On the Pressures and the Flux of Energy in a Gas according to Maxwell's Kinetic Theory, I

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NOTE. Chapter I was commenced in the summer of 1954; Chapter VI was completed in the summer of 1955. These parts of the work were supported by contract from the Office of Naval Research and the Army Office of Ordnance Research to Indiana University.

While all material in Part I of this memoir has been discussed and revised jointly, the main ideas in the analysis in §§5–14, including the fundamental theorem (7.2) and its proof in §13, are due to IKENBERRY; to TRUESDELL, the iterative schemes in §§2, 4, 16–18, and 20–22, the proof of effectiveness in §13, and most of the exposition.

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1. Objectives and summary. This memoir treats solely the analytical problem of solving the equations of MAXWELL's kinetic theory. The physical worth and physical implications of the theory itself, recently the subject of much discussion elsewhere, are not searched here. Most of our calculations are restricted to the case of Maxwellian molecules.¹

Let not the reader who glances at the following pages lose heart at their adamantine aspect. Truly, in many cases the *results* are elaborate, but not only are the *ideas* simple and immediate, even the *calculations* leading to the long formulæ are easy, especially in contrast to everything done previously in this subject. Clarity is our goal. Neither any advanced mathematics nor any knowledge of physics beyond a firm grasp of the usual formal structure of the Maxwellian kinetic theory is required in order to follow every step.

The memoir consists in six loosely connected studies, printed as a single work in two parts. Each chapter may be read independently of the others except for the general preliminaries in §2 (and §§7 and 14 for Chapter VI).

Our first aim is to construct a method of formal solution of the equations of transfer which yields results of the type associated with the names of CHAPMAN, ENSKOG, and BURNETT yet is simple enough to be surveyed and grasped both in principle and in detail. This aim is achieved in Chapter I. Special forms or series for the distribution function, physical claims, arguments of "order", and other so-called approximations are scrupulously avoided. Assuming only the existence and differentiability of the moments we construct a simple, explicit, formal iterative scheme. The idea of the method, which is called *Maxwellian iteration*, can be mastered from the material in §2 alone. The equilibrium solution is taken for the zeroth iterate; the first iterates yield the Newton-Navier-Stokes formula for the pressures and the Newton-Fourier formula for the flux of energy; the second iterate for the pressures, which follows almost without additional calculation, is essentially the Burnett formula. The rest of Chapter I is devoted to construction of the n^{th} Maxwellian iterate. It includes proof that the scheme is effective at the n^{th} stage and presents explicit formulæ for the third iterates to the pressures and the flux of energy, $P_{ij}^{(3)}$ and $h_i^{(3)}$.

The key to the formal calculations is the introduction of the *spherical moments* $P_{2r|s}$, defined in §5. These are certain linear combinations of the ordinary moments $p_{i_1 i_2 \dots i_q}$ of order $q = 2r + s$. The corresponding equations of transfer have a particularly simple form, which is stated in §7 and proved in §13. This form is a differential equation

$$\dot{P}_{2r|s} + \dots = -\mathfrak{C}_{2r|s} P_{2r|s} + Q_{2r|s}, \quad (1.1)$$

¹Three portions are valid more generally:

1. The classical material in eqq. (3.1)–(4.2).
2. The definition of the spherical moments and hence the eqq. (5.1)–(6.9).
3. The method of calculating collision integrals, eqq. (13.1)–(13.16). Cf. also §56.

where the dots stand for a certain bilinear form in the spherical moments of orders q and $q - 1$ and the spatial gradients of the moments of orders 1 and 2 plus a linear form in the gradients of the spherical moments of order $q + 1$; $\mathfrak{C}_{2r|s}$ is a constant; and $Q_{2r|s}$ is a bilinear form in the spherical moments of orders less than q . In these equations, *only a single spherical moment of order q occurs linearly*. By using these particular equations of transfer we achieve a thorough easing of the algebraic side of the problem, a simplification which becomes more and more useful as higher moments are considered.

Chapter II discusses variants of the iterative scheme introduced in Chapter I. These concern the choice both of the iterative equations and of the zeroth iterate. Alteration of the former allows a connection with GRAD's method of moments. Alteration of the latter permits approximation about a state of free molecular flow, precluding universal formulæ for the pressures and the flux of energy, whose relation to the kinematical variables and the thermostatic state varies from one problem to another. As an example we consider free unidirectional expansion.

All these methods, like Maxwellian iteration, in essence seek solutions of a differential system by means of repeated differentiations. Such methods we call *differential iterations*. In the context of the kinetic theory, in most applications they fall under the *classical approach*, which seeks to express the pressures and the flux of energy by universal formulæ, valid for all conditions of flow, in terms of the density, pressure, stream velocity, and their derivatives of all orders with respect to space and time—as we shall say, in terms of the *gross condition* of the gas. Examples of the classical approach but not of differential iteration are the methods of HILBERT, ENSKOG, and CHAPMAN.

Chapter III attempts to cast light both on the analytical side of the kinetic theory itself and on the classical approach to it. Consider first the latter. Chapters I and II wrought the extremely simple method of differential iteration for getting formulæ of the classical type: To solve a system of differential equations, it employs infinite differentiation. This method, while dimly foreseen in MAXWELL's work, seems most implausible and has never before been put forward. Obviously it gives no possibility for satisfying arbitrary initial conditions. If it converges to a solution at all, that solution will necessarily be only a special one. While the method of differential iteration thus is useless for getting the general solution, it may nevertheless turn out to be just the thing for special problems where a particular solution is specially singled out in some way. Perhaps this is the case in the kinetic theory. Perhaps there is a particular solution which in some sense is approximate to all members of a general class, and perhaps differential iteration with properly selected zeroth iterate can yield just this solution in preference to all others. If so, differential iteration solves the famous *methodological paradox* of the kinetic theory, for it eliminates the troublesome problem of foreknowledge, at least in principle, of the initial values of the higher moments.

To put these ideas on a rigorous basis would justify one aspect of the classical approach and might establish it in general.

The infinite system of equations of moments in the kinetic theory is so elaborate in form that attack on this fundamental problem is now beyond us. Perforce we rest content with analysis of a mathematical model. This model is an infinite sequence of ordinary differential equations, in which pressure, viscosity (μ), differential iteration, and the Navier-Stokes approximation have each their counterparts. For the model, moreover, not only can the n^{th} differential iterate be calculated explicitly but so also can the general exact solution of the initial value problem. For the model we obtain the following precise results:

1. For a very broad class of initial values, there is a *common asymptotic solution* as $\mu \rightarrow 0+$, for any fixed positive t and fixed initial conditions. This asymptotic form is a *particular exact solution*, being in fact the only steady solution for which the higher moments vanish identically, or, if we prefer, the only solution which is an analytic function of μ near $\mu = 0$. The Navier-Stokes approximation is the leading term in the power series expansion of this solution.
2. For a somewhat narrower class of initial values, the asymptotic solution of No. 1 is also the *limiting form* of an arbitrary solution as $t \rightarrow \infty$ when μ and the initial conditions are held fixed.
3. For many kinds of differential iteration, the Navier-Stokes formula is the first iterate.
4. There are simple conditions sufficient that differential iteration converge; its limit, if a solution, is a particular solution determined by the choice of the zeroth iterates.
5. For a broad class of zeroth iterates, uniformly bounded constants being included, properly selected differential iteration converges for sufficiently small values of μ and t , and *its limit is the asymptotic solution*.

These results, which are proved in §25 and §26, would seem to foster great hope for the method of differential iteration and hence for results of the classical type in the kinetic theory. In fact, however, the unusual nature of differential iteration and the caution with which it must be treated is revealed at the end of §26:

6. In a special case, the asymptotic solution is the Navier-Stokes formula. In this case differential iteration begins with the Navier-Stokes formula as first iterate and ultimately converges back to it. *Every "higher approximation" is worse than the first.*

These results are proved only for the model. It is not impossible, however, that they may indicate what is to be expected in the kinetic theory and may give rise to further researches. On the analytical side, they introduce a new branch of the theory of differential equations: *asymptotic solution by infinite differentiation*. For mechanics in general, the existence of the asymptotic solution

is quite extraordinary, since it is the first example of a phenomenon of the edge or boundary-layer type to exhibit the confluence of an *infinite number of initial conditions* in the limit as a parameter vanishes. For the kinetic theory, our theorems suggest *a precise and accurate meaning for the results of the classical approach*, but also they give warning that *convergence may be so slow as to negate certain claims*, based usually on beliefs concerning the mean free path, which have been alleged in favor of the so-called higher approximations. If a result like No. 6 is valid for the kinetic theory, it explains the now well known failure of the first few "approximations" by the Chapman-Enskog process to yield results as near to experimental data on the thickness of shock waves and on the absorption and dispersion of ultrasonic waves as are those which follow from the Navier-Stokes equations (for review and references, cf. [1954, 1]; see also §55).

Chapter IV sketches two new methods of integral iteration for solving the general problem of initial values. Formulæ of the classical type do not follow from these methods. The simpler of the two is illustrated for the case of shearing flow, where definite results are easily obtained. No proof of convergence is attempted.

Chapter V points out that for Maxwellian molecules it is possible to find exact solutions in which all moments of order greater than one are constant in space but changing in time. The solution is given in all detail for the case of simple shearing flow, the classical example used for discussing the meaning of the shear viscosity in the theories of fluids. This is the *first exact solution for any dissipative flow* ever obtained in the kinetic theory. It furnishes a counter-example to nearly every one of the views on this subject expressed or implied by the authors of the classical researches:

I. *There is no "normal" solution*, independent of initial conditions, which is approached by an arbitrary solution of the present class. Rather, there is indeed a part of the solution which is rapidly damped, but the remainder or *dominant part* p_{ii}^D is not steady and is not independent of the initial conditions (§33).

II. The ratios p_{ii}^D/p^D of the dominant pressure components to the dominant scalar pressure are functions only of the gross condition of the gas and thus may be compared with the results of various schemes within the classical approach. The phenomenon is governed by a single dimensionless parameter, the *truncation number* $\mathfrak{T} \equiv \mu K/p$, where K is the rate of shearing. The process called *atemporal Maxwellian iteration*, which was introduced in §15 and which appears to be equivalent (with a certain reservation) to the Chapman-Enskog process, is *proved to converge if $\mathfrak{T} < \frac{1}{3}\sqrt{2}$ and to diverge if $\mathfrak{T} > \frac{1}{3}\sqrt{2}$* . When it converges, *its limit functions are the ratios p_{ii}^D/p^D of the dominant parts of the exact solution*. This result contradicts the common view that the power series in the viscosity which result from the classical approach are only asymptotic (§36).

III. We investigate for the absolute pressures p_{ii}^D as functions of time the possibility of an asymptotic solution as $\mu \rightarrow 0+$. For the problem of shearing flow, such an asymptotic series is easy to obtain, but *only the first term, yielding the Navier-Stokes approximation, is independent of the initial conditions*. Thus, by counter-example, *no method within the classical approach can yield in general correct absolute pressures beyond the first non-vanishing asymptotic term*. This is the main theorem of Part II (§37).

IV. We investigate the details of the solution of the problem of shearing flow by atemporal Maxwellian iteration. We prove that the Navier-Stokes solution for the shearing pressure is indeed correct to the first order in $\mathfrak{T}$, that the Burnett solution (which yields the same shearing pressure) indeed gives normal pressures correct to the second order in $\mathfrak{T}$. From the main theorem stated in the preceding paragraph it follows that no method within the classical approach can improve these results by obtaining generally correct further terms in the expansion in powers of $\mathfrak{T}$. It is commonly believed that “solutions” obtained by the classical approach tend to settle into correctness with time. This view is reversed by the proof that for the present problem, after a sufficient time has elapsed, *the “solution” according to any finite step of the Chapman-Enskog process becomes steadily a worse approximation* and in fact is exponentially infinitely larger or infinitely smaller than *any* exact solution as $t \rightarrow \infty$ (§§38-42).

V. We are able to fix the precise position of the solution corresponding to the limit $n \rightarrow \infty$ for atemporal Maxwellian iteration. This limit solution is not an asymptotic formula. Instead, it is *a particular exact solution*, in fact the unique solution for which the damped part is zero. This solution has the correct behavior as a function of time. That this is so indicates the slow convergence of the classical methods: For any finite n , the dependence of the “solution” on time is infinitely in error as $t \rightarrow \infty$, although in the limit $n \rightarrow \infty$ the dependence on time becomes correct. The magnitude of the limit solution is very special, being the result of peculiarly adjusted initial conditions. Moreover, there is nothing to recommend these particular initial conditions as in any way especially appropriate, so that the partiality of the classical approach appears to be conferred upon a merely analytic simplification: The limit solution is the steadiest possible, *i.e.*, the only monotone solution (§42).

The results of this research divest the classical approach to the kinetic theory, as presented for example in BOLTZMANN’S lectures and in the treatise of CHAPMAN & COWLING [1952, 1] and simplified and extended in Chapters I and II of the present memoir, of most of the relevance it was once fancied to have.

VI. It is commonly believed that there is in any flow problem a condition of ultimate rarefaction called “free molecular flow”, in which collisions are so infrequent as to be negligible. It has long been known that for the strict interpretation of this concept within Maxwell’s theory, solutions exist only in a few degenerate cases (locally Maxwellian solutions). However, it has been believed

that the concept of free molecular flow should have meaning in the asymptotic sense as τ , the time of relaxation, becomes infinite, just as the Navier-Stokes solution is valid in the limit $\mathfrak{T} \rightarrow 0$. It has been expected that an asymptotic solution in powers of τ^{-1} , independent of the initial conditions, represents free molecular flow and higher approximations about this state. For the problem of shearing flow, we construct such an expansion, but even the leading term is *not* independent of initial conditions. Thus *the Maxwellian kinetic theory does not in general include the phenomenon of free molecular flow* (§46).

The results obtained from the model in Chapter III and from the exact solution for shearing in Chapter V complement one another. The model is oversimplified, but the treatment of it is entirely general and deeply illuminating. Shearing flow is a very special and in many ways degenerate case, but the equations of the theory are in no way molested, and the solution is exact and uncontroversial. In the model, we set upon the great methodological paradox and conquer it, but, necessarily, only for the model. Shearing flow, on the other hand, is a particular problem where because of the special defining conditions the paradox virtually fails to manifest itself. On the whole, the results gotten from the model are shown by the solution for shearing flow to be unduly flattering both to the kinetic theory and to the classical approach. This can be explained in part. The model corresponds to the infinite system of the equations of moments, *excluding the five field equations of continuum mechanics*. If we make a corresponding break in the discussion of the exact solution for shearing, we pull up at the Conclusion II above: For the *pressure ratios*, as $t \rightarrow \infty$ there is a common asymptotic form which is independent of the initial conditions, and to these ratios atemporal Maxwellian iteration converges for small $\mathfrak{T}$. The in the main seriously wrong results which follow from the classical approach come only after this point: The correct asymptotic forms when put into the correct field equations yield incorrect solutions. This should not, in the end, be hard to understand: The process, being akin to differentiating an asymptotic expansion, is illegitimate. Here we add one more to the many recent examples of the great error which can result from neglecting "small" terms in differential equations.

By calculating the dissipation of energy, we show that according to the kinetic theory it is possible for a gas in the first few instants of shearing flow to give out gross work rather than using it up in deformation. That is, even though the H-theorem is satisfied, the thermostatic entropy may decrease for a while. After lapse of sufficient time, however, dissipation will set in and prevail, and as $t \rightarrow \infty$ the entropy principle and the H-theorem become equivalent. This example constitutes *the first rigorous proof that kinetic entropy and thermostatic entropy are in general distinct* (§50).

A brief discussion of the exact values of the third moments reveals that for small enough values of $\mathfrak{T}$ these moments relax to zero, but for large $\mathfrak{T}$ a flux of energy across the flow is amplified. When the same problem is approached by GRAD's thirteen moment approximation, this instability is not predicted, and in

fact *the rightness of results from GRAD's equations is shown to be confined to small values of $\mathfrak{T}$* (§51).

Chapter VI presents a proof that in the case when all moments are spatially homogeneous, *they approach with time the moments of a Maxwellian distribution* (§52). The solution of this famous open problem is given by an easy explicit calculation based on the fundamental theorem asserted in §7 and proved in §13. In fact, the rate of approach to equilibrium may be estimated, and it is proved that *no moment tends to its equilibrium value more slowly than $e^{-\frac{2}{3}t/\tau}$* (§54).

Chapter VII compares the results obtained from various methods in the kinetic theory. The integral iteration proposed in Chapter IV turns out to be the only known method which when applied to shearing flow exactly and effectively yields the true solution of Chapter V. While most of the explicit calculations in this memoir are carried out only for Maxwellian molecules, §56 takes note of this limitation and explains means of removing it.

In addition to their meaning for the kinetic theory, the results here have a certain significance for pure probability. MAXWELL's kinetic theory furnished historically the first example of a measure whose defect from invariance is specified by a particular mechanism. The collision integral, however, is not only special but also extremely complex. Thus the kinetic theory, despite its priority, has had little influence on the development of the ideas of probability. The results claimed in the kinetic theory up to now, while perhaps physically plausible, have failed of the rigor necessary in work which is to carry conviction among mathematicians. Although in the present paper Part I and Chapter IV, apart from the fundamental theorem and details concerning collision integrals in §§7-8, 13-14, are formal, Chapters III, V, and VI are entirely rigorous. The ideas there and the examples of what may indeed follow from a fully specified stochastic process in mechanics may deserve the notice of those who would study the dissipation which may result in mean from an apparently conservative milieu.

CHAPTER I. *Maxwellian iteration*

2. The basic idea of Maxwellian iteration, and the start of the calculation.

While the molecular view of matter suggests that it may be desirable to replace continuum models by something more accurate, and indeed this idea is regularly encountered in physical conversation, MAXWELL proved that his formal kinetic theory of gases offers no such possibility.¹ This theory, intended to describe the probable motion of a moderately rarefied monatomic gas, is based upon a function $F(\mathbf{x}, \boldsymbol{\xi}, t)$ giving the number density of molecules at the place $\mathbf{x}$ and time t with velocity $\boldsymbol{\xi}$; the temporal evolution of this *distribution function* is determined by BOLTZMANN's equation or by MAXWELL's equation of transfer. MAXWELL showed that, with suitable definitions of pressure tensor p_{ij} and flux of energy h_i in terms of mean molecular actions, field equations having precisely the same form as those of continuum mechanics are exact consequences of his theory [1867, eqq. (74), (76), (94)] [1879, eqq. (39), (41)].

For those who understand this consistency, the fundamental problem of the Maxwellian kinetic theory has been to relate p_{ij} and h_i to the motion of the gas as a whole. Let bars denote *mean* or *expected* values:

$$n\bar{Q} \equiv \int_{\infty} QF \, d\boldsymbol{\xi}, \quad n \equiv \int_{\infty} F \, d\boldsymbol{\xi}. \quad (2.1)$$

With $u_i \equiv \bar{\xi}_i$, $c_i \equiv \xi_i - u_i$, $\rho \equiv mn$, where m = mass of a molecule, put

$$p_{i_1 i_2 \dots i_n} \equiv \overline{\rho c_{i_1} c_{i_2} \dots c_{i_n}} = m \int_{\infty} c_{i_1} c_{i_2} \dots c_{i_n} F \, d\boldsymbol{\xi} \quad (2.2)$$

for the n^{th} moment of F . Then p_{ij} is the second moment, while $h_i = \frac{1}{2}p_{iij}$. Although various formal methods seeking to determine F were proposed by BOLTZMANN, HILBERT, CHAPMAN, and ENSKOG, these if successful would yield more information than could be used for physical interpretation. It is no wonder that these methods are so complicated formally as to obscure their rationale and to engender uncertainty over what really are the "approximations" they employ. Nevertheless, the only ultimate use to which these elaborate methods have been put is to determine forms for p_{ij} and h_i . Most of the work has been directed toward finding approximate expressions for p_{ij} and h_i as functions of p , ρ , u_i , and their derivatives with respect to place and time—as we shall say, functions of the *gross condition* of the gas.

With his usual acumen, MAXWELL [1879, §§4, 10–14] sketched the beginnings of a method of finding p_{ij} and h_i by successive approximations without² first

¹ Except in respect to the entropy inequality of thermodynamics. See §50.

² True, MAXWELL began [1879, eq. (22)] by assuming a special functional form for F , but he made no attempt to determine the numerical coefficients in that form, using it only to infer certain relations among the higher moments in first approximation. In the present work we

determining F . While this method is relatively simple, nevertheless the formal details were elaborate enough that MAXWELL did not publish them; his actual reckonings must be guessed from slight indications which provide the suggestion for the iterative scheme that we shall formulate explicitly in Chapter I of this memoir. While our work is done in MAXWELL's shadow, it is no mere translation of what his papers contain or imply. Rather, we carry to its natural conclusion what seems to us MAXWELL's basic idea, an idea which he himself exploited only in part and disguised by unnecessary and in fact ultimately complicating approximations.

