*Plane Strain of a Net Formed by Inextensible Cords**

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1. Introduction. In this paper we consider the mechanics of a plane net formed by two families of parallel inextensible cords. We assume that, in the undeformed state of the net, the cords are straight. We assume also that the cords are perfectly flexible and thin in comparison with the distance between adjacent cords of a family, and that the deformation is such that when the net is deformed no two cords of a family are brought into contact. Furthermore, we assume that intersecting cords of the two families cannot move relative to each other at their points of intersection, as a result of the deformation. Such a net provides an idealisation of certain types of textile fabric.

The scope of the present paper is restricted to plane deformations of the net, although it is intended in a later paper to discuss the mechanics of the deformation when this restriction is removed[†]. It will be appreciated that, in certain cases, the predicted deformations, although kinematically and statically possible, may nevertheless not be realised because of their instability.

In Secs. 2 and 3, we consider the pure homogeneous deformation of a sheet of the ideal fabric and it is shown that, as a result of the constraints imposed by the inextensibility of the cords, the principal directions for the pure homogeneous deformation bisect the angles between the cords.

In Sec. 4 it is shown that, for a general plane deformation of the sheet, the displacements undergone by the various points of the sheet must satisfy differential equations of the hyperbolic type[‡].

In the subsequent sections it is shown how these equations may be solved for

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† It appears that certain geometrical aspects of the latter problem have been discussed by TCHEBICHEF (C. R. Ass. Française pour l'Avancement des Sciences, 7^e session, p. 154 (1878)). I have not been able, however, to find more than a brief abstract of his work.

‡ See also, A. Voss, Math. Ann. **19**, 1 (1882).

various types of boundary condition. Thus, in Sec. 5 we consider that the displacements are specified on the edges of a sheet of our ideal fabric and it is shown how the resulting displacements may be determined throughout the sheet. In Secs. 6 to 10 the equations of motion, boundary conditions and stress-deformation relations are discussed and in Sec. 11 it is shown how these may be used in order to determine the deformation which results from the application of specified edge tractions to a sheet of the fabric of arbitrary contour.

Throughout the paper, the analysis is carried out on the assumption that the net can be treated as a plane continuum which is homogeneous in its undeformed state and has two directions of inextensibility. Consequently, the results of the analysis will provide only average values for the displacements and cord tensions taken over regions large compared with the distance between adjacent cords in each family.

FIG. 1

2. Pure Homogeneous Deformation. We consider a plane sheet of the ideal fabric described in the introduction, consisting of two plane sets of inextensible cords intersecting at an angle 2α . We assume that the cords in each set are uniformly spaced and that the distance between adjacent cords in each set is d .

Now, let us consider a rectangle of this fabric, the edges of which are parallel to the bisectors of the angles between the cords, to be subjected to a pure homogeneous deformation with principal directions parallel to the edges of the rectangle, as shown in Fig. 1. We take as a reference system, a rectangular Cartesian coordinate system x_1, x_2 parallel to the bisectors of the angles between the two sets of cords. If X_1, X_2 are the coordinates of a point of the fabric before deformation and x_1, x_2 are its coordinates after the pure homogeneous deformation, then we have

$$x_1 = \lambda_1 X_1 \quad \text{and} \quad x_2 = \lambda_2 X_2, \quad (2.1)$$

where λ_1 and λ_2 are constants—the extension ratios for the pure homogeneous deformation. It is readily seen that as a result of this pure homogeneous deformation, the angle between the two sets of cords is changed to 2β , where

$$\cos \beta = \lambda_1 \cos \alpha \quad \text{and} \quad \sin \beta = \lambda_2 \sin \alpha. \quad (2.2)$$

It follows from (2.2) that λ_1 and λ_2 are related by the equation

$$\lambda_1^2 \cos^2 \alpha + \lambda_2^2 \sin^2 \alpha = 1. \quad (2.3)$$

3. General Plane Deformation; Kinematic Considerations. We shall consider a plane sheet of the fabric described in the previous sections to be subjected to a deformation in which each point of the sheet is displaced in the plane of the sheet. We take as our reference system a rectangular Cartesian coordinate system x_1, x_2 in the plane of the sheet, the axes of which do not necessarily bisect the angles between the two sets of cords. Suppose that in a deformation of the sheet points which are initially at X_1, X_2 in the coordinate system x_1, x_2 move to x_1, x_2 in the same coordinate system. Then, the deformation undergone by the sheet may be completely described by the functional dependences of x_1 and x_2 on X_1 and X_2 , thus:

$$x_1 = x_1(X_1, X_2) \quad \text{and} \quad x_2 = x_2(X_1, X_2). \quad (3.1)$$

If dS is an element of length in the undeformed sheet joining the points at (X_1, X_2) and $(X_1 + dX_1, X_2 + dX_2)$ and ds is the corresponding element of length in the deformed sheet joining (x_1, x_2) and $(x_1 + dx_1, x_2 + dx_2)$, then

$$(dS)^2 = (dX_1)^2 + (dX_2)^2$$

and

$$(ds)^2 = (dx_1)^2 + (dx_2)^2. \quad (3.2)$$

We have, from equations (3.1),

$$dx_1 = \frac{\partial x_1}{\partial X_1} dX_1 + \frac{\partial x_1}{\partial X_2} dX_2 \quad \text{and} \quad dx_2 = \frac{\partial x_2}{\partial X_1} dX_1 + \frac{\partial x_2}{\partial X_2} dX_2. \quad (3.3)$$

