Addition Theorems for General Legendre and Gegenbauer Functions

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Introduction

Outline of the paper. In this paper we continue some earlier investigations [15, 16, 17, 18, 19] of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (2\mu + 1)x^{-1}\frac{\partial u}{\partial x} + (2\nu + 1)y^{-1}\frac{\partial u}{\partial y} + [\lambda - k^2(x^2 + y^2)]u = 0 \quad (1)$$

or of one of its special cases, seeking expansions of the general solution as a series of particular solutions constituting a complete set. The method of our previous papers consisted in setting up a continuous and linear one-to-one correspondence

$$f(z) = u\left(\frac{z}{2}, -\frac{iz}{2}\right) \quad (2)$$

between a class of analytic functions of one variable, regular at the origin, and a class C of regular solutions of (1). By Taylor's theorem every solution in C can then be expanded as a series in the solutions $u_n(x, y)$ corresponding to the functions $f(z) = z^n$. These "normal solutions" were identified in the general case of equation (1) as products of Whittaker functions and Jacobi polynomials; in special instances they reduce to products of Bessel functions and Gegenbauer polynomials ($2\mu + 1 = 0, k = 0$ [15, 16]), Jacobi polynomials ($\lambda = 0, k = 0$ [19]) or Gegenbauer polynomials ($2\mu + 1 = 0, \lambda = 0, k = 0$ [17]). The application of our expansion theorems to special solutions of (1) yielded in each case a number of functional relations (mostly addition theorems and generating functions) involving various types of hypergeometric functions.

In the present paper we consider a special case of (1) which was already discussed in [19], namely

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (2\mu + 1)x^{-1}\frac{\partial u}{\partial x} + (2\nu + 1)y^{-1}\frac{\partial u}{\partial y} = 0. \quad (3)$$

While in [19] and in the other papers mentioned above the class C consisted in each case of functions regular at $x = y = 0$, we now seek expansions of solutions of (3) regular at $x = c, y = 0$, where c is a complex constant different from zero. It will be shown that the class C of these solutions (with some trivial exceptions) again admits a one-to-one mapping onto the class of analytic functions $f(w)$ regular at $w = 0$. If one chooses

$$f(w) = (1 - w)^{-\mu-\nu-1} u\left(\frac{c}{1 - w}, -\frac{icw}{1 - w}\right) \quad (4)$$

as the connecting relation, the normal solutions corresponding to $f(w) = w^n$ can be represented by certain combinations of Legendre functions and Gegenbauer

polynomials called Legendre-Gegenbauer (L.G.) functions. In view of (4) the problem of finding the expansion of an arbitrary solution in C in terms of these L.G. functions is again reduced to the problem of finding the Taylor expansion of a given function of one variable.

After establishing some preparatory lemmas, we formulate in the first part of the paper the expansion theorem which forms the basis for all later developments. The second part is devoted to applications. Here the most interesting results are obtained by applying the expansion theorem to L.G. functions defined with respect to a different point of reference c . This yields three very general and yet relatively simple expansions of an arbitrary Legendre or Gegenbauer function in terms of products of Legendre functions and Gegenbauer polynomials, expansions which contain as special and limiting cases not only the classical addition theorems of the Legendre functions of the first and second kinds with upper index zero (proved here under less restrictive conditions than by HOBSON [3, 20]), but also the addition theorem of the Gegenbauer polynomials, GEGENBAUER'S and GRAF'S addition theorems of the Bessel functions and some expansions which can be considered as generalizations of NEUMANN'S expansion for $(\xi - \zeta)^{-1}$ in terms of Legendre functions. Another application of the expansion theorem gives rise to a new series for the product of two Bessel functions, while a third application (to the normal solutions of [19]) yields another generalization of NEUMANN'S formula. In an appendix we list some integral relations which can be inferred from our expansions.

Notations. In the notation of special functions (which here are mostly Legendre functions) we follow ERDÉLYI [1] (in particular, chapter III), and thus indirectly HOBSON. We also retain the factor $e^{-i\mu\pi}$ in Hobson's definition of the Legendre function of the second kind, although most of the formulas involving this function would simplify if that factor were omitted.

For the expression $(\vartheta^2 - 1)^{\frac{1}{2}}$, where ϑ stands for any variable denoted by a small Greek letter or by an aggregate of such letters, we have found it convenient to introduce a special symbol by putting

$$\vartheta' = (\vartheta^2 - 1)^{\frac{1}{2}}. \quad (5)$$

If ϑ is restricted to the complex plane cut between the points -1 and $+1$ we mean by ϑ' the branch of (5) for which $\vartheta \sim \vartheta'$ as $\vartheta \rightarrow \infty$. If ϑ is unrestricted we mean by the symbol ϑ' the full algebraic function (5) which is unambiguous on an appropriate two-sheeted Riemann surface. We shall call the sheet where $\vartheta' \sim \vartheta$ for large ϑ the upper sheet and the sheet where $\vartheta' \sim -\vartheta$ the lower sheet of the Riemann surface.

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Part I. THE EXPANSION THEOREM

1. Definitions

Characteristic coordinates. Let c be a complex constant, $c \neq 0$. We define the coordinates (w, w^*) in the complex (x, y) -space by

$$w = \frac{z - c}{z + c}, \quad w^* = \frac{z^* - c}{z^* + c}, \quad (6)$$

where

$$z = x + iy, \quad z^* = x - iy. \quad (7)$$

Since x and y are envisaged as independent complex variables, the variables w and w^* are also independent in general. They are conjugate complex if both x and y are real. The inverse transformation to (6) is

$$x = c \frac{1 - ww^*}{(1 - w)(1 - w^*)}, \quad y = -ic \frac{w - w^*}{(1 - w)(1 - w^*)}. \quad (8)$$

Special domains. For the sake of brevity we introduce special symbols for certain domains which will occur frequently.

In the w -plane or in the w^* -plane we denote by $\mathbb{R}_r$ the interior of the circle with radius r and center at the origin.

In the (w, w^*) -space we denote by $\mathfrak{B}_r$ the bicylinder $\mathbb{R}_r \times \mathbb{R}_r$, *i. e.* the set of all points (w, w^*) for which $|w| < r$, $|w^*| < r$. Two subdomains of $\mathfrak{B}_r$ are also of importance:

(i) the domain $\mathfrak{S}_r$, which is obtained from $\mathfrak{B}_r$ by removing all points for which ww^* is real and ≥ 1 ;

(ii) the domain $\mathfrak{G}_r$, which is obtained from $\mathfrak{S}_r$ by removing also all points for which either w or w^* is real and ≥ 1 .

Clearly,

$$\mathfrak{G}_r \subset \mathfrak{S}_r \subset \mathfrak{B}_r,$$

and if $r \leq 1$,

$$\mathfrak{G}_r = \mathfrak{S}_r = \mathfrak{B}_r.$$

In the general case it is easy to see that both $\mathfrak{G}_r$ and $\mathfrak{S}_r$ are star-shaped with respect to $(0, 0)$ and therefore simply connected. If $r \geq 1$,

$$\mathfrak{B}_1 \subset \mathfrak{G}_r, \quad (9)$$

and for every s such that $r^{-1} \leq s \leq r$,

$$\mathfrak{R}_s \times \mathfrak{R}_{1/s} \subset \mathfrak{S}_r. \quad (10)$$

By (6) there corresponds to every domain in the (w, w^*) -space a certain domain in the (x, y) -space. The symbols $\mathfrak{B}_r$, $\mathfrak{G}_r$ and $\mathfrak{S}_r$ will also be used to denote these corresponding domains.

Bipolar coordinates. We assume

$$|\arg w| < \pi, \quad |\arg w^*| < \pi \quad (11)$$

and define a further set of variables by

$$\xi = \frac{1 + ww^*}{2\sqrt{ww^*}}, \quad \eta = \frac{w + w^*}{2\sqrt{ww^*}}, \quad (12)$$

where $\sqrt{ww^*} = \sqrt{w} \sqrt{w^*}$, the latter square roots having their principal values. In view of (11) also $|\arg \sqrt{ww^*}| < \pi$, $|\arg \sqrt{w/w^*}| < \pi$, so that ξ and η do not assume real values between $-\infty$ and -1 . On the Riemann surfaces of the functions $(\xi^2 - 1)^{\frac{1}{2}}$ and $(\eta^2 - 1)^{\frac{1}{2}}$ we have, respectively,

$$\xi' = \frac{1 - ww^*}{2\sqrt{ww^*}}, \quad \eta' = \frac{w - w^*}{2\sqrt{ww^*}}. \quad (13)$$

It is easy to verify that for ξ' the $\frac{\text{upper}}{\text{lower}}$ sheet of the Riemann surface is represented by $|ww^*| \leq 1$ and for η' by $|w/w^*| \leq 1$. We also record the relation

$$\xi - \eta = \frac{(1 - w)(1 - w^*)}{2\sqrt{ww^*}}. \quad (14)$$

It is well known that for $w^* = \bar{w}$ the curves $\arg w = \text{const}$ in the real (x, y) -plane are the circles passing through $(\pm c, 0)$ and the curves $|w| = \text{const}$ are the circles orthogonal to these. Since ξ depends on $\sqrt{ww^*}$ only and η on $\sqrt{w/w^*}$ only, the circles through $(\pm c, 0)$ are also characterized by $\eta = \text{const}$ and the orthogonal circles by $\xi = \text{const}$. The coordinates (ξ, η) must thus be related to the bipolar coordinates associated with the poles $(\pm c, 0)$. We do not use the familiar definitions of these coordinates by circular and hyperbolic functions, because our present definitions are easier to work with for complex values of x and y ¹. From (12) and (13) we find

$$\begin{aligned} ww^* &= (\xi - \xi')^2 = (\xi + \xi')^{-2}, \\ w/w^* &= (\eta - \eta')^2 = (\eta + \eta')^{-2}. \end{aligned} \quad (15)$$

¹ Although they will not be used in the following, we might mention the direct relations between (ξ, η) and (x, y) . Writing $r^2 = x^2 + y^2$, we have

$$\xi = \frac{r^2 + c^2}{\sqrt{(r^2 + c^2)^2 - 4c^2 x^2}}, \quad \eta = \frac{r^2 - c^2}{\sqrt{(r^2 - c^2)^2 + 4c^2 x^2}}.$$

The class $C(\mathfrak{D})$. Values of (μ, ν) satisfying the condition

$$\nu \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots \quad (16)$$

will be called *admissible values* of the parameters.

If $\mathfrak{D}$ is a domain in the (w, w^*) -space containing the point $(0, 0)$ and if (μ, ν) are admissible values of the parameters, we denote by $C(\mathfrak{D}) = C_{\mu, \nu}(\mathfrak{D})$ the class of all solutions $u(x, y) = U(w, w^*)$ of (3) which are analytic in $\mathfrak{D}$ and even functions of y (or symmetric in w and w^*).

2. Legendre-Gegenbauer functions

Definition. Let $L_\nu^\mu(\xi)$ represent any linear combination of $P_\nu^\mu(\xi)$ and $Q_\nu^\mu(\xi)$, and let $C_\kappa^\lambda(\eta)$ denote the Gegenbauer function. The following lemma is easily proved by formal calculation (see also [19]):

Lemma 1. For arbitrary κ the function

$$g(\xi, \eta) = (\xi - \eta)^{\mu+\nu+1} \xi^{\nu-\mu} L_{\nu+\kappa}^\mu(\xi) C_\kappa^{\nu+\frac{1}{2}}(\eta), \quad (17)$$

considered as a function of x and y , is a solution of (3) in every domain of regularity.

The case of arbitrary κ and $L_{\nu+\kappa}^\mu$ in (17) will be taken up in part II, §5. In part I we need only the case where $L_{\nu+\kappa}^\mu$ is the Legendre function of the second kind and κ is a non-negative integer. We define the Legendre-Gegenbauer function of order n for $n = 0, 1, 2, \dots$ and for admissible values of the parameters by

$$g_n(\xi, \eta) = \alpha_n (\xi - \eta)^{\mu+\nu+1} \xi^{\nu-\mu} Q_{\nu+n}^\mu(\xi) C_n^{\nu+\frac{1}{2}}(\eta); \quad (18)$$

the constant

$$\alpha_n = \frac{2^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\Gamma(\mu + \nu + 1) \sqrt{\pi}} \frac{(\nu + n + \frac{1}{2})n!}{(\mu + \nu + 1)_n} \quad (19)$$

serves as a normalizer.