Write p , ρ , and $\theta \equiv mp/(k\rho)$ for the pressure, density, and temperature, k being the Boltzmann constant. Put

$$\mu \equiv \frac{1}{3} \sqrt{\frac{2m}{G}} \frac{mp}{\rho A_2} = \frac{1}{3} \sqrt{\frac{2m}{G}} \frac{k\theta}{A_2}, \quad (2.3)$$

where G is the constant of proportionality between the intermolecular force and the reciprocal 5th power of the distance, and $A_2 = 1.37 \dots$ is a numerical constant which was evaluated by MAXWELL [1867, eq. (16)] (see §7 below). The following equations are exact for Maxwellian molecules:³

$$-\frac{p}{\mu} (p_{ii} - p\delta_{ii}) = \dot{p}_{ii} + p_{ij}u_{k,k} + 2p_{k(i}u_{j),k} + p_{ijk,k}, \quad (2.4)$$

$$-\frac{p}{2\mu} (3p_{ijk} - 2h_{(i}\delta_{jk)}) = \dot{p}_{ijk} + p_{ijk}u_{k,k} + p_{ijk l,l} - \frac{3}{\rho} p_{(i}p_{k)l,l} + 3p_{l(i}u_{k),l}, \quad (2.5)$$

where the dot denotes the material derivative,

$$(\dot{}) \equiv \frac{D()}{Dt} \equiv \frac{\partial}{\partial t}() + ()_{,i}u_i, \quad (2.6)$$

and a parenthesis around s subscripts is the usual notation for a sum over the $s!$ permutations of the indices, divided by $s!$. Put

$$\begin{aligned} P_{ii} &\equiv p_{ii} - p\delta_{ii}, \\ E &\equiv u_{k,k}, \\ E_{ii} &\equiv \frac{1}{2}(u_{i,i} + u_{i,i}) - \frac{1}{3}E\delta_{ii}. \end{aligned} \quad (2.7)$$

introduce these relations among moments by an iterative scheme, otherwise making no reference to F .

The idea of determining p_{ij} and h_i without making any attempt to find F was introduced anew by GRAD [1949, 1, §5], who, while using equations known to MAXWELL, proposed a different scheme of successive approximation. In §17 we shall discuss GRAD's method in relation to various others.

³ MAXWELL did not write out all the terms in these equations, but his remarks make it clear that he had derived them [1867, pp. 68, 74][1879, §§10–11]. They were first published by GRAD [1949, 1, eqq. (5.11), (5.13), (5.14)].

If we make use of these definitions and of the equation of energy,

$$3\dot{p} + 5pE + 2P_{ij}E_{ij} + 2h_{i,i} = 0, \quad (2.8)$$

we may put (2.4) into the more complicated but for our purposes more convenient form

$$-\frac{p}{\mu} P_{ij} = 2pE_{ij} + \dot{P}_{ij} + P_{ij}E - \frac{2}{3}P_{kl}E_{kl}\delta_{ij} + 2P_{k(i}u_{j),k} + (p_{ijk} - \frac{2}{3}h_k\delta_{ij})_{,k}. \quad (2.9)$$

These exact equations suggest several schemes of iteration. Both the heuristic theory of O.-E. MEYER and certain general considerations in fluid mechanics recommend the possibility of a power series in μ . Such series will be our first objective. Let superscript (n) denote the n^{th} iterate, and put⁴

$$-\frac{p}{\mu} P_{ij}^{(n+1)} \equiv 2pE_{ij} + \dot{P}_{ij}^{(n)} + P_{ij}^{(n)}E - \frac{2}{3}P_{kl}^{(n)}E_{kl}\delta_{ij} + 2P_{k(i}^{(n)}u_{j),k} + (p_{ijk}^{(n)} - \frac{2}{3}h_k^{(n)}\delta_{ij})_{,k}, \quad (2.10)$$

$$\begin{aligned} -\frac{p}{2\mu} (3p_{ijk}^{(n+1)} - 2h_{(i}^{(n+1)}\delta_{jk)}) &\equiv \dot{p}_{ijk}^{(n)} + p_{ijk}^{(n)}E \\ &+ p_{ijkl,l}^{(n)} - \frac{3p}{\rho} p_{,(i}\delta_{jk)} - \frac{3}{\rho} p_{,(i}P_{jk}^{(n)} \\ &- \frac{3}{\rho} p\delta_{(ij}P_{k)l,l}^{(n)} - \frac{3}{\rho} P_{(ij}^{(n)}P_{k)l,l}^{(n)} + 3p_{l(i}^{(n)}u_{k),l}. \end{aligned} \quad (2.11)$$

To complete the definition of the iterative scheme, we must specify:

1. The zeroth iterates $p_{ij}^{(0)}$, $p_{ijk}^{(0)}$, etc.

2. The method of calculating the iterates of the fourth moments, $p_{ijkl}^{(n)}$.

It is the second requirement which presents itself under one guise or another in every attempt at integration in the kinetic theory, and which makes our present method interesting and different from other types of iteration.

We can make a start before facing this second requirement. To do so, we decide to let the zeroth iterates be the values appropriate to equilibrium, in the following sense:

Initiation agreement. For $p_{i_1 i_2 \dots i_n}^{(0)}$ we take the expression for $p_{i_1 i_2 \dots i_n}$ in terms of the thermodynamic state which holds in equilibrium:

$$\begin{aligned} p_{ij}^{(0)} &= p\delta_{ij}, & P_{ij}^{(0)} &= 0, \\ p_{ijk}^{(0)} &= 0, & h_i^{(0)} &= 0, \\ p_{ijkl}^{(0)} &= 3\rho \left(\frac{k\theta}{m}\right)^2 \delta_{(ij}\delta_{kl)}, & \dots \end{aligned} \quad (2.12)$$

⁴ The beginning of such an iterative scheme was suggested by CHAPMAN & COWLING [1952, 1, Note G], but they did not develop it or state the various alternatives within it.

These relations follow easily from a recurrence relation for the moments of a Maxwellian distribution:⁵

$$\begin{aligned} p_{i_1} &= 0, & p_{i_1 i_2} &= p \delta_{i_1 i_2}, \\ p_{i_1 i_2 \dots i_n} &= (n-1) \frac{k\theta}{m} p_{(i_1 i_2 \dots i_{n-2} \delta_{i_{n-1} i_n})}. \end{aligned} \quad (2.13)$$

In virtue of this agreement, we may begin the iteration and put $n = 0$ in (2.10) and (2.11), thus obtaining

$$P_{ij}^{(1)} = -2\mu E_{ij}, \quad (2.14)$$

$$h_i^{(1)} = -\alpha\mu\theta_{,i}, \quad (2.15)$$

$$p_{ijk}^{(1)} = -\frac{6}{5}\alpha\mu\theta_{,(i}\delta_{jk)}, \quad (2.16)$$

where we have set

$$\alpha \equiv \frac{15}{4} \frac{k}{m} = \frac{15p}{4\rho\theta}. \quad (2.17)$$

Thus the first step yields the Newton-Navier-Stokes formula for the pressures and the Newton-Fourier law for the flux of energy; the viscosity is μ and the conductivity is $\alpha\mu$.

To calculate $P_{ij}^{(2)}$, we put $n = 1$ in (2.10). By (2.14)–(2.16) follows immediately

$$\begin{aligned} P_{ij}^{(2)} &= -2\mu E_{ij} + \frac{2\mu}{p} (\mu E_{ij})^\cdot + \frac{2\mu^2}{p} EE_{ij} + \frac{4\mu^2}{p} E_{k(i} u_{j),k} - \frac{4\mu^2}{3p} E_{ki} E_{kl} \delta_{ij} \\ &\quad + \frac{\mu}{\rho\theta} [3(\mu\theta_{,(i}\delta_{j)}) - (\mu\theta_{,k})_{,k} \delta_{ij}]. \end{aligned} \quad (2.18)$$

Apart from a different numerical coefficient [*cf.* §19 below], MAXWELL [1879, eq. (53)] published the term in brackets and doubtless had obtained terms of the same general form as the rest, though with different numerical coefficients; various of the other terms have been calculated by various authors, and the entire formula is usually associated with the name of BURNETT.⁶

⁵ One such relation is derived in [1950, 1, Appendix 6], but we prefer that given here because it yields symmetric $p_{i_1 i_2 \dots i_n}$ whether or not $p_{i_1 i_2 \dots i_{n-2}}$ is expressed as a function of p and ρ .

⁶ The form given by CHAPMAN & COWLING [1952, 1, eqq. 15.3.6, 15.41.1, 15.41.3] (there is an error in the formula as printed in [1939]) for Maxwellian molecules is identical with ours but for two exceptions: (1) Instead of the material derivative $\dot{E}_{ij}$, CHAPMAN & COWLING obtain $D_0 E_{ij}/Dt$, where D_0/Dt is a certain “approximation” to the material derivative (*cf.* [1952, 1, §7.14]). This difference of results, which is noted also in [1952, 1, Note G], is important; see §§15–16 below. (2) We have $\dot{\mu} = \dot{\theta} d\mu/d\theta$; using this formula, to reduce our result (2.18) to that given by CHAPMAN & COWLING we must replace $\dot{\theta}$ by $-\frac{2}{3}\theta\dot{\theta}$, its value in adiabatic motion (*cf.* §15 below).

It is only fair to remark here that the far more elaborate methods of ENSKOG and others are not simplified in principle when applied to Maxwellian molecules, while the extension of Maxwellian iteration to non-Maxwellian molecules requires not only more laborious calculation but also a basically more complicated scheme (see §56).

To calculate $h_i^{(2)}$ by (2.11), we need to know the first iterate $p_{ikk}^{(1)}$ to the contracted fourth moments. This cannot be obtained without further equations or further assumptions. It is here that we break off this introduction and turn to general formulations of methods suggested by the simple ideas we have begun to explore.

3. Additional notation. Quantities associated with the second of a pair of molecules are denoted by an asterisk; quantities associated with the outcome of an encounter, by primes; and the accents when attached to function symbols refer to their arguments: $F' \equiv F(\xi')$, *etc.* We set

$$2\mathbf{v} \equiv \xi^* - \xi = \mathbf{c}^* - \mathbf{c}, \quad 2\mathbf{w} \equiv \mathbf{c}^* + \mathbf{c}. \quad (3.1)$$

Conservation of momentum and energy requires $\mathbf{w} = \mathbf{w}'$, $v = v'$. Let ϵ be the angle between the plane of $\mathbf{v}$ and $\mathbf{v}'$ and the plane containing $\mathbf{v}$ and a direction fixed in space. Suppose ξ fixed; then all possible encounters will be included by letting ϵ increase from 0 to 2π ; in this process $\mathbf{v}'$ rotates around $\mathbf{v}$ with fixed magnitude $v' = v$ while subtending a constant angle¹ ϕ from $\mathbf{v}$. To include all possible encounters experienced by a molecule of velocity ξ at $\mathbf{x}$, we integrate over a plane normal to $\mathbf{v}$ and passing through $\mathbf{x}$. In this plane, let b , ϵ be polar co-ordinates. Then b is the distance from $\mathbf{x}$ to the initial asymptote of the second molecular trajectory. A knowledge of the law of intermolecular force enables us to calculate b as a function of ϕ and v .

Let $Q = Q(\mathbf{x}, \xi, t)$. Then

$$\mathcal{C}(Q) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi^* \int_0^{\infty} b db \int_0^{2\pi} d\epsilon \delta Q F F^* v, \quad (3.2)$$

with δQ having either of the forms

$$\delta Q = Q^{*'} + Q' - Q^* - Q \quad \text{or} \quad \delta Q = 2(Q^{*'} - Q^*), \quad (3.3)$$

gives the mean rate of change of Q due to collisions.

4. An equation of transfer and the iterative scheme. All considerations in this paper rest on an equation of transfer for a function $Q = Q(\mathbf{c}) = Q(\xi - \mathbf{u})$. This equation of transfer, which may be derived easily from any of the usual formulations of Maxwell's kinetic theory (*e.g.* [1952, 1, §3.12]), is

$$\mathcal{L}(Q) = m\mathcal{C}(Q), \quad (4.1)$$

where $\mathcal{C}(Q)$ is defined by (3.2) and where

$$\mathcal{L}(Q) \equiv (\rho \bar{Q})' + \rho \bar{Q} E + \overline{\rho c_i Q_{,i} u_{i,i}} - \overline{Q_{,i} p_{ii,i}} + (\rho c_i \bar{Q})_{,i}. \quad (4.2)$$

(This equation does not imply the equation of momentum, which has been used to derive it. Cf. (6.2), (8.2), and (15.6).)

¹ In terms of MAXWELL's θ , we have $\phi = \pi - 2\theta$.

Although the equation of transfer is valid for all molecular models, we shall use it only for Maxwellian molecules. For these, as MAXWELL showed, $vbdb/d\phi$ is independent of v , and hence if Q is a polynomial of degree r in $\mathbf{c}$, $\mathcal{C}(Q)$ may be expressed as a polynomial in the mean values of certain other polynomials of degrees not greater than r . To find the form of the result, knowledge of the form of F is unnecessary. For example, as MAXWELL showed [1867, eqq. (40)–(43)],

$$m\mathcal{C}(c_i c_j) = -\frac{p}{\mu} (p_{ij} - p\delta_{ij}), \quad (4.3)$$

$$m\mathcal{C}(c_i c_j c_k) = -\frac{p}{2\mu} (3p_{ijk} - 2h_{(i}\delta_{jk)}). \quad (4.4)$$

Substitution of these results into (4.1) easily yields (2.4) and (2.5), on which all of the analysis of §2 was founded. *There, as in all the rest of this paper, no series expansion or other special form for F was assumed; for all of our analysis, only the existence and differentiability of the moments is needed.*

The method introduced in §2 may now be formalized in part as the

Iterative scheme:

$$m\mathcal{C}^{(n+1)}(Q) \equiv \mathcal{L}^{(n)}(Q). \quad (4.5)$$

That is, the $n + 1^{\text{st}}$ iterate for $m\mathcal{C}(Q)$ is obtained by substituting n^{th} iterates in the expression for $\mathcal{L}(Q)$.

To render the iterative scheme definite, what remains is to specify the functions Q . Examples are furnished by our earlier iterative equations (2.10) and (2.11). Examination of (2.10) shows that to calculate the $n + 1^{\text{st}}$ iterate $P_{ij}^{(n+1)}$ to P_{ij} we must know the n^{th} iterate $p_{ijk}^{(n)}$ to p_{ijk} . Subsequent examination of (2.11) shows that to calculate the n^{th} iterate $p_{ijk}^{(n)}$ to p_{ijk} we require the $n - 1^{\text{st}}$ iterate $p_{ijkl}^{(n-1)}$ to p_{ijkl} . Thus we need an iterative scheme for the fourth moments.

We might extrapolate the scheme started above by taking $Q = c_i c_j c_k c_l$, thus obtaining an equation for the fourth moments,¹ but it turns out that $\mathcal{C}(c_i c_j c_k c_l)$ is not proportional to p_{ijkl} . Rather, $\mathcal{C}(c_i c_j c_k c_l)$ is a complicated sum of fourth and products of second moments. For $Q = c_{i_1} c_{i_2} \cdots c_{i_n}$, such complications increase with n ; while not defeating the method, they are both undesirable and partially unnecessary. This, apparently, was observed by MAXWELL, who suggested the introduction of homogeneous spherical harmonics.² Besides lightening somewhat the labor of calculating $\mathcal{C}(Q)$, the choice of homogeneous spherical harmonics in $\mathbf{c}$ for $Q(\mathbf{c})$ facilitates also the calculation of some of the higher iterates.

¹ Such an equation is suggested by GRAD [1949, 1, §4], but he does not obtain its explicit form.

² The advantages of an expansion of the distribution function in spherical harmonics have been emphasized by JAFFÉ [1930, §2].

5. Spherical moments. Although any of various basic sets of homogeneous spherical harmonics in $\mathbf{c}$ may be used, in this paper we employ the following:

$$Y(\mathbf{c}) \equiv 1, \quad (5.1)$$

$$Y_i(\mathbf{c}) \equiv c_i, \quad (5.2)$$

$$Y_{ij}(\mathbf{c}) \equiv c_i c_j - \frac{1}{3} c^2 \delta_{ij}, \quad (5.3)$$

$$Y_{ijk}(\mathbf{c}) \equiv c_i c_j c_k - \frac{3}{5} c^2 c_{(i} \delta_{jk)}, \quad (5.4)$$

$$Y_{ijkl}(\mathbf{c}) \equiv c_i c_j c_k c_l - \frac{6}{7} c^2 Y_{(ij}(\mathbf{c}) \delta_{kl)} - \frac{1}{5} c^4 \delta_{(ij} \delta_{kl)}, \quad (5.5)$$

$$Y_{ijklm}(\mathbf{c}) \equiv c_i c_j c_k c_l c_m - \frac{10}{9} c^2 Y_{(ijk}(\mathbf{c}) \delta_{lm)} - \frac{3}{7} c^4 Y_{(i}(\mathbf{c}) \delta_{jk} \delta_{lm)}. \quad (5.6)$$

These are the first members of a sequence whose general construction has been given by IKENBERRY [1956]. They form the components of symmetric tensors which when contracted on any pair of indices reduce to zero. Further advantageous properties of these spherical harmonics will appear below in this section and in §7.

Set $Y_{2r|s} \equiv c^{2r} Y_s(\mathbf{c}), \quad (5.7)$

where s represents a set of s indices, $i_1 i_2 \cdots i_s$. The number $q \equiv 2r + s$ is the *order* of $Y_{2r|s}$. It is known that any homogeneous polynomial of degree q can be expressed as a linear combination of the functions $Y_{2r|s}$ of order q . The formulæ (5.1)–(5.6) furnish the examples of this theorem which will be needed in our calculations below.

We introduce the *spherical moments* $P_{2r|s}$ by the following definition:

$$P_{2r|s} \equiv \rho \bar{Y}_{2r|s} = m \int_{\infty} c^{2r} Y_s(\mathbf{c}) F d\mathbf{c}. \quad (5.8)$$

In circumstances where confusion is either legitimate or out of the question, we may refer to the $P_{2r|s}$ simply as *moments*. The number $2r + s$ is the *order* of $P_{2r|s}$. Both for $Y_{2r|s}$ and for $P_{2r|s}$, we shall not write the s for a vacuous set, nor the $2r|$ when $r = 0$. The quantities $P_{2r|s}$ include and generalize P_{ij} as defined by (2.7)₁. Any $P_{2r|s}$ of order q is a symmetric isotropic tensor function of the q^{th} moments, and when contracted on any pair of indices it reduces to zero. The spherical moments which we shall use in this chapter are expressed in terms of the moments as follows:

$$\begin{aligned} P &= p, & P_i &= 0, & P_{ij} &= p_{ij} - p \delta_{ij}, & P_{2|} &= 3p = 3nk\theta, \\ P_{ijk} &= p_{ijk} - \frac{3}{5} p_{ll(i} \delta_{jk)}, & P_{2|i} &= p_{iij} = 2h_i, \\ P_{ijkl} &= p_{ijkl} - \frac{6}{7} p_{mm(ij} \delta_{kl)} + \frac{3}{35} p_{rrmm} \delta_{(ij} \delta_{kl)}, \\ P_{2|ij} &= p_{ijkk} - \frac{1}{3} p_{kkll} \delta_{ij}, & P_{4|} &= p_{iiii}, \\ P_{ijklm} &= p_{ijklm} - \frac{10}{9} p_{rr(ijk} \delta_{lm)} + \frac{5}{21} p_{iirrr(ij} \delta_{kl)} \delta_{lm)}, \\ P_{2|ijk} &= p_{llijk} - \frac{3}{5} p_{llmm(i} \delta_{jk)}, & P_{4|i} &= p_{kkjji}. \end{aligned} \quad (5.9)$$

From the theorem stated in the second paragraph of this section it follows that the q^{th} moment $p_{i_1 i_2 \dots i_q}$ can be expressed as a linear isotropic tensor function of the spherical moments $P_{2r|s}$ of order q . For the moments of orders 2, 3, 4, and 5 the required formulæ* may be read off from (5.1)–(5.6):

$$\begin{aligned}
 p_{ij} &= P_{ij} + \frac{1}{3}P_{2|i}\delta_{ij} , \\
 p_{ijk} &= P_{ijk} + \frac{3}{5}P_{2|(i}\delta_{jk)} , \quad p_{iji} = P_{2|i} , \\
 p_{ijkl} &= P_{ijkl} + \frac{9}{7}P_{2|(ij}\delta_{kl)} + \frac{1}{5}P_{4|i}\delta_{(ij}\delta_{kl)} , \\
 p_{ijkk} &= P_{2|ij} + \frac{1}{3}P_{4|i}\delta_{ij} , \quad p_{iiij} = P_{4|i} , \\
 p_{ijklm} &= P_{ijklm} + \frac{10}{9}P_{2|(ijk}\delta_{lm)} + \frac{3}{7}P_{4|(i}\delta_{jk}\delta_{lm)} , \\
 p_{ijkll} &= P_{2|ijk} + \frac{3}{5}P_{4|(i}\delta_{jk)} , \quad P_{ijkkk} = P_{4|i} .
 \end{aligned} \tag{5.10}$$

By the initiation agreement of §2, we are to take as zeroth iterates $P_{2r|s}^{(0)}$ those functions of the thermostatic state to which $P_{2r|s}$ reduces in the case of a Maxwellian distribution. By (2.13), $P_{2r|s}^{(0)}$ is a symmetric isotropic tensor which when contracted on any index reduces to zero. This fact yields the former alternative¹ in the formula

$$P_{2r|s}^{(0)} = \begin{cases} 0 & \text{if } s \neq 0, \\ (2r+1)!!\rho\left(\frac{p}{\rho}\right)^r & \text{if } s = 0, \end{cases} \tag{5.11}$$

while the latter alternative is an easy consequence of (2.13).