Substituting from (3.3) in (3.2), we have

$$(ds)^2 = g_{11}(dX_1)^2 + 2g_{12}(dX_1 dX_2) + g_{22}(dX_2)^2, \quad (3.4)$$

where

$$g_{11} = \left(\frac{\partial x_1}{\partial X_1} \right)^2 + \left(\frac{\partial x_2}{\partial X_1} \right)^2, \quad g_{22} = \left(\frac{\partial x_1}{\partial X_2} \right)^2 + \left(\frac{\partial x_2}{\partial X_2} \right)^2$$

and

$$g_{12} = \frac{\partial x_1}{\partial X_1} \frac{\partial x_1}{\partial X_2} + \frac{\partial x_2}{\partial X_1} \frac{\partial x_2}{\partial X_2}. \quad (3.5)$$

Now, if (l_1, l_2) are the direction-cosines in the coordinate system x_1, x_2 of the element of length dS in the undeformed sheet, we have

$$l_1 = dX_1/dS \quad \text{and} \quad l_2 = dX_2/dS. \quad (3.6)$$

From (3.4) and (3.5), we obtain

$$\left(\frac{ds}{dS} \right)^2 = g_{11} l_1^2 + 2g_{12} l_1 l_2 + g_{22} l_2^2. \quad (3.7)$$

If equations (3.1) take the form

$$x_1 = \lambda_1 X_1 \quad \text{and} \quad x_2 = \lambda_2 X_2, \quad (3.8)$$

where λ_1 and λ_2 are constants, then the deformation is pure homogeneous and the principal directions for the deformation are parallel to the axes x_1 and x_2 . It can readily be shown that this deformation is not a possible deformation of the sheet unless the axes x_1 and x_2 bisect the angles between the sets of cords. Let (l'_1, l'_2) and (l''_1, l''_2) be the direction-cosines of the two sets of cords in the coordinate system x_1, x_2 , in the undeformed state of the sheet. Then, since the cords are inextensible, we have, from (3.7),

$$g_{11} l_1'^2 + 2g_{12} l'_1 l'_2 + g_{22} l_2'^2 = 1$$

and

$$g_{11} l_1''^2 + 2g_{12} l''_1 l''_2 + g_{22} l_2''^2 = 1. \quad (3.9)$$

Also, from equations (3.8) and (3.5) we have

$$g_{11} = \lambda_1^2, \quad g_{22} = \lambda_2^2 \quad \text{and} \quad g_{12} = 0. \quad (3.10)$$

Introducing (3.10) into equations (3.9), we have

$$\lambda_1^2 l_1'^2 + \lambda_2^2 l_2'^2 = 1$$

and

$$\lambda_1^2 l_1''^2 + \lambda_2^2 l_2''^2 = 1. \quad (3.11)$$

We have also,

$$l_1'^2 + l_2'^2 = l_1''^2 + l_2''^2 = 1. \quad (3.12)$$

From equations (3.11) and (3.12), we obtain

$$l_1'^2 = l_1''^2 = \frac{1 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}$$

and

$$l_2'^2 = l_2''^2 = \frac{1 - \lambda_1^2}{\lambda_2^2 - \lambda_1^2}. \quad (3.13)$$

Whence, we have

$$l'_1 = \pm l''_1 \quad \text{and} \quad l'_2 = \pm l''_2. \quad (3.14)$$

It follows that the principal directions for the pure homogeneous deformation must bisect the angles between the two sets of cords.

Returning now to the arbitrary plane deformation described by equations (3.1), we note that the deformation of any infinitesimal element of area of the sheet may be considered to be the resultant of a rotation of the element about an axis normal to the plane of the sheet, followed by a pure homogeneous deformation of the element. From our result that the only possible pure homogeneous deformation is one for which the principal directions bisect the angles between the two sets of cords, it follows that at each point of the sheet the principal directions in the deformed sheet bisect the angles between the two sets of cords. It will, of course, be appreciated that the magnitudes of the principal extension ratios and the angle through which the infinitesimal element of area is rotated may vary from point to point of the sheet.

These variations are, however, limited by the necessity that the quantities g_{11} , g_{12} and g_{22} , defined by equations (3.5), which may be regarded as components of a two-dimensional strain tensor in the coordinate system x_1 , x_2 shall satisfy certain compatibility conditions.

4. The Effect of the Inextensible Cords. In the plane sheet of fabric under consideration, we have, in the undeformed state, two sets of parallel inextensible cords intersecting at an angle 2α . We choose the axes of our reference co-ordinate system (x_1 , x_2) parallel to the bisectors of the angles between the two sets of cords. Then, the direction cosines of the two sets of cords are $(\cos \alpha, \sin \alpha)$ and $(\cos \alpha, -\sin \alpha)$. Since the cords are inextensible, we have, from equation (3.9),

$$g_{11} \cos^2 \alpha + 2g_{12} \cos \alpha \sin \alpha + g_{22} \sin^2 \alpha = 1$$

and

$$g_{11} \cos^2 \alpha - 2g_{12} \cos \alpha \sin \alpha + g_{22} \sin^2 \alpha = 1. \quad (4.1)$$

Whence,

$$g_{11} \cos^2 \alpha + g_{22} \sin^2 \alpha = 1 \quad (4.2)$$

and

$$g_{12} = 0. \quad (4.3)$$

With the definition of g_{11} , g_{22} and g_{12} given in equations (3.5), we have, from (4.2) and (4.3),

$$\frac{\partial x_1}{\partial X_1} \frac{\partial x_1}{\partial X_2} + \frac{\partial x_2}{\partial X_1} \frac{\partial x_2}{\partial X_2} = 0$$

and

$$\left[\left(\frac{\partial x_1}{\partial X_1} \right)^2 + \left(\frac{\partial x_2}{\partial X_1} \right)^2 \right] \cos^2 \alpha + \left[\left(\frac{\partial x_1}{\partial X_2} \right)^2 + \left(\frac{\partial x_2}{\partial X_2} \right)^2 \right] \sin^2 \alpha = 1. \quad (4.4)$$