Fundamental properties. We now express $g_n(\xi, \eta)$ in terms of w and w^* . Expressing the Q -function by a hypergeometric series² of argument $(\xi - \xi')^2$ and the Gegenbauer polynomial by a terminating hypergeometric series³ of argument η^{-2} , we find that all numerical factors cancel and that the function

$$G_n(w, w^*) = g_n(\xi, \eta)$$

² [1], eq. 3.2(45).

³ [1], eq. 3.15(18).

has the representation

$$G_n(w, w^*) = [(1-w)(1-w^*)]^{\mu+\nu+1} {}_2F_1 \left[\begin{matrix} \mu + \frac{1}{2}, \mu + \nu + 1 + n; \\ \nu + n + \frac{3}{2} \end{matrix}; ww^* \right] \cdot (w + w^*)^n {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; \\ \frac{1}{2} - \nu - n \end{matrix}; \frac{4ww^*}{(w + w^*)^2} \right]. \quad (20)$$

Here the product of the last two factors is a polynomial in w and w^* . The first hypergeometric function is regular as long as ww^* lies in the plane cut between $+1$ and $+\infty$, i. e. for $(w, w^*) \in \mathfrak{S}_\infty$. The factor $[(1-w)(1-w^*)]^{\mu+\nu+1}$ is regular for $(w, w^*) \in \mathfrak{G}_\infty$. Since $\mathfrak{G}_\infty \subset \mathfrak{S}_\infty$, we have thus proved

Lemma 2. *For all admissible values of the parameters,*

$$G_n(w, w^*) \in C(\mathfrak{G}_\infty), \quad n = 0, 1, 2, \dots$$

If in place of $G_n(w, w^*)$ we consider the function

$$H_n(w, w^*) = [(1-w)(1-w^*)]^{-\mu-\nu-1} G_n(w, w^*), \quad (21)$$

the restriction arising from the first factor in (20) can be dropped. Noting that for $w^* = 0$ the two hypergeometric functions in (20) reduce to 1, we arrive at

Lemma 3. *The function $H_n(w, w^*)$ defined by (21) is regular in $\mathfrak{S}_\infty$, and*

$$H_n(w, 0) = w^n, \quad n = 0, 1, 2, \dots \quad (22)$$

Equiconvergence property. We next turn to a simple equiconvergence property of the L.G. functions, which links the convergence of a series of such functions to that of an associated power series.

Lemma 4. *Let $r > 0$ and let*

$$f(w) = \sum_{n=0}^{\infty} a_n w^n$$

be regular in $\mathfrak{R}_r$. Then for fixed admissible values of the parameters the series

$$H(w, w^*) = \sum_{n=0}^{\infty} a_n H_n(w, w^*) \quad (23)$$

converges uniformly in every closed subregion $\mathfrak{D}$ of $\mathfrak{S}_r$.

Proof. Let $r' < r$ be such that $\mathfrak{D} \subset \mathfrak{B}_{r'}$ and let $r' < r'' < r$. We have proved elsewhere⁴ that if $(w, w^*) \in \mathfrak{B}_{r'}$,

$$\left| (w + w^*)^n {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; \\ \frac{1}{2} - \nu - n \end{matrix}; \frac{4ww^*}{(w + w^*)^2} \right] \right| < K \left| \frac{n!}{(\nu + \frac{1}{2})_n} \right| r''^n,$$

⁴ [16], p. 286.

where K depends on r' , r'' and ν , but not on w , w^* and n . Furthermore, if c is not zero or a negative integer,

$$\lim_{n \rightarrow \infty} {}_2F_1 \left[\begin{matrix} a, b; \\ c + n \end{matrix} \middle| z \right] = 1$$

uniformly for all points z of a closed region not containing any points on the real axis between 1 and $+\infty$. Under the transformation

$$z_1 = \frac{z}{z - 1}$$

any such region transforms into one with the same property. Hence, as $n \rightarrow \infty$ and if $(w, w^*) \in \mathfrak{D}$,

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} \mu + \frac{1}{2}, \mu + \nu + 1 + n; \\ \nu + n + \frac{3}{2} \end{matrix} \middle| ww^* \right] \\ &= (1 - ww^*)^{-\mu - \frac{1}{2}} {}_2F_1 \left[\begin{matrix} \frac{1}{2} + \mu, \frac{1}{2} - \mu; \\ \nu + n + \frac{3}{2} \end{matrix} \middle| \frac{ww^*}{ww^* - 1} \right] \rightarrow (1 - ww^*)^{-\mu - \frac{1}{2}}. \end{aligned}$$

Thus the terms of (23) are dominated in $\mathfrak{D}$ for sufficiently large n by the terms of the convergent series

$$K \left| (1 - ww^*)^{-\mu - \frac{1}{2}} \right| \sum_{n=0}^{\infty} \left| a_n \frac{n!}{(\nu + \frac{1}{2})_n} \right| r''^n,$$

and Lemma 4 follows.

As a corollary we obtain

Lemma 5. *The function*

$$G(w, w^*) = [(1 - w)(1 - w^*)]^{\mu + \nu + 1} H(w, w^*),$$

where H is given by (23), is in $C(\mathfrak{G}_r)$, and

$$(1 - w)^{-\mu - \nu - 1} G(w, 0) = f(w).$$

Proof. The first statement follows from the uniform convergence proved in Lemma 4 and from Lemma 2. The second statement is an immediate consequence of Lemma 3.

By Lemma 5 the class of analytic functions regular in a circle $\mathfrak{R}_r$ is mapped on a set of functions contained in $C(\mathfrak{G}_r)$. We will show next that this set is in fact the class $C(\mathfrak{G}_r)$ itself⁵.

⁵ Similar mappings for general classes of elliptic equations with regular coefficients and also for certain singular equations have been considered by BERGMAN (see e.g. [12]) and VEKUA [10].

3. Uniqueness of the solutions of (3).

Lemma 6. *Let $G(w, w^*)$ be in $C(\mathfrak{D})$ for some admissible values of the parameters and for some neighborhood $\mathfrak{D}$ of $(0, 0)$. Let*

$$G(w, 0) = 0 \quad (24)$$

in $\mathfrak{D}$. Then

$$G(w, w^*) = 0$$

identically in $\mathfrak{D}$.

Proof. Upon substitution of the variables (w, w^*) in (3) we find for the function

$$H(w, w^*) = [(1 - w)(1 - w^*)]^{-\mu-\nu-1} G(w, w^*)$$

the differential equation

$$\begin{aligned} \frac{\partial^2 H}{\partial w \partial w^*} - \frac{\nu + \frac{1}{2}}{w - w^*} \left(\frac{\partial H}{\partial w} - \frac{\partial H}{\partial w^*} \right) \\ - \frac{\nu + \frac{1}{2}}{1 - ww^*} \left[w \frac{\partial H}{\partial w} + w^* \frac{\partial H}{\partial w^*} + (\mu + \nu + 1)H \right] = 0. \end{aligned} \quad (25)$$

Evidently it suffices to show that $H(w, 0) = 0$ implies $H(w, w^*) = 0$ in some neighborhood of $w = w^* = 0$. Since $G(w, w^*)$ is regular in $\mathfrak{D}$ and symmetric in w and w^* , we can write

$$H(w, w^*) = \sum_{m=0}^{\infty} \sum_{n=0}^m c_{m,n} w^{m-n} w^{*n}, \quad (26)$$

where

$$c_{m,m-n} = c_{m,n}. \quad (27)$$

We call $s + t$ the weight of the monomial $w^s w^{*t}$. Substituting (26) in (25) and multiplying by $(1 - ww^*)(w - w^*)$, we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^m c_{m,n} [n(m - n + \nu + \frac{1}{2}) w^{m-n} w^{*n-1} \\ - (m - n)(n + \nu + \frac{1}{2}) w^{m-n-1} w^{*n} + R_{m-1}] = 0, \end{aligned} \quad (28)$$

where R_{m-1} denotes terms of higher weight than $m - 1$. We prove that $c_{m,n} = 0$ for all values of m and n by induction with respect to the weight m . By (24), $c_{m,0} = c_{m,m} = 0$ for $m = 0, 1, 2, \dots$ and hence in particular $c_{0,0} = 0$. We assume that

$$c_{k,n} = 0 \quad \text{for} \quad n = 0, \dots, k \quad \text{and} \quad k = 0, \dots, m - 1 \quad (29)$$

and consider in (28) the terms of weight $m - 1$. Then the terms denoted by R_{m-1} will be multiplied by coefficients $c_{k,n}$ with $k < m$ which are zero by hypothesis. Since the coefficient of each monomial must vanish separately, we are left with the recurrence relation

$$c_{m,n} n(m - n + \nu + \tfrac{1}{2}) - c_{m,n-1}(m - n - 1)(n + \nu - \tfrac{1}{2}) = 0, \\ n = 1, \dots, m.$$

Since $c_{m,0} = 0$ and $m - n + \nu + \frac{1}{2} \neq 0$ for admissible values of the parameters, it follows that $c_{m,n} = 0$ for $n = 1, 2, \dots, m - 1$. Since $c_{m,m} = 0$ by (27), (29) is true for $k = m$ and hence for all values of k .

4. The expansion theorem.

Theorem. Let $r > 0$ and let $\mathfrak{D}$ be a domain in the (w, w^*) -space containing the set $\{w^* = 0, |w| < r\}$ in its interior. Let

$$G(w, w^*) \in C(\mathfrak{D})$$

for some admissible values of the parameters, and set

$$(1 - w)^{-\mu-\nu-1} G(w, 0) = \sum_{n=0}^{\infty} a_n w^n. \quad (30)$$

Then the series

$$S(w, w^*) = \sum_{n=0}^{\infty} a_n G_n(w, w^*) \quad (31)$$

converges in $\mathfrak{G}_r$ (uniformly in every closed subdomain) to $G(w, w^*)$ or, if $\mathfrak{G}_r \not\subset \mathfrak{D}$, to the analytic continuation of $G(w, w^*)$ from $\mathfrak{D}$ into $\mathfrak{G}_r$.

Remark. That in the case $\mathfrak{G}_r \not\subset \mathfrak{D}$ the analytic continuation from $\mathfrak{D}$ into $\mathfrak{G}_r$ exists, is part of the theorem. Since $\mathfrak{G}_r$ is simply connected, this continuation is unique.

Proof. Since (30) is regular in $\mathfrak{R}_r$, $S(w, w^*)$ represents a function in $C(\mathfrak{G}_r)$. Now let $\mathfrak{G}_{r'} \subset \mathfrak{D} \cap \mathfrak{G}_r (r' > 0)$. Both $G(w, w^*)$ and $S(w, w^*)$ are in $C(\mathfrak{G}_{r'})$, therefore the same is true for the function

$$D(w, w^*) = G(w, w^*) - S(w, w^*).$$

For $w^* = 0$,

$$D(w, 0) = G(w, 0) - (1 - w)^{\mu+\nu+1} \sum_{n=0}^{\infty} a_n w^n = 0,$$

thus $D(w, w^*) = 0$ in $\mathfrak{G}_{r'}$ by Lemma 6. Since $G(w, w^*)$ coincides with $S(w, w^*)$ on the open set $\mathfrak{G}_{r'}$, $S(w, w^*)$ represents the analytic continuation of $G(w, w^*)$ from $\mathfrak{G}_{r'}$ into $\mathfrak{G}_r$. This completes the proof.

If the function $G(w, w^*)$ is given in terms of the variables a and y , then the series (30) can be formed directly from the function $u(x, y) = G(w, w^*)$ by expansion. Thus,

$$(1-w)^{-\mu-\nu-1} u\left(\frac{c}{1-w}, -\frac{icw}{1-w}\right) = \sum_{n=0}^{\infty} a_n w^n.$$

The series (31) will be called the L.G. expansion of the function $G(w, w^*)$ and the power series (30) will be termed (in slight generalization of the usual meaning of this expression) the generating function of this L.G. expansion.