We are now in a position to give a formal definition of

Maxwellian iteration: Adopting the initiation agreement of §2, effect the iterative scheme (4.5) by taking the functions Q as $Y_{2r|s}$ in order of increasing $2r+s$ and, for fixed $2r+s$, in order of increasing s .

To calculate the Maxwellian iterates, we need the exact values of $\mathcal{L}(Y_{2r|s})$ and $\mathcal{C}(Y_{2r|s})$. While both these quantities are algebraic expressions in the $P_{2r|s}$ and their derivatives, unfortunately we are not able to give the general explicit forms for these expressions, and thus we cannot exhibit the general formula of iteration. However, it is not difficult to calculate explicitly and for increasing order as many of the quantities $\mathcal{L}(Y_{2r|s})$ and $\mathcal{C}(Y_{2r|s})$ as are needed to obtain any desired iterates $P_{ii}^{(n)}$ and $h_i^{(n)}$. In this memoir we shall calculate $P_{ii}^{(3)}$ and $h_i^{(3)}$. The $\mathcal{L}(Y_{2r|s})$ necessary to this end are listed in the next section; the necessary $\mathcal{C}(Y_{2r|s})$ in §8. The method of calculating $\mathcal{L}(Y_{2r|s})$ is evident

*Note added in proof: The status of this decomposition is indicated by Theorem (5.7.F) of H. WEYL, *The classical groups*, Princeton (1939).

¹ We are not aware of any published statement or proof of the easy theorem concerning isotropic tensors which we use here: A completely symmetric isotropic tensor with vanishing trace is a zero tensor.

from (4.2) and the completeness of the set $Y_{2r|s}$. The method of calculating $\mathcal{C}(Y_{2r|s})$ is not obvious and will be explained in full generality in §13.

6. A list of certain $\mathcal{L}(Y_{2r|s})$. Substituting the explicit formulæ (5.2)–(5.7) into the definition (4.2), by straightforward but increasingly lengthy calculations we get

$$\mathcal{L}(1) = \dot{\rho} + \rho E, \quad (6.1)$$

$$\mathcal{L}(Y_i) = 0, \quad (6.2)$$

$$\mathcal{L}(c^2) = 3\dot{p} + 5pE + 2P_{ij}E_{ij} + 2h_{i,i}, \quad (6.3)$$

$$\begin{aligned} \mathcal{L}(Y_{ij}) = & \dot{P}_{ij} + P_{ij}E + 2P_{k(i}u_{j),k} + 2pE_{ij} \\ & - \frac{2}{3}P_{kl}E_{kl}\delta_{ij} + P_{ijk,k} - \frac{4}{15}h_{k,k}\delta_{ij} + \frac{4}{5}h_{(i,i)}, \end{aligned} \quad (6.4)$$

$$\begin{aligned} \mathcal{L}(c^2 Y_i) = & 2\dot{h}_i + \frac{10}{3}h_i E + 2P_{ijk}E_{jk} + 2h_i u_{i,j} + \frac{8}{5}h_i E_{ii} \\ & - \frac{5p}{\rho} p_{ij,i} - \frac{2P_{ij}}{\rho} p_{jk,k} + P_{2|ij,i} + \frac{1}{3}P_{4|i,i}, \end{aligned} \quad (6.5)$$

$$\begin{aligned} \mathcal{L}(Y_{ijk}) = & \dot{P}_{ijk} + P_{ijk}E + 3P_{l(ij}u_{k),l} - \frac{8}{5}P_{lm(i}\delta_{jk)}E_{lm} \\ & + \frac{12}{5}h_{(i}E_{jk)} - \frac{24}{5}h_l E_{l(i}\delta_{jk)} - \frac{3}{\rho}P_{(ij}p_{k),l} + \frac{6}{5\rho}P_{l(i}\delta_{jk)}p_{lm,m} \\ & + P_{ijk,l} - \frac{8}{35}\delta_{(ij}P_{2|k)l,l} + \frac{3}{7}P_{2|(ij,k)}, \end{aligned} \quad (6.6)$$

$$\mathcal{L}(c^4) = \dot{P}_{4|} + \frac{7}{3}P_{4|}E + 4P_{2|ij}E_{ij} - \frac{8h_i}{\rho} p_{ij,i} + P_{4|i,i}, \quad (6.7)$$

$$\begin{aligned} \mathcal{L}(c^2 Y_{ij}) = & \dot{P}_{2|ij} + \frac{5}{3}P_{2|ij}E + 2P_{ijk}E_{kl} + 2P_{2|k(i}u_{j),k} \\ & + \frac{8}{7}P_{2|k(i}E_{j)k} - \frac{22}{5}P_{2|kl}E_{kl}\delta_{ij} + \frac{14}{15}P_{4|}E_{ij} \\ & - \frac{2}{\rho}P_{ijk}p_{kl,l} - \frac{28}{5\rho}h_{(i}p_{j)k,k} + \frac{28}{15\rho}h_k p_{kl,l}\delta_{ij} \\ & + P_{2|ijk,k} - \frac{2}{15}P_{4|k,k}\delta_{ij} + \frac{2}{5}P_{4|(i,j)}, \end{aligned} \quad (6.8)$$

$$\begin{aligned} \mathcal{L}(c^4 Y_i) = & \dot{P}_{4|i} + \frac{7}{3}P_{4|i}E + P_{4|j}u_{i,j} + 4P_{2|ijk}E_{jk} \\ & + \frac{8}{5}P_{4|j}E_{ji} - \frac{7}{3\rho}P_{4|}p_{ij,i} - \frac{4}{\rho}P_{2|ij}p_{jk,k} \\ & + P_{4|ij,i} + \frac{1}{3}P_{6|i,i}. \end{aligned} \quad (6.9)$$

The calculation of these expressions is routine and too lengthy to print. $\mathcal{L}(Y_{ijkl})$ and $\mathcal{L}(c^2 Y_{ijk})$, which are not listed, we shall not need except when their arguments are zeroth iterates; for these arguments, we shall show in §9 that they vanish.

7. Statement of the fundamental theorem. Principles of calculation of the collision integrals $m\mathcal{C}(Y_{2r|s})$. In order to calculate $m\mathcal{C}(Y_{2r|s})$, we form a systematic procedure based on the last suggestions made by MAXWELL [1879, Notes added after §1 and §5]¹. The calculations are increasingly lengthy as the order of $Y_{2r|s}$ increases, and the discovery of shorter methods would be desirable. As MAXWELL [1879, Note added after §15] stated that he had performed the calculations up to order six, it is possible that his genius radically shortened the work.

One of the methods used by MAXWELL in evaluating $\mathcal{C}(Q)$ starts with the substitution of $\mathbf{w} + \mathbf{v}'$ for $\mathbf{c}^{*'}$, $\mathbf{w} - \mathbf{v}'$ for $\mathbf{c}'$, $\mathbf{w} + \mathbf{v}$ for $\mathbf{c}^*$, and $\mathbf{w} - \mathbf{v}$ for $\mathbf{c}$ in δQ , followed by the expansion of δQ in terms of spherical harmonics in $\mathbf{v}$ and $\mathbf{v}'$. MAXWELL made use of the constancy of $v' = v$ and of $\mathbf{w}$ as $\mathbf{v}'$ swings around $\mathbf{v}$, and of the integration formula²

$$\int_0^{2\pi} Y_s(\mathbf{v}') d\epsilon = 2\pi Y_s(\mathbf{v}) P_s(\cos \phi), \quad (7.1)$$

where $P_s(\cos \phi)$ is the Legendre polynomial. For molecules which repel each other as the inverse fifth power of the distance, MAXWELL calculated numerically the needed quadratures over ϕ . Finally, he replaced $2\mathbf{v}$ and $2\mathbf{w}$ by $\mathbf{c}^* - \mathbf{c}$ and $\mathbf{c}^* + \mathbf{c}$, respectively, and performed the integrations over $\mathbf{c}$ and $\mathbf{c}^*$. MAXWELL did not publish even the results of the lengthier of these calculations. From his cryptic remarks [1879, Notes added after §1 and §5] we are led to conjecture for Maxwellian molecules the following *fundamental theorem*:

$$m\mathcal{C}(Y_{2r|s}) = -\mathfrak{C}_{2r|s} P_{2r|s} + Q_{2r|s}, \quad (7.2)$$

where

$$Q_{2r|s} = \sum_{r_1, r_2, s_1, s_2} A_{r_1 r_2 s_1 s_2} P_{2r_1|s_1} P_{2r_2|s_2}, \quad (7.3)$$

$$2r_1 + s_1 \geq 2r_2 + s_2 > 0, \quad 2r_1 + s_1 + 2r_2 + s_2 = 2r + s.$$

The quantity $Q_{2r|s}$ is a bilinear form in spherical moments of order less than $2r + s$; the sum of the orders in each product is $2r + s$. Since s_1 and s_2 stand for groups of tensorial indices $i_1 i_2 \dots i_{s_1}$, $i_{s_1+1} i_{s_1+2} \dots i_{s_1+s_2}$, respectively, the summation in (7.3) is partly symbolic and in fact extends over an equal number of unwritten dummy indices. The importance of the fundamental theorem (7.2) is that on the right-hand side occurs *only a single moment of order $2r + s$* , namely, $P_{2r|s}$, rather than a linear combination of such moments. The coefficient $\mathfrak{C}_{2r|s}$ is the same for every component of the tensor $Y_{2r|i_1 i_2 \dots i_s}$.

¹ Another interesting method is invented by GRAD [1949, 1, Appendix 3].

² This formula holds for any spherically symmetric molecular model. It is generalized by (13.12) below.

As will be proved in §13, the conjecture (7.2) is correct, with $\mathfrak{C}_{2r|s}$ given by³

$$\begin{aligned}\mathfrak{C}_{2r|s} &= \frac{n}{2^{r-s}} \sum_{k=0}^s \sum_{j=0}^r [1 + (-1)^{i+k}] \binom{s}{k} a_{rj} \mathfrak{B}_{jk}, \\ &= 4\pi n v \int_0^\pi [1 + \delta_{r0} \delta_{s0} - \cos^{2r+s} \tfrac{1}{2}\phi] P_s(\cos \tfrac{1}{2}\phi) \\ &\quad - \sin^{2r+s} \tfrac{1}{2}\phi P_s(\sin \tfrac{1}{2}\phi) b \frac{db}{d\phi} d\phi,\end{aligned}\tag{7.4}$$

in the former of which

$$a_{rj} = \frac{(2j+1)2^{r+1}(r!)^2}{(r+j+1)!(r-j)!}, \quad r \geq j, \tag{7.5}$$

$$\mathfrak{B}_{jk} = 2\pi v \int_0^\pi [1 - P_j(\cos \phi) P_k(\cos \phi)] b \frac{db}{d\phi} d\phi. \tag{7.6}$$

$\mathfrak{C}_{2r|s}$ may be calculated, for given r and s , by numerical integration.

Although the form (7.2) is valid only for Maxwellian molecules, the coefficients $\mathfrak{C}_{2r|s}$ as defined by (7.4) occur also in the study of other molecular models. The following analysis through (7.13) is valid for all models.

Each $\mathfrak{C}_{2r|s}$ may be expressed as a linear combination of certain $\mathfrak{C}_{q|}$ by using the explicit formula for the polynomial $P_s(x)$ [1927, §15.1]:

$$P_s(x) = \sum_{k=0}^{\lfloor \frac{1}{2}s \rfloor} \frac{(-1)^k}{2^s} \binom{s}{k} \binom{2s-2k}{s} x^{s-2k}. \tag{7.7}$$

Substituting this formula in (7.4)₂ and using the fact that $P_s(1) = 1$ yields the result:

$$\mathfrak{C}_{2r|s} = \sum_{k=0}^{\lfloor \frac{1}{2}s \rfloor} \frac{(-1)^k}{2^s} \binom{s}{k} \binom{2s-2k}{s} \mathfrak{C}_{2r+2s-2k|}. \tag{7.8}$$

The formula just derived seems to offer the greatest convenience for numerical quadrature, since it expresses each $\mathfrak{C}_{2r|s}$ as a finite sum of quadratures in each of which no non-trivial Legendre polynomial occurs. Thus calculation of $\mathfrak{C}_{2r|}$ may be effected by using only tables of circular functions and of $bdb/d\phi$. A different reduction, not possessing this advantage but perhaps preferable for formal work, was hinted by MAXWELL [1879, Note added after §1]. Out of respect for his views, we discover and demonstrate this alternative reduction, which consists in expressing the coefficients $\mathfrak{B}_{jk}$ in (7.4)₁ as linear combinations of the $\mathfrak{B}_{0k}$. Writing $\mathfrak{B}_k \equiv \mathfrak{B}_{0k}$, we find, for example, that $\mathfrak{B}_{11} = \frac{2}{3}\mathfrak{B}_2$. To get a general expression for $\mathfrak{B}_{jk}$, we start with [1927, Ch. XV, Ex. 11]

$$P_i(x)P_k(x) = \sum_{l=0}^{\text{Min}(i,k)} a_{jkl} P_{j+k-2l}(x), \tag{7.9}$$

³ $\mathfrak{C}_{2r|s}$ is proportional to the eigenvalue λ_{rs} of CHANG & UHLENBECK [1952, 3, eq. (24)].

where

$$a_{ikl} = \frac{D_{i-l} D_l D_{k-l}}{D_{j+k-l}} \cdot \frac{2j + 2k - 4l + 1}{2j + 2k - 2l + 1}, \quad (7.10)$$

$$D_l \equiv \frac{(2l-1)!!}{l!}. \quad (7.11)$$

Setting $x = 1$, since $P_l(1) = 1$ we obtain from (7.9)

$$1 = \sum_{l=0}^{\text{Min}(j,k)} a_{ikl} \quad (7.12)$$

for any fixed j and k . Hence, upon substituting from (7.9) and (7.12) into (7.6), we obtain

$$\mathfrak{B}_{jk} = \sum_{l=0}^{\text{Min}(j,k)} a_{ikl} \mathfrak{B}_{j+k-2l}. \quad (7.13)$$

From (7.4)₁ it is evident that to calculate $\mathfrak{G}_{2r|s}$ we need the $\mathfrak{B}_{ik}$ only when $j + k$ is even.

From MAXWELL'S work it follows that for Maxwellian molecules

$$\mathfrak{B}_{jk} = \sqrt{\frac{G}{2m}} \mathfrak{A}_{jk}, \quad \mathfrak{A}_{jk} \equiv 2\pi \int_{\phi=0}^{\phi=\pi} [1 - P_j(\cos \phi) P_k(\cos \phi)] \alpha \, d\alpha, \quad (7.14)$$

where

$$v b \, db = \sqrt{\frac{G}{2m}} \alpha \, d\alpha, \quad (7.15)$$

and $\alpha = \alpha(\phi)$ is to be calculated by inverting

$$\phi = 2 \int_0^{\tau(\alpha)} \left[1 - \chi^2 - \frac{1}{2} \left(\frac{\chi}{\alpha} \right)^4 \right]^{-1/2} d\chi, \quad (7.16)$$

$\tau(\alpha)$ being the single positive root of

$$1 - \tau^2 - \frac{1}{2} \left(\frac{\tau}{\alpha} \right)^4 = 0. \quad (7.17)$$

The substitutions

$$\alpha^4 = 2 \cot^2 2\theta, \quad \chi = \sqrt{1 - \tan^2 \theta} \cos \psi \quad (7.18)$$

reduce (7.16) to the form

$$\phi = 2 \sqrt{\cos 2\theta} K(\sin \theta), \quad (7.19)$$

where $K(k)$ is the complete elliptic integral of the second kind. Substitution of (7.18) into (7.14) results in

$$\mathfrak{A}_{jk} = 2\pi \sqrt{2} \int_0^{\frac{1}{2}\pi} [1 - P_j(\cos \phi) P_k(\cos \phi)] \csc^2 2\theta \, d\theta. \quad (7.20)$$

The $\mathfrak{U}_{ik}$ are dimensionless numbers which by means of (7.14)₁ and (7.13) may be expressed as linear combinations of the $\mathfrak{U}_{0k}$.

In the present notation, (2.3) reads

$$\mu = \frac{p}{n\mathfrak{B}_2}. \quad (7.21)$$

Each $\mathfrak{B}_{ij}$ differs from $\mathfrak{B}_2$ by a dimensionless factor and hence is inversely proportional to μ . Evidently $\mathfrak{B}_0 = 0$. Mr. HARRISON HANCOCK of the Naval Research Laboratory has calculated three of the $\mathfrak{B}$'s by quadrature, his results being⁴

$$\begin{aligned} \mathfrak{B}_1 &= \sqrt{\frac{G}{2m}} \cdot 2.6511 \dots, \\ \mathfrak{B}_2 &= 3\sqrt{\frac{G}{2m}} \cdot 1.3703 \dots, \\ \mathfrak{B}_4 &= 2\sqrt{\frac{G}{2m}} \cdot 4.9087 \dots. \end{aligned} \quad (7.22)$$

8. A list of exact values of the collision integrals $m\mathfrak{C}(Y_{2r|s})$ of the first five orders, with a specimen of their calculation. Equation (7.2) may be used for a partial check on detailed calculations which are necessary to determine the non-linear terms in $\mathfrak{C}(Y_{2r|s})$. Results of these calculations, which shall presently be explained through an example, are:

$$m\mathfrak{C}(1) = 0, \quad (8.1)$$

$$m\mathfrak{C}(Y_i) = 0, \quad (8.2)$$

$$m\mathfrak{C}(c^2) = 0, \quad (8.3)$$

$$m\mathfrak{C}(Y_{ii}) = -n\mathfrak{B}_2 P_{ii}, \quad (8.4)$$

$$m\mathfrak{C}(c^2 Y_i) = -\frac{4}{3}n\mathfrak{B}_2 h_i, \quad (8.5)$$

$$m\mathfrak{C}(Y_{ijk}) = -\frac{3}{2}n\mathfrak{B}_2 P_{ijk}, \quad (8.6)$$

$$m\mathfrak{C}(c^4) = -\frac{2n\mathfrak{B}_2}{3\rho} (\rho P_{A_1} - 15p^2 + P_{ii}P_{ii}), \quad (8.7)$$

$$m\mathfrak{C}(c^2 Y_{ii}) = -\frac{7n\mathfrak{B}_2}{6\rho} (\rho P_{2|ii} - pP_{ii} + \frac{4}{7}[P_{ik}P_{ki} - \frac{1}{3}P_{kl}P_{kl}\delta_{ii}]), \quad (8.8)$$

$$m\mathfrak{C}(Y_{ijkl}) = -\frac{1}{4}n(6\mathfrak{B}_2 + \mathfrak{B}_4)P_{ijkl} + \frac{3n}{4\rho} (2\mathfrak{B}_2 - \mathfrak{B}_4) \quad (8.9)$$

$$\cdot (P_{(ij}P_{kl)} - \frac{4}{7}P_{m(i}\delta_{jk}P_{l)m} + \frac{2}{35}P_{mn}P_{mn}\delta_{(ij}\delta_{kl)}),$$

⁴ The numbers following the radicals are those MAXWELL denoted by A_1 , A_2 , and $A_2 + A_4$, respectively, but our $\mathfrak{B}$'s are not his B 's. MAXWELL's table for the inversion of (7.19) contains only one error of more than 1', but from it he calculated $A_1 = 2.6595$, $A_2 = 1.3682$.

$$m\mathfrak{C}(c^4 Y_i) = -\frac{n\mathfrak{B}_2}{\rho} (\rho P_{4|i} - 8ph_i + \frac{4}{3}P_{ijk}P_{jk} + \frac{12}{5}h_i P_{ij}), \quad (8.10)$$

$$\begin{aligned} m\mathfrak{C}(c^2 Y_{ijk}) &= -\frac{n}{14} (19\mathfrak{B}_2 + 2\mathfrak{B}_4)P_{2|ijk} - \frac{9n}{7\rho} (\mathfrak{B}_2 - \mathfrak{B}_4)pP_{ijk} \\ &\quad - \frac{3n}{14\rho} (8\mathfrak{B}_2 - \mathfrak{B}_4)(P_{l(i}P_{jk)l} - \frac{2}{5}P_{lm}P_{lm(i}\delta_{jk)}) \\ &\quad + \frac{9n}{35\rho} (13\mathfrak{B}_2 - 6\mathfrak{B}_4)(h_{(i}P_{jk)} - \frac{2}{5}h_l P_{l(i}\delta_{jk)}). \end{aligned} \quad (8.11)$$

To the best of our knowledge, the exact collision integrals (8.7)–(8.11) have not been obtained previously (except possibly by MAXWELL, *cf.* §7).