We re-write these equations as

$$2 \left(\frac{\partial x_1}{\partial X_1} \frac{\partial x_1}{\partial X_2} + \frac{\partial x_2}{\partial X_1} \frac{\partial x_2}{\partial X_2} \right) \cos \alpha \sin \alpha = 0$$

and

$$\left[\left(\frac{\partial x_1}{\partial X_1} \right)^2 \cos^2 \alpha + \left(\frac{\partial x_1}{\partial X_2} \right)^2 \sin^2 \alpha \right] + \left[\left(\frac{\partial x_2}{\partial X_1} \right)^2 \cos^2 \alpha + \left(\frac{\partial x_2}{\partial X_2} \right)^2 \sin^2 \alpha \right] = 1. \quad (4.5)$$

Adding these two equations, we have

$$\left(\frac{\partial x_1}{\partial X_1} \cos \alpha + \frac{\partial x_1}{\partial X_2} \sin \alpha \right)^2 + \left(\frac{\partial x_2}{\partial X_1} \cos \alpha + \frac{\partial x_2}{\partial X_2} \sin \alpha \right)^2 = 1. \quad (4.6)$$

Subtracting the equations (4.5) we have

$$\left(\frac{\partial x_1}{\partial X_1} \cos \alpha - \frac{\partial x_1}{\partial X_2} \sin \alpha \right)^2 + \left(\frac{\partial x_2}{\partial X_1} \cos \alpha - \frac{\partial x_2}{\partial X_2} \sin \alpha \right)^2 = 1. \quad (4.7)$$

Writing

$$X_1 = Y_1 \cos \alpha \quad \text{and} \quad X_2 = Y_2 \sin \alpha \quad (4.8)$$

and

$$Y_1 = Z_1 + Z_2 \quad \text{and} \quad Y_2 = Z_1 - Z_2, \quad (4.9)$$

we may re-write equations (4.6) and (4.7) as

$$\left(\frac{\partial x_1}{\partial Z_1} \right)^2 + \left(\frac{\partial x_2}{\partial Z_1} \right)^2 = 1$$

and

$$\left(\frac{\partial x_1}{\partial Z_2} \right)^2 + \left(\frac{\partial x_2}{\partial Z_2} \right)^2 = 1. \quad (4.10)$$

Differentiating the first of equations (4.10) with respect to Z_2 and the second with respect to Z_1 , we obtain

$$\frac{\partial x_1}{\partial Z_1} \frac{\partial^2 x_1}{\partial Z_1 \partial Z_2} + \frac{\partial x_2}{\partial Z_1} \frac{\partial^2 x_2}{\partial Z_1 \partial Z_2} = 0$$

and

$$\frac{\partial x_1}{\partial Z_2} \frac{\partial^2 x_1}{\partial Z_1 \partial Z_2} + \frac{\partial x_2}{\partial Z_2} \frac{\partial^2 x_2}{\partial Z_1 \partial Z_2} = 0. \quad (4.11)$$

Eliminating $\partial^2 x_2 / \partial Z_1 \partial Z_2$ from equations (4.11), we obtain

$$\left(\frac{\partial x_2}{\partial Z_2} \frac{\partial x_1}{\partial Z_1} - \frac{\partial x_2}{\partial Z_1} \frac{\partial x_1}{\partial Z_2} \right) \frac{\partial^2 x_1}{\partial Z_1 \partial Z_2} = 0. \quad (4.12)$$

Whence,

$$\frac{\partial^2 x_1}{\partial Z_1 \partial Z_2} = 0, \quad (4.13)$$

or

$$\frac{\partial x_2}{\partial Z_2} \frac{\partial x_1}{\partial Z_1} - \frac{\partial x_2}{\partial Z_1} \frac{\partial x_1}{\partial Z_2} = 0. \quad (4.14)$$

From (4.14), (4.8) and (4.9), we have

$$\frac{\partial x_1}{\partial X_1} \frac{\partial x_2}{\partial X_2} - \frac{\partial x_1}{\partial X_2} \frac{\partial x_2}{\partial X_1} = 0. \quad (4.15)$$

Now the expression on the left-hand side of equation (4.15) is the ratio between the area of an element of the sheet in its deformed and undeformed states. Since this cannot be zero for a real material, we need only consider the result (4.13).

In a similar manner we can show that

$$\frac{\partial^2 x_2}{\partial Z_1 \partial Z_2} = 0. \quad (4.16)$$

5. Solution of the Equation. Equation (4.13) has the general solution

$$x_1 = f_1(Z_1) + g_1(Z_2), \quad (5.1)$$

where f_1 and g_1 are arbitrary functions. Similarly, equation (4.16) has the general solution

$$x_2 = f_2(Z_1) + g_2(Z_2), \quad (5.2)$$

where f_2 and g_2 are arbitrary functions. However, in view of the relations (4.10), f_1 and f_2 are related and g_1 and g_2 are related by the equations

$$f_1'^2 + f_2'^2 = 1 \quad \text{and} \quad g_1'^2 + g_2'^2 = 1, \quad (5.3)$$

where f_1', f_2', g_1' and g_2' denote $df_1(Z_1)/dZ_1$, $df_2(Z_1)/dZ_1$, $dg_1(Z_2)/dZ_2$ and $dg_2(Z_2)/dZ_2$ respectively.