Part II. APPLICATIONS

5. Applications to Legendre and Gegenbauer functions.

5.1. Preliminaries. Although some of the formal manipulations of this section may appear unduly complicated, the underlying idea is simple. We shall first consider the general solution (17) of (3), where however w and w^* and consequently ξ and η are now defined with the aid of a constant $c_1 \neq c$. For this solution we shall establish the generating function of its L.G. expansion, using certain earlier results of TRUESDELL [9] and the author [16], and then the L.G. expansion itself. We then shall single out three special cases of (17), which lead to particularly simple results, and give various transformations of their L.G. expansions. The resulting relations are believed to be new. They contain as special cases a great number of special addition theorems and related expansions for Legendre and Gegenbauer functions and make it thus possible to develop this branch of the theory of Legendre functions in a systematic way⁶.

We select a complex constant c_1 such that $c_1 \neq c$, $c_1 \neq 0$ and put

$$w_1 = \frac{z - c_1}{z + c_1}, \quad w_1^* = \frac{z^* - c_1}{z^* + c_1}, \quad (32)$$

where z and z^* are again given by (7). Observing the same conventions as in §1, we set for $|\arg w_1| < \pi$, $|\arg w_1^*| < \pi$,

$$\xi_1 = \frac{1 + w_1 w_1^*}{2\sqrt{w_1 w_1^*}}, \quad \eta_1 = \frac{w_1 + w_1^*}{2\sqrt{w_1 w_1^*}}, \quad (33)$$

and have similarly to (13) and (14)

$$\xi_1' = \frac{1 - w_1 w_1^*}{2\sqrt{w_1 w_1^*}}, \quad \eta_1' = \frac{w_1 - w_1^*}{2\sqrt{w_1 w_1^*}}, \quad \xi_1 - \eta_1 = \frac{(1 - w_1)(1 - w_1^*)}{2\sqrt{w_1 w_1^*}}, \quad (34)$$

where ξ_1' and η_1' are again single-valued on their respective Riemann surfaces.

We now express ξ_1 and η_1 in terms of the variables w and w^* defined by (6) with the old c . Using the abbreviations

$$\zeta = \frac{c^2 + c_1^2}{c^2 - c_1^2} \quad (35)$$

⁶ Other systematic accounts of the addition theorems, leading to less general results, have been given by HOBSON [3] (see also [20]), R. LAGRANGE [4] and LENSE [5]. Some aspects of the addition theorems are also considered by SNOW [7], and, for the Gegenbauer polynomials, in the chapter on spherical harmonics of [1].

and

$$\tilde{\omega} = \sqrt{1 + 2\zeta w + w^2}, \quad \tilde{\omega}^* = \sqrt{1 + 2\zeta w^* + w^{*2}}, \quad (36)$$

(the square roots being $+1$ for $w = 0$ and $w^* = 0$), we find

$$\xi_1 = \frac{w + w^* + (1 + ww^*)\zeta}{\tilde{\omega}\tilde{\omega}^*}, \quad \eta_1 = \frac{1 + ww^* + (w + w^*)\zeta}{\tilde{\omega}\tilde{\omega}^*}. \quad (37)$$

From this we derive

$$\begin{aligned} \xi_1' &= \zeta' \frac{1 - ww^*}{\tilde{\omega}\tilde{\omega}^*}, & \eta_1' &= \zeta' \frac{w - w^*}{\tilde{\omega}\tilde{\omega}^*}, \\ \xi_1 - \eta_1 &= (\zeta - 1) \frac{(1 - w)(1 - w^*)}{\tilde{\omega}\tilde{\omega}^*}. \end{aligned} \quad (38)$$

For later reference we note that for $w^* = 0$

$$\begin{aligned} \xi_1 &= \frac{w + \zeta}{\tilde{\omega}}, & \eta_1 &= \frac{1 + w\zeta}{\tilde{\omega}}, & \xi_1' &= \zeta' \frac{1}{\tilde{\omega}}, & \eta_1' &= \zeta' \frac{w}{\tilde{\omega}}, \\ \xi_1 - \eta_1 &= (\zeta - 1) \frac{1 - w}{\tilde{\omega}}, \end{aligned} \quad (39)$$

and for $w = w^* = 0$,

$$\begin{aligned} \xi_1 &= \zeta, & \eta_1 &= 1, \\ \xi_1' &= \zeta', & \eta_1' &= 0, \\ \xi_1 - \eta_1 &= \zeta - 1. \end{aligned} \quad (40)$$

5.2. The addition theorems, complex form.

The general addition theorem. From Lemma 1 it follows that for $|\arg(\zeta - 1)| < \pi$ the function

$$G^{(\alpha)}(w, w^*) = \text{const.} (\xi_1 - \eta_1)^{\mu+\nu+1} \xi_1'^{-\mu} L_{\nu+\alpha}^{\mu}(\xi_1) C_{\alpha}^{\nu+\frac{1}{2}}(\eta_1) \quad (41)$$

is in $C(\mathfrak{D})$ for admissible values of the parameters and for a sufficiently small neighborhood $\mathfrak{D}$ of $(w, w^*) = (0, 0)$. Indeed, it satisfies (3), is symmetric in w and w^* and satisfies the condition of regularity at $w = w^* = 0$, since $\xi_1 - \eta_1$ and ξ_1' are regular and different from zero there and since the Legendre and the Gegenbauer function are regular⁷ at $\xi_1 = \zeta$ and $\eta_1 = 1$. Hence $G^{(\alpha)}(w, w^*)$

⁷ It is not possible to include the Gegenbauer function of the second kind in the definition of $G^{(\alpha)}(w, w^*)$, since this function is in general singular at $\eta_1 = 1$. We also must exclude certain isolated values of α for which $C_{\alpha}^{\nu+\frac{1}{2}}$ is undefined.

can be expanded into a L.G. series. In the following the constant in (41) will be fixed so that

$$G^{(\alpha)}(0, 0) = L_{\nu+\alpha}^{\mu}(\zeta) C_{\alpha}^{\nu+\frac{1}{2}}(1). \quad (42)$$

It is known from the expansion theorem that the L.G. expansion will converge in the largest domain $\mathfrak{G}_r$ into which $G^{(\alpha)}(w, w^*)$ can be continued analytically, starting from the point $w = w^* = 0$. We shall use the symbol $G^{(\alpha)}(w, w^*)$ also to denote the continued function. It should be noted, however, that in the large the definition (41) of $G^{(\alpha)}(w, w^*)$ may fail to hold, since it may be necessary in the process of continuation to cross the cuts from $-\infty$ to 1 which must be made in the plane of the argument of the Legendre functions as well as the cuts in the Riemann surfaces of ξ_1 and η_1 .

We now proceed to find the largest possible r in $\mathfrak{G}_r$. In view of the expansion theorem this number is identical with the radius of convergence of the power series of the generating function of the L.G. expansion of $G^{(\alpha)}(w, w^*)$. In the neighborhood of $w = 0$ we have

$$\begin{aligned} H^{(\alpha)}(w, 0) &= (1 - w)^{-\mu-\nu-1} G^{(\alpha)}(w, 0) \\ &= \tilde{\omega}^{-\nu-1} L_{\nu+\alpha}^{\mu} \left(\frac{w + \zeta}{\tilde{\omega}} \right) C_{\alpha}^{\nu+\frac{1}{2}} \left(\frac{1 + w\zeta}{\tilde{\omega}} \right). \end{aligned} \quad (43)$$

To expand this, we write

$$H^{(\alpha)}(w, 0) = H_1(w)H_2(w), \quad (44)$$

where

$$H_1(w) = \tilde{\omega}^{-\nu-\alpha-1} L_{\nu+\alpha}^{\mu} \left(\frac{w + \zeta}{\tilde{\omega}} \right), \quad (45)$$

$$H_2(w) = \tilde{\omega}^{\alpha} C_{\alpha}^{\nu+\frac{1}{2}} \left(\frac{1 + w\zeta}{\tilde{\omega}} \right), \quad (46)$$

and compute the Taylor expansion of $H^{(\alpha)}(w, 0)$ by Cauchy multiplication of the Taylor expansions of H_1 and H_2 . If r_1 and r_2 are the respective radii of convergence, the resulting series will converge for $r = \text{Min}(r_1, r_2)$.

The expansion of H_1 can be effected with the aid of TRUESEL's theory of the F -equation⁸. The result is

$$H_1(w) = \sum_{n=0}^{\infty} \frac{(\nu + \alpha - \mu + 1)_n}{n!} (-w)^n L_{\nu+\alpha+n}^{\mu}(\zeta). \quad (47)$$

To determine r_1 , we observe that for $n \rightarrow \infty$

$$Q_{\nu+n}^{\mu}(\zeta) \sim \text{const. } n^{\mu-\frac{1}{2}}(\zeta - \zeta')^n,$$

$$P_{\nu+n}^{\mu}(\zeta) \sim \text{const. } n^{\mu-\frac{1}{2}}(\zeta + \zeta')^n,$$

⁸ For the particular results needed here, see [9], p. 85, eq. (12), and p. 87, eq. (23).

where μ , ν and ζ are fixed and the constants are independent⁹ of n . It follows that if $L_{\nu+\alpha}^\mu$ is a multiple of $Q_{\nu+\alpha}^\mu$, then r_I is

$$r_I = |\zeta - \zeta'|^{-1} = |\zeta + \zeta'|;$$

while if $L_{\nu+\alpha}^\mu$ contains a multiple of $P_{\nu+\alpha}^\mu$, then r_I is

$$r_{II} = |\zeta + \zeta'|^{-1} = |\zeta - \zeta'|.$$

For the expansion of $H_2(w)$ we can use a formula which was given by us previously¹⁰. It follows that

$$H_2(w) = C_{\alpha}^{\nu+1}(1) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{(2\nu+1)_n} (-w)^n C_n^{\nu+1}(\zeta), \quad (48)$$

the radius of convergence being r_{II} except for nonnegative integral values of α , in which case the series terminates.

The coefficient of $(-w)^n$ in the Taylor expansion of $H^{(\alpha)}(w, 0)$ thus proves to be

$$L_{\nu,\alpha,n}^\mu(\zeta) = C_{\alpha}^{\nu+1}(1) \sum_{k=0}^n \frac{(-\alpha)_k (\nu + \alpha - \mu + 1)_{n-k}}{(2\nu+1)_k (n-k)!} L_{\nu+\alpha+n-k}^\mu(\zeta) C_k^{\nu+1}(\zeta), \quad (49)$$

and we obtain by the expansion theorem

$$G^{(\alpha)}(w, w^*) = \sum_{n=0}^{\infty} (-)^n L_{\nu,\alpha,n}^\mu(\zeta) G_n(w, w^*). \quad (50)$$

If $L_{\nu+\alpha}^\mu$ is a multiple of $Q_{\nu+\alpha}^\mu$ and if α is a nonnegative integer, this series converges in $\mathfrak{G}_n$; in all other cases the domain of convergence is $\mathfrak{G}_{II}$.

Three special addition theorems. Of particular interest are the cases where one of the two factors in (44) reduces to a constant.

For $H_2(w)$ this happens if and only if $\alpha = 0$, and we then have

$$H_2(w) = 1.$$

Consequently,

$$L_{\nu,0,n}^\mu(\zeta) = \frac{(\nu - \mu + 1)_n}{n!} L_{\nu+n}^\mu(\zeta). \quad (51)$$

We now define two special cases of $G^{(0)}$ which will receive particular attention in the sequel. In the neighborhood of $w = w^* = 0$ we put

$$G_I(w, w^*) = \text{const. } (\xi_1 - \eta_1)^{\mu+\nu+1} \xi_1'^{-\mu} Q_{\nu}^\mu(\xi_1), \quad (52)$$

$$G_{II}(w, w^*) = \text{const. } (\xi_1 - \eta_1)^{\mu+\nu+1} \xi_1'^{-\mu} P_{\nu}^\mu(\xi_1); \quad (53)$$

⁹ See [1], §3.9.1.

¹⁰ See [16], eq. (50). For nonnegative integral α this expansion had been given earlier by BRAFMAN [13].

the constants are fixed by

$$G_I(0, 0) = Q_\nu^\mu(\zeta), \quad G_{II}(0, 0) = P_\nu^\mu(\zeta),$$

and in the large the definition is made by analytic continuation. By (50) and (51) we have the two expansions

$$G_I(w, w^*) = \sum_{n=0}^{\infty} \frac{(\nu - \mu + 1)_n}{n!} (-)^n Q_{\nu+n}^\mu(\zeta) G_n(w, w^*), \quad (54)$$

$$G_{II}(w, w^*) = \sum_{n=0}^{\infty} \frac{(\nu - \mu + 1)_n}{n!} (-)^n P_{\nu+n}^\mu(\zeta) G_n(w, w^*), \quad (55)$$

which are uniformly convergent in every closed subdomain of $\mathfrak{G}_I$ and $\mathfrak{G}_{II}$, respectively.

