$ in (4.5) as $-\mathfrak{C}_{2r|s} P_{2r|s}^{(n+1)} + Q_{2r|s}^{(n)}$, where $Q_{2r|s}^{(n)}$ is the result of replacing each $P_{2u|v}$ in $Q_{2r|s}$ by $P_{2u|v}^{(n)}$. This method has the disadvantage of presenting us with an infinite series which must be summed at each stage of the iteration. Since to obtain the coefficients in this series requires laborious evaluation of certain quadratures, this method is impractical. It seems natural to modify it by truncating the infinite series (7.3) after the moments of order $2r + s + n - 1$. Then each step in the iterative process will employ not only the equations of transfer for moments

² I suspect that if MAXWELL's notes and unpublished calculations have been preserved, they may contain much that is of value.

of order 1 greater than the last but also terms in the series (7.3) involving moments of order 1 greater. For example, we shall have

$$\begin{aligned} Q_{ij}^{(1)} &= A_{ijklmn} P_{kl}^{(0)} P_{mn}^{(0)} = 0, \\ Q_{ij}^{(2)} &= A_{ijklmn} P_{kl}^{(1)} P_{mn}^{(1)} = \mu^2 A_{ijklmn} E_{kl} E_{mn}, \end{aligned} \quad (56.1)$$

these formulæ being especially simple because $P_i \equiv 0$.

By such devices the iterative methods of this memoir may be adapted to such problems as the calculation of viscosity and thermal conductivity for general molecular models.

57. Toward a phenomenological theory of gas flow. In reviewing the matters presented above we may make some curious comments. Chapter V shows that much of the earlier work has erred in trying to force out from the kinetic theory results similar to those of classical hydrodynamics. Much of the present memoir is an essay in applying the methods and concepts of modern non-linear continuum mechanics to a field previously untouched by them. Thence, in particular, grew our emphasis on exact solutions illustrating non-linear interaction, on semi-inverse methods, on models, on limiting values as a parameter vanishes, on rigorous statements regarding convergence and precise estimates of error. On the other hand, in looking back on my work on the Maxwellian fluid [1951, 1], to the criticism in §16 I should now add that my definition of a fluid was unduly influenced by a desire to get within continuum mechanics formulæ of the type resulting from the Chapman-Enskog process in the kinetic theory. In this memoir we have seen (§§15–16) that the appearance of $f_{i,i}$ and of certain vorticity terms in the formulae for p_{ij} and h_i is unnecessary and reflects a deficiency in the Chapman-Enskog method rather than a property of true solutions according to the kinetic theory. Moreover, we have seen in Ch. V that expressions for p_{ij} and h_i in terms of the gross condition can yield results whose correctness is at best very limited.

If we were now to begin afresh with a phenomenological theory of materials including gases, I believe we should combine two ideas which already have been employed separately:

1. The constitutive equations, both for p_{ij} and for h_i , should be functional equations (*e.g.* differential equations) rather than algebraic equations.
2. The entries in the two sets of constitutive equations should be the same.

The former idea has been given its most recent and perfect application, for differential constitutive equations, in NOLL's theory of hygrosteric materials [1955, 1]. NOLL's work, however, is incomplete in that it refers only to the pressures and takes no account of the flux of energy. Thus it is not suitable for application to gas flows (*cf.* footnote 3 of §31). To this end it should be generalized by use of the second idea, which is but an extension of that I introduced earlier under the name of *Brillouin's principle* [1951, 1, §19].

The simplest theory of the desired type would rest on nine constitutive equations which are differential equations of first order in time and space. It would not include the Maxwellian kinetic theory, but necessarily it would include and generalize GRAD's "thirteen moment approximation" (§17). More elaborate theories could be constructed along the same lines. For example, it now seems urgent to study exact, non-linear theories of the BOLTZMANN-VOLTERRA type.

When we take into account the recent progress in continuum mechanics, the now well known failings of certain aspects of the kinetic theory and of the mathematical methods previously employed in it, and the results both positive and negative in the present memoir, the usefulness of a new phenomenological theory is evident. At the very least, by introducing appropriate material parameters it could extend to polyatomic gases whatever results on gas flows are now correctly established in the Maxwellian theory of monatomic gases. There is at present no evidence in experiment or in theory sufficient to support any positive claim either for continuum mechanics or for the Maxwellian kinetic theory as being the closer to a physically correct description for streaming gases.

The general lines along which the new continuum theory could be constructed will now be clear to anyone adept in modern continuum mechanics. The details are certain to be elaborate.

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INDEX OF SPECIAL SYMBOLS

Only those symbols used in more than one section are listed. In some cases these symbols are defined and used in a different sense within a single section.