We see from equations (4.8) and (4.9) that if $Z_1 = \text{const.}$

$$X_1 \sin \alpha + X_2 \cos \alpha = \text{const.} \quad (5.4)$$

and if $Z_2 = \text{const.}$

$$X_1 \sin \alpha - X_2 \cos \alpha = \text{const.} \quad (5.5)$$

Thus, $Z_1 = \text{const.}$ represents the family of cords which is inclined at the angle $-\alpha$ to the X_1 axis and $Z_2 = \text{const.}$ represents the family of cords which is inclined at the angle α to the X_1 axis.

The two families of straight lines $Z_1 = \text{const.}$ and $Z_2 = \text{const.}$ are the families of characteristics for each of the hyperbolic differential equations (4.13) and (4.16).

FIG. 2

In Fig. 2, $ABCD$ is a parallelogram bounded by the characteristic lines $Z_1 = Z_1'$, $Z_1 = Z_1''$, $Z_2 = Z_2'$ and $Z_2 = Z_2''$. It is noted that the diagonals of this parallelogram are parallel to the coordinate axes x_1 and x_2 . If x_1 is specified on BC , *i.e.* if $x_1(Z_1, Z_2')$ is given in the range $Z_1 = Z_1'$ to Z_1'' , then, from (5.1), we have

$$x_1(Z_1, Z_2') = f_1(Z_1) + g_1(Z_2'). \quad (5.6)$$

Again, if x_1 is specified on AB , *i.e.* $x_1(Z_1', Z_2)$ is given in the range $Z_2 = Z_2'$ to Z_2'' , then, from (5.1)

$$x_1(Z_1', Z_2) = f_1(Z_1') + g_1(Z_2). \quad (5.7)$$

Adding (5.6) and (5.7), we have

$$x_1(Z_1, Z_2') + x_1(Z_1', Z_2) = f_1(Z_1) + g_1(Z_2) + f_1(Z_1') + g_1(Z_2'), \quad (5.8)$$

for Z_1 in the range Z_1' to Z_1'' and Z_2 in the range Z_2' to Z_2'' . Now, at the point B , $x_1 = x_1(Z_1', Z_2)$ and, from (5.1), we have

$$x_1(Z_1', Z_2) = f_1(Z_1') + g_1(Z_2). \quad (5.9)$$

From (5.8) and (5.9), we obtain

$$f_1(Z_1) + g_1(Z_2) = x_1(Z_1, Z_2') + x_1(Z_1', Z_2) - x_1(Z_1', Z_2'). \quad (5.10)$$

Thus, from (5.1), x_1 is determined throughout the parallelogram $ABCD$ in terms of the known quantities on the right-hand side of equation (5.10).

From (5.6) and (5.7), we see that if $x_1(Z_1, Z'_2)$ and $x_1(Z'_1, Z_2)$ are specified, f'_1 and g'_1 are determined by

$$f'_1 = \frac{dx_1(Z_1, Z'_2)}{dZ_1} \quad \text{and} \quad g'_1 = \frac{dx_1(Z'_1, Z_2)}{dZ_2} \quad (5.11)$$

throughout the parallelogram $ABCD$. Thus, from (5.3), we have

$$f'_2 = \pm(1 - f_1'^2)^{\frac{1}{2}} \quad \text{and} \quad g'_2 = \pm(1 - g_1'^2)^{\frac{1}{2}}. \quad (5.12)$$

Two of the four possible combinations of signs in (5.12) will be excluded by the condition that

$$\frac{\partial x_1}{\partial X_1} \frac{\partial x_2}{\partial X_2} - \frac{\partial x_1}{\partial X_2} \frac{\partial x_2}{\partial X_1} > 0, \quad (5.13)$$

$$\text{i.e.} \quad \frac{1}{2 \sin \alpha \cos \alpha} \left(\frac{\partial x_1}{\partial Z_2} \frac{\partial x_2}{\partial Z_1} - \frac{\partial x_1}{\partial Z_1} \frac{\partial x_2}{\partial Z_2} \right) > 0, \quad (5.14)$$

$$\text{i.e.} \quad \frac{1}{2 \sin \alpha \cos \alpha} (g'_1 f'_2 - f'_1 g'_2) > 0. \quad (5.15)$$

For example, if, taking both of the positive signs in (5.12), the condition (5.15) is satisfied, then it cannot be satisfied if both of the negative signs are taken. Again, if the condition (5.15) is satisfied, when in (5.12) the negative sign is taken for f'_2 and the positive sign for g'_2 , then it cannot be satisfied if the positive sign is taken for f'_2 and the negative for g'_2 .

It is, of course, apparent that $f_1'^2$ and $g_1'^2$, given by equations (5.11), must each be less than unity. If this were not the case, f'_2 and g'_2 would be imaginary.

We thus see that if x_1 is specified on AB and BC , then it is completely determined throughout the parallelogram $ABCD$. There is, however, an ambiguity of sign in the determination of f'_2 and g'_2 which implies that we cannot determine x_2 unambiguously throughout the parallelogram $ABCD$.

However, if x_2 is specified on AB by $x_2 = x_2(Z'_1, Z_2)$ and on BC by $x_2 = x_2(Z_1, Z'_2)$ then x_2 is uniquely determined throughout the parallelogram $ABCD$. It is evident that x_2 can be specified on AB and BC only in such a way that the relations (5.3) and (5.15) are satisfied.

Now, we shall consider that both x_1 and x_2 are specified at all values of Z_1, Z_2 corresponding to points on the curve ABC shown in Fig. 3. It is assumed that at no point on ABC is the tangent to the curve parallel to a characteristic line. It will be shown that x_1 and x_2 are then determined throughout the region bounded by the curve ABC and the characteristic lines AP and CP passing through A and C .