In order to reduce the function $H_1(w)$ to a constant, we set $\alpha = -\mu - \nu - 1$ and

$$L_{-\mu-1}^\mu(\xi_1) = \text{const. } P_{-\mu-1}^\mu(\xi_1). \quad (56)$$

This is an admissible choice, since¹¹ P_λ^μ can be expressed as a linear combination of P_λ^μ and Q_λ^μ . We then find that apart from a constant factor, $H_1(w)$ is equal to

$$\tilde{\omega}^\mu P_{-\mu-1}^\mu \left(\frac{w + \zeta}{\tilde{w}} \right) = \frac{\zeta^\mu}{2^\mu \Gamma(1 + \mu)},$$

which is independent of w . A third case of (41) which possesses a simple L.G. expansion is thus given by $G^{(-\mu-\nu-1)}$ with the choice (56) for $L_{-\mu-1}^\mu$; we define (in the large again by analytic continuation)

$$G_{III}(w, w^*) = \text{const. } (\xi_1 - \eta_1)^{\mu+\nu+1} C_{-\mu-\nu-1}^{\nu+\frac{1}{2}}(\eta_1) \quad (57)$$

and normalize by

$$G_{III}(0, 0) = 1.$$

This normalization has the advantage that the restriction $|\arg(\zeta - 1)| < \pi$ which had to be imposed so far can be dropped. Normalizing $H_2(w)$ accordingly, we find

$$G_{III}(w, w^*) = \sum_{n=0}^{\infty} \frac{(\mu + \nu + 1)_n}{(2\nu + 1)_n} (-)^n C_n^{\nu+\frac{1}{2}}(\zeta) G_n(w, w^*). \quad (58)$$

Since ζ is unrestricted, the domain of validity is now $\mathfrak{G}_{III}$, where

$$r_{III} = \text{Min } |\zeta \pm \zeta'|.$$

5.3. The addition theorems, first real form.

¹¹ [1], eq. 3.3(5).

Statement of the expansions in terms of real variables. Transforming to ξ and η we find

$$\xi_1 = \frac{\zeta\xi + \eta}{\Omega}, \quad \eta_1 = \frac{\xi + \eta\zeta}{\Omega} \quad (59)$$

and

$$\xi'_1 = \frac{\zeta'\xi'}{\Omega}, \quad \eta'_1 = \frac{\eta'\zeta'}{\Omega}, \quad \xi_1 - \eta_1 = \frac{(\zeta - 1)(\xi - \eta)}{\Omega}, \quad (60)$$

where¹²

$$\begin{aligned} \Omega &= [(\zeta\xi + \eta)^2 - \zeta'^2 \xi'^2]^{\frac{1}{2}} \\ &= [(\xi + \eta\zeta)^2 - \eta'^2 \zeta'^2]^{\frac{1}{2}}, \end{aligned} \quad (61)$$

the square roots being such that $\Omega \sim \xi$ for large values of ξ . Using these relations and the definition of the function $G^{(\alpha)}$, we can write the general addition theorem (50) in the following form:

$$\begin{aligned} \Omega^{-\nu-1} L_{\nu+\alpha}^{\mu} \left(\frac{\zeta\xi + \eta}{\Omega} \right) C_{\alpha}^{\nu+\frac{1}{2}} \left(\frac{\xi + \eta\zeta}{\Omega} \right) &= e^{-i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\Gamma(\mu + \nu + 1) \sqrt{\pi}} \\ &\cdot \sum_{n=0}^{\infty} \frac{(\nu + n + \frac{1}{2}) n!}{(\mu + \nu + 1)_n} (-)^n L_{\nu,\alpha,n}^{\mu}(\zeta) Q_{\nu+n}^{\mu}(\xi) C_n^{\nu+\frac{1}{2}}(\eta), \end{aligned} \quad (62)$$

where $L_{\nu,\alpha,n}^{\mu}(\zeta)$ is given by (49). Similarly we obtain for the special addition theorems

$$\begin{aligned} \Omega^{-\nu-1} Q_{\nu}^{\mu} \left(\frac{\zeta\xi + \eta}{\Omega} \right) \\ = e^{i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1) \sqrt{\pi}} \sum_{n=0}^{\infty} (\nu + \frac{1}{2} + n) (-)^n Q_{\nu+n}^{\mu}(\zeta) Q_{\nu+n}^{-\mu}(\xi) C_n^{\nu+\frac{1}{2}}(\eta), \end{aligned} \quad (63)$$

$$\begin{aligned} \Omega^{-\nu-1} P_{\nu}^{\mu} \left(\frac{\zeta\xi + \eta}{\Omega} \right) \\ = e^{i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1) \sqrt{\pi}} \sum_{n=0}^{\infty} (\nu + \frac{1}{2} + n) (-)^n P_{\nu+n}^{\mu}(\zeta) Q_{\nu+n}^{-\mu}(\xi) C_n^{\nu+\frac{1}{2}}(\eta), \end{aligned} \quad (64)$$

¹² The identities

$$\begin{aligned} (1 + 2\zeta w + w^2)(1 + 2\zeta w^* + w^{*2}) \\ = [w + w^* + (1 + ww^*)\zeta]^2 - (\zeta^2 - 1)(1 - ww^*)^2 \\ = [(w + w^*)\zeta + 1 + ww^*]^2 - (\zeta^2 - 1)(w - w^*)^2 \end{aligned}$$

are used in the course of the computation.

$$\Omega^{-\mu-\nu-1} C_{\mu-\nu}^{\nu+\frac{1}{2}} \left(\frac{\xi + \eta\zeta}{\Omega} \right) = e^{-i\mu\pi} \frac{2^{-\nu+1} \xi'^{-\mu}}{\Gamma(\mu - \nu + 1) \Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(\nu + \frac{1}{2} + n)(-)^n n!}{(2\nu + 1)_n} Q_{\nu+n}^{\mu}(\xi) C_n^{\nu+\frac{1}{2}}(\eta) C_n^{\nu+\frac{1}{2}}(\zeta). \quad (65)^{13}$$

In the last four equations the function symbols on the left must be interpreted as representing the analytic continuations of the functions defined by them in the conventional way at $w = w^* = 0$, *i. e.* at $\xi = \infty$. Similarly for $|ww^*| > 1$ the functions $Q_{\nu+n}^{\pm\mu}(\xi)$ on the right are to be continued into the lower sheet of the Riemann surface of ξ' . Since the coordinate functions on the right are here essentially the functions $H_n(w, w^*)$ associated with $G_n(w, w^*)$ by (21), the domains of validity $\mathfrak{G}_{r_i}$ given for (50), (54), (55) and (58) can now be replaced by the corresponding domains $\mathfrak{S}_{r_i}$. Thus (63), (64) and (65) are valid in $\mathfrak{S}_{r_I}$, $\mathfrak{S}_{r_{II}}$ and $\mathfrak{S}_{r_{III}}$, respectively, and (62) is valid in $\mathfrak{S}_{r_I}$, if $L_{\nu+\alpha}^{\mu}$ is a multiple of $Q_{\nu+\alpha}^{\mu}$ and $C_{\alpha}^{\nu+\frac{1}{2}}$ is a polynomial, and in $\mathfrak{S}_{r_{III}}$, if this is not the case. During the process of continuation the point (w, w^*) must stay within the appropriate domain $\mathfrak{S}_{r_i}$.

Domains of validity in terms of ξ, η, ζ . It suffices to discuss the domain $\mathfrak{S}_{r_I}$, $\mathfrak{S}_{r_{II}}$ and $\mathfrak{S}_{r_{III}}$.

(i) $\mathfrak{S}_{r_I}$. Since $|\arg(\zeta - 1)| < \pi$, ζ' is defined uniquely, and

$$r_I = |\zeta + \zeta'| > 1.$$

In view of (15) the condition $|ww^*|^{\frac{1}{2}} < r_I$ immediately yields

$$|\xi - \xi'| < |\zeta + \zeta'|. \quad (66)$$

This is not only satisfied in the entire upper sheet of the Riemann surface of ξ' , but also in a certain part of the lower sheet. The condition that ww^* must not be real and ≥ 1 excludes real values of $\xi \geq 1$ on the lower sheet. To find the conditions on η , let us fix

$$p = \xi - \xi'.$$

We then have

$$\eta = \frac{1}{2} \left(\frac{w}{p} + \frac{p}{w} \right), \quad \text{i.e.} \quad \frac{w}{p} = \eta \pm \eta'.$$

Since $|w| < r_I$, $|w^*| = |p^2/w| < r_I$, we must have

$$\left| \frac{\xi - \xi'}{\zeta + \zeta'} \right| < \left| \frac{w}{p} \right| < \left| \frac{\zeta + \zeta'}{\xi - \xi'} \right|$$

¹³ For simplification of the formulas, $Q_{\nu}^{\mu}(\xi)$ has been expressed in terms of $Q_{\nu}^{\mu}(\xi)$ in (63) and (64). In (65) use has been made of the linear relation connecting $C_{\mu-\nu-1}^{\nu+\frac{1}{2}}$ and $C_{\mu-\nu}^{\nu+\frac{1}{2}}$ (see [1], eq. 3.15(31)).

or

$$|\eta \pm \eta'| < \left| \frac{\zeta + \zeta'}{\xi - \xi'} \right|. \quad (67)$$

(ii) $\mathfrak{S}_{\text{II}}$. In this case

$$r_{\text{II}} = |\zeta - \zeta'| < 1,$$

and $|ww^*|^{\frac{1}{2}} < r_{\text{II}}$ leads to

$$|\xi - \xi'| < |\zeta - \zeta'|, \quad (68)$$

which holds only in a certain part of the upper sheet of ξ' . For η we obtain in the same manner as above

$$|\eta \pm \eta'| < \left| \frac{\zeta - \zeta'}{\xi - \xi'} \right|. \quad (69)$$

(iii) $\mathfrak{S}_{\text{III}}$. Here ζ is arbitrary and also η is arbitrary, since for given η it is possible to find within any $\mathfrak{S}_{\text{III}}$ values w and w^* such that the second relation (12) holds. We denote by q a value of

$$\sqrt{w/w^*} = \eta \pm \eta'$$

whose modulus is ≤ 1 . Then

$$\xi = \frac{1}{2} \left(wq + \frac{1}{wq} \right)$$

and, since $|wq| < r_{\text{III}} \leq 1$ and because ξ must be in the upper sheet,

$$wq = \xi - \xi'.$$

Since $|w| < \text{Min } |\zeta \pm \zeta'|$, we find

$$|\xi - \xi'| < |\eta \pm \eta'| |\zeta \pm \zeta'|, \quad (70)$$

where on the right all possible combinations of signs have to be taken.

The above conditions can be rendered more intuitive geometrically if use is made of the following simple property of confocal ellipses:

For any complex number c , let $E(c)$ denote the ellipse with foci at ± 1 and passing through c , let ϑ be in the upper sheet of ϑ' and let a be a complex number with $|a| < 1$. Then the two following statements are equivalent:

(a) ϑ lies in the $\begin{matrix} \text{exterior} \\ \text{interior} \end{matrix}$ of $E\left(\frac{a + a^{-1}}{2}\right)$;

(b) $|\vartheta - \vartheta'| \leq |a|$.

Using this, we find that

$$\begin{aligned} (i) \quad \mathfrak{S}_{rI} \text{ is characterized by} \\ \left. \begin{aligned} \zeta: & |\arg(\zeta - 1)| < \pi, \\ \xi: & \text{in the upper sheet of the Riemann surface of } \xi', \text{ or in the lower sheet} \\ & \text{in the interior of } E(\zeta) \text{ and not real with } |\xi| \geq 1, \\ \eta: & \text{in the interior of } E(\xi\zeta + \xi'\zeta'); \end{aligned} \right\} \quad (71) \end{aligned}$$

$$\begin{aligned} (ii) \quad \mathfrak{S}_{rII} \text{ is characterized by} \\ \left. \begin{aligned} \zeta: & |\arg(\zeta - 1)| < \pi, \\ \xi: & \text{in the exterior of } E(\zeta), \\ \eta: & \text{in the interior of } E(\xi\zeta - \xi'\zeta'); \end{aligned} \right\} \quad (72) \end{aligned}$$

$$\begin{aligned} (iii) \quad \mathfrak{S}_{rIII} \text{ is characterized by} \\ \left. \begin{aligned} \eta: & \text{arbitrary;} \\ \zeta: & \text{arbitrary;} \\ \xi: & \text{in the exterior of both } E(\eta\zeta - \eta'\zeta') \text{ and } E(\eta\zeta + \eta'\zeta'). \end{aligned} \right\} \quad (73) \end{aligned}$$

Analytic continuation. We do not discuss in detail the analytic continuation of the functions on the left of (62), (63), (64) and (65) for complex values of ξ , η and ζ . As a matter of practical importance it should however be mentioned that for real and positive values of the arguments of the Legendre and Gegenbauer functions and for real values of the arguments of the Gegenbauer polynomials the expansions are valid as stated whenever the conditions of convergence are satisfied.