Although the calculations for $\mathfrak{C}(c^4)$ are not indicative of the length of those made in obtaining some of the preceding expressions, they illustrate well the general method. Starting with (3.1) and (3.3)₁, we have

$$\begin{aligned} \delta c^4 &= c^{*4} + c'^4 - c^{*4} - c^4, \\ &= (\mathbf{v}' + \mathbf{w}')^4 + (\mathbf{v}' - \mathbf{w}')^4 - (\mathbf{v} + \mathbf{w})^4 - (\mathbf{v} - \mathbf{w})^4, \\ &= 8(\mathbf{v}' \cdot \mathbf{w})^2 - 8(\mathbf{v} \cdot \mathbf{w})^2, \end{aligned} \quad (8.12)$$

since $v' = v$, $\mathbf{w}' = \mathbf{w}$, and $w' = w$. Then by (5.3)

$$\begin{aligned} \delta c^4 &= 8w_i w_j (v'_i v'_j - v_i v_j), \\ &= 8w_i w_j (Y_{ij}(\mathbf{v}') - Y_{ij}(\mathbf{v})). \end{aligned} \quad (8.13)$$

Using (7.1), we get

$$\int_0^{2\pi} d\epsilon \delta c^4 = 16\pi \{P_2(\cos \phi) - 1\} w_i w_j Y_{ij}(\mathbf{v}), \quad (8.14)$$

whence by use of (7.6) and (3.1) follow

$$\begin{aligned} \int_0^\infty v b db \int_0^{2\pi} d\epsilon \delta c^4 &= -8\mathfrak{B}_2 w_i w_j Y_{ij}(\mathbf{v}), \\ &= 8\mathfrak{B}_2 [\frac{1}{3}w^2 v^2 - (\mathbf{v} \cdot \mathbf{w})^2], \\ &= -\frac{2}{3}\mathfrak{B}_2 [\frac{1}{2}(c^4 + c^{*4}) - 2c^2 c^{*2} + (\mathbf{c} \cdot \mathbf{c}^*)^2]. \end{aligned} \quad (8.15)$$

Putting this last expression into (3.2) yields at once

$$m\mathfrak{C}(c^4) = -\frac{2}{3}\mathfrak{B}_2 \left(nP_{4|} - \frac{18}{m} p^2 + \frac{1}{m} p_{ii} p_{ii} \right), \quad (8.16)$$

equivalent to (8.7). In the simplicity of this final step our example is not typical: Generally the result of the integration over $\mathbf{c}$ and $\mathbf{c}^*$ is an expression in the moments $p_{i_1 i_2 \dots i_l}$ which has to be put in terms of the $P_{2r|}$ by means of (5.10).

It is important to notice that equating (8.1)–(8.8) and (8.10), respectively, to (6.1)–(6.9) yields nine *exact equations of transfer*. The first is

$$\dot{\rho} + \rho E = 0, \quad (8.17)$$

the equation of mass. The second, replacing the momentum equation already employed to derive (4.1), is $0 = 0$. The third is (2.8), the equation of energy; the fourth is (2.9); the fifth and sixth are equivalent to (2.5) and will replace it in subsequent considerations, while the remainder are new exact equations needed to calculate higher iterates.

9. The quantities $\mathfrak{L}^{(0)}(Y_{2r|s})$. The zeroth iterates $P_{2r|s}^{(0)}$ are given by (5.11). Substituting these into the right-hand sides of (6.4)–(6.9) yields

$$\mathfrak{L}^{(0)}(Y_{ii}) = 2pE_{ii}, \quad (9.1)$$

$$\mathfrak{L}^{(0)}(c^2 Y_i) = \frac{4}{3}\alpha p \theta_{,i}, \quad (9.2)$$

$$\mathfrak{L}^{(0)}(Y_{ijk}) = 0, \quad (9.3)$$

$$\mathfrak{L}^{(0)}(c^4) = 20 \frac{p^2}{\rho} \dot{\beta}, \quad (9.4)$$

$$\mathfrak{L}^{(0)}(c^2 Y_{ii}) = 14 \frac{p^2}{\rho} E_{ii}, \quad (9.5)$$

$$\mathfrak{L}^{(0)}(c^4 Y_i) = \frac{56\alpha p^2}{3\rho} \theta_{,i}. \quad (9.6)$$

The quantity β in (9.4) is given by

$$\beta \equiv \frac{3}{2} \log p \rho^{-5/3} \quad (9.7)$$

and is proportional to the specific entropy.

We can show that

$$\mathfrak{L}^{(0)}(Y_{ijkl}) = 0, \quad (9.8)$$

$$\mathfrak{L}^{(0)}(c^2 Y_{ijk}) = 0 \quad (9.9)$$

without explicit calculation of $\mathfrak{L}(Y_{ijkl})$ and $\mathfrak{L}(c^2 Y_{ijk})$. Restricting attention to zeroth iterates, we wish $\mathfrak{L}(Q)$ for $Q = Y_4$ and $Q = Y_{2|3}$. By (4.2), we need $\bar{Q}$, $c_i \bar{Q}_{,i}$, $\bar{Q}_{,i}$, and $c_i \bar{Q}$. By (5.11), $\bar{Q} = 0$. From the general theory of spherical harmonics it follows that $c_i Y_{4,i}$ is a linear combination of functions $Y_{2r|s}$ of order 4 and with $s > 0$. Hence $c_i Y_{4,i}$ is a linear combination of functions $P_{2r|s}$ with $s > 0$. By (5.11), for the zeroth iterates each term in this combination is zero. The same reasoning can be applied to the remaining terms in $\mathfrak{L}(Q)$, whence follows (9.8) and (9.9). In fact, $\mathfrak{L}^{(0)}(Y_{2r|s}) = 0$ if $s \geq 3$.

10. The first iterates $P_{2r|s}^{(1)}$. By (4.5), the first iterates are to be calculated from the formula $m \mathfrak{C}^{(1)}(Q) = \mathfrak{L}^{(0)}(Q)$. We carry out the calculation when Q

is taken as one of the eight $Y_{2r|s}$ considered in the foregoing section. That is, on the right-hand sides of (8.4)–(8.11) we substitute $P_{2r|s}^{(1)}$ for $P_{2r|s}$ and equate the results to the right-hand sides of (9.1)–(9.6), (9.8) and (9.9). We obtain the following system of equations:

$$-n\mathfrak{B}_2 P_{ij}^{(1)} = 2pE_{ij}, \quad (10.1)$$

$$-n\mathfrak{B}_2 h_i^{(1)} = \alpha p \theta_{,i}, \quad (10.2)$$

$$P_{ijk}^{(1)} = 0, \quad (10.3)$$

$$-n\mathfrak{B}_2(\rho P_{4|}^{(1)} - 15p^2 + P_{ij}^{(1)} P_{ij}^{(1)}) = 30p^2 \dot{\beta}, \quad (10.4)$$

$$-n\mathfrak{B}_2(\rho P_{2|ij}^{(1)} - pP_{ij}^{(1)} + \frac{4}{7}[P_{ik}^{(1)} P_{kj}^{(1)} - \frac{1}{3}P_{kl}^{(1)} P_{kl}^{(1)} \delta_{ij}]) = 12p^2 E_{ij}, \quad (10.5)$$

$$-(6\mathfrak{B}_2 + \mathfrak{B}_4)P_{ijkl}^{(1)} + \frac{3}{\rho}(2\mathfrak{B}_2 - \mathfrak{B}_4) \quad (10.6)$$

$$\cdot (P_{(ij}^{(1)} P_{kl)}^{(1)} - \frac{4}{3}P_{m(i}^{(1)} \delta_{jk} P_{l)m}^{(1)} + \frac{2}{35}P_{mn}^{(1)} P_{mn}^{(1)} \delta_{(ij} \delta_{kl)}) = 0,$$

$$-n\mathfrak{B}_2(\rho P_{4|i}^{(1)} - 8ph_i^{(1)} + \frac{4}{3}P_{ijk}^{(1)} P_{jk}^{(1)} + \frac{12}{5}h_i^{(1)} P_{ij}^{(1)}) = \frac{59}{3}\alpha p^2 \theta_{,i}, \quad (10.7)$$

$$\begin{aligned} & -\frac{1}{2}(19\mathfrak{B}_2 + 2\mathfrak{B}_4)P_{2|ijk}^{(1)} - \frac{9}{\rho}(\mathfrak{B}_2 - \mathfrak{B}_4)pP_{ijk}^{(1)} \\ & - \frac{3}{2\rho}(8\mathfrak{B}_2 - \mathfrak{B}_4)(P_{l(i}^{(1)} P_{jk)l}^{(1)} - \frac{2}{5}P_{lm}^{(1)} P_{lm(i}^{(1)} \delta_{jk)}) \\ & + \frac{9}{5\rho}(13\mathfrak{B}_2 - 6\mathfrak{B}_4)(h_{(i}^{(1)} P_{jk)}^{(1)} - \frac{2}{5}h_l^{(1)} P_{l(i}^{(1)} \delta_{jk)}) = 0. \end{aligned} \quad (10.8)$$

These are algebraic equations for the iterates $P_{2r|s}^{(1)}$. In each, the maximum of $2r + s$ occurs in only one term, and that term is a linear one. Therefore, we may solve these equations for the first iterates $P_{2r|s}^{(1)}$ in turn in increasing order, and by inspection. Using the notation (7.19), we may write the results as follows:

$$P_{ij}^{(1)} = -2\mu E_{ij}, \quad (10.9)$$

$$h_i^{(1)} = \frac{1}{2}P_{2|i}^{(1)} = -\alpha\mu\theta_{,i}, \quad (10.10)$$

$$P_{ijk}^{(1)} = 0, \quad (10.11)$$

$$P_{4|}^{(1)} = \frac{1}{\rho}(15p^2 - 4\mu^2 E_{ij} E_{ij} - 30\mu p \dot{\beta}), \quad (10.12)$$

$$P_{2|ij}^{(1)} = -\frac{14\mu p}{\rho} E_{ij} - \frac{16\mu^2}{7\rho} (E_{ki} E_{jk} - \frac{1}{3}E_{kl} E_{kl} \delta_{ij}), \quad (10.13)$$

$$P_{ijkl}^{(1)} = \frac{12\mu^2}{\rho} \left(\frac{2\mathfrak{B}_2 - \mathfrak{B}_4}{6\mathfrak{B}_2 + \mathfrak{B}_4} \right) (E_{(ij} E_{kl)} - \frac{4}{3}E_{m(i} \delta_{jk} E_{l)m} + \frac{2}{35}E_{mn} E_{mn} \delta_{(ij} \delta_{kl)}), \quad (10.14)$$

$$P_{4|i}^{(1)} = -\frac{80\alpha\mu p}{3\rho} \theta_{,i} - \frac{24\alpha\mu^2}{5\rho} E_{ii} \theta_{,i} , \quad (10.15)$$

$$P_{2|ijk}^{(1)} = \frac{36\alpha\mu^2}{5\rho} \left(\frac{13\mathfrak{B}_2 - 6\mathfrak{B}_4}{19\mathfrak{B}_2 + \mathfrak{B}_4} \right) (E_{(ij} \theta_{,k)} - \frac{2}{5} E_{l(i} \delta_{jk)} \theta_{,l}) . \quad (10.16)$$

11. The second iterates. The second iterates are to be calculated from $m\mathfrak{C}^{(2)}(Q) = \mathfrak{L}^{(1)}(Q)$. To obtain $P_{ij}^{(2)}$, we have only to substitute the first iterates (10.9)–(10.11) into the right-hand side of (6.4) and equate the result to $m\mathfrak{C}^{(2)}(Y_{ij})$, which by (8.4) is equal to $-n\mathfrak{B}_2 P_{ij}^{(2)}$. The outcome for $P_{ij}^{(2)}$ is identical with (2.18).

To obtain $h_i^{(2)}$, we substitute the first iterates (10.9)–(10.13) into the right-hand side of (6.5) to obtain $\mathfrak{L}^{(1)}(c^2 Y_i)$. The resulting expression is equated to $m\mathfrak{C}^{(2)}(c^2 Y_i)$, which by (8.5) is equal to $-\frac{4}{3}n\mathfrak{B}_2 h_i^{(2)}$. The outcome, after some rearrangement, is

$$\begin{aligned} h_i^{(2)} = & -\alpha\mu\theta_{,i} + \frac{3\alpha\mu}{2p} (\mu\theta_{,i})^\cdot + \frac{\alpha\mu^2}{2p} (5\theta_{,i}E + 3\theta_{,i}u_{i,i} + 8\theta_{,i}E_{ii}) + \frac{3\mu}{\rho} (\mu E_{ii})_{,i} \\ & - \frac{3\mu^2}{p\rho} E_{ii}p_{,i} + \frac{15\mu}{2\rho\theta} (\mu\theta\dot{\theta})_{,i} \\ & + \frac{12\mu}{7\rho} \left(\frac{\mu^2}{\rho} E_{ii}E_{jk} \right)_{,k} + \frac{3\mu}{7\rho} \left(\frac{\mu^2}{\rho} E_{jk}E_{ik} \right)_{,i} + \frac{6\mu^2}{p\rho} E_{ii}(\mu E_{ik})_{,k} . \end{aligned} \quad (11.1)$$

In this and the following results the derivatives of μ may be calculated explicitly from (2.3):

$$\dot{\mu} = \frac{d\mu}{d\theta} \dot{\theta} = \mu(\log \theta)^\cdot, \quad \mu_{,i} = \mu(\log \theta)_{,i} , \quad (11.2)$$

but these formulae do not enable any ultimate condensation of the terms.

To obtain $P_{ijk}^{(2)}$, put (10.9), (10.11), (10.13), and (10.14) into (6.6), thus getting $\mathfrak{L}^{(1)}(Y_{ijk})$. This expression is then equated to $m\mathfrak{C}^{(2)}(Y_{ijk})$, which by (8.6) is equal to $-\frac{2}{3}n\mathfrak{B}_2 P_{ijk}^{(2)}$. Thus follows

$$\begin{aligned} P_{ijk}^{(2)} = & \frac{6\mu^2}{\rho\theta} (\theta_{,i}E_{jk} - \frac{2}{5}\theta_{,l}E_{l(i}\delta_{jk)}) \\ & + 4\mu \left\{ \left(\frac{\mu}{\rho} E_{(ij} \right)_{,k)} - \frac{2}{5} \left(\frac{\mu}{\rho} \delta_{(ij} E_{k)l} \right)_{,l} \right\} \\ & + \frac{8\mu^2}{\rho p} \{ E_{(ij}(\mu E_{k)l})_{,l} - \frac{2}{5} E_{l(i}\delta_{jk)}(\mu E_{lm})_{,m} \} \\ & - \frac{64\mu}{245p} \left\{ \frac{\mu^2}{\rho} \delta_{(ij} (E_{k)l} E_{lm} - \frac{1}{3} \delta_{k)m} E_{rs} E_{rs} \right\}_{,m} \\ & + \frac{32\mu}{49p} \left\{ \left(\frac{\mu^2}{\rho} E_{l(i} E_{j)l} \right)_{,k)} - \frac{1}{3} \left(\frac{\mu^2}{\rho} E_{rs} E_{rs} \right)_{,i} \delta_{jk} \right\} \\ & - \frac{8\mu}{p} \left\{ \frac{\mu^2}{\rho} \left(\frac{2\mathfrak{B}_2 - \mathfrak{B}_4}{6\mathfrak{B}_2 + \mathfrak{B}_4} \right) (E_{(ij} E_{k)l} - \frac{4}{7} E_{m(i} \delta_{jk)} E_{l)m} + \frac{2}{35} E_{rs} E_{rs} \delta_{(ij} \delta_{k)l}) \right\}_{,l} , \end{aligned} \quad (11.3)$$

where the subscript notation $l\{ijlk\}$ stands for the result of summing over l , then symmetrizing with respect to i, j , and k .

To obtain $P_{41}^{(2)}$, we substitute (10.9), (10.10), (10.12), (10.13), and (10.15) into the right-hand side of (6.7). On the right-hand side of (8.7), replace P_{41} by $P_{41}^{(2)}$ and P_{ij} by $P_{ij}^{(2)}$, the latter being given by (2.18). The iterative equation $m\mathfrak{C}^{(2)}(c^4) = \mathfrak{L}^{(1)}(c^4)$ is obtained by equating these two right-hand sides. Solution for $P_{41}^{(2)}$ yields

$$\begin{aligned} P_{41}^{(2)} = & \frac{15p^2}{\rho} - \frac{30\mu p}{\rho} \dot{\beta} + \frac{45\mu}{\rho\theta} (\mu\theta\dot{\beta})^\cdot + \frac{105\mu^2}{\rho} \dot{\beta}E \\ & + \frac{84\mu^2}{\rho} E_{ij}E_{ij} - \frac{1}{\rho} P_{ij}^{(2)}P_{ij}^{(2)} - \frac{45\mu^2}{\rho^2\theta} \theta_{,i}p_{,i} + \frac{150\mu p}{\rho^2\theta^2} (\mu\theta\theta_{,i})_{,i} \\ & + \frac{96\mu^3}{7\rho p} E_{ij}E_{ik}E_{ki} - \frac{24\mu^2}{5\rho p} (\mu E_{ij})_{,i}\theta_{,j} + \frac{27\mu^2}{\rho\theta} \left(\frac{\mu\theta_{,i}}{\rho}\right)_{,i} E_{ii}. \end{aligned} \quad (11.4)$$

To get $P_{21ij}^{(2)}$, we substitute (10.9)–(10.16) into (6.8) to obtain $\mathfrak{L}^{(1)}(c^2Y_{ij})$, then substitute (2.18) into (8.8) to obtain $m\mathfrak{C}^{(2)}(c^2Y_{ij})$, equate the two, and solve for $P_{21ij}^{(2)}$, obtaining

$$\begin{aligned} P_{21ij}^{(2)} = & -\frac{14\mu p}{\rho} E_{ij} + \frac{14\mu}{\rho} (\mu E_{ij})^\cdot + \frac{32\mu^2}{\rho} \dot{\beta}E_{ij} \\ & + \frac{14\mu^2}{\rho} EE_{ij} + \frac{4\mu^2}{21\rho} \{147E_{k(i}u_{j),k} + 72E_{ki}E_{jk} - 73E_{kl}E_{kl}\delta_{ij}\} \\ & - \frac{4}{21\rho} \{3P_{ki}^{(2)}P_{jk}^{(2)} - P_{kl}^{(2)}P_{kl}^{(2)}\delta_{ij}\} \\ & - \frac{6\mu^2}{\rho^2\theta} \{3\theta_{,i}p_{,i} - \theta_{,k}p_{,k}\delta_{ij}\} + \frac{80\mu^2 p}{7\rho^2\theta^2} \{3\theta_{,i}\theta_{,i} - \theta_{,k}\theta_{,k}\delta_{ij}\} \\ & + \frac{87\mu p}{7\rho^2\theta} \{3(\mu\theta_{,i})_{,i} - (\mu\theta_{,k})_{,k}\delta_{ij}\} + \frac{32\mu}{49\rho p} \{\mu^2(3E_{ki}E_{jk} - E_{kl}E_{kl}\delta_{ij})\}^\cdot \\ & + \frac{256\mu^3 E}{147\rho p} \{3E_{ki}E_{jk} - E_{kl}E_{kl}\delta_{ij}\} + \frac{64\mu^3}{49\rho p} \{3E_{kl}E_{l(i} - E_{rs}E_{rs}\delta_{k(i)}u_{j),k} \\ & + \frac{16\mu^3}{1715\rho p} \{240E_{ki}E_{jl} + 263E_{ij}E_{kl} - 220E_{km}E_{lm}\delta_{ij}\}E_{kl} \\ & + \frac{72\mu}{35\rho\theta} \left\{3\left(\frac{\mu^2}{\rho} \theta_{,k}E_{k(i)}\right)_{,i} - \left(\frac{\mu^2}{\rho} \theta_{,k}E_{kl}\right)_{,l}\delta_{ij}\right\} \\ & + \frac{12\mu^2}{\rho\theta} \{3\theta_{,i}(\mu E_{jk})_{,k} - \theta_{,k}(\mu E_{kl})_{,l}\delta_{ij}\} - \frac{12\mu}{7p} P_{ijkl}^{(1)}E_{kl} - \frac{6\mu}{7p} P_{21ijk,k}^{(1)}. \end{aligned} \quad (11.5)$$

12. The third iterates for the pressures and the flux of energy. The third iterate to P_{ij} is given by $-n\mathfrak{B}_2P_{ij}^{(3)} = m\mathfrak{C}^{(3)}(Y_{ij}) = \mathfrak{L}^{(2)}(Y_{ij})$. By (6.4) we get

$$P_{ij}^{(3)} = -\frac{\mu}{p} \{ \dot{P}_{ij}^{(2)} + P_{ij}^{(2)} E + 2P_{k(i)u_j, k}^{(2)} + 2pE_{ij} - \frac{2}{3}P_{kl}^{(2)} E_{kl} \delta_{ij} + P_{ijk, k}^{(2)} - \frac{4}{15}h_{k, k}^{(2)} \delta_{ij} + \frac{4}{5}h_{(i, j)}^{(2)} \}, \quad (12.1)$$

where $P_{ij}^{(2)}$, $h_i^{(2)}$, and $P_{ijk}^{(2)}$ are given explicitly by (2.18), (11.1), and (11.3).