Let the curve ABC have the equation

$$\varphi(Z_1, Z_2) = 0. \quad (5.16)$$

Then,

$$\frac{\partial \varphi}{\partial Z_1} dZ_1 + \frac{\partial \varphi}{\partial Z_2} dZ_2 = 0, \quad (5.17)$$

where $\partial \varphi / \partial Z_1 \neq 0$ and $\partial \varphi / \partial Z_2 \neq 0$ on the curve ABC

FIG. 3

Let $x_1 = x'_1$ and $x_2 = x'_2$ on ABC . Then, from (5.1), we have

$$x'_1 = f_1(Z_1) + g_1(Z_2)$$

and

$$x'_2 = f_2(Z_1) + g_2(Z_2), \quad (5.18)$$

if Z_1 and Z_2 are related by (5.16). Thus,

$$dx'_1 = f'_1(Z_1) dZ_1 + g'_1(Z_2) dZ_2$$

and

$$dx'_2 = f'_2(Z_1) dZ_1 + g'_2(Z_2) dZ_2, \quad (5.19)$$

if Z_1 and Z_2 are related by (5.16) and dZ_1 and dZ_2 are related by (5.17).

Eliminating dZ_2 from (5.17) and (5.19), and employing the notation

$$a_1 = \frac{\partial x'_1}{\partial Z_1}, \quad a_2 = \frac{\partial x'_2}{\partial Z_1} \quad \text{and} \quad \mu = \frac{\partial \varphi / \partial Z_1}{\partial \varphi / \partial Z_2}, \quad (5.20)$$

we obtain

$$a_1 = f'_1 - \mu g'_1 \quad \text{and} \quad a_2 = f'_2 - \mu g'_2. \quad (5.21)$$

With equations (5.3), we have four equations for the determination of f'_1 , g'_1 , f'_2 and g'_2 at each point of the curve ABC .

From equations (5.21) and the first of equations (5.3), we obtain

$$(a_1 + \mu g_1')^2 + (a_2 + \mu g_2')^2 = 1. \quad (5.22)$$

With the second of equations (5.3), this yields

$$2\mu(a_1 g_1' + a_2 g_2') = \nu, \quad (5.23)$$

where

$$\nu = 1 - \mu^2 - (a_1^2 + a_2^2). \quad (5.24)$$

Employing the relation (5.23) to substitute for g_2' in the second of equations (5.3), we obtain

$$4\mu^2(a_1^2 + a_2^2) g_1'^2 - 4\mu \nu a_1 g_1' + (\nu^2 - 4\mu^2 a_2^2) = 0. \quad (5.25)$$

Equation (5.25) yields

$$g_1' = \frac{\nu a_1 \pm a_2 \chi}{2\mu(a_1^2 + a_2^2)} \quad (5.26)$$

where

$$\chi = [-\nu^2 + (a_1^2 + a_2^2) 4\mu^2]^{\frac{1}{2}} \quad (5.27)$$

If

$$g_1' = \frac{\nu a_1 + a_2 \chi}{2\mu(a_1^2 + a_2^2)}, \quad (5.28)$$

it follows, from (5.23) and (5.21), that

$$g_2' = \frac{\nu a_2 - a_1 \chi}{2\mu(a_1^2 + a_2^2)},$$

$$f_1' = a_1 + \frac{\nu a_1 + a_2 \chi}{2(a_1^2 + a_2^2)}$$

and

$$f_2' = a_2 + \frac{\nu a_2 - a_1 \chi}{2(a_1^2 + a_2^2)}. \quad (5.29)$$

Similarly, if

$$g_1' = \frac{\nu a_1 - a_2 \chi}{2\mu(a_1^2 + a_2^2)}, \quad (5.30)$$

it follows that

$$g_2' = \frac{\nu a_2 + a_1 \chi}{2\mu(a_1^2 + a_2^2)},$$

$$f_1' = a_1 + \frac{\nu a_1 - a_2 \chi}{2(a_1^2 + a_2^2)}$$

and

$$f_2' = a_2 + \frac{\nu a_2 + a_1 \chi}{2(a_1^2 + a_2^2)} \quad (5.31)$$

If f_1' , g_1' , f_2' and g_2' are given by equations (5.28) and (5.29), then the condition (5.15) becomes

$$\frac{\chi}{4\mu \sin \alpha \cos \alpha} > 0. \quad (5.32)$$

Again, if f_1' , g_1' , f_2' and g_2' are given by equations (5.30) and (5.31), then the condition (5.15) becomes

$$-\frac{\chi}{4\mu \sin \alpha \cos \alpha} > 0. \quad (5.33)$$

It is clear that the conditions (5.32) and (5.33) cannot be satisfied simultaneously. Thus, subject to the condition (5.15), equations (5.28) and (5.29) or (5.30) and (5.31) determine f_1' , f_2' , g_1' and g_2' uniquely at each point on the curve ABC . It follows, from a theorem in the theory of hyperbolic equations, that x_1 and x_2 are uniquely determined throughout the region bounded by the curve ABC and the characteristic lines PA and PC .