5.4. The addition theorems, second real form.

New variables. The process of analytic continuation mentioned in the last section can be rendered more explicit if WHIPPLE's transformation¹⁴ is applied to the various Legendre functions. In doing so we can also exhibit the relationship between our addition theorems and the classical addition theorems of the Legendre functions of order zero.

In order to simplify our formulas we introduce a new set of variables (ρ , σ , τ) defined as follows:

$$\rho = \frac{\xi}{\xi'}, \quad \sigma = \frac{\eta}{\eta'}, \quad \tau = \frac{\zeta}{\zeta'}. \quad (74)$$

This transformation is reciprocal, for we have

$$\xi = \frac{\rho}{\rho'}, \quad \eta = \frac{\sigma}{\sigma'}, \quad \zeta = \frac{\tau}{\tau'}. \quad (75)$$

We also note the relations

$$\xi'\rho' = \eta'\sigma' = \zeta'\tau' = 1. \quad (76)$$

¹⁴ See [1], §3.3.1.

We now have

$$\begin{aligned}\Omega &= \frac{(\rho\tau + \rho'\tau'\eta)'}{\rho'\tau'} \\ &= \frac{(\sigma\tau + \sigma'\tau'\xi)'}{\sigma'\tau'}\end{aligned}\quad (77)$$

and hence

$$\begin{aligned}\rho_1 &\equiv \frac{\xi_1}{\xi'_1} = \rho\tau + \rho'\tau'\eta, \\ \sigma_1 &\equiv \frac{\eta_1}{\eta'_1} = \sigma\tau + \sigma'\tau'\xi.\end{aligned}\quad (78)$$

In the case of the general addition theorem (62) there is some doubt how Whipple's transformation should be applied, and the resulting formulas are in any case unwieldy. However, in the case of the three special addition theorems the results are simple.

The two Legendre expansions. In order to be able to apply Whipple's transformation to (63) and (64) we have to impose, in addition to the conditions of convergence as given by (71) and (72), the further conditions

$$\operatorname{Re} \xi > 0, \quad \operatorname{Re} \zeta > 0. \quad (79)$$

The variable ξ , however, may still vary on both sheets of the Riemann surface of ξ' , as long as this is compatible with (79). If ν is replaced by $\nu - \frac{1}{2}$ and μ by $-\mu - \frac{1}{2}$, and then Whipple's transformation is applied in (63) and (64), the result is

$$\begin{aligned}\rho_1'^{-\nu} P_\mu^{-\nu}(\rho_1) &= 2^\nu \Gamma(\nu) \rho'^{-\nu} \tau'^{-\nu} \\ &\cdot \sum_{n=0}^{\infty} (\nu + n)(-)^n (\mu + \nu + 1)_n (\nu - \mu)_n P_\mu^{-\nu-n}(\rho) P_\mu^{-\nu-n}(\tau) C_n^\nu(\eta),\end{aligned}\quad (80)$$

$$\begin{aligned}\rho_1'^{-\nu} Q_\mu^{-\nu}(\rho_1) &= 2^\nu \Gamma(\nu) \rho'^{-\nu} \tau'^{-\nu} \\ &\cdot \sum_{n=0}^{\infty} (\nu + n)(-)^n (\mu + \nu + 1)_n (\nu - \mu)_n P_\mu^{-\nu-n}(\rho) Q_\mu^{-\nu-n}(\tau) C_n^\nu(\eta).\end{aligned}\quad (81)$$

To these expansions we may add the two following somewhat simpler formulae:

$$\rho_1'^{-\nu} P_\mu^\nu(\rho_1) = 2^\nu \Gamma(\nu) \rho'^{-\nu} \tau'^{-\nu} \sum_{n=0}^{\infty} (\nu + n) P_\mu^{-\nu-n}(\rho) P_\mu^{\nu+n}(\tau) C_n^\nu(\eta), \quad (82)$$

$$\rho_1'^{-\nu} Q_\mu^\nu(\rho_1) = 2^\nu \Gamma(\nu) \rho'^{-\nu} \tau'^{-\nu} \sum_{n=0}^{\infty} (\nu + n) P_\mu^{-\nu-n}(\rho) Q_\mu^{\nu+n}(\tau) C_n^\nu(\eta). \quad (83)$$

The last relation follows from (81) by switching the sign of the upper index of the Q -function according to [1], eq. 3.3(2); the first relation is obtained by multiplying

$$(80) \text{ by } \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu - \nu + 1)} \quad \text{and} \quad (81) \text{ by } \frac{2}{\pi} e^{-i\nu\pi} \sin \nu\pi,$$

adding, and using [1], eq. 3.3(6). In all four expansions the function symbols on the left must be interpreted as representing the analytic continuations of the functions defined by them in the conventional way near $w = w^* = 0$, *i.e.* near $\rho_1 = \tau$ (see below).

We now state the conditions of convergence for the above series in terms of ρ , τ and η . We first observe that the transformation $\rho = \xi/\xi'$ maps the portion $\operatorname{Re} \xi > 0$ of the Riemann surface of the function $f(\xi) = \xi'$ on the schlicht ρ -plane cut between -1 and $+1$. The values with $\operatorname{Re} \xi > 0$ in the upper sheet are mapped on $\operatorname{Re} \rho > 0$, and the values with $\operatorname{Re} \xi > 0$ in the lower sheet on $\operatorname{Re} \rho < 0$, the image of the cut being the line $\operatorname{Re} \rho = 0$. The condition that on the lower sheet we must exclude the values which are real and in absolute value ≥ 1 leads to the exclusion of values of ρ between -1 and $-\infty$, so that the ρ -plane is cut between $+1$ and $-\infty$. The same remarks apply *mutatis mutandis* to the transformation $\tau = \zeta/\zeta'$. The conditions (71) and (72) now lead immediately to the following conditions of convergence:

(i) for (80):

$$\left. \begin{aligned} \tau: \operatorname{Re} \tau > 0, \quad \tau \neq \text{real and } \leq 1, \\ \rho: \left| \frac{\rho - 1}{\rho + 1} \right| < \left| \frac{\tau + 1}{\tau - 1} \right|, \quad \rho \neq \text{real and } \leq 1, \\ \eta: \text{in the interior of } E\left(\frac{\rho\tau + 1}{\rho'\tau'}\right); \end{aligned} \right\} \quad (84)$$

(ii) for (81), (82) and (83):

$$\left. \begin{aligned} \tau: \operatorname{Re} \tau > 0, \quad \tau \neq \text{real and } \leq 1, \\ \rho: \left| \frac{\rho - 1}{\rho + 1} \right| < \left| \frac{\tau - 1}{\tau + 1} \right| \text{ (this implies } \operatorname{Re} \rho > 0), \quad \rho \neq \text{real and } \leq 1, \\ \eta: \text{in the interior of } E\left(\frac{\rho\tau - 1}{\rho'\tau'}\right). \end{aligned} \right\} \quad (85)$$

We turn next to the analytic continuation of the functions on the left. Our aim is to exhibit these functions explicitly for values of (ρ, τ, η) within the respective domains of convergence. We can confine our discussion to the two expansions (80) and (81), since the remaining two formulae can be obtained from (80) and (81) by linear combination.

We first assume that $\operatorname{Re} \rho \geq 0$. For a set of given values (ρ, τ, η) lying in the

domain of convergence we perform the analytic continuation in two steps. First, keeping $\eta = \text{const.} = 1$, we move the point ρ_1 from $\rho_1 = \tau$ to $\rho_1 = \rho\tau + \rho'\tau'$. Then, keeping ρ at its final value, we move η into the given position. We shall show that in the first step the cut between $-\infty$ and $+1$ in the ρ_1 -plane is not crossed. Indeed, under the given conditions we can parametrize ρ and τ as follows:

$$\rho = \frac{1}{2}\left(u + \frac{1}{u}\right), \quad \tau = \frac{1}{2}\left(v + \frac{1}{v}\right),$$

where $\text{Re } u \geq 0$, $|u| < 1$ and $\text{Re } v > 0$, $|v| < 1$. One easily finds

$$\rho' = \frac{1}{2}\left(u - \frac{1}{u}\right), \quad \tau' = \frac{1}{2}\left(v - \frac{1}{v}\right)$$

and hence

$$\rho\tau + \rho'\tau' = \frac{1}{2}\left(uv + \frac{1}{uv}\right). \quad (86)$$

Since $|uv| < 1$, $|\arg uv| < \pi$, the point (86) cannot lie on or cross the real axis between $-\infty$ and $+1$. We proceed now to the second step, keeping ρ fixed and allowing η to vary as indicated in (84) and (85). If η is in the interior of $E[(\rho\tau - 1)/(\rho'\tau')]$, then the point $\rho_1 = \rho\tau + \rho'\tau'\eta$ varies in the interior of the ellipse with foci at $\rho\tau \pm \rho'\tau'$ and passing through the point $+1$. If η is in the interior of $E[(\rho\tau + 1)/(\rho'\tau')]$, then ρ_1 varies in the interior of the ellipse with the same foci, but passing through -1 . Since $\mathfrak{S}_{\pi\pi} \subset \mathfrak{S}_{\pi}$, the domain (85) is contained in the domain (84), and the ellipse passing through -1 contains the ellipse passing through $+1$ in its interior. In particular it contains the point $+1$. Hence it has no point in common with the portion of the real axis between $-\infty$ and -1 . Since for the function $\rho_1'^{-\nu} P_{\mu}^{-\nu}(\rho_1)$ the ρ_1 -plane is cut only between $-\infty$ and -1 , we have the following result:

Theorem I. *The expansion (80) is valid as it stands for $\text{Re } \rho \geq 0$, $\text{Re } \tau > 0$ and ρ and τ not real and ≤ 1 , provided that the point $\rho_1 = \rho\tau + \rho'\tau'\eta$ lies in the interior of the ellipse with foci at $\rho\tau \pm \rho'\tau'$ and passing through -1 .*

Since it lies in the interior of the ellipse passing through -1 , the ellipse passing through $+1$ also does not contain any points on the real axis between $-\infty$ and -1 . It may however intersect the real axis between -1 and $+1$. Since in the case of the function $\rho_1'^{-\nu} Q_{\mu}^{-\nu}(\rho_1)$ the cut must be extended to $\rho_1 = 1$, it may be necessary to continue the function on the left of (81) across the cut. The formulae for the analytic continuation of the Q -function across the cut between -1 and $+1$ are well known¹⁵. If $Q_{\mu}^{-\nu}(\rho_1, 1\pm)$ denotes the value of the analytic continuation of $Q_{\mu}^{-\nu}(\rho_1)$, when ρ_1 crosses the cut in the downward upward

¹⁵ See [1], eq. 3.3(20).

direction, then we have

$$Q_{\mu}^{-\nu}(\rho_1, 1\pm) = e^{\mp i\nu\pi} Q_{\mu}^{-\nu}(\rho_1) \mp i\pi e^{-i\nu\pi} P_{\mu}^{-\nu}(\rho_1).$$

We can thus formulate the following theorem concerning (81):

Theorem II. Let $\operatorname{Re} \tau > 0$, ρ and τ not real and ≤ 1 ,

$$\left| \frac{\rho - 1}{\rho + 1} \right| < \left| \frac{\tau - 1}{\tau + 1} \right|,$$

and let E be the ellipse with foci at $\rho\tau \pm \rho'\tau'$ and passing through $+1$. Let $\rho_1 = \rho\tau + \rho'\tau'\eta$ be in the interior of E . Then the series on the right of (81) converges to

$$\rho_1'^{-\nu} Q_{\mu}^{-\nu}(\rho_1),$$

if the interior of E has no point in common with the segment of the real axis between -1 and $+1$, or if ρ_1 lies on the same side of the real axis as the point $\rho\tau + \rho'\tau'$. If the interior of E intersects the segment and if ρ_1 is not on the same side of the real axis as $\rho\tau + \rho'\tau'$, then the series converges to

$$\rho_1'^{-\nu} \{ e^{-2i\nu\pi} Q_{\mu}^{-\nu}(\rho_1) - i\pi e^{-2i\nu\pi} P_{\mu}^{-\nu}(\rho_1) \},$$

if $\operatorname{Im} \rho_1 < 0$, and to

$$\rho_1'^{-\nu} \{ e^{2i\nu\pi} Q_{\mu}^{-\nu}(\rho_1) + i\pi P_{\mu}^{-\nu}(\rho_1) \},$$

if $\operatorname{Im} \rho_1 > 0$ ¹⁶.