The third iterate to h_i is given by $-\frac{4}{3}n\mathfrak{B}_2 h_i^{(3)} = m\mathfrak{C}^{(3)}(c^2 Y_i) = \mathfrak{L}^{(2)}(c^2 Y_i)$. By (6.5) we get

$$h_i^{(3)} = -\frac{3\mu}{4p} \left\{ 2\dot{h}_i^{(2)} + \frac{10}{3}h_i^{(2)} E + 2P_{ijk}^{(2)} E_{jk} + 2h_i^{(2)} u_{i, i} + \frac{8}{5}h_i^{(2)} E_{ii} - \frac{5p}{\rho} P_{ii, i}^{(2)} - \frac{2P_{ij}^{(2)}}{\rho} P_{jk, k}^{(2)} - \frac{5p}{\rho} p_{, i} - \frac{2P_{ij}^{(2)}}{\rho} p_{, i} + P_{2|ii, i}^{(2)} + \frac{1}{3}P_{4|, i}^{(2)} \right\}, \quad (12.2)$$

where $P_{ij}^{(2)}$, $h_i^{(2)}$, $P_{ijk}^{(2)}$, $P_{2|ii}^{(2)}$, and $P_{4|}^{(2)}$ are given explicitly by (2.18), (11.1), (11.3), (11.4), and (11.5).

Thus our program of obtaining $P_{ij}^{(3)}$ and $h_i^{(3)}$ explicitly and by straightforward algebraic substitutions is complete. (For another method, see §19.)

13. Criticism; proof of effectiveness; proof of the fundamental theorem.

We note three important properties of the foregoing results:

A. All terms in $P_{2r|s}^{(n)}$ whose degree in μ does not exceed n are identical with the corresponding terms in $P_{2r|s}^{(n+m)}$ for all $m \geq 0$.

B. Terms in $P_{2r|s}^{(n)}$ of degree equal to or less than n in μ remain unaltered in the higher iterates $P_{2r|s}^{(n+m)}$, but in general permanent terms of degree $n + m$ are obtained only with $P_{2r|s}^{(n+m)}$.

C. Deletion or alteration of the terms in $P_{2r|s}^{(n)}$ of degree equal to or exceeding n in μ does not affect the terms in $P_{2r|s}^{(n+m)}$ of degree equal to or less than $n + m$ in μ .

In the statements just made it is assumed that the derivatives of μ have been expressed explicitly in terms of μ by means of (11.2). The correctness of Statement A will become more apparent in §19. The validity of Statements B and C may be seen from the following considerations: Recall that $P_{2r|s}^{(n+1)}$ is obtained by transferring the bilinear form $Q_{2r|s}$ in (7.2) from the left-hand side of the equation $m\mathfrak{C}^{(n+1)}(Y_{2r|s}) = \mathfrak{L}^{(n)}(Y_{2r|s})$ to the right-hand side and thereafter multiplying both sides of the resulting equation by $-1/\mathfrak{C}_{2r|s}$, which is proportional to μ . Terms of degree equal to or less than $n - 1$ in the right-hand side of the resulting equation become terms of degree equal to or less than n in $P_{2r|s}^{(n+1)}$. But these are the terms which became the terms of degree equal to or less than n in $P_{2r|s}^{(n)}$. Also, terms of degree greater than n in μ in the right-hand side of the equation for $P_{2r|s}^{(n+1)}$ do not affect terms of degree less than or equal to $n + 1$ in $P_{2r|s}^{(n+1)}$. Induction completes the proof. Q.E.D.

Statement B expresses a disadvantage of the method. A second criticism, which becomes increasingly significant as higher iterates are reached, is:

D. $P_{2r|s}^{(n+1)}$ is a sum of the iterate $P_{2r|s}^{(n)}$ and further quantities; much of the labor and pencil work in the computation of $P_{2r|s}^{(n)}$ from the equation $m\mathcal{C}^{(n)}(Y_{2r|s}) = \mathcal{L}^{(n-1)}(Y_{2r|s})$ is repeated in the computation of $P_{2r|s}^{(n+1)}$ from the equation $m\mathcal{C}^{(n+1)}(Y_{2r|s}) = \mathcal{L}^{(n)}(Y_{2r|s})$.

This defect is completely avoided in the method to be given in §19.

We now prove that the method illustrated in the preceding sections is effective: By a finite number of differentiations and algebraic operations it enables us to calculate $p_{i_1 i_2 \dots i_m}^{(n)}$, for any given m and n , the result being always a function only of the gross condition of the gas.

First, since the $Y_{2r|s}$ of order q constitute a complete set of homogeneous polynomials of degree q , the quantity $c_i c_{i_2} \dots c_{i_q}$ may be expressed as a linear combination of the $Y_{2r|s}$ of order q . By the definitions (2.2) and (5.8), $p_{i_1 i_2 \dots i_q}$ may be expressed as a linear combination of the $P_{2r|s}$ of order q . (For examples, see (5.10)). By definition, $p_{i_1 i_2 \dots i_q}^{(n)}$ is the same linear combination of the $P_{2r|s}^{(n)}$. Hence it suffices to calculate $P_{2r|s}^{(n)}$.

Second, from the aforestated completeness of the set $Y_{2r|s}$ of order q it follows that $Y_{2r|s, i}$, $c_i Y_{2r|s, i}$, and $c_i Y_{2r|s}$ are linear combinations of the $Y_{2u|v}$ of orders $q-1$, q , and $q+1$, respectively. By (4.2) and (5.8), $\mathcal{L}(Y_{2r|s})$ is a function only of $u_{i, j}$, of the $P_{2r|s}$ of orders 2, $q-1$, q , $q+1$, and of the first derivatives of these $P_{2r|s}$. Hence, by definition, $\mathcal{L}^{(n)}(Y_{2r|s})$ is a function only of $u_{i, j}$, of the $P_{2r|s}^{(n)}$ of orders 2, $q-1$, q , $q+1$, and of the first derivatives of these $P_{2r|s}^{(n)}$.

Third, for the time being assume the validity of (7.2). Then, by definition, $m\mathcal{C}^{(n+1)}(Y_{2r|s}) = -\mathfrak{C}_{2r|s} P_{2r|s}^{(n+1)} + \text{a function of } p, \rho, \mu, \text{ and the } P_{2u|v}^{(n+1)} \text{ of order less than } q$.

Hence the iterative equation $m\mathcal{C}^{(n+1)}(Y_{2r|s}) = \mathcal{L}^{(n)}(Y_{2r|s})$ yields a *linear* equation for $P_{2r|s}^{(n+1)}$, an equation of the form $\mathfrak{C}_{2r|s} P_{2r|s}^{(n+1)} = \text{a function of } p, \rho, \mu, u_{i, j}, \text{ the } P_{2u|v}^{(n+1)} \text{ of lesser order, and the } P_{2u|v}^{(n)} \text{ of orders 2, } q-1, q, q+1, \text{ and the first derivatives of these latter.}$ By (7.6) and the inequality $|P_i(\cos \phi)| \leq 1$, each non-zero summand in (7.4)₁ is positive; hence $\mathfrak{C}_{2r|s} > 0$ if $2r + s \geq 3$. Thus we can solve for $P_{2r|s}^{(n+1)}$.

We have proved the **inductive lemma**. *Once the collision integral $\mathcal{C}(Y_{2r|s})$ is evaluated, a finite number of algebraic operations yields $P_{2r|s}^{(n+1)}$ as a function of the gross condition, the $P_{2u|v}^{(n+1)}$ of lower order, the $P_{2u|v}^{(n)}$ of order not more than one greater, and the derivatives of these latter.*

This lemma enables us to determine how many iterates are needed to calculate $P_{2r|s}^{(n)}$. First, we need the n^{th} iterates to certain P 's of order $< q$. Next, we need the $n-1^{\text{st}}$ iterates to P 's of order $q+1$; the $n-2^{\text{nd}}$ iterates to P 's of order $q+2$; $\dots$ the zeroth iterates to P 's of order $q+n$. Thus to calculate $P_{2r|s}^{(n)}$, we carry out the following steps:

- 1. Calculate $\mathcal{L}(Y_{2u|v})$ and $\mathcal{C}(Y_{2u|v})$ for all Y 's of order $\leq n + 2r + s - 1$.
0. Using the results of step -1, write out $m\mathcal{C}^{(l+1)}(Y_{2u|v}) = \mathcal{L}^{(l)}(Y_{2u|v})$ for these Y 's.

1. Using (4.11), calculate the corresponding $P_{2u|v}^{(1)}$ for orders $\leq n + 2r + s - 1$.
2. Using the results of step 1, calculate the corresponding $P_{2u|v}^{(2)}$ for orders $\leq n + 2r + s - 2$.

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- n. Using the results of step $n - 1$, calculate $P_{2r|s}^{(n)}$.

At each step, the number of iterates to be determined decreases.

By induction we conclude the **theorem of effectiveness**: Suppose the collision integrals $m\mathcal{C}(Y_{2u|v})$ for all $Y_{2u|v}$ of orders not greater than $q + n$ are known explicitly. Then a finite number of differentiations and algebraic operations yields $P_{2r|s}^{(n)}$, where $2r + s = q$, as an explicit function of the gross condition.

That for Maxwellian molecules the collision integral for any polynomial of degree n can be expressed exactly as a polynomial in the moments of degree not greater than n was indicated by MAXWELL and was illustrated for the first few $Y_{2r|s}$ in §8. All that remains to complete our proof is to strengthen this assertion of MAXWELL by proving the more specific **fundamental theorem** (7.2), which now we proceed to do.

In $Q(\mathbf{c}^*) = (c^*)^{2r} Y_s(\mathbf{c}^*)$, we replace $\mathbf{c}^*$ by $\mathbf{v} + \mathbf{w}$, obtaining

$$Q(\mathbf{c}^*) = Q(\mathbf{v} + \mathbf{w}) = \Theta^r(v, w, \chi) Y_s(\mathbf{v} + \mathbf{w}), \quad (13.1)$$

where χ is the angle between $\mathbf{v}$ and $\mathbf{w}$, and

$$\Theta(v, w, \chi) \equiv v^2 + w^2 + 2vw \cos \chi. \quad (13.2)$$

Θ^r can then be expanded as a finite sum of Legendre polynomials:

$$\Theta^r(v, w, \chi) = \frac{1}{2} \sum_{j=0}^r A_{rj}(v, w) P_j(\cos \chi), \quad (13.3)$$

where¹

$$A_{rj}(v, w) = (2j + 1) \int_{-1}^{+1} \Theta^r(v, w, \chi) P_j(\cos \chi) d \cos \chi \quad (13.4)$$

is a homogeneous polynomial of total degree $2r$ in v and w . We may also expand $Y_s(\mathbf{v} + \mathbf{w})$ in a finite sum of the form

$$Y_s(\mathbf{v} + \mathbf{w}) = \sum_{k=0}^s Z_{sk}(\mathbf{w}) Y_k(\mathbf{v}), \quad (13.5)$$

since $Y_s(\mathbf{v} + \mathbf{w})$, considered as a function of $\mathbf{v}$, is a non-homogeneous spherical harmonic in $\mathbf{v}$. The coefficients $Z_{sk}(\mathbf{w})$ are homogeneous polynomials of degree $s - k$ in $\mathbf{w}$. The terms in the expansion (13.5) may separately be written in the form

¹ An explicit expression for A_{rj} is given by CHAPMAN & COWLING [1952, 1, eq. A 1.2.3], but, as we shall find, it is not needed for the present purposes.

$$Z_{sk}(\mathbf{w}) Y_k(\mathbf{v}) = \frac{1}{k!} \left(v_i \frac{\partial}{\partial w_i} \right)^k Y_s(\mathbf{w}). \quad (13.6)$$

For $Q(\mathbf{c}^*)$ we obtain, since $v' = v$, $\mathbf{w}' = \mathbf{w}$, and $w' = w$, $\Theta'(v, w, \chi') Y_s(\mathbf{v}' + \mathbf{w})$, where χ' is the angle between $\mathbf{v}'$ and $\mathbf{w}$. For an expansion of $Q(\mathbf{c}^*)$, we need only replace χ by χ' and $\mathbf{v}$ by $\mathbf{v}'$ wherever χ and $\mathbf{v}$, respectively, occur.

We take δQ as given by $(3.3)_2$ rather than $(3.3)_1$, so obtaining the intermediate formula

$$\delta Y_{2r|s} = \sum_{i=0}^r \sum_{k=0}^s A_{ri}(v, w) Z_{sk}(\mathbf{w}) [P_i(\cos \chi') Y_k(\mathbf{v}') - P_i(\cos \chi) Y_k(\mathbf{v})]. \quad (13.7)$$

This expression is to be put into (3.2).

For all spherically symmetric molecules the integration with respect to ϵ , the azimuthal angle in the co-ordinate plane normal to $\mathbf{v}$, may be carried out explicitly. For if ϕ is the angle between $\mathbf{v}$ and $\mathbf{v}'$, for spherically symmetric molecules ϕ does not depend on ϵ . But from spherical trigonometry we have

$$\cos \chi' = \cos \chi \cos \phi + \sin \chi \sin \phi \cos \epsilon, \quad (13.8)$$

so that by a basic expansion in Legendre polynomials follows [1939, eq. A. 2.2]

$$P_i(\cos \chi') = P_i(\cos \chi) P_i(\cos \phi) + 2 \sum_{l=0}^i b_{il} P_i^{(l)}(\cos \phi) P_i^{(l)}(\cos \chi) \cos l\epsilon, \quad (13.9)$$

$$b_{il} = \frac{(j-l)!}{(j+l)!} (1 - \delta_{l0}).$$

It is well known [1931, §77 and 94] that the general solid harmonic $S_k(r, \phi, \epsilon)$ of degree k is

$$r^k \left[c_{k0} P_k(\cos \phi) + \sum_{p=1}^k (c_{kp} \cos p\epsilon + d_{kp} \sin p\epsilon) P_k^{(p)}(\cos \phi) \right], \quad (13.10)$$

where r, ϕ , and ϵ are spherical co-ordinates, c_{kp} and d_{kp} are $2k$ arbitrary constants, and $c_{k0} = S_k(1, 0, \epsilon) =$ value of S_k at the pole of the unit sphere. Hence, taking the polar axis of the spherical coordinate system in the direction of $\mathbf{v}$, by applying (13.10) to each member of the set $Y_k(\mathbf{v}')$, we get an expansion of the form

$$Y_k(\mathbf{v}') = Y_k(\mathbf{v}) P_k(\cos \phi) + \sum_{p=1}^k [c_{kp}(\mathbf{v}) \cos p\epsilon + d_{kp}(\mathbf{v}) \sin p\epsilon] P_k^{(p)}(\cos \phi). \quad (13.11)$$

If we multiply (13.9) and (13.11) together and then integrate with respect to ϵ , we get at once

$$\begin{aligned} \int_0^{2\pi} Y_k(\mathbf{v}') P_i(\cos \chi') d\epsilon &= 2\pi Y_k(\mathbf{v}) P_i(\cos \chi) P_k(\cos \phi) P_i(\cos \phi) \\ &+ 2\pi \sum_{l=0}^{\min(j,k)} c_{kl}(\mathbf{v}) b_{il} P_i^{(l)}(\cos \chi) P_k^{(l)}(\cos \phi) P_i^{(l)}(\cos \phi). \end{aligned} \quad (13.12)$$

This formula, valid for all spherically symmetric molecules, includes and generalizes (7.1). Putting (13.12) into (13.7) yields

$$\begin{aligned} \int_0^{2\pi} \delta Y_{2r|s} d\epsilon = & -2\pi \sum_{j=0}^r \sum_{k=0}^s A_{rj}(v, w) Z_{sk}(\mathbf{w}) \\ & \cdot \left\{ Y_k(\mathbf{v}) P_i(\cos \chi) [1 - P_i(\cos \phi) P_k(\cos \phi)] \right. \\ & \left. - \sum_{l=0}^{\text{Min}(j, k)} c_{kl}(\mathbf{v}) b_{il} P_i^{(l)}(\cos \chi) P_k^{(l)}(\cos \phi) P_i^{(l)}(\cos \chi) \right\}. \end{aligned} \quad (13.13)$$

We are now to integrate with respect to ϕ . In this integration the quantities $\mathbf{v}$, $\mathbf{w}$, and χ are constant. Thus follows

$$\begin{aligned} v \int_0^\infty b db \int_0^{2\pi} \delta Y_{2r|s} d\epsilon \\ = - \sum_{j=0}^r \sum_{k=0}^s A_{rj}(v, w) Z_{sk}(\mathbf{w}) \left\{ \mathfrak{B}_{jk} Y_k(\mathbf{v}) P_i(\cos \chi) \right. \\ \left. + \sum_{l=0}^{\text{Min}(j, k)} c_{kl}(\mathbf{v}) b_{il} \mathfrak{B}_{jk}^{(l)} P_i^{(l)}(\cos \chi) \right\}, \\ \equiv C[Y_{2r|s}], \end{aligned} \quad (13.14)$$

say, where

$$\begin{aligned} \mathfrak{B}_{jk} &\equiv 2\pi v \int_0^\infty \{1 - P_i(\cos \phi) P_k(\cos \phi)\} b db, \\ \mathfrak{B}_{jk}^{(l)} &\equiv -2\pi v \int_0^\infty P_i^{(l)}(\cos \phi) P_k^{(l)}(\cos \phi) b db. \end{aligned} \quad (13.15)$$

For laws of force such that ϕ is a continuously differentiable monotone function of b , the form (7.6) for $\mathfrak{B}_{jk}$ follows. In general, the $\mathfrak{B}_{jk}$ and $\mathfrak{B}_{jk}^{(l)}$ are certain functions of v .

Now

$$\mathcal{C}(Y_{2r|s}) = \int_\infty d\mathbf{c} \int_\infty d\mathbf{c}^* FF^* C[Y_{2r|s}], \quad (13.16)$$

where $C[Y_{2r|s}]$ is the function of $\mathbf{v}$ and $\mathbf{w}$ given by the finite sum (13.14). The remaining work consists in replacing $\mathbf{v}$ and $\mathbf{w}$ by $\frac{1}{2}(\mathbf{c}^* - \mathbf{c})$ and $\frac{1}{2}(\mathbf{c}^* + \mathbf{c})$, respectively, and then carrying out the integrations. This is extremely laborious, even when all functions occurring are expanded in series of powers of $\mathbf{c}$ and $\mathbf{c}^*$ and these series are multiplied together and arranged so that the results of the integrations are sums of products of moments.

We shall carry out this process only for Maxwellian molecules, for which the $\mathfrak{B}_{jk}$ and $\mathfrak{B}_{jk}^{(l)}$ become constants independent of v , and at that we shall rest content with calculating the term containing the spherical moment $P_{2r|s}$ of highest order. This term is obtained from (13.14) by adding the value obtained by putting

$\mathbf{v} = \mathbf{w} = \frac{1}{2}\mathbf{c}^*$ to the value obtained by putting $\mathbf{v} = -\frac{1}{2}\mathbf{c}$, $\mathbf{w} = +\frac{1}{2}\mathbf{c}$, since these values are the terms of highest degree in $\mathbf{c}^*$ and $\mathbf{c}$, respectively. For $\mathbf{v} = \mathbf{w} = \frac{1}{2}\mathbf{c}^*$ we get $\chi = 0$, $P_i(\cos \chi) = 1$, and $P_i^{(l)}(\cos \chi) = 0$, $l \neq 0$. For $\mathbf{v} = -\mathbf{w} = -\frac{1}{2}\mathbf{c}$, we get $\chi = \pi$, $P_i(\cos \chi) = (-1)^i$, and $P_i^{(l)}(\cos \chi) = 0$, $l \neq 0$. Thus the terms in (13.12) which contain the coefficients $\mathfrak{B}_{jk}^{(l)}$, $l \neq 0$, do not contribute to the desired leading term. For the terms which do contribute, we make use of the following identities:

$$A_{ri}(\tfrac{1}{2}\mathbf{c}^*, \tfrac{1}{2}\mathbf{c}^*)P_i(1) = a_{ri}(\mathbf{c}^*)^{2r}/2^r, \quad (13.17)$$

$$A_{ri}(\tfrac{1}{2}\mathbf{c}, \tfrac{1}{2}\mathbf{c})P_i(-1) = (-1)^i a_{ri}\mathbf{c}^{2r}/2^r, \quad (13.18)$$

$$Z_{sk}(\tfrac{1}{2}\mathbf{c}^*)Y_k(\tfrac{1}{2}\mathbf{c}^*) = \binom{s}{k} Y_s(\mathbf{c}^*)/2^s, \quad (13.19)$$

$$Z_{sk}(\tfrac{1}{2}\mathbf{c})Y_k(-\tfrac{1}{2}\mathbf{c}) = (-1)^k \binom{s}{k} Y_s(\mathbf{c})/2^s, \quad (13.20)$$

where

$$a_{ri} = (2j+1) \int_{-1}^{+1} (1+x)^r P_i(x) dx. \quad (13.21)$$

The first two of the preceding identities, (13.17) and (13.18), follow readily from (13.2)–(13.4), the preceding remarks, and the relation $P_i(-x) = (-1)^i P_i(x)$. For the latter two, namely (13.19) and (13.20), we return to (13.6). Recalling that $Z_{sk}(\mathbf{c})$ and $Y_k(\mathbf{c})$ are homogeneous in $\mathbf{c}$ of degrees $s-k$ and k , respectively, we get

$$\begin{aligned} Z_{sk}(\tfrac{1}{2}\mathbf{c})Y_k(\tfrac{1}{2}\mathbf{c}) &= \frac{1}{2^s} Z_{sk}(\mathbf{c})Y_k(\mathbf{c}), \\ &= \frac{1}{k!2^s} \left(x_i \frac{\partial}{\partial c_i} \right)^k Y_s(\mathbf{c}) \Big|_{\mathbf{x}=\mathbf{c}}. \end{aligned} \quad (13.22)$$

The identity (13.19) follows from this as a consequence of a general property of homogeneous polynomials. By noting that $Y_k(-\mathbf{c}) = (-1)^k Y_k(\mathbf{c})$, we readily obtain (13.20) also.