Symbol	Name or Explanation	Equation of definition or section of first appearance
A		(33.12), (33.13)
A_2		(2.3)
$\mathfrak{A}_{jk}$		(7.14)
$\mathfrak{A}_k \equiv \mathfrak{A}_{0k}$		7
$\mathfrak{A}$		(33.6)
a_{rj}		(7.5), (13.21)
a_{jkl}		(7.10)
B		(33.12), (33.13)
$\mathfrak{B}_{jk}$		(7.6)
$\mathfrak{B}_k \equiv \mathfrak{B}_{0k}$		7
b	Polar co-ordinate for encounter	3
C		(33.12), (33.13)
$\mathfrak{C}$	Collision operator	(3.2)
$\mathfrak{C}_{2r s}, \mathfrak{C}_{2r } \equiv \mathfrak{C}_{2r 0}, \mathfrak{C}_s \equiv \mathfrak{C}_{0 s}$		(7.2)
$c_i, \mathbf{c}, c$	relative molecular velocity	2
E, E_{ij}	expansion, deviatoric rate of deformation	(2.7)
F	molecular distribution function	2
f_i	extrinsic force per unit mass	15
G	force constant for Maxwellian molecules	2
h_i	flux of energy	2
$I_{2r s}, J_{2r s}$		(13.28)
$\mathfrak{I}$		(33.10)
K	rate of shearing	(30.1), (33.1)
$K(k)$	complete elliptic integral	7
k	Boltzmann constant	2
$\mathcal{L}$	differential transfer operator	(4.2)
m	molecular mass	2
P_s	Legendre polynomial	(7.7)
P_{ii}	pressure deviator	(2.7), (5.8)
$P_{2r s} = P_{2r i_1 i_2 \dots i_s}$	spherical moment	(5.8)
$P_{2r } \equiv P_{2r 0}, P_s \equiv P_{0 s}$		(5.8)
p	mean pressure	2
p_0		(33.14)
p_{ii}	pressure tensor	2

$p_{i_1 i_2 \dots i_n}$	n^{th} moment	(2.2)
$p^{\text{D}}, p_{ij}^{\text{D}}, p^{\text{O}}, p_{ij}^{\text{O}}$		(33.15)
$Q_{2r s}$		(7.2)
$\mathfrak{K}$		(33.10)
r		(33.9)
$\mathfrak{T}$	truncation number	(30.3), (33.2)
$u_i, \mathbf{u}$	stream velocity	2
$2\mathbf{v}, 2v$	relative velocity before encounter	(3.1)
$2\mathbf{w}, 2w$	specific momentum of molecular pair . .	(3.1)
$\mathbf{x}$	position in the gas	2
$Y, Y_i, Y_{ij}, \dots$	special spherical harmonics	(5.1)–(5.6)
$Y_s(\mathbf{c}) = Y_{i_1 i_2 \dots i_s}$	special spherical harmonics	(5.1)–(5.6)
$Y_{2r s} = Y_{2r i_1 i_2 \dots i_s}, Y_{2r } \equiv Y_{2r 0}, Y_s \equiv Y_{0 s}$		(5.7)
α	ratio of thermal conductivity to viscosity .	(2.17)
$\alpha(\phi)$		(7.16)
β	proportional to specific entropy	(9.7)
δ		(3.3)
ϵ	encounter angle	3
η	specific entropy	(15.1)
θ	temperature	2
$\lambda_i^{2r s}$		(52.1)
μ	viscosity	(2.3), (7.21)
ξ_i, ξ	molecular velocity	2
Π_{ij}	initial pressure ratios	(33.16)
$\pi_{ij}, \pi_{ij}^{\text{D}}$	relative pressure ratios	(33.16)
$\varpi, \varpi^{\text{D}}, \varpi_{ij}^{\text{D}}$	absolute pressure ratios	(33.16)
ρ	mass density of the gas	2
τ	time of relaxation	(30.2), (31.7)
ϕ	encounter angle	3
$()^{\cdot}$ or $(\dot{})$	material derivative	(2.6)
$()'$ up to §30	evaluation after encounter	3
$()'$ from §30 onward	derivative	
$()^*$	quantity associated with oncoming molecule	3
$()^{(n)}$	n^{th} Maxwellian iterate (also used for some other schemes in Ch. II)	2, 4
$()^{(0)}$	zero th Maxwellian iterate (unless otherwise specified)	(2.12), (5.11)
$()^{(n)}$	n^{th} iterate by the scheme of §20	20
$()^{\{n\}}$	n^{th} atemporal Maxwellian iterate	15
$^{[n]}()$	solution corresponding to n^{th} atemporal Maxwellian iterate	38
$()^{[n]}$	n^{th} iterate by powers	19
$()^{\langle n \rangle}$	n^{th} integral iterate	(28.3)

INDEX OF SPECIAL TERMS

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