The expansion (81) has thus been taken care of. It remains to discuss the case where $\operatorname{Re} \rho < 0$ in (80). We note that¹⁷

$$P^{-\nu-n}(\rho) = e^{\mp i\nu\pi} P_{\mu}^{-\nu-n}(-\rho) - \frac{2}{\pi} e^{i\nu\pi} \sin \pi(\nu - \mu) Q_{\mu}^{-\nu-n}(-\rho),$$

where the upper or lower sign has to be taken according as $\operatorname{Im} \rho \gtrless 0$. Thus on the right of (80) we can express $P_{\mu}^{-\nu-n}(\rho)$ linearly in terms of $P_{\mu}^{-\nu-n}(-\rho)$ and $Q_{\mu}^{-\nu-n}(-\rho)$ with coefficients which are independent of n . Since

$$\operatorname{Re}(-\rho) > 0, \quad (-\rho)' = \rho',$$

and

$$\left| \frac{\tau - 1}{\tau + 1} \right| < \left| \frac{(-\rho) - 1}{(-\rho) + 1} \right|,$$

the two resulting series can be summed according to Theorem I and Theorem II. (For the application of Theorem II the roles of σ and τ have to be interchanged.) Thus we can find the sum explicitly also in the case $\operatorname{Re} \rho < 0$. It seems unnecessary to write down the resulting formulae, which are complicated in view of the many cases which have to be distinguished.

¹⁶ If $\operatorname{Im} \rho_1 = 0$, either of the limits from above and from below can be taken.

¹⁷ See [1], eq. 3.3(10).

It should again be noted that for real and positive values of ρ and τ all expansions are valid as they stand provided only that the conditions for convergence are satisfied.

The Gegenbauer expansion. In order to apply Whipple's transformation to (65) we first express the Gegenbauer functions in terms of Legendre functions of the first kind¹⁸ and obtain after some simplification

$$\Omega^{-\mu-1} P_{\mu}^{-\nu} \left(\frac{\xi + \eta \zeta}{\Omega} \right) = e^{-i\mu\pi} \frac{\Gamma(2\nu + 1)}{\Gamma(\mu + \nu + 1)} \xi'^{-\mu} \cdot \sum_{n=0}^{\infty} \frac{(2\nu + 1 + 2n)(2\nu + 1)_n}{n!} (-)^n Q_{\nu+n}^{\mu}(\xi) P_{\nu+n}^{-\nu}(\eta) P_{\nu+n}^{-\nu}(\zeta). \quad (87)$$

Supposing

$$\operatorname{Re} \eta > 0, \quad \operatorname{Re} \zeta > 0, \quad (88)$$

we apply Whipple's transformation to the P -functions, replace $\mu + \frac{1}{2}$ by μ and ν by $\nu + \frac{1}{2}$, and obtain

$$(\xi' \sigma' \tau')^{\mu+\nu+1} \sigma_1'^{-\mu} Q_{\nu}^{\mu}(\sigma_1) = ie^{-2\nu\pi i} \xi'^{\nu+3/2} \sigma'^{\nu+1} \tau'^{\nu+1} \cdot \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{(2\nu + 2 + 2n)(-)^n}{\Gamma(2\nu + 2 + n)n!} Q_{\nu+\frac{1}{2}+n}^{\mu-\frac{1}{2}}(\xi) Q_{\nu+1+n}^{\nu}(\sigma) Q_{\nu+1+n}^{\nu}(\tau). \quad (89)$$

The substitution (74) maps the domains $\operatorname{Re} \eta > 0$ and $\operatorname{Re} \zeta > 0$, respectively, on the whole σ -plane and the whole τ -plane, each cut between -1 and $+1$, and the following conditions for convergence of (89) follow from (73):

$$\left. \begin{aligned} \sigma: & \text{exterior to } E(0)^{19} \\ \tau: & \text{exterior to } E(0), \\ \xi: & \text{exterior to both } E\left(\frac{\sigma\tau + 1}{\sigma'\tau'}\right) \text{ and } E\left(\frac{\sigma\tau - 1}{\sigma'\tau'}\right). \end{aligned} \right\} \quad (90)$$

The function on the left of (89) is again defined by analytic continuation starting from $w = w^* = 0$, i.e. $\xi = \infty$. It is single-valued near $\xi = \infty$, since it can be written in the form

$$\left(\frac{\xi' \sigma' \tau'}{\sigma_1} \right)^{\mu+\nu+1} \sigma_1'^{-\mu} \sigma_1^{\mu+\nu+1} Q_{\nu}^{\mu}(\sigma_1); \quad (91)$$

here the product of the last three terms is single-valued if σ_1 is in the exterior of $E(0)$, and the same is true for the first factor if both σ_1 and ξ are in the exterior of $E(0)$. For any fixed value of σ , (91) is thus defined unambiguously for sufficiently large values of ξ . If now σ is kept fixed and ξ varies in the exterior of

¹⁸ See [1], eq. 3.15(4).

¹⁹ $E(0)$ is the closed straight line segment joining -1 and $+1$.

$E[(\sigma\tau + 1)/(\sigma'\tau')]$, the point $\sigma_1 = \sigma\tau + \sigma'\tau'\xi$ varies in the exterior of the ellipse with foci $\sigma\tau \pm \sigma'\tau'$ and passing through the point -1 . If ξ varies in the exterior of $E[(\sigma\tau - 1)/(\sigma'\tau')]$, σ_1 varies in the exterior of the ellipse with the same foci, but passing through -1 . Since ξ must be exterior to both $E[(\sigma\tau \pm 1)/(\sigma'\tau')]$, σ_1 is confined to the domain $\mathfrak{D}$ exterior to both ellipses with foci at $\sigma\tau \pm \sigma'\tau'$ and passing through $+1$ and -1 . This domain does not contain the points -1 and $+1$ and therefore, being the complement of a closed convex domain, has no point in common with $E(0)$. Hence (91) is defined unambiguously in the whole of $\mathfrak{D}$, and we have proved the following result:

Theorem III. *If σ and τ are not in the real interval $(-1, +1)$, and if $\sigma_1 = \sigma\tau + \sigma'\tau'\xi$ lies in the domain exterior to both ellipses with foci at $\sigma\tau \pm \sigma'\tau'$ and passing through the points -1 and $+1$, respectively, then the series on the right of (89) converges to (91).*

5.5. Special values of μ and ν . For practical purposes and in order to exhibit relations to earlier work we list in this and the next section some of the more important special cases of our above expansions. This list is not intended to be complete; as the principal expansions depend on five parameters, it could easily be augmented.

The classical addition theorems of the Legendre functions. In view of²⁰

$$\lim_{\nu \rightarrow 0} \Gamma(\nu)(\nu + n)C_n^\nu(\eta) = \epsilon_n T_n(\eta),$$

where $T_n(\eta) = \cos(n \cos^{-1} \eta)$ is the Tchebychef polynomial, we get from (82) and (83) for $\nu \rightarrow 0$ the two expansions

$$P_\mu(\rho\tau + \rho'\tau'\eta) = \sum_{n=0}^{\infty} \epsilon_n P_\mu^{-n}(\rho) P_\mu^n(\tau) T_n(\eta), \quad (92)$$

$$Q_\mu(\rho\tau + \rho'\tau'\eta) = \sum_{n=0}^{\infty} \epsilon_n P_\mu^{-n}(\rho) Q_\mu^n(\tau) T_n(\eta), \quad (93)$$

also valid under the conditions of Theorems I and II. These are the classical²¹ addition theorems of the Legendre functions. Using [1], eqs. 3.3(2) and 3.6(12), we obtain in a similar way from (89)

$$Q_\nu(\sigma\tau + \sigma'\tau'\xi) = 2e^{-2i\nu\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(2\nu + 2 + n)} Q_\nu^{n+1}(\sigma) Q_\nu^{n+1}(\tau) (\xi - \xi')^{n+1},$$

valid under the conditions of Theorem III²².

The addition theorem of the Gegenbauer polynomials. Since every Legendre

²⁰ $\epsilon_0 = 1, \epsilon_n = 2, n = 1, 2, \dots$

²¹ First stated by HEINE ([2], p. 312 and p. 333) without precise conditions of validity; proved by WHITTAKER & WATSON for real η and $\operatorname{Re} \rho > 0, \operatorname{Re} \tau > 0, \operatorname{Re} \rho_1 > 0$; by HOBSON ([3], chapter VIII) for real η and $\operatorname{Re} \rho > 0, \operatorname{Re} \tau > 0$; and by R. LAGRANGE [4] under the conditions of our Theorems I and II.

²² An equivalent formula was stated by HEINE ([2], p. 339) and proved by HOBSON [3] for $\xi \geq 1, \sigma$ and τ purely imaginary and ν integral. HOBSON's final formula is vitiated by several misprints. Another equivalent result was given by SNOW ([7], p. 275).

function of the first or second kind can be expressed in terms of Gegenbauer functions of the first and second kind²³, we can restate the expansions (80) and (81) in terms of Gegenbauer functions. For (80) the result is the well known addition theorem of the Gegenbauer polynomials²⁴, now proved without any restriction on the lower index. From (81) there results an expansion of similar form for the Gegenbauer function of the second kind. Both formulas are more complicated than the corresponding ones for Legendre functions.

The addition theorems of the Bessel functions. With the aid of the relation

$$\mu^\nu P_\nu^{-\mu} \left(1 + \frac{t^2}{2\mu^2} \right) \rightarrow I_\nu(t) \quad \text{as} \quad \mu \rightarrow \infty,$$

where I_ν denotes the modified Bessel function, we can derive from the expansions of §5.3 and §5.4 several well known relations involving Bessel functions. Putting in (80) and (82)

$$\rho = 1 + \frac{r^2}{2\mu^2}, \quad \tau = 1 + \frac{t^2}{2\mu^2},$$

letting $\mu \rightarrow \infty$ and writing

$$R = \sqrt{r^2 + 2st\eta + t^2},$$

we obtain the formulae

$$R^{-\nu} I_\nu(R) = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) r^{-\nu} I_{\nu+n}(r) t^{-\nu} I_{\nu+n}(t) C_n^\nu(\eta) \quad (95)$$

and

$$R^{-\nu} I_{-\nu}(R) = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) r^{-\nu} I_{\nu+n}(r) t^{-\nu} I_{-\nu-n}(t) C_n^\nu(\eta), \quad (96)$$

which are equivalent to GEGENBAUER's addition theorems for the Bessel functions²⁵. In a similar way, by applying Whipple's transformation to the Q -function, putting $\xi = \mu/r$ and letting $\mu \rightarrow \infty$, we obtain from (65)

$$e^{-r\eta\xi} \eta'^{-\nu+\frac{1}{2}} \xi'^{-\nu+\frac{1}{2}} I_{\nu-\frac{1}{2}}(r\eta'\xi') = \frac{\Gamma(\nu)}{\Gamma(\nu + \frac{1}{2})} \sqrt{\frac{2}{r}} \cdot \sum_{n=0}^{\infty} \frac{(\nu + n)n!(-)^n}{(2\nu)_n} I_{\nu+n}(r) C_n^\nu(\eta) C_n^\nu(\xi). \quad (97)$$

The last formula also is due essentially to GEGENBAUER²⁶.

²³ See [1], eq. 3.15(4), and, for the Gegenbauer function of the second kind, WATSON [22]. (The corresponding formula 3.16(32) in [1] is misprinted.)