We have so far assumed that the tangent to the arc ABC is nowhere parallel to a characteristic line. Let us consider the case when the region BD of the arc is parallel to one of the families of characteristics, as shown in Fig. 4. We assume that x_1 and x_2 are specified on $ABDC$. Since x_1 and x_2 are specified on the arc DC , they can be uniquely determined throughout the region bounded by the arc DC and the characteristics QC and QD . x_1 and x_2 are thus uniquely determined on QD . Again, since x_1 and x_2 are specified on the arc AB , they are uniquely determined throughout the region bounded by arc AB and the characteristic lines AR and BR . Hence x_1 and x_2 are uniquely determined on AR and BR . Since x_1 and x_2 are specified on BD and have been determined on QD , it is seen, from the discussion given earlier in this section, that x_1 and x_2 are uniquely determined throughout the parallelogram bounded by the characteristics QD ,

BD , QS and BS . Since x_1 and x_2 are now known on RS and AR they can be uniquely determined throughout the region bounded by the characteristics PS , SR , RA and AP . It is thus seen that if x_1 and x_2 are specified on $ABDC$, then they can be uniquely determined throughout the region bounded by $ABDC$ and the characteristic lines PA and PC .

FIG. 4

6. The Definition of the Stress Components. Let us assume that a sheet of the material is subjected to a plane strain by applied forces. The stress components t_{11} , t_{22} , t_{12} and t_{21} at a point (x_1, x_2) of the deformed sheet in the rectangular Cartesian coordinate system (x_1, x_2) may be defined in the following manner:

- (i) t_{11} and t_{12} are the components in the positive directions of the x_1 and x_2 axes respectively, of the force per unit length, measured in the deformed state, exerted across an element of length at (x_1, x_2) normal to the x_1 axis by the material on the positive side of the element on the material on the negative side of the element;
- (ii) t_{21} and t_{22} are the components in the positive directions of the x_1 and x_2 axes respectively, of the force per unit length, measured in the deformed state, exerted across an element of length at (x_1, x_2) normal to the x_2 axis by the material on the positive side of the element on the material on the negative side of the element.

It can readily be shown in the manner adopted in classical elasticity theory, that

$$t_{12} = t_{21}.$$

7. The Stress Components for Pure Homogeneous Deformation. Let us consider a sheet of the fabric described in §2 to be located with the bisectors of the angles between its cords parallel to the axes of a rectangular Cartesian coordinate system x_1, x_2 . Let us suppose that the sheet is subjected to a pure homogeneous deformation described by equations (2.1). Suppose that, in the deformed sheet, τ_1 and τ_2 are the tensions per cord in the sets of cords which, in the undeformed state, are inclined at angles α and $-\alpha$ respectively to the x_1 axis. It can readily

be shown that the stress components t_{11} , t_{22} and t_{12} are given by

$$t_{11} = \frac{\cos \alpha}{\lambda_2 d} (\tau_1 + \tau_2) \cos \beta,$$

$$t_{22} = \frac{\sin \alpha}{\lambda_1 d} (\tau_1 + \tau_2) \sin \beta$$

and

$$t_{12} = \frac{\cos \alpha}{\lambda_2 d} (\tau_1 - \tau_2) \sin \beta. \quad (7.1)$$

With the relation (2.2), we have

$$t_{11} = \frac{\lambda_1 \cos^2 \alpha}{\lambda_2 d} (\tau_1 + \tau_2),$$

$$t_{22} = \frac{\lambda_2 \sin^2 \alpha}{\lambda_1 d} (\tau_1 + \tau_2)$$

and

$$t_{12} = \frac{\sin \alpha \cos \alpha}{d} (\tau_1 - \tau_2). \quad (7.2)$$

8. The Stress Components for a General Plane Deformation. Now, let us consider a sheet of the fabric to be subjected to a general plane deformation. Consider a point of the sheet which is at (X_1, X_2) and (x_1, x_2) , in the rectangular Cartesian coordinate system (x_1, x_2) , in the undeformed and deformed states of the sheet respectively. Let y_1, y_2 be a rectangular Cartesian coordinate system, the axes of which bisect the angles between the cords in the deformed sheet at (x_1, x_2) . Then, it has been pointed out in §3, that if we consider an element of area of the sheet at (X_1, X_2) in the undeformed state, it may be considered that, in the deformation, the element undergoes a translation in which (X_1, X_2) moves to (x_1, x_2) , a rotation in which the axes (x_1, x_2) are brought into parallelism with the axes y_1, y_2 and a pure homogeneous deformation with principal directions y_1 and y_2 . It is clear that the stress at the point considered is that associated with this pure homogeneous deformation. If the stress components in the coordinate system y_1, y_2 are denoted $t'_{11}, t'_{22}, t'_{12}$, then, from (7.2), they are given by

$$t'_{11} = \frac{\lambda_1 \cos^2 \alpha}{\lambda_2 d} (\tau_1 + \tau_2), \quad t'_{22} = \frac{\lambda_2 \sin^2 \alpha}{\lambda_1 d} (\tau_1 + \tau_2)$$

and

$$t'_{12} = \frac{\sin \alpha \cos \alpha}{d} (\tau_1 - \tau_2), \quad (8.1)$$

where τ_1 and τ_2 are the tensions in the two cords passing through the point considered and λ_1, λ_2 are the principal extension ratios at that point.

Let the axis y_1 have direction-cosines (m_1, n_1) and let the axis y_2 have direction-cosines (m_2, n_2) in the coordinate system (x_1, x_2) . Then,

$$m_1 = \frac{1}{\lambda_1} \frac{\partial x_1}{\partial X_1} \quad \text{and} \quad n_1 = \frac{1}{\lambda_1} \frac{\partial x_2}{\partial X_1}.$$

Also,

$$m_2 = \frac{1}{\lambda_2} \frac{\partial x_1}{\partial X_2} \quad \text{and} \quad n_2 = \frac{1}{\lambda_2} \frac{\partial x_2}{\partial X_2}. \quad (8.2)$$

Since, $m_1^2 + n_1^2 = 1$ and $m_2^2 + n_2^2 = 1$, we obtain

$$\lambda_1^2 = g_{11} \quad \text{and} \quad \lambda_2^2 = g_{22}, \quad (8.3)$$

where g_{11} and g_{22} are defined as in equations (3.5).