The proof of the fundamental theorem (7.2) with the form (7.4)₁ for $\mathfrak{C}_{2r|s}$, apart from the evaluation (7.5) for a_{ri} , follows immediately from (13.14)–(13.20).

We turn now to the proof of (7.5). We substitute Rodrigues' formula [1927, §15.11] for $P_i(x)$ in (13.21) and then integrate by parts j times. By noting that, each time, the integrated part vanishes at the limits $x = -1$ and $x = +1$, we obtain

$$a_{ri} = \begin{cases} 0, & r < j, \\ \frac{2j+1}{2^j} \binom{r}{j} \int_{-1}^{+1} (1+x)^r (1-x)^j dx, & r \geq j \end{cases} \quad (13.23)$$

The definite integral is a beta function; expressing it in terms of gamma functions, we obtain (7.5).

Finally, we establish the integral expression (7.4)₂ for $\mathfrak{E}_{2r|s}$. To do so, we begin by noting that, in virtue of (13.21),

$$\sum_{i=0}^r a_{ri} P_i(\cos \phi) = 2(1 + \cos \phi)^r = 2^{r+1} \cos^{2r} \frac{1}{2} \phi, \quad (13.24)$$

from which, since $P_i(-x) = (-1)^i P_i(x)$,

$$\sum_{i=0}^r (-1)^i a_{ri} P_i(\cos \phi) = 2(1 - \cos \phi)^r = 2^{r+1} \sin^{2r} \frac{1}{2} \phi. \quad (13.25)$$

From these we obtain, using $P_i(1) = 1$,

$$\sum_{i=0}^r a_{ri} = 2^{r+1}, \quad \sum_{i=0}^r (-1)^i a_{ri} = 2^{r+1} \delta_{r0}. \quad (13.26)$$

Upon substituting the expression (7.6) for $\mathfrak{B}_{ik}$ into the right-hand member of (7.4)₁ and using (13.24)–(13.26), we obtain

$$\mathfrak{E}_{2r|s} = I_{2r|s} + J_{2r|s}, \quad (13.27)$$

where

$$\begin{aligned} I_{2r|s} &= \frac{4\pi n v}{2^s} \int_0^\pi \sum_{k=0}^s \binom{s}{k} [1 - \cos^{2r} \frac{1}{2} \phi P_k(\cos \phi)] b \frac{db}{d\phi} d\phi, \\ J_{2r|s} &= \frac{4\pi n v}{2^s} \int_0^\pi \sum_{k=0}^s (-1)^k \binom{s}{k} [\delta_{r0} - \sin^{2r} \frac{1}{2} \phi P_k(\cos \phi)] b \frac{db}{d\phi} d\phi. \end{aligned} \quad (13.28)$$

To obtain (7.4)₂ we must effect the summations indicated in the integrands. By using the binomial theorem and the Schläfli integral representation [1927, §15.12]:

$$P_k(x) = \frac{1}{(2\pi i)2^k} \oint_C \frac{(t^2 - 1)^k}{(t - x)^{k+1}} dt, \quad (13.29)$$

where C is a contour which encircles the point x once counter-clockwise, we obtain, by a little rearrangement,

$$\sum_{k=0}^s \binom{s}{k} P_k(x) = \frac{1}{2\pi i} \oint_C \frac{1}{t - x} \left[1 + \frac{t^2 - 1}{2(t - x)} \right]^s dt. \quad (13.30)$$

The substitutions $t = (2x')^{\frac{1}{2}} t' - 1$, $x = x' - 1$ transform the right-hand side into

$$\frac{(2x')^{\frac{1}{2}s}}{2^s(2\pi i)} \oint_C \frac{(t'^2 - 1)^s}{[t' - (\frac{1}{2}x')^{\frac{1}{2}}]^{s+1}} dt' = (2x')^{\frac{1}{2}s} P_s(\sqrt{\frac{1}{2}x'}). \quad (13.31)$$

With $x = \cos \phi$, $x' = 2 \cos^2 \frac{1}{2} \phi$, this gives

$$\sum_{k=0}^s \binom{s}{k} P_k(\cos \phi) = 2^s \cos^s \frac{1}{2} \phi P_s(\cos \frac{1}{2} \phi). \quad (13.32)$$

On the other hand, with $x = -\cos \phi$, $x' = 2 \sin^2 \frac{1}{2}\phi$, it gives

$$\sum_{k=0}^s (-1)^k \binom{s}{k} P_k(\cos \phi) = 2^s \sin^s \frac{1}{2}\phi P_s(\sin \frac{1}{2}\phi). \quad (13.33)$$

By using these last two formulæ and the binomial theorem in (13.28), we obtain

$$\begin{aligned} I_{2r|s} &= 4\pi n v \int_0^\pi [1 - \cos^{2r+s} \frac{1}{2}\phi P_s(\cos \frac{1}{2}\phi)] b \frac{db}{d\phi} d\phi, \\ J_{2r|s} &= 4\pi n v \int_0^\pi [\delta_{r0}\delta_{s0} - \sin^{2r+s} \frac{1}{2}\phi P_s(\sin \frac{1}{2}\phi)] b \frac{db}{d\phi} d\phi. \end{aligned} \quad (13.34)$$

Our desired result (7.4)₂ now follows from (13.27).

14. Bounds for $\mathfrak{C}_{2r|s}$. Since the $\mathfrak{C}_{2r|s}$ determine rates of decay of deviations from a uniform state (*cf.* Chapter VI), it is desirable to obtain bounds for these coefficients.¹

The integrand in (7.4)₂ is positive at almost all points of the interval $0 \leq \phi \leq \pi$ except for the three cases: $r = s = 0$, $r = 1$ and $s = 0$, and $r = 0$ and $s = 1$, for each of which it vanishes identically. In all cases but these, which need not be considered further in the present discussion, it follows that $\mathfrak{C}_{2r|s} > 0$. Since $|P_s(x)| \leq 1$, it is readily seen from (7.4)₂ that

$$\mathfrak{C}_{2r|s} \geq \mathfrak{C}_{q|}, \quad q = 2r + s, \quad (14.1)$$

where $\mathfrak{C}_{q|}$, for odd as well as for even values of q , is defined by

$$\mathfrak{C}_{q|} \equiv 4\pi n v \int_0^\pi (1 + \delta_{q0} - \cos^q \frac{1}{2}\phi - \sin^q \frac{1}{2}\phi) b \frac{db}{d\phi} d\phi. \quad (14.2)$$

Since the integrand in (14.2) is an increasing function of q at all points of the interval $0 < \phi < \pi$, we have

$$\mathfrak{C}_{q|} > \mathfrak{C}_{q'|} \quad \text{for } q > q'. \quad (14.3)$$

Numerical values for the $\mathfrak{C}_{2r|s}$ when $2r + s = 2$ or 3 and for $\mathfrak{C}_{q|}$ may be read off from (8.3)–(8.7). From these and from (14.1) and (14.3) we obtain the following general bound:

$$\mathfrak{C}_{2r|s} \geq \mathfrak{C}_{2|1} = \frac{2}{3} n \mathfrak{B}_2 = \frac{2}{3} \frac{p}{\mu} \quad (14.4)$$

for all r and s except in the three previously noted special cases when $\mathfrak{C}_{2r|s} = 0$.

So as to facilitate obtaining an increasing lower bound for $\mathfrak{C}_{q|}$, we set²

$$\begin{aligned} x &= 2\theta, \\ y &= \frac{1}{2}\pi - \frac{1}{2}\phi = \frac{1}{2}\pi - \sqrt{\cos x} K(\sin \frac{1}{2}x), \end{aligned} \quad (14.5)$$

¹ CHANG & UHLENBECK [1952, 3, App. III] have noted that $\mathfrak{C}_{2r|s} \geq \mathfrak{C}_{2r|}$ and have inferred, by arguments which we do not fully justify, that $\mathfrak{C}_{2r|} \rightarrow \infty$ as $r \rightarrow \infty$.

² Our y is MAXWELL'S θ (*cf.* footnote 1 of §3).

where the last form follows by (7.19). Putting these into (14.2) we get

$$\begin{aligned}\mathfrak{E}_a &= I_a + J_a, \\ I_a &= c \int_0^{\frac{1}{2}\pi} (1 - \cos^a y) \csc^2 x \, dx, \\ J_a &= c \int_0^{\frac{1}{2}\pi} (\delta_{a0} - \sin^a y) \csc^2 x \, dx,\end{aligned}\tag{14.6}$$

where

$$c \equiv 2\pi n \sqrt{G/m}.\tag{14.7}$$

Preparatory to deriving a lower bound for $\mathfrak{E}_a$, we note certain properties of the function $y(x)$, which are easily verified from the definition (14.5)₂ :

1. In $0 \leq x \leq \frac{1}{2}\pi$, $y(x)$ is monotonically increasing, and $y(0) = 0$, $y(\frac{1}{2}\pi) = \frac{1}{2}\pi$.
2. For small x we have

$$y(x) = \frac{3\pi}{32} x^2 + \frac{23\pi}{2048} x^4 + O(x^6).\tag{14.8}$$

3. $y''(x) > 0$ for $0 \leq x \leq \frac{1}{2}\pi$.

We now assert the following **lemma**:

$$\frac{3\pi}{32} x^2 < y(x) < x \quad \text{for } 0 < x < \frac{1}{2}\pi.\tag{14.9}$$

The right-hand inequality is immediate from properties 1 and 3. To prove the left-hand inequality, we compare $y(x)$ with the parabola $z = 3\pi x^2/32$. By property 2, we have $y \geq z$ for small x ; since $y(\frac{1}{2}\pi) = \frac{1}{2}\pi$ and since y is left-continuous at $\frac{1}{2}\pi$, we have $y > z$ for x near to $\frac{1}{2}\pi$. Since z is convex and, by property 3, y is convex, the lemma follows. Q.E.D.

In order to obtain a lower bound for I_a , we begin by noting that $1 - \cos^a y$ is a monotone increasing function of y as y varies from 0 to $\frac{1}{2}\pi$. Thus, if we take any x_0 such that $0 < x_0 \leq \frac{1}{2}\pi$ and any y_0 such that $y_0 \leq y(x_0)$, by property 1 we get

$$\begin{aligned}I_a &> c(1 - \cos^a y_0) \int_{x_0}^{\frac{1}{2}\pi} \csc^2 x \, dx, \\ &> c(1 - \cos^a y_0) \cot x_0.\end{aligned}\tag{14.10}$$

For a fixed x_0 , we should maximize the right-hand side of (14.10) by taking $y_0 = y(x_0)$. However, so as to avoid difficult calculations, we abandon search for the best possible bound and rest content with the result of a rather arbitrary choice of y_0 and x_0 , from which a suitable lower bound emerges.

Take any positive $a < \frac{1}{2}q$ and hold it fixed. Take y_0 determined uniquely by the conditions

$$\sin^2 y_0 = \frac{2a}{q}, \quad 0 < y_0 < \frac{1}{2}\pi.\tag{14.11}$$

Then $\cos^a y_0$ increases monotonically with q and approaches e^{-a} as $q \rightarrow \infty$. Thus from (14.10) follows

$$I_a > c(1 - e^{-a}) \cot x_0, \quad (14.12)$$

provided we choose x_0 so that $y(x_0) \geq y_0$ and $a < \frac{1}{2}q$. From (14.11) and the general inequality $\sin y > \frac{2}{\pi}y$ for $0 < y < \frac{1}{2}\pi$ we get

$$y_0 < \frac{\pi}{2} \sqrt{\frac{2a}{q}}. \quad (14.13)$$

Hence, from (14.9) follows $y(x_0) > y_0$ if $\frac{3\pi}{32} x_0^2 = \frac{\pi}{2} \sqrt{\frac{2a}{q}}$; equivalently,

$$x_0 = \sqrt[4]{\frac{512a}{9q}}. \quad (14.14)$$

However, in order to satisfy the requirement $0 < x_0 \leq \frac{1}{2}\pi$ we must now replace the original stipulation $0 < a < \frac{1}{2}q$ by the stronger inequality

$$0 < a \leq \frac{9}{512}(\frac{1}{2}\pi)^4 q = 0.1070q. \quad (14.15)$$

Substituting (14.14) into (14.12), we obtain

$$I_a > c(1 - e^{-a}) \cot \sqrt[4]{\frac{512a}{9q}}. \quad (14.16)$$

For a given q , the bound (14.16) holds for any a satisfying (14.15).

A lower bound for J_a may be obtained from the inequality $\sin y < \sin x$, which is a direct consequence of (14.9). For $q > 2$ we have

$$\begin{aligned} 0 > J_a &> -c \int_0^{\frac{1}{2}\pi} \sin^{a-2} x \, dx \\ &> -\frac{c\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{q-1}{2}\right)}{\Gamma(\frac{1}{2}q)}. \end{aligned} \quad (14.17)$$

This bound vanishes with $q^{-\frac{1}{2}}$ as $q \rightarrow \infty$.

Substitution of (14.17) and (14.16) in (14.6)₁ yields the final result:

$$\mathfrak{C}_{a1} > c(1 - e^{-a}) \cot \sqrt[4]{\frac{512a}{9q}} - \frac{c\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{q-1}{2}\right)}{\Gamma(\frac{1}{2}q)}, \quad (14.18)$$

where a is any number satisfying (14.15). In view of the remark following (14.10), there is no point in determining the best possible a . The bound (14.18) suffices to show that as $q \rightarrow \infty$, $\mathfrak{C}_{a1}$ also approaches infinity at least as fast as $q^{\frac{1}{2}}$.

By a calculation not presented here we have derived an upper bound for $\mathfrak{C}_{a1}$ which also is ultimately proportional to $q^{\frac{1}{2}}$. It is also possible to obtain an asymptotic form proportional to $q^{\frac{1}{2}}$.

CHAPTER II. *Other differential iterations*

15. Comparison with the results of the Chapman-Enskog process (iteration for the rate of change of entropy and deformation). The CHAPMAN-ENSKOG process has been carried as far as $p_{ij}^{(2)}$ and $h_i^{(2)}$. (For the former, due to BURNETT, see [1952, 1, eqq. 15.3.6, 15.41.1, and 15.41.3]; the latter was obtained by CHAPMAN & COWLING [1939, eqq. 15.3.4, 15.4.4, 15.4.5, 15.4.6] and was corrected by CHANG & UHLENBECK [1948], 4, §III, *cf.* [1952, 1, eqq. 15.3.4, 15.4.4, 15.4.5]). The results are similar to (2.18) and (11.1) but not quite the same (*cf.* footnote 6 of §2).

To explain the difference, recall first that the CHAPMAN-ENSKOG process employs a systematic elimination of all time derivatives in terms of space derivatives.¹ Its results are expressions for p_{ij} and h_i in terms of p , ρ , u_i and their space derivatives of all orders, and of first and higher space derivatives of the extrinsic force f_i —as we shall say, of the *static gross condition* of the gas. The method of Maxwellian iteration may be adjusted to give results which appear to be the same. For reduction of the gross condition to the static gross condition, two types of time derivatives, those of the thermostatic state p , ρ and those of the velocity u_i , must be eliminated.

Now by means of the continuity equation (8.17), we may eliminate $\dot{\rho}$ without any approximation. The equation of energy (2.8) expresses $\dot{\theta}$ in terms of p_{ij} and h_i as well as the static gross condition. Thus if we have for p_{ij} and h_i iterates which are functions only of the static gross condition, a similar expression for $\dot{\theta}$ follows. Any of the numerous forms of the equation of energy may be employed. Noticing that the specific entropy η is given by

$$\eta \equiv \frac{3k}{2m} \log p \rho^{-5/3} + \text{const.} = \frac{k}{m} \beta + \text{const.}, \quad (15.1)$$

we prefer to write (2.8) in the form

$$p\dot{\beta} = -P_{ij}E_{ij} - h_{i,i}. \quad (15.2)$$

We then set up the iterative equation

$$p\dot{\beta}^{(n)} \equiv -P_{ij}^{(n)}E_{ij} - h_{i,i}^{(n)}. \quad (15.3)$$

Thus

$$\dot{\beta}^{(0)} = 0, \quad (15.4)$$

the adiabatic approximation,² while

$$p\dot{\beta}^{(1)} = 2\mu E_{ij}E_{ij} + \alpha(\mu\theta_{,i})_{,i}, \quad \text{etc.} \quad (15.5)$$

¹ Such an idea has been mentioned also in continuum mechanics [1952, 2, §78].

² This was used in the equivalent form $\dot{\theta}^{(0)} = -\frac{2}{3}\theta E$ for comparison of (2.18) with Burnett's equations in footnote 6 of §2.

When (15.3) is adjoined to the iterative equations of Chapter I, we shall replace the superscript (n) by $\{n\}$ and shall call the process *atemporal Maxwellian iteration*. With it we get such simplifications as $\mathfrak{L}^{(0)}(c^4) = 0$ in place of (9.4). For the solution of a special problem of gas flow, in §36 we shall prove the convergence of this process.

Elimination of time derivatives of the velocity may be effected by a similar use of the equation of momentum:

$$\rho \dot{u}_i = -p_{,i} - P_{ii,i} + \rho f_i. \quad (15.6)$$

Hence, exactly,

$$\begin{aligned} (u_{i,i})^\cdot &= \partial_i u_{i,i} + u_{i,jk} u_{k,j} \\ &= (\dot{u}_i)_{,i} - u_{i,k} u_{k,i} \\ &= -\left(\frac{p_{,i}}{\rho}\right)_{,i} - \left(\frac{1}{\rho} P_{ik,k}\right)_{,i} + f_{i,i} - u_{i,k} u_{k,i}. \end{aligned} \quad (15.7)$$

Thus we are led to set³

$$(u_{i,i})^{\cdot(n+1)} \equiv -\left(\frac{1}{\rho} P_{ik,k}^{(n)}\right)_{,i} - \left(\frac{p_{,i}}{\rho}\right)_{,i} + f_{i,i} - u_{i,k} u_{k,i}. \quad (15.8)$$

From (15.5) and (15.8) we get, for example,

$$\begin{aligned} (\mu E_{ii})^{\cdot(1)} &= \dot{\mu}^{(1)} E_{ii} + \mu \dot{E}_{ii}^{(1)}, \\ &= \frac{2}{3} \mu (\dot{\beta} + \dot{\rho}/\rho)^{(1)} E_{ii} + \mu \dot{E}_{ii}^{(1)}, \\ &= \mu \left\{ \frac{2}{3} \left[\frac{2\mu}{p} E_{kl} E_{kl} + \frac{\alpha}{p} (\mu \theta_{,k})_{,k} - E \right] E_{ii} - \left[\left(\frac{p_{,i}}{\rho}\right)_{,i} - \frac{1}{3} \left(\frac{p_{,k}}{\rho}\right)_{,k} \delta_{ii} \right] \right. \\ &\quad \left. + f_{(i,i)} - \frac{1}{3} f_{k,k} \delta_{ii} - [u_{k,(i} u_{i),k} - \frac{1}{3} u_{l,k} u_{k,l} \delta_{ii}] \right\}. \end{aligned} \quad (15.9)$$

Second, in writings on the Chapman-Enskog process the phrase “to the present order of approximation” or similar locutions occur frequently. In so far as any rationale for the concept of “order” there employed can be discerned, it seems to mean that quantities multiplied by powers of μ greater than n are to be neglected in the expressions for $p_i^{(n)}$ and $h_i^{(n)}$ (see §13 and §19).

The above described modifications reduce our formulæ (2.18) and (11.1) to the corresponding results obtained from the Chapman-Enskog process, for Maxwellian molecules.