²⁴ For this and for other generalizations of a more algebraic character see Chapter XI in [1].

²⁵ See [11], p. 365. In a similar fashion HEINE obtained the special case $\nu = 0$ from the classical addition theorem of the Legendre functions.

²⁶ See [11], p. 370. The formula is also a special case of an expansion for the product of two Whittaker functions which we have given in [18].

By similar, if more roundabout procedures we could derive the addition theorems of GRAF and SOMMERFELD²⁷ from the case $\nu + \frac{1}{2} = 0$ of our general addition theorem (62).

Cases with elementary sum. In this paragraph we list some of the special cases of the first real form of the addition theorems where the functions on the left can be expressed in terms of elementary functions. We remind the reader that

$$\begin{aligned}\Omega &= [(\zeta\xi + \eta)^2 - \zeta'^2 \xi'^2]^{\frac{1}{2}} \\ &= [(\xi + \eta\zeta)^2 - \eta'^2 \zeta'^2]^{\frac{1}{2}}\end{aligned}$$

and that all sums are to be interpreted as analytic continuations starting from $w = w^* = 0$, *i.e.* from $\xi = \infty$. We indicate the domains of convergence only in terms of $\mathfrak{S}_{\text{rI}}$, $\mathfrak{S}_{\text{rII}}$ and $\mathfrak{S}_{\text{rIII}}$ and refer to §5.3 for the characterisation of these domains by ξ , η and ζ .

For $\mu = \nu = 0$ we obtain from (63) a generalisation of a formula of FROBENIUS²⁸, namely

$$\frac{1}{2\Omega} \log \frac{\xi\zeta + \eta + \Omega}{\xi\zeta + \eta - \Omega} = \sum_{n=0}^{\infty} (2n+1)(-)^n Q_n(\zeta) Q_n(\xi) P_n(\eta), \quad (98)$$

which is valid in $\mathfrak{S}_{\text{rI}}$. From (64) we have for $\mu = \nu = 0$ the corresponding generalisation of NEUMANN's series, valid in $\mathfrak{S}_{\text{rII}}$:

$$\Omega^{-1} = \sum_{n=0}^{\infty} (2n+1)(-)^n Q_n(\xi) P_n(\eta) P_n(\zeta). \quad (99)$$

If we set $\mu = -\nu$ in the same formula and express the P -functions in terms of Gegenbauer polynomials, again the function on the left becomes elementary and the result is a generalisation of (99), namely

$$\Omega^{-2\nu-1} = e^{-i\nu\pi} \frac{2^{-\nu} \xi'^{-\nu}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(2\nu+1+2n)(-)^n n!}{(2\nu+1)_n} Q_{\nu+n}(\xi) C_n^{\nu+\frac{1}{2}}(\eta) C_n^{\nu+\frac{1}{2}}(\zeta). \quad (100)$$

The same formula is obtained by putting $\mu = \nu$ in (65). It follows that (100) and thus (99) is valid in $\mathfrak{S}_{\text{rII}}$ as well as in $\mathfrak{S}_{\text{rIII}}$. If we set $\mu = 0$ in (65), the Gegenbauer function on the left again can be expressed in elementary terms and we get another generalisation of (99):

$$\begin{aligned}\Omega^{-1}(\Omega + \xi + \eta\zeta)^{-\nu} &= \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\nu+1)} \\ &\cdot \sum_{n=0}^{\infty} \frac{(2\nu+1+2n)(-)^n n!}{(2\nu+1)_n} Q_{\nu+n}(\xi) C_n^{\nu+\frac{1}{2}}(\eta) C_n^{\nu+\frac{1}{2}}(\zeta).\end{aligned} \quad (101)$$

²⁷ See [11], p. 359.

²⁸ See [7], p. 76.

The domain of convergence is $\mathfrak{S}_{r\text{III}}$; an equivalent result was proved under more stringent conditions by OSSICINI²⁹.

We mention that the case $\mu = \frac{1}{2}$ (or $-\frac{1}{2}$) in (63) yields nothing but a disguised form of the classical generating function of the Gegenbauer polynomials³⁰.

5.6. Special values of η .

Cowling's expansions. For $\eta = 0$ we have

$$\Omega = (\xi^2 + \zeta^2 - 1)^{\frac{1}{2}}.$$

Since $C_n^{\nu+\frac{1}{2}}(0)$ is zero for odd values of n and can be expressed in terms of factorials for even values, the expansions (63) and (64) then become

$$\begin{aligned} & (\xi^2 + \zeta^2 - 1)^{-(\nu+1)/2} Q_\nu^\mu \left(\frac{\xi\zeta}{(\xi^2 + \zeta^2 - 1)^{\frac{1}{2}}} \right) \\ &= e^{i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} \sum_{n=0}^{\infty} \frac{(\nu + \frac{1}{2} + 2n)(\nu + \frac{1}{2})_n (-)^n}{n!} Q_{\nu+2n}^\mu(\zeta) Q_{\nu+2n}^{-\mu}(\xi) \end{aligned} \quad (102)$$

and

$$\begin{aligned} & (\xi^2 + \zeta^2 - 1)^{-(\nu+1)/2} P_\nu^\mu \left(\frac{\xi\zeta}{(\xi^2 + \zeta^2 - 1)^{\frac{1}{2}}} \right) \\ &= e^{-i\mu\pi} \frac{2^{\mu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} \sum_{n=0}^{\infty} \frac{(\nu + \frac{1}{2} + 2n)(\nu + \frac{1}{2})_n (-)^n}{n!} Q_{\nu+2n}^\mu(\zeta) Q_{\nu+2n}^{-\mu}(\xi). \end{aligned} \quad (103)$$

These expansions are valid in $\mathfrak{S}_{r\text{I}}$ and $\mathfrak{S}_{r\text{II}}$, respectively (the restrictions on η being disregarded). For integral μ and ν they have been given by COWLING [14] who expressed the coefficients in a less explicit way. The corresponding expansion derived from (65),

$$\begin{aligned} & (\xi^2 + \zeta^2 - 1)^{-(\mu+\nu+1)/2} C_{\mu-\nu}^{\nu+\frac{1}{2}} \left(\frac{\xi}{(\xi^2 + \zeta^2 - 1)^{\frac{1}{2}}} \right) = e^{-i\mu\pi} \frac{2^{-\nu+1} \xi'^{-\mu}}{\Gamma(\mu - \nu + 1) \Gamma(\nu + 1)} \\ & \cdot \sum_{n=0}^{\infty} \frac{(\nu + \frac{1}{2} + 2n)(\frac{1}{2})_n n! (-)^n}{(\nu + 1)_n} Q_{\nu+2n}^\mu(\xi) C_n^{\nu+\frac{1}{2}}(\eta), \end{aligned} \quad (104)$$

seems to be new.

Further generalisations of Frobenius' and Neumann's expansions. For $\eta = \pm 1$ we have

$$\Omega = \xi \pm \zeta,$$

²⁹ See [21], where also a number of special cases are given.

³⁰ This was apparently not realized by SNOW [7], who on p. 81 writes down and proves this expansion in terms of Q -functions.

and $C_n^{\nu+1}(\eta)$ can again be expressed in terms of factorials. Thus for $\eta = -1$ the expansions (63) and (64) reduce to

$$(\xi - \zeta)^{-\nu-1} Q_\nu^\mu \left(\frac{\xi\zeta - 1}{\xi - \zeta} \right) = e^{i\mu\pi} \frac{2^{\nu+1}\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\nu - \mu + 1)} \cdot \sum_{n=0}^{\infty} \frac{(\nu + \frac{1}{2} + n)(2\nu + 1)_n}{n!} Q_{\nu+n}^{-\mu}(\xi) Q_{\nu+n}^\mu(\zeta) \quad (105)$$

and

$$(\xi - \zeta)^{-\nu-1} P_\nu^\mu \left(\frac{\xi\zeta - 1}{\xi - \zeta} \right) = e^{i\mu\pi} \frac{2^{\nu+1}\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\nu - \mu + 1)} \cdot \sum_{n=0}^{\infty} \frac{(\nu + \frac{1}{2} + n)(2\nu + 1)_n}{n!} Q_{\nu+n}^{-\mu}(\xi) P_{\nu+n}^\mu(\zeta). \quad (106)$$

The domains of convergence are again $\mathfrak{S}_{rI}$ and $\mathfrak{S}_{rII}$, without the conditions pertaining to η . The analytic continuation of the functions on the left starts at ζ .

Similarly we obtain from (65) for $\eta = 1$ the following formula valid in $\mathfrak{S}_{rIII}$:

$$(\xi - \zeta)^{-\mu-\nu-1} = e^{-i\mu\pi} \frac{2^\nu\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu + \nu + 1)} \xi'^{-\mu} \cdot \sum_{n=0}^{\infty} (2\nu + 2n + 1) Q_{\nu+n}^\mu(\xi) C_n^{\nu+\frac{1}{2}}(\zeta). \quad (107)$$

A number of special cases of this formula are known in several connections. For $\mu = -\nu$ the formula is a special instance of SZEGÖ's generalisation of Neumann's formula³¹, which is in turn obtained by putting $\mu = \nu = 0$. For $\mu = \frac{1}{2}$ the formula is equivalent to the generating function of the Gegenbauer polynomials. The case $\nu = -\frac{1}{2}$ is also known³².

6. An application of the expansion theorem to Bessel functions

Another solution of (3), different from the type considered above, is the function

$$g_\zeta(x, y) = (\zeta x)^{-\mu} Z_\mu(\zeta x) (\zeta y)^{-\nu} I_\nu(\zeta y),$$

where Z_μ denotes any linear combination of J_μ and Y_μ and ζ is an arbitrary complex number not on the negative real axis. In terms of ξ and η this solution can be written

$$g_\zeta(x, y) = \left(\frac{\xi\xi'}{\xi - \eta} \right)^{-\mu} Z_\mu \left(\frac{\xi\xi'}{\xi - \eta} \right) \left(\frac{\xi\eta'}{\xi - \eta} \right)^{-\nu} J_\nu \left(\frac{\xi\eta'}{\xi - \eta} \right). \quad (108)$$

³¹ See [8], Theorem 9.2.1.

³² See [6], p. 214.

One easily verifies that for admissible values of μ and ν the function

$$G_{\zeta}(w, w^*) = g_{\zeta}(x, y) \quad (109)$$

is in $C(\mathfrak{D})$ for a neighborhood $\mathfrak{D}$ of $w = w^* = 0$ and has therefore a L.G. expansion. Since

$$G_{\zeta}(w, 0) = \left(\frac{\zeta}{1-w}\right)^{-\mu} Z_{\mu}\left(\frac{\zeta}{1-w}\right) \left(\frac{\zeta w}{1-w}\right)^{-\nu} J_{\nu}\left(\frac{\zeta w}{1-w}\right) \quad (110)$$

is regular for $|w| < 1$, this expansion can be expected to converge uniformly in every closed subdomain of $\mathfrak{S}_1 = \mathfrak{R}_1 \times \mathfrak{R}_1$ to the analytic continuation of (108) starting from $w = w^* = 0$, *i.e.* from $\xi = \infty$.