The stress components t_{11} , t_{22} and t_{12} in the coordinate system x_1, x_2 may be obtained from the relations

$$t_{11} = t'_{11} m_1^2 + 2t'_{12} m_1 m_2 + t'_{22} m_2^2,$$

$$t_{22} = t'_{11} n_1^2 + 2t'_{12} n_1 n_2 + t'_{22} n_2^2$$

and

$$t_{12} = t'_{11} m_1 n_1 + t'_{22} m_2 n_2 + t'_{12}(m_1 n_2 + m_2 n_1). \quad (8.4)$$

Substituting in (8.4) from (8.1), (8.2) and (8.3), we have

$$t_{11} = \frac{1}{\lambda_1 \lambda_2 d} \left[\tau_1 \left(\frac{\partial x_1}{\partial X_1} \cos \alpha + \frac{\partial x_1}{\partial X_2} \sin \alpha \right)^2 + \tau_2 \left(\frac{\partial x_1}{\partial X_1} \cos \alpha - \frac{\partial x_1}{\partial X_2} \sin \alpha \right)^2 \right]$$

$$t_{22} = \frac{1}{\lambda_1 \lambda_2 d} \left[\tau_1 \left(\frac{\partial x_2}{\partial X_1} \cos \alpha + \frac{\partial x_2}{\partial X_2} \sin \alpha \right)^2 + \tau_2 \left(\frac{\partial x_2}{\partial X_1} \cos \alpha - \frac{\partial x_2}{\partial X_2} \sin \alpha \right)^2 \right]$$

and

$$t_{12} = \frac{1}{\lambda_1 \lambda_2 d} \left[\tau_1 \left(\frac{\partial x_1}{\partial X_1} \cos \alpha + \frac{\partial x_1}{\partial X_2} \sin \alpha \right) \left(\frac{\partial x_2}{\partial X_1} \cos \alpha + \frac{\partial x_2}{\partial X_2} \sin \alpha \right) \right. \\ \left. + \tau_2 \left(\frac{\partial x_1}{\partial X_1} \cos \alpha - \frac{\partial x_1}{\partial X_2} \sin \alpha \right) \left(\frac{\partial x_2}{\partial X_1} \cos \alpha - \frac{\partial x_2}{\partial X_2} \sin \alpha \right) \right]. \quad (8.5)$$

Employing the relations (4.8) and (4.9), equations (8.5) may be re-written as

$$t_{11} = \frac{1}{\lambda_1 \lambda_2 d} \left[\tau_1 \left(\frac{\partial x_1}{\partial Z_1} \right)^2 + \tau_2 \left(\frac{\partial x_1}{\partial Z_2} \right)^2 \right],$$

$$t_{22} = \frac{1}{\lambda_1 \lambda_2 d} \left[\tau_1 \left(\frac{\partial x_2}{\partial Z_1} \right)^2 + \tau_2 \left(\frac{\partial x_2}{\partial Z_2} \right)^2 \right]$$

and

$$t_{12} = \frac{1}{\lambda_1 \lambda_2 d} \left[\tau_1 \frac{\partial x_1}{\partial Z_1} \frac{\partial x_2}{\partial Z_1} + \tau_2 \frac{\partial x_1}{\partial Z_2} \frac{\partial x_2}{\partial Z_2} \right]. \quad (8.6)$$

With equations (5.1) and (5.2) and the expressions (3.10) for λ_1 and λ_2 , equations (8.6) become

$$t_{11} = \frac{\sin 2\alpha}{d(f_2' g_1' - f_1' g_2')} (\tau_1 f_1'^2 + \tau_2 g_1'^2),$$

$$t_{22} = \frac{\sin 2\alpha}{d(f_2' g_1' - f_1' g_2')} (\tau_1 f_2'^2 + \tau_2 g_2'^2)$$

and

$$t_{12} = \frac{\sin 2\alpha}{d(f_2' g_1' - f_1' g_2')} (\tau_1 f_1' f_2' + \tau_2 g_1' g_2'). \quad (8.7)$$

9. The Equations of Motion. The equations of motion take the form

$$\frac{\partial t_{ij}}{\partial x_j} + F_i = \frac{\partial^2 x_i}{\partial t^2} \quad (9.1)$$

where F_i are the components in the direction of the x_i axis of the surface forces acting on the fabric per unit area measured in the deformed state, ρ is the mass of the fabric per unit area measured in the deformed state and t denotes time. We shall restrict ourselves to static problems so that $\partial^2 x_i / \partial t^2 = 0$. Equation (9.1) then becomes

$$\frac{\partial t_{ij}}{\partial x_j} + F_i = 0. \quad (9.2)$$

This equation may be re-written as

$$\frac{\partial t_{ij}}{\partial Z_k} \frac{\partial Z_k}{\partial x_j} + F_i = 0. \quad (9.3)$$

Now, it can readily be shown that

$$\frac{\partial Z_1}{\partial x_1} = \frac{1}{\Delta} \frac{\partial x_2}{\partial Z_2}, \quad \frac{\partial Z_1}{\partial x_2} = -\frac{1}{\Delta} \frac{\partial x_1}{\partial Z_2}, \quad \frac{\partial Z_2}{\partial x_1} = -\frac{1}{\Delta} \frac{\partial x_2}{\partial Z_1}, \quad \frac{\partial Z_2}{\partial x_2} = \frac{1}{\Delta} \frac{\partial x_1}{\partial Z_1}, \quad (9.4)$$

where

$$\Delta = \frac{\partial x_1}{\partial Z_1} \frac{\partial x_2}{\partial Z_2} - \frac{\partial x_1}{\partial Z_2} \frac{\partial x_2}{\partial Z_1}. \quad (9.5)$$