MAXWELL [1879, Note added after §15] calculated certain terms in $p_i^{(3)}$ and

³ The case $n = 0$ corresponds to the operator D_0/Dt of CHAPMAN & COWLING [1952, 1, §§7.14, 15.2].

$p_{ij}^{(4)}$, but he did not publish their precise forms. CHANG & UHLENBECK [1948, 4, end of §III] assert that they also have calculated certain of these terms, but their results are published only in part [1948, 4, §IIIB]. Thus, so far as we know, our formulæ for $p_{ij}^{(3)}$ and $h_i^{(3)}$ are the first to be published. General forms for these expressions, with undetermined numerical coefficients, have been obtained with little essential reference to the kinetic theory by M. BRILLOUIN [1900]; with none by TRUESDELL [1948, 2][1951, 1].

In regard to the results of the Chapman-Enskog process, we observe that the time derivatives of p , ρ , and u_i would seem to be as readily available for experimental measurement as, say, their higher space derivatives, and there appears to be no physical ground for attempting to express one set of derivatives in terms of the others. Mathematically, the time derivatives of ρ , u_i , and p are given by the equations of mass, momentum, and energy, and it would seem sufficient to let these equations stand as exact laws of motion. In Maxwellian iteration no elimination of time derivatives is necessary, and we have introduced (15.3) and (15.8) here only so as to compare our results with those of other authors.

The exact equation (15.2) shows that (15.3) is especially inappropriate in regions such as shock layers where a moving element of the gas as a whole experiences a rapid change of entropy.⁴

The replacement of the material derivative of the velocity gradient by other expressions seems equally unnecessary and even more questionable. For example, the iterates $(u_{i,j})^{(n)}$ do not necessarily vanish when $u_{i,j} = \text{const}$. The appearance of the field of extrinsic force f_i in expressions for the pressures and the flux of energy is strange and often seems unnatural on first contact with the kinetic theory. It arises only from the unnecessary additional iterative scheme (15.8).

Finally, a glance at (15.9) shows that elimination of the simple term $(\mu E_{ij})'$ is bought at the cost of extreme complexity even for the first iterate. A more serious objection will be recalled in the next section.

16. On the invariance of the iterates. TRUESDELL [1948, 2] [1951, 1, §30] has pointed out the strange, and indeed anomalous, role of the vorticity in BURNETT's equations: At a point and time where the thermostatic state is constant and the gas as a whole is suffering no deformation, according to the formula as printed by CHAPMAN & COWLING a non-zero rigid rotation gives rise to non-zero tangential pressures. While the physical relevance of TRUESDELL's conclusion

⁴ No like objection can be made to the process in general: in a shock layer the first spatial derivatives of the velocity field are large, but there is no reason to expect the higher derivatives also to be large, except possibly in a relatively small part of the layer. The failure of Burnett's equations to describe the shock layer [1953, 1][1954, 2] may be due in part to their unnecessary neglect of entropy changes. It might be worthwhile to calculate the thickness of a shock layer from our equations (2.18) and (11.1). For the theory of the shock layer according to Burnett's equations, see [1951, 2].

has been denied (*cf.* the references cited in [1951, 1, §30]), the conclusion itself is the result of a simple substitution and cannot be controverted: The terms $u_{k,(i}u_{j),k} - \frac{1}{3}u_{k,i}u_{i,k}\delta_{ij}$ in (15.9) do not vanish in a rigid rotation, and hence in such a rotation at uniform temperature, pressure, and extrinsic force the quantity $(\mu E_{ij})^{(1)}$ is not zero even though $E_{ij} = 0$. No such consequence follows from our formula (2.18) for $P_{ij}^{(2)}$, since it contains the exact material derivative $(\mu E_{ij})^{\cdot}$ rather than the "approximation" $(\mu E_{ij})^{(1)}$. It thus appears that TRUESDELL's criticism applies properly to the CHAPMAN-ENSKOG process of solution rather than to the kinetic theory itself.

There is, however, a deeper question of invariance concerning the time rate of change of the pressure tensor. The *exact* equation (2.4) for the second moments involves the time rate $\dot{p}_{ij}$ only in the combination $\dot{p}_{ij} + p_{ij}u_{k,k} + 2p_{k(i}u_{j),k}$. It has been shown [1953, 2, §55 bis] [1955, 1, §4] [1955, 5] from general considerations in continuum mechanics that for proper dynamical invariance just such a combination must occur if the time rate $\dot{p}_{ij}$ is to appear at all in the constitutive equations of a material. The iterative equation (2.10), however, destroys this invariance because it replaces P_{ij} by two different expressions, $P_{ij}^{(n+1)}$ and $P_{ij}^{(n)}$, on the left-hand and right-hand sides. Thus for no finite $n > 1$ can $P_{ij}^{(n)}$ exhibit proper dynamical invariance. However, if the iterative scheme converges to a limit $P_{ij}^{(\infty)}$, and if $\lim \dot{P}_{ij}^{(n)} = (P_{ij}^{(\infty)})^{\cdot}$, then by (2.10) $P_{ij}^{(\infty)}$ will exhibit the required dynamical invariance.

TRUESDELL [1951, 1] has formulated within continuum mechanics the theory of the "Maxwellian fluid", in which p_{ij} and h_i depend on gradients of pressure, temperature, and velocity. The remarks concerning time derivatives in the first paragraph of this section reconcile the kinetic theory with TRUESDELL's definition of a Maxwellian fluid after the objectionable vorticity terms are removed [1951, 1, §21 and §30]. For this general fluid TRUESDELL has written out the forms of the terms multiplied by μ^2 and by μ^3 , in view of which he has criticised results of the Chapman-Enskog type for the special relations among the coefficients which they imply in the case of the terms in μ^2 . The results of the present memoir would permit comparison of the terms in μ^3 , revealing many more special relations.

However, such comparison would be premature. As was pointed out in a letter of 20 December 1951 from Professor T. G. COWLING, TRUESDELL's original treatment did not impose as many restrictions as are necessary for proper dynamical invariance. Since then the general theory of the invariance appropriate to the constitutive equations of continuum mechanics has become a field of vigorous and earnest inquiry (*cf.* [1953, 2, §55 bis]). Important contributions have been made by NOLL [1955, 1, §4], by RIVLIN & ERICKSEN [1955, 2], and by COTTER & RIVLIN [1955, 5], but further enlightenment is necessary before TRUESDELL's theory of the Maxwellian fluid can be put on a sound footing. It is certain that in the ultimate form the number of material coefficients will be intermediate between that originally proposed by TRUESDELL and that obtained from the kinetic theory by atemporal Maxwellian iteration. *Cf.* §57.

17. Relation to Grad's method of moments. The method of GRAD [1949, 1, §§4–5] begins (in effect) with the system of exact equations of transport for the moments $p_{i_1 i_2 \dots i_q}$, equivalent to the system for the spherical moments $P_{2r|s}$ used in this memoir.¹ Instead of attempting to express p_{ij} and h_i in terms of the gross condition, GRAD truncates the infinite system of moment equations at the q^{th} stage by replacing the $q + 1^{\text{st}}$ moments by more or less arbitrary expressions in terms of moments of lower order. Thus results a determinate system of partial differential equations for the first q moments. Subject to suitable initial and boundary conditions, a solution of these equations is then to be found. A part of this solution will consist of expressions for p_{ij} and h_i as functions of place and time, appropriate to the particular problem solved.

In principle, GRAD's method is attractive in that it employs $q - 1$ (or $q - 2$) of the exact equations of moments and alters only the q^{th} (or q^{th} and $q - 1^{\text{st}}$). By taking q as large as we please, we make use of as many exact equations as desired. However, the equations themselves are so elaborate that solutions cannot be found, even for small q , and it is very difficult to extract any information whatever from GRAD's formalism.

The truncation method advocated by GRAD, as might be expected, is to suppose the $q + 1^{\text{st}}$ moments related to the $q - 1^{\text{st}}$ moments (and possibly also the q^{th} moments related among themselves) by a relation which is exactly valid in the case of a Maxwellian distribution. There are, of course, many such relations. One of these is (2.13), but this is not the one selected by GRAD, who prefers the relation which results by annulling the coefficients of q^{th} order in the expansion of the distribution function in a series of Hermite polynomials. When $q = 4$, this relation is²

$$p_{ijkl} = \frac{3k\theta}{m} (2p_{(ij}\delta_{kl)} - p\delta_{(ij}\delta_{kl)}). \quad (17.1)$$

The exact equations (8.17), (15.6), (2.8), and (2.4), supplemented by the result of substituting (17.1) into (2.5), are the equations of GRAD's "twenty moment approximation". The additional assumption

$$P_{ijk} = 0 \quad (17.2)$$

reduces this set to the equations of GRAD's "thirteen moment approximation".

If we truncate the equations of moments according to some rule such as GRAD's,

¹ That GRAD uses Hermite coefficients rather than moments or spherical moments is inessential in the present connection.

² Following is a table for interconversion of some of GRAD's notations with ours:

GRAD's paper

P_{ij} , p_{ij} , p
 S_{ijk} , s_{ijk} , S_i
 Q_{ijkl} , q_{ij} , q

The present paper

p_{ij} , P_{ij} , p
 p_{ijk} , P_{ijk} , $P_{2|i} = 2h_i$
 p_{ijkl} , $P_{2|ij}$, $\frac{1}{3}P_{4i}$

we may subsequently apply a scheme analogous to Maxwellian iteration. For the first few iterates, the results are much the same as for the exact equations, but for the higher iterates they are not. This is to be expected, for in GRAD's method no information whatever may be extracted from the discarded equations of higher moments, while in Maxwellian iteration each stage makes some use, even if limited, of one more set of equations of transfer than was used at the preceding stage. Such equations as are used in GRAD's method, on the other hand, are used in a far more complete way, for they are treated as partial differential equations to be solved analytically. It is certainly not possible to decide *a priori* whether either the one method or the other is justified, let alone to compare their worth.

We conclude this section by applying an analogue of Maxwellian iteration to the twenty moment and thirteen moment approximations, so as to give a specimen of the results. The iterative scheme for the twenty moment equations retains (2.10) unchanged but replaces (2.11) by

$$\begin{aligned}
 -\frac{p}{2\mu} (3p_{ijk}^{(n+1)} - 2h_{(i}^{(n+1)} \delta_{jk)}) &\equiv \dot{p}_{ijk}^{(n)} + p_{ijk}^{(n)} E \\
 &- \frac{3}{\rho} p_{,i} P_{jk}^{(n)} - \frac{3p}{\rho} \delta_{(ij} P_{k)l,l}^{(n)} - \frac{3}{\rho} P_{(ij}^{(n)} P_{k)l,l}^{(n)} \\
 &+ 3p_{l(i}^{(n)} u_{k),l} + \frac{4}{3}\alpha[(\theta P_{(ij),k}^{(n)} + \delta_{(ij}(\theta P_{k)l,l}^{(n)} + p\theta_{,i} \delta_{jk})].
 \end{aligned} \tag{17.3}$$

The results (2.14)–(2.16) and (2.18) for the iterates $P_{ij}^{(1)}$, $h_i^{(1)}$, $p_{ijk}^{(1)}$, and $P_{ij}^{(2)}$ hold unchanged; the difference between the methods begins to appear at $h_i^{(2)}$, for which we get now

$$\begin{aligned}
 h_i^{(2)} &= -\alpha\mu\theta_{,i} + \frac{3\alpha\mu}{2p} (\mu\theta_{,i})_{\cdot} + \frac{\alpha\mu^2}{2p} (5\theta_{,i}E + 3\theta_{,i}u_{i,i} + 8\theta_{,i}E_{ii}) \\
 &+ \frac{3\mu}{\rho} (\mu E_{ii})_{,i} - \frac{3\mu^2}{\rho p} E_{ii}p_{,i} + \frac{6\mu^2}{\rho p} E_{ii}(\mu E_{ik})_{,k}
 \end{aligned} \tag{17.4}$$

in place of (11.1). The difference between (17.4) and (11.1) quite apparently arises from shortening (10.12) and (10.13) to

$$P_{4i}^{(1)} = \frac{15p^2}{\rho}, \tag{17.5}$$

$$P_{2ii}^{(1)} = \frac{7p}{\rho} P_{ii}^{(1)} = -\frac{14\mu p}{\rho} E_{ii}.$$

For the thirteen moment equations, the iterative scheme replacing (2.10) and (2.11) is

$$\begin{aligned}
-\frac{p}{\mu} P_{ij}^{(n+1)} &\equiv 2pE_{ij} + \dot{P}_{ij}^{(n)} + P_{ij}^{(n)}E \\
&\quad - \frac{2}{3}P_{kl}^{(n)}E_{kl}\delta_{ij} + 2P_{k(i}^{(n)}u_{j),k} + \frac{4}{5}(h_{(i,i)} - \frac{1}{3}h_{k,k}\delta_{ii}),
\end{aligned}
\tag{17.6}$$

$$\begin{aligned}
-\frac{p}{\mu} h_i^{(n+1)} &\equiv \alpha p\theta_{,i} + \frac{3}{2}\dot{h}_i^{(n)} + \frac{5}{2}h_i^{(n)}E + \frac{7}{5}\alpha(\theta P_{ij}^{(n)})_{,i} + \frac{6}{5}h_i^{(n)}E_{ii} \\
&\quad + \frac{2}{3}h_i^{(n)}u_{i,i} - \frac{3}{2\rho}p_{,i}P_{ii}^{(n)} - \frac{15}{4}\frac{p}{\rho}P_{ij,i}^{(n)} - \frac{3}{2\rho}P_{ij}^{(n)}P_{ik,k}^{(n)}.
\end{aligned}
\tag{17.7}$$

The results agree as far as $P_{ij}^{(2)}$ and $h_i^{(2)}$ with those obtained from the iterative scheme (2.10) and (17.3) for the twenty moment equations. This is so because (17.5), (17.6), and (10.11) are compatible with (17.1) and (17.2).

18. A scheme using the maximum available information. In Maxwellian iteration we write down and employ the iterative equations in a definite sequence, but the first time any one equation is used we put zeroth iterates on the right-hand side; the next time, first iterates, *etc.* If *all* the equations were used at once, we should then be taking all moments in zeroth iterate as those appropriate to a Maxwellian distribution and hence in effect would be taking the Maxwellian distribution as zeroth approximation to the distribution function F , as has often been done by others working in the kinetic theory. However, our assumptions are in fact weaker,¹ since to calculate $P_{ij}^{(n)}$ for any finite n the zeroth iterates to only a finite number of P_{2r+1} are used, and thus no unique form for the zeroth iterate to F is implied except in the limit as $n \rightarrow \infty$. We now pursue this idea by setting up still weaker assumptions, in two stages.

We consider first a variant in which each successive iterate is used as soon as it is calculated. A formal prescription for this scheme could be written out but would be so complicated as to obscure the in fact very simple algorithm. We shall rest content with giving the beginning steps, whence the idea is readily grasped.

¹ Here we remark a curious quality of differential iteration in the equations of transfer: *Each step introduces a new assumption (viz., initial iterates for moments of order one greater).* For example, our derivation of the Newton-Navier-Stokes formula in §2 employs zeroth iterates for the second and third moments only, making *no assumption whatever* regarding the moments of higher order. To derive the second iterate (2.18), we have added the assumption (2.12)₈ regarding the fourth moments. A different choice of zeroth iterates for the fourth moments would yield a different formula for $P_{ij}^{(2)}$ but leave the Newton-Navier-Stokes formula unaltered. Thus the iterative possibilities ramify: *Corresponding to any given $P_{ij}^{(n)}$, infinitely many different $P_{ij}^{(n+1)}$ are possible.* This suggests another possible explanation for the failure of Burnett's equations (*cf.* footnote 4 of §15): The implied restriction of the fourth moments to get this second approximation may bring us further from the appropriate solution than is the first approximation without this restriction.

The equations of transfer $m\mathcal{C}(Y_{2r|s}) = \mathcal{L}(Y_{2r|s})$ are to be ordered as before, and for our first consideration we retain the initiation agreement (2.12). There is no change in calculating $P_{ij}^{(1)}$, for which we get (2.14). Calculation of $h_i^{(1)}$, as before, is based on the result of equating (8.5) to (6.5):

$$-\frac{4}{3}\frac{p}{\mu}h_i = 2\dot{h}_i + \frac{1}{3}\theta h_i E + 2P_{ijk}E_{jk} + 2h_{ij}u_{i,j} \quad (18.1)$$

$$+ \frac{8}{5}h_{ij}E_{ij} - \frac{5p}{\rho}p_{ii,j} - \frac{2P_{ij}}{\rho}p_{jk,k} + P_{2|ij,j} + \frac{1}{3}P_{4|,i}.$$

In Maxwellian iteration, to calculate $h_i^{(1)}$ we replace *all* quantities on the right by their zeroth iterates. Instead, we now replace them by *the highest iterates we already have*: P_{ijk} and h_i and $P_{2|ij}$ by zero, as before, and $P_{4|}$ by $5!!p(p/\rho)^2$, as before, but P_{ij} by $-2\mu E_{ij}$. This gives

$$h_i^{(1)} = -\alpha\mu\theta_{,i} - \frac{15}{2}\frac{\mu}{\rho}(\mu E_{ii})_{,i} - \frac{3\mu^2}{p\rho}E_{ii}p_{,i} + \frac{6\mu^2}{p\rho}E_{ij}(\mu E_{jk})_{,k} \quad (18.2)$$

in place of the Newton-Fourier law (2.15). Comparison with (11.1) shows that the present scheme obtains at the first stage some, but not all, of the terms obtained at the second stage in Maxwellian iteration.

At the next step the gain in terms is far greater, for instead of (2.18) we get

$$\frac{p}{\mu}P_{ii}^{(2)} = -2pE_{ii} + (2\mu E_{ii})' + 2\mu E_{ij}E + 4\mu E_{k(i}u_{j),k} \quad (18.3)$$

$$- \frac{4}{3}\mu E_{kl}E_{kl}\delta_{ij} - P_{ijk,k}^{(1)} + \frac{4}{15}h_{k,k}^{(1)}\delta_{ij} - \frac{4}{5}h_{(i,j)}^{(1)},$$

where $h_i^{(1)}$ is given by (18.2) and where

$$\frac{p}{\mu}P_{ijk}^{(1)} = -\frac{8}{5}h_{(i}^{(1)}E_{jk)} + \frac{1}{2}\frac{8}{5}h_l^{(1)}E_{l(i}\delta_{jk)} + \frac{2}{\rho}P_{(ij}^{(1)}p_{k)l,l} - \frac{4}{5\rho}P_{l(i}^{(1)}\delta_{jk)}p_{l m,m}^{(1)}. \quad (18.4)$$

For our second consideration, we extend the idea just used so as to apply to the initiation as well. After we get $P_{ii}^{(1)} = -2\mu E_{ii}$ as usual, when we come to need the zeroth iterate for the fourth moments, we discard (2.12)₅ and instead, taking (2.13)₃ as our guide, begin with

$$p_{ijkl}^{(0)} = \frac{3k\theta}{m}p_{(ij}^{(1)}\delta_{kl)}, \quad (18.5)$$

$$= \frac{4}{5}\alpha\theta[p\delta_{(ij}\delta_{kl)} - 2\mu E_{(ij}E_{kl)}].$$

This amounts to replacing $P_{2|ii}^{(0)} = 0$ by

$$P_{2|ii}^{(0)} = \frac{7p}{2\rho}P_{ij}^{(1)} = -\frac{7\mu p}{\rho}E_{ij} \quad (18.6)$$

and adds a term $(21\mu/(4\rho\theta)) (\mu\theta E_{ii})_{,i}$ to the right-hand member of (18.2). In this way, we get at the first stage part of one more term which Maxwellian iteration yields in its entirety at the second stage.

19. Iteration by powers. We write the equation of transfer in the form

$$m\mathfrak{C}(Q) = \lambda\mathfrak{L}(Q), \quad (19.1)$$

where λ is a parameter afterwards to be set equal to one. Taking $Q = Y_{2r|s}$, we suppose formally

$$P_{2r|s} = \sum_{n=0}^{\infty} \lambda^n P_{2r|s}^{[n]}, \quad (19.2)$$

substitute in (19.1), equate coefficients of like powers of λ , and afterwards set $\lambda = 1$.

From (19.1) and (19.2) we see that the zeroth iterates $P_{2r|s}^{[0]}$ must be the solutions $P_{2r|s}$ of

$$\mathfrak{C}(Y_{2r|s}) = 0. \quad (19.3)$$

Since this equation is satisfied for all Y 's if and only if the distribution is Maxwellian (*e.g.* [1950, 1, §27]), the $P_{2r|s}^{[0]}$ must¹ be appropriate to a Maxwellian distribution and hence $P_{2r|s}^{[0]} = P_{2r|s}^{(0)}$, where $P_{2r|s}^{(0)}$ is given by (4.11).