It remains to determine the coefficients of this expansion. The generating function is

$$\begin{aligned} H_{\zeta}(w, 0) &= (1-w)^{-\mu-\nu-1} G_{\zeta}(w, 0) \\ &= \zeta^{-\mu-\nu} w^{-\nu} \frac{1}{1-w} J_{\nu}\left(\frac{\zeta w}{1-w}\right) Z_{\mu}\left(\frac{\zeta}{1-w}\right). \end{aligned} \quad (111)$$

We shall show that the coefficients of the Taylor expansion of this function can be represented as finite linear combinations of Bessel functions and polynomials in ζ . Using a special case of Gegenbauer's addition theorem³³ we have

$$Z_{\mu}\left(\frac{\zeta}{1-w}\right) = (\zeta w)^{-\mu} 2^{\mu} \Gamma(\mu) \sum_{m=0}^{\infty} (-)^m \frac{(\mu+m)(2\mu)_m}{m!} Z_{\mu+m}(\zeta) J_{\mu+m}\left(\frac{\zeta w}{1-w}\right).$$

Inserting this into (111) and expanding the Bessel functions in terms of the common argument, we find

$$\begin{aligned} H_{\zeta}(w, 0) &= \frac{\zeta^{-\mu} 2^{-\nu} \Gamma(\mu)}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{m=0}^{\infty} \frac{(-)^m (\mu+m)(2\mu)_m}{m! (\mu+1)_m} \left(\frac{\zeta}{2}\right)^m Z_{\mu+m}(\zeta) w^m \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{(-)^k (\mu+\nu+1+m+k)_k}{(\nu+1)_k (\mu+1+m)_k k!} \left(\frac{\zeta}{2}\right)^k \left(\frac{w}{1-w}\right)^{2k}. \end{aligned}$$

By using the binomial series and rearranging, the sum with respect to k can be transformed into

$$\sum_{r=0}^{\infty} \frac{(\mu+\nu+1+m)_r}{r!} w^r {}_2F_3 \left[\begin{matrix} -\frac{r}{2}, \frac{1-r}{2}; & -\zeta^2 \\ \nu+1, \mu+m+1, \mu+\nu+m+1 \end{matrix} \right].$$

After some simplification we thus obtain

$$H_{\zeta}(w, 0) = \frac{\zeta^{-\mu} 2^{-\nu}}{\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(\mu+\nu+1)_r}{r!} Z_{\mu,\nu,r}(\zeta) w^r, \quad (112)$$

³³ See [11], p. 366.

where

$$Z_{\mu,\nu,r}(\zeta) = \sum_{m=0}^r \frac{(2\mu)_m (-r)_m}{(\mu)_m (\mu + \nu + 1)_m m!} \left(\frac{\zeta}{2}\right)^m \\ {}_2F_3 \left[\begin{matrix} \frac{m-r}{2}, & \frac{l+m-r}{2}, & -\zeta^2 \\ \nu+1, & \mu+m+1, & \mu+\nu+m+1 \end{matrix} \right] Z_{\mu+m}(\zeta). \quad (113)$$

The expansion theorem now yields

$$\left(\frac{\zeta\eta'}{\xi-\eta}\right)^{-\nu} J_{\nu} \left(\frac{\zeta\eta'}{\xi-\eta}\right) Z_{\mu} \left(\frac{\zeta\xi'}{\xi-\eta}\right) = e^{-i\mu\pi} \frac{\Gamma(\nu + \frac{1}{2})(\xi - \eta)^{\nu+1}}{\sqrt{\pi} \Gamma(\mu + \nu + 1) \Gamma(\nu + 1)} \\ \cdot \sum_{n=0}^{\infty} (2\nu + 2n + 1) Z_{\mu,\nu,n}(\zeta) Q_{\nu+n}^{\mu}(\xi) C_n^{\nu+\frac{1}{2}}(\eta). \quad (114)$$

In this equation the function symbols on the left must be interpreted as representing the analytic continuations of the functions defined by them in the conventional way at $w = w^* = 0$, *i.e.* at $\xi = \infty$. (For values of ζ with $\operatorname{Re} \zeta > 0$ this remark is unnecessary.) A discussion of the condition $(w, w^*) \in \mathfrak{S}_1$ similar to that of §5.3 shows that this expansion is valid for arbitrary values of ξ not on the real axis between -1 and $+1$ and for $|\eta \pm \eta'| > |\xi - \xi'|$, *i.e.* for η in the interior of $E(\xi)$.

Putting in (114) $Z_{\mu} = J_{\mu}$, multiplying by $\zeta^{-\mu}$ and letting $\zeta \rightarrow 0$ yields (107), since in $Z_{\mu,\nu,r}(\zeta)$ all terms except the first vanish.

7. An application to the hypergeometric function

A further solution of (3) which satisfies the conditions of the expansion theorem is the function³⁴

$$g^{\alpha}(x, y) = (x^2 + y^2)^{\alpha} {}_2F_1 \left[\begin{matrix} -\alpha, & \mu + \nu + \alpha + 1; & \frac{y^2}{x^2 + y^2} \\ \nu + 1 \end{matrix} \right] \\ = \left(\frac{\xi + \eta}{\xi - \eta}\right)^{\alpha} {}_2F_1 \left[\begin{matrix} -\alpha, & \mu + \nu + \alpha + 1; & -\frac{\eta'^2}{\xi^2 - \eta^2} \\ \nu + 1 \end{matrix} \right] \quad (115) \\ = \left(\frac{\xi'}{\xi - \eta}\right)^{2\alpha} {}_2F_1 \left[\begin{matrix} -\alpha, & -\mu - \alpha; & \eta'^2/\xi'^2 \\ \nu + 1 \end{matrix} \right].$$

It is readily verified that

$$G^{\alpha}(w, w^*) = g^{\alpha}(x, y)$$

³⁴ See [18], §2.

is regular in $\mathfrak{S}_1$. The generating function

$$H^\alpha(w, 0) = (1 - w)^{-\mu-\nu-1} G^\alpha(w, 0)$$

is given by

$$\begin{aligned} H^\alpha(w, 0) &= (1 - w)^{-\mu-\nu-2\alpha-1} {}_2F_1 \left[\begin{matrix} -\alpha, & -\mu - \alpha; & w^2 \\ & \nu + 1 \end{matrix} \right] \\ &= (1 + w)^{\mu+\nu+2\alpha+1} {}_2F_1 \left[\begin{matrix} \nu + \alpha + 1, & \mu + \nu + \alpha + 1; & w^2 \\ & \nu + 1 \end{matrix} \right] \end{aligned} \quad (116)$$

and can be represented in elementary terms for certain values of α ; in particular we have

$$\begin{aligned} H^0(w, 0) &= (1 - w)^{-\mu-\nu-1}, \\ H^{-\nu-1}(w, 0) &= (1 + w)^{\mu-\nu-1}, \\ H^{-\mu-\nu-1}(w, 0) &= (1 + w)^{-\mu-\nu-1}. \end{aligned} \quad (117)$$

For general α one easily finds

$$H^\alpha(w, 0) = \sum_{n=0}^{\infty} \frac{(\mu + \nu + 2\alpha + 1)_n}{n!} w^n \cdot {}_4F_3 \left[\begin{matrix} -\frac{n}{2}, & \frac{1-n}{2}, & -\alpha, -\mu - \alpha; \\ -\frac{\mu + \nu + 2\alpha + n}{2}, & \frac{-\mu - \nu - 2\alpha + n + 1}{2}, & \nu + 1 \end{matrix} \right]. \quad (118)$$

This leads to the L.G. expansion

$$\begin{aligned} \left(\frac{\xi'}{\xi - \eta} \right)^{2\alpha} {}_2F_1 \left[\begin{matrix} -\alpha, & -\mu - \alpha; & \eta'^2/\xi'^2 \\ & \nu + 1 \end{matrix} \right] \\ = e^{i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\nu + \mu + 1)} (\xi - \eta)^{\mu+\nu+1} \xi'^{-\mu} \\ \cdot \sum_{n=0}^{\infty} \frac{(\nu + n + \frac{1}{2})(\mu + \nu + 2\alpha + 1)_n}{(\mu + \nu + 1)_n} {}_4F_3[\dots] Q_{\nu+n}^\mu(\xi) C_n^{\nu+\frac{1}{2}}(\eta), \end{aligned} \quad (119)$$

where the hypergeometric sum ${}_4F_3$ has the same parameters as in (118). The conditions of validity are identical with those given for (114).

The cases where $H^\alpha(w, 0)$ is elementary lead once again to the generalized Neumann expansion (107).

Appendix: A list of definite integrals.

In this appendix we list some definite integrals which follow from our expansions by the orthogonality relations of the Gegenbauer polynomials. It is evident that these formulas contain a large number of known formulas as special cases; we do not attempt, however, to establish individual links.³⁵

We have from §5.3, supposing $\operatorname{Re} \nu > -\frac{1}{2}$ for convergence,

$$\begin{aligned} \int_{-1}^1 \Omega^{-\nu-1} L_{\nu+\alpha}^\mu \left(\frac{\zeta \xi + \eta}{\Omega} \right) C_{\alpha}^{\nu+\frac{1}{2}} \left(\frac{\xi + \eta \zeta}{\Omega} \right) (1 - \eta^2)^\nu d\eta \\ = e^{-i\mu\pi} \frac{2^{1-\nu} \Gamma(2\nu + 1 + \alpha)}{\Gamma(\mu + \nu + 1) \Gamma(\nu + \frac{1}{2}) \Gamma(\alpha + 1)} L_{\nu+\alpha}^\mu(\zeta) Q_\nu^\mu(\xi), \end{aligned} \quad (120)$$

$$\int_{-1}^1 \Omega^{-\nu-1} Q_\nu^\mu \left(\frac{\zeta \xi + \eta}{\Omega} \right) (1 - \eta^2)^\nu d\eta = e^{i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} Q_\nu^\mu(\zeta) Q_\nu^{-\mu}(\xi), \quad (121)$$

$$\int_{-1}^1 \Omega^{-\nu-1} P_\nu^\mu \left(\frac{\zeta \xi + \eta}{\Omega} \right) (1 - \eta^2)^\nu d\eta = e^{i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} P_\nu^\mu(\zeta) Q_\nu^{-\mu}(\xi), \quad (122)$$

$$\begin{aligned} \int_{-1}^1 \Omega^{-\mu-\nu-1} C_{\mu-\nu}^{\nu+\frac{1}{2}} \left(\frac{\xi + \eta \zeta}{\Omega} \right) (1 - \eta^2)^\nu d\eta \\ = e^{-i\mu\pi} \frac{2^{-\nu+1} \sqrt{\pi}}{\Gamma(\mu - \nu + 1) \Gamma(\nu + \frac{1}{2})} \xi'^{-\mu} Q_\nu^\mu(\xi). \end{aligned} \quad (123)$$

We note that

$$\begin{aligned} \Omega &= [(\zeta \xi + \eta)^2 - \zeta'^2 \xi'^2]^{\frac{1}{2}} \\ &= [(\xi + \eta \zeta)^2 - \eta'^2 \zeta'^2]^{\frac{1}{2}}. \end{aligned}$$

From §5.4 we get ($\operatorname{Re} \nu > -1$)

$$\int_{-1}^1 \rho_1'^{-\nu} P_\mu^\nu(\rho_1) (1 - \eta^2)^{\nu-\frac{1}{2}} d\eta = 2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2}) \rho'^{-\nu} P_\mu^{-\nu}(\rho) \tau'^{-\nu} P_\mu^\nu(\tau), \quad (124)$$

$$\int_{-1}^1 \rho_1'^{-\nu} Q_\mu^\nu(\rho_1) (1 - \eta^2)^{\nu-\frac{1}{2}} d\eta = 2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2}) \rho'^{-\nu} P_\mu^{-\nu}(\rho) \tau'^{-\nu} Q_\mu^\nu(\tau), \quad (125)$$

³⁵ We may rightfully claim to have proved these integral relations, as well as all their special cases, since at no place in our investigation have we used integrals. In this respect our analysis differs from that of most previous investigators who, following JACOBI, make an integral representation the starting point of their proofs of the addition theorems.

where

$$\rho_1 = \rho\tau + \rho'\tau'\eta.$$

The main results of §6 and §7 yield, if $\operatorname{Re} \nu > -\frac{1}{2}$,

$$\begin{aligned} \int_{-1}^1 (\xi\eta')^{-\nu} J_\nu\left(\frac{\xi\eta'}{\xi-\eta}\right) Z_\mu\left(\frac{\xi\xi'}{\xi-\eta}\right) \frac{(1-\eta^2)^\nu}{(\xi-\eta)^{\mu+\nu+1}} d\eta \\ = \frac{2e^{-i\mu\pi}}{\Gamma(\mu+\nu+1)} Z_\mu(\xi) Q_\nu^\mu(\xi), \end{aligned} \quad (126)$$

$$\begin{aligned} \int_{-1}^1 \left(\frac{\xi'}{\xi-\eta}\right)^{2\alpha} {}_2F_1\left[\begin{matrix} -\alpha, & -\mu-\alpha; & \frac{\eta'^2}{\xi'^2} \\ \nu+1 \end{matrix}\right] \frac{(1-\eta^2)^\nu}{(\xi-\eta)^{\mu+\nu+1}} d\eta \\ = e^{-i\mu\pi} \frac{2^{\nu+1} \Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} \xi'^{-\mu} Q_\nu^\mu(\xi). \end{aligned} \quad (127)$$

It seems worth noticing that the functions on the left of (123) and (127) are independent of ξ and α , respectively.

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