To calculate the $P_{2r|s}^{[1]}$, we substitute appropriate Q into (19.1), use the explicit expressions of §6 and §8 for $\mathfrak{L}(Q)$ and $m\mathfrak{C}(Q)$, use (19.2) for the $P_{2r|s}$ which appear, and equate coefficients of the first power of λ . From (6.4)–(6.9), (9.7), (9.8), and (8.4)–(8.11) we thus derive

$$-n\mathfrak{B}_2 P_{ii}^{[1]} = 2pE_{ii}, \quad (19.4)$$

$$-n\mathfrak{B}_2 h_i^{[1]} = \alpha p\theta_{,i}, \quad (19.5)$$

$$P_{ijk}^{[1]} = 0, \quad (19.6)$$

$$-n\mathfrak{B}_2 \rho P_{4i}^{[1]} = 30p^2\dot{\beta}, \quad (19.7)$$

$$n\mathfrak{B}_2(\rho P_{2|ij}^{[1]} - pP_{ij}^{[1]}) = 12pE_{ij}, \quad (19.8)$$

$$P_{ijkl}^{[1]} = 0, \quad (19.9)$$

$$-n\mathfrak{B}_2(\rho P_{4|i}^{[1]} - 8ph_i^{[1]}) = \frac{5}{3}\alpha p^2\theta_{,i}, \quad (19.10)$$

$$(19\mathfrak{B}_2 + 2\mathfrak{B}_4)P_{2|ijk}^{[1]} + 18(\mathfrak{B}_2 - \mathfrak{B}_4)pP_{ijk}^{[1]} = 0. \quad (19.11)$$

¹ Note that the iterative schemes based on $m\mathfrak{C}^{(n+1)}(Q) = \mathfrak{L}^{(n)}(Q)$ do not force us to use any particular initiation: The initiation agreement of §2 is but one possibility. This greater freedom of these schemes is exploited in §18 and, more strikingly, in §22. The scheme of the present section in some ways recalls ENSKOG's method.

In obtaining (19.9) and (19.11), we noted first that results similar to (9.8) and (9.9) remain valid here.

The solutions of (19.4)–(19.11), expressed in our previous notations, are:

$$P_{ij}^{[1]} = -2\mu E_{ij} , \quad (19.12)$$

$$h_i^{[1]} = -\alpha\mu\theta_{,i} , \quad (19.13)$$

$$P_{ijk}^{[1]} = 0, \quad (19.14)$$

$$P_{4|}^{[1]} = -\frac{30\mu p}{\rho} \dot{\beta}, \quad (19.15)$$

$$P_{2|i}^{[1]} = -\frac{14\mu p}{\rho} E_{ii} , \quad (19.16)$$

$$P_{ijkl}^{[1]} = 0, \quad (19.17)$$

$$P_{4|i}^{[1]} = -\frac{80\alpha\mu p}{3\rho} \theta_{,i} , \quad (19.18)$$

$$P_{2|ijk}^{[1]} = 0. \quad (19.19)$$

Comparison of the values of $P_{2r|s}^{[0]} + P_{2r|s}^{[1]}$ with the $P_{2r|s}^{(1)}$ of §10 shows that the difference consists of the terms in $P_{2r|s}^{(1)}$ of degree higher than the first in μ .

The iterative scheme begun here could be modified by writing $\dot{p}$ in the form

$$\dot{p} = \sum_{n=0}^{\infty} \lambda^n \dot{p}^{[n]} \quad (19.20)$$

and by using this and (19.2) in the equation of energy (2.8) to obtain

$$\dot{p}^{[0]} = -\frac{5}{3}pE, \quad (19.21)$$

$$\dot{p}^{[n]} = -\frac{2}{3}(E_{ij}P_{ij}^{[n]} + h_{i,i}^{[n]}), \quad n > 0. \quad (19.22)$$

This would result in the replacement of $\dot{\beta}$ in (19.15) and subsequent equations by $\dot{\beta}^{[0]} = 0$. It is of note that (19.2) need not be used in the iterative scheme, except for finding $\dot{p}^{[n]}$ after finding $P_{ij}^{[n]}$ and $h_i^{[n]}$. The considerations of §15 concerning this variation in the iterative procedure apply here also. We shall continue with the procedure already started.

By using $Q = Y_{ij}, c^2 Y_i, Y_{ijk}, c^4$, and $c^2 Y_{ij}$ in (19.1) and equating coefficients of λ^2 on the two sides of the resulting equations, we obtain the following:

$$P_{ij}^{[2]} = \frac{2\mu}{p} (\mu E_{ij})^\cdot + \frac{2\mu^2}{p} EE_{ij} + \frac{4\mu^2}{p} E_{k(i} u_{j),k} \quad (19.24)$$

$$- \frac{4}{3} \frac{\mu^2}{p} E_{kl} E_{kl} \delta_{ij} + \frac{\mu}{\rho\theta} \{3(\mu\theta_{, (i})_{, j)} - (\mu\theta_{, k})_{, k} \delta_{ij}\},$$

$$h_i^{[2]} = \frac{3\alpha\mu}{2p} (\mu\theta_{, i})^\cdot + \frac{\alpha\mu^2}{2p} (5\theta_{, i} E + 3\theta_{, i} u_{i, i} + 8\theta_{, i} E_{ii}) \quad (19.25)$$

$$+ \frac{3\mu}{\rho} (\mu E_{ii})_{, i} - \frac{3\mu^2}{\rho p} E_{ii} p_{, i} + \frac{15\mu}{2\rho\theta} (\mu\theta\dot{\beta})_{, i},$$

$$P_{ijk}^{[2]} = \frac{6\mu^2}{\rho\theta} (\theta_{, (i} E_{jk)} - \frac{2}{5} \theta_{, l} E_{l(i} \delta_{jk)}) \quad (19.26)$$

$$+ 4\mu \left\{ \left(\frac{\mu}{\rho} E_{(ij)} \right)_{, k} - \frac{2}{5} \left(\frac{\mu}{\rho} \delta_{(ij} E_{k)l} \right)_{, l} \right\},$$

$$P_{41}^{[2]} = \frac{45\mu}{\rho\theta} (\mu\theta\dot{\beta})^\cdot + \frac{105\mu^2}{\rho} \dot{\beta} E + \frac{80\mu^2}{\rho} E_{ii} E_{ii} \quad (19.27)$$

$$- \frac{45\mu^2}{\rho^2\theta} \theta_{, i} p_{, i} + \frac{150\mu p}{\rho^2\theta} (\mu\theta\theta_{, i})_{, i},$$

$$P_{21ij}^{[2]} = \frac{14\mu}{\rho} (\mu E_{ij})^\cdot + \frac{32\mu^2}{\rho} \dot{\beta} E_{ij} + \frac{14\mu^2}{\rho} EE_{ij} \quad (19.28)$$

$$+ \frac{4\mu^2}{7\rho} (49E_{k(i} u_{j),k} + 20E_{ki} E_{jk} - 23E_{kl} E_{kl} \delta_{ij})$$

$$- \frac{6\mu^2}{\rho^2\theta} (3\theta_{, (i} p_{, j)} - \theta_{, k} p_{, k} \delta_{ij}) + \frac{80\mu^2 p}{7\rho^2\theta^2} (3\theta_{, i} \theta_{, j} - \theta_{, k} \theta_{, k} \delta_{ij})$$

$$+ \frac{87\mu p}{7\rho^2\theta} \{3(\mu\theta_{, (i})_{, j)} - (\mu\theta_{, k})_{, k} \delta_{ij}\}.$$

To obtain $P_{ij}^{[3]}$ and $h_i^{[3]}$, we equate coefficients of λ^3 in the equations $m\mathcal{C}(Y_{ij}) = \lambda\mathcal{L}(Y_{ij})$ and $m\mathcal{C}(c^2 Y_i) = \lambda\mathcal{L}(c^2 Y_i)$, using (19.2) for the P_{2r1s} . The expression for $P_{ij}^{[3]}$ is

$$P_{ij}^{[3]} = -\frac{\mu}{p} \{\dot{P}_{ij}^{[2]} + P_{ij}^{[2]} E + 2P_{k(i}^{[2]} u_{j),k} - \frac{2}{3} P_{kl}^{[2]} E_{kl} \delta_{ij} \quad (19.29)$$

$$+ P_{ijk,k}^{[2]} - \frac{4}{15} h_{k,k}^{[2]} \delta_{ij} + \frac{4}{5} h_{(i,i)}^{[2]}\},$$

into which the right-hand members of (19.24)–(19.26) are to be substituted for $P_{ij}^{[2]}$, $h_i^{[2]}$ and $P_{ijk}^{[2]}$. The expression for $h_i^{[3]}$ is

$$\begin{aligned} h_i^{[3]} = & -\frac{3\mu}{4\rho} \left\{ 2h_i^{[2]} + \frac{1}{3}h_i^{[2]}E + 2P_{ijk}^{[2]}E_{ik} \right. \\ & + 2h_i^{[2]}u_{i,i} + \frac{8}{3}h_i^{[2]}E_{ii} - \frac{5p}{\rho}P_{ii,i}^{[2]} \\ & \left. - \frac{2}{\rho}P_{ii}^{[1]}P_{ik,k}^{[1]} + P_{2|ii,i}^{[2]} - \frac{2}{\rho}P_{ii}^{[2]}p_{,i} + \frac{1}{3}P_{4|i,i}^{[2]} \right\}, \end{aligned} \quad (19.30)$$

into which the right-hand members of (19.12) and (19.24)–(19.28) are to be substituted for $P_{ii}^{[1]}$, $P_{ij}^{[2]}$, $h_i^{[2]}$, $P_{ijk}^{[2]}$, $P_{4|i}^{[2]}$, and $P_{2|ii}^{[2]}$.

It is to be noted that if the derivatives of μ are eliminated by (11.2) the ratio $P_{2r|s}^{[n]}/\mu^n$ becomes independent of μ . That this is quite generally the case appears by examination of (19.1) and (19.2): The term $-\mathfrak{C}_{2r|s}P_{2r|s}$ in $m\mathfrak{C}(Y_{2r|s})$ contains μ^{-1} as a factor, whence the iterative scheme defined by (19.1) and (19.2) arranges $P_{2r|s}$ in groups of terms according to the powers of μ .

We may now justify more fully characteristic A (§13) of the Maxwellian iteration process merely by noting that the sum of the terms in $P_{2r|s}^{(n+m)}$ of degree not exceeding n in μ is equal to the sum $\sum_{i=0}^n P_{2r|s}^{(i)}$ and hence is equal to the sum of the terms in $P_{2r|s}^{(n)}$ of degree not exceeding n .

20. A scheme which treats algebraically like terms on an equal footing. All the previous schemes distinguish in essence between the left-hand and right-hand sides of the equation $m\mathfrak{C}(Q) = \mathfrak{L}(Q)$. If we were to seek a reason for this discrimination, we should probably notice that μ occurs only in $m\mathfrak{C}(Q)$, not in $\mathfrak{L}(Q)$, and we might think of our iterative procedures as appropriate to the limit $\mu \rightarrow 0$ (*cf.* §§23, 26). If we abandon any such idea, it becomes unnatural to treat the same quantity in different ways because it happens to occur on different sides of the equation. The idea of the iterative schemes is to replace higher moments and time derivatives by approximations, and to do this no separation of like terms is necessary. For example, from (2.9) we might be led to set up

$$\begin{aligned} -\left(\frac{p}{\mu} + E\right)P_{ii}^{((n+1))} + \frac{2}{3}P_{kl}^{((n+1))}E_{kl}\delta_{ii} - 2P_{k(i}^{((n+1))}u_{j),k} \\ = 2pE_{ii} + \dot{P}_{ii}^{((n))} + (p_{ijk}^{((n))} - \frac{2}{3}h_k^{((n))}\delta_{ij})_{,k}, \end{aligned} \quad (20.1)$$

in which all occurrences of P_{ij} except that in the difficult term $\dot{P}_{ii}$ are placed on an equal footing. For $n = 0$ we get

$$-\left(\frac{p}{\mu} + E\right)P_{ii}^{((1))} + \frac{2}{3}P_{kl}^{((1))}E_{kl}\delta_{ii} - 2P_{k(i}^{((1))}u_{j),k} = 2pE_{ii}. \quad (20.2)$$

(20.1) is an inhomogeneous system of six linear equations for the six $P_{ij}^{((n+1))}$; it could be reduced to five independent equations by elimination of $P_{xx}^{((1))}$, say, by means of the equation $P_{ii}^{((n+1))} = 0$, which is easily seen to be consistent with (20.1). For purposes of symmetry in obtaining the solution by Cramer's rule, it is more convenient to retain all six equations (20.1). That the determinant of the coefficients does not vanish identically is shown by noting that p/μ appears once and once only in each row and each column of the matrix of the coefficients, so that the determinant is of the sixth degree in p/μ . The numerator in the solution for any one of the $P_{ij}^{((n+1))}$ by Cramer's rule is a polynomial of the fifth degree in p/μ . Thus, in general, from (20.1) follows a solution of the type

$$\frac{P_{ij}^{((n+1))}}{\mu} = \frac{N_{ij}^{((n+1))}}{D}, \quad (20.3)$$

where $N_{ij}^{((n+1))}$ is a polynomial of the fifth degree in μ with coefficients which are linear functions of the components $\dot{P}_{ij}^{((n))} + (p_{ijk}^{((n))} - \frac{2}{3}h_k^{((n))}\delta_{ij})_{,k}$ and polynomial functions of the gross condition, while D is a scalar polynomial of the sixth degree in μ with coefficients which are polynomial functions of the gross condition only. If time derivatives are eliminated as in §15, the result, in general, is a formula for $P_{ij}^{((n+1))}$ as a rational function of μ . Similar iterative schemes may be set up for the higher moments. For example,

$$\begin{aligned} -\left(\frac{4}{3}\frac{p}{\mu} + \frac{1}{3}E\right)h_i^{((n+1))} - 2(u_{i,j} + \frac{4}{3}E_{ij})h_j^{((n+1))} \\ = -\frac{5p}{\rho}p_{ij,i}^{((n))} - \frac{2}{\rho}P_{ij}^{((n))}p_{ik,k}^{((n))} + P_{2|ij,i}^{((n))} + \frac{1}{3}P_{4|i,i}^{((n))}, \end{aligned} \quad (20.4)$$

an inhomogeneous system of three linear equations; for $n = 0$ (20.4) reduces to

$$-\left(\frac{4}{3}\frac{p}{\mu} + \frac{1}{3}E\right)h_i^{((1))} - 2(u_{i,j} + \frac{4}{3}E_{ij})h_j^{((1))} = \frac{4}{3}\alpha\theta_{,i}. \quad (20.5)$$

To achieve a general tensorial solution of the iterative equations may be difficult, and we shall not attempt it. We pause only to remark that the present method, since its results are *rational functions rather than polynomials in μ* , offers the possibility of approximations uniformly valid for all degrees of rarefaction, out of the question with perturbation series of the type obtained in the previous sections. That this hope is not baseless will be shown in §35, where it will turn out that in application to the special case of shearing flow, the present method is the best yet obtained.

21. Iteration in the equations of moments. In §2 we used the equation of energy to convert (2.4) into the form (2.9) before setting up a corresponding

iterative scheme. The reason for the transformation was to introduce the term $2pE_{ij}$ on the right-hand side and so to obtain the Navier-Stokes formula for $P_{ij}^{(1)}$. Suppose instead we begin with the original equation (2.4) for the second moments. Taking

$$p_{ij}^{(0)} = p\delta_{ij}, \quad p_{ijk}^{(0)} = 0 \quad (21.1)$$

as appropriate to equilibrium, by setting up an iterative scheme corresponding to (2.4) we get

$$P_{ij}^{(1)} = -2\mu E_{ij} - \frac{2}{3}\mu\dot{\beta}\delta_{ij}. \quad (21.2)$$

This formula appears to contradict the relation $P_{ii} = 0$, which holds by definition. The trace of the right-hand side of the exact equation (2.4) vanishes in virtue of the equation of energy, but in the iterative process we have substituted certain zeroth iterates whose traces do not vanish. One way to look at the matter is to expect that $\lim_{n \rightarrow \infty} P_{ij}^{(n)}$ will have vanishing trace, even though $P_{ij}^{(n)}$ does not.¹ Another is to replace (21.1)₂ by a zeroth iterate so adjusted that

$$3\dot{p} + 5pE + p_{ijk,k}^{(0)} = 0, \quad (21.3)$$

or, by (2.8),

$$p_{ijk,k}^{(0)} = 2P_{ij}^{(0)}E_{ij} + 2h_{i,i}^{(0)} = 2h_{i,i}^{(0)}. \quad (21.4)$$

There are various ways of satisfying (21.4); for example, take

$$p_{ijk}^{(0)} = \frac{8}{3}h_{(i}^{(0)}\delta_{jk)} = -\frac{8}{3}\alpha\mu\theta_{,(i}\delta_{jk)}. \quad (21.5)$$

Then (21.2) is replaced by

$$P_{ij}^{(1)} = -2\mu E_{ij} - \frac{2}{3}\mu\dot{\beta}\delta_{ij} - \frac{2\alpha\mu}{5p} [2(\mu\theta_{,(i),i)} + (\mu\theta_{,k}),_k\delta_{ij}], \quad (21.6)$$

and the trace of the right-hand side is now zero if the thermostatic state is that appropriate to the flow of heat in the fluid considered at rest. Curiously, it is the last term in the present result, rather than that in the Burnett equation or (2.18), which MAXWELL calculated in his study of the stresses arising from inequality of temperature (*cf.* the remark following (2.18)).

22. Iteration from free molecular flow. The preceding six sections have considered variants in the iterative equations, but in all cases the zeroth iterates were taken as those appropriate in some sense to equilibrium. We now consider an essentially different initiation agreement.

¹ Such an idea is at least no less plausible than the argument of "order" which would have to be used if one were to try to justify the failure of invariance noted in §§15–16 for the Chapman-Enskog process.

It is natural to begin with free molecular flow, appropriate to a very rarefied gas. In such a flow subject to no extrinsic force, if $F(\mathbf{x}, \xi)$ is the distribution function when $t = 0$ then at time t we have

$$F(\mathbf{x}, \xi, t) = F(\mathbf{x} - \xi t, \xi). \quad (22.1)$$

The modification is now to replace (5.11) by taking for $P_{2r|s}^{(0)}$ the values appropriate to free molecular flow. In general, the functional equation (22.1) alone does not yield any particularly simple form for $P_{2r|s}^{(0)}$, and we shall rest content to illustrate the method by a special case.

Consider a unidirectional free expansion: For $t < 0$ the gas on the left-hand side of the plane $x = 0$ is in equilibrium, but at $t = 0$ it is suddenly allowed to expand freely into the initially vacuous region $x > 0$. J. KELLER [1948, 1, §5] has studied this case and has shown that (22.1) yields

$$\frac{\rho}{\rho_0} = E\left(\sqrt{b} \frac{x}{t}\right), \quad u \equiv u_x = -\frac{e^{-bx^2/t^2}}{2\sqrt{\pi b} E(\sqrt{b} x/t)}, \quad (22.2)$$

$$p = p_0 \frac{\rho}{\rho_0} + \frac{1}{3\rho} u\left(\frac{x}{t} - u\right),$$

where

$$E(x) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-\eta^2} d\eta, \quad b \equiv \frac{m}{2k\theta_0} = \frac{\rho_0}{2p_0}, \quad (22.3)$$

and where p_0 , ρ_0 , θ_0 is the thermostatic state when $t < 0$, $x < 0$. It is easy to show also that

$$P_{xx} = -\frac{2}{3}\rho u\left(u - \frac{x}{t}\right) = -2P_{yy} = -2P_{zz}, \quad (22.4)$$

$$P_{ij} = 0 \quad \text{if } i \neq j,$$

$$p_{xxx} = -\rho u\left(\frac{p_0}{\rho_0} - \frac{x^2}{t^2} + 3u \frac{x}{t} - 2u^2\right), \quad (22.5)$$

while all other p_{ijk} vanish.

We now take the values (22.4) and (22.5) for $P_{ij}^{(0)}$ and $p_{ijk}^{(0)}$ in (2.10). After considerable rearrangement follows¹

$$P_{xx}^{(1)} = -\frac{8}{3}\mu\left(1 - \frac{\theta_0}{\theta}\right)u_{,x}, \quad (22.6)$$

¹ KELLER forces (22.4)₁ into agreement with the Navier-Stokes formula by asserting that it implies a viscosity equal to $k\theta_0\rho l/(3m)$. (In KELLER's paper the numerical factor in equation (18) should be $\frac{1}{3}$ rather than $\frac{1}{2}$). Such an interpretation, according to which the viscosity depends on time as well as the thermostatic state, is not consistent with the usual notion of viscosity. While he discusses a method of integration based on free molecular flow as first approximation, KELLER does not effectuate the method in general or in any special case.

where μ is given by (2.3). According to this result the Navier-Stokes formula again emerges, but the viscosity instead of being μ is now $2\mu(1 - \theta_0/\theta)$. It must be recalled, however, that u and $\theta \equiv mp/(k\rho)$ are definite functions of x and t , given by (22.2). No universal relation between P_{ij} and E_{ij} can emerge by the method given here. The result (22.6) cannot be expected to be appropriate to other flows.

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