With equations (5.1) and (5.2), we obtain

$$\frac{\partial Z_1}{\partial x_1} = \frac{1}{\Delta} g'_2, \quad \frac{\partial Z_1}{\partial x_2} = -\frac{1}{\Delta} g'_1, \quad \frac{\partial Z_2}{\partial x_1} = -\frac{1}{\Delta} f'_2, \quad \frac{\partial Z_2}{\partial x_2} = \frac{1}{\Delta} f'_1, \quad (9.6)$$

where

$$\Delta = f'_1 g'_2 - f'_2 g'_1. \quad (9.7)$$

Introducing (9.6) into (9.3), we obtain

$$\begin{aligned} g'_2 \frac{\partial t_{11}}{\partial Z_1} - g'_1 \frac{\partial t_{12}}{\partial Z_1} - f'_2 \frac{\partial t_{11}}{\partial Z_2} + f'_1 \frac{\partial t_{12}}{\partial Z_2} + \Delta F_1 &= 0, \\ g'_2 \frac{\partial t_{12}}{\partial Z_1} - g'_1 \frac{\partial t_{22}}{\partial Z_1} - f'_2 \frac{\partial t_{12}}{\partial Z_2} + f'_1 \frac{\partial t_{22}}{\partial Z_2} + \Delta F_2 &= 0. \end{aligned} \quad (9.8)$$

Employing the expressions (8.7) for t_{ij} in (9.8), we obtain

$$\frac{\partial}{\partial Z_1} (f'_1 \tau_1) + \frac{\partial}{\partial Z_2} (g'_1 \tau_2) - \frac{d\Delta F_1}{\sin 2\alpha} = 0$$

and

$$\frac{\partial}{\partial Z_1} (f'_2 \tau_1) + \frac{\partial}{\partial Z_2} (g'_2 \tau_2) - \frac{d\Delta F_2}{\sin 2\alpha} = 0. \quad (9.9)$$

10. Alternative Forms for the Equations of Equilibrium. We shall assume that no body forces are acting, so that $F_i = 0$. Then, equation (9.9) becomes

$$\frac{\partial}{\partial Z_1} (f'_1 \tau_1) + \frac{\partial}{\partial Z_2} (g'_1 \tau_2) = 0$$

and

$$\frac{\partial}{\partial Z_1} (f'_2 \tau_1) + \frac{\partial}{\partial Z_2} (g'_2 \tau_2) = 0. \quad (10.1)$$

The first of these equations must have a solution of the form

$$f'_1 \tau_1 = \frac{\partial \varphi}{\partial Z_2}, \quad g'_1 \tau_2 = -\frac{\partial \varphi}{\partial Z_1}, \quad (10.2)$$

where φ is a function of Z_1 and Z_2 . Similarly, the second of equations (10.1) must have a solution of the form

$$f'_2 \tau_1 = \frac{\partial \psi}{\partial Z_2}, \quad g'_2 \tau_2 = -\frac{\partial \psi}{\partial Z_1}, \quad (10.3)$$

where ψ is a function of Z_1 and Z_2 . From (10.2) and (10.3) we obtain

$$f'_2 \frac{\partial \varphi}{\partial Z_2} = f'_1 \frac{\partial \psi}{\partial Z_2} \quad \text{and} \quad g'_2 \frac{\partial \varphi}{\partial Z_1} = g'_1 \frac{\partial \psi}{\partial Z_1}. \quad (10.4)$$

Whence

$$f'_2 \varphi = f'_1 \psi + h \quad \text{and} \quad g'_2 \varphi = g'_1 \psi + k, \quad (10.5)$$

where h is a function of Z_1 only and k is a function of Z_2 only.

From equations (10.5), we obtain

$$(f'_2 g'_1 - f'_1 g'_2) \varphi = h g'_1 - k f'_1$$

and

$$(f'_2 g'_1 - f'_1 g'_2) \psi = h g'_2 - k f'_2. \quad (10.6)$$

Whence,

$$\begin{aligned} \tau_2 &= - \frac{1}{g'_1} \frac{\partial \varphi}{\partial Z_1} = - \frac{1}{g'_2} \frac{\partial \psi}{\partial Z_1} \\ &= \frac{h'(f'_1 g'_2 - f'_2 g'_1) + h(f''_2 g'_1 - f''_1 g'_2) + k(f''_1 f'_2 - f''_2 f'_1)}{(f'_2 g'_1 - g'_2 f'_1)^2} \end{aligned}$$

and

$$\begin{aligned} \tau_1 &= \frac{1}{f'_1} \frac{\partial \varphi}{\partial Z_2} = \frac{1}{f'_2} \frac{\partial \psi}{\partial Z_2} \\ &= \frac{k'(f'_1 g'_2 - f'_2 g'_1) + k(g''_1 f'_2 - g''_2 f'_1) + h(g''_2 g'_1 - g''_1 g'_2)}{(f'_2 g'_1 - g'_2 f'_1)^2} \end{aligned} \quad (10.7)$$

We may also write equations (10.7) in the form

$$\tau_2 = \frac{\partial}{\partial Z_1} \left[\frac{h}{f'_1 g'_2 - f'_2 g'_1} \right] + \frac{k(f''_1 f'_2 - f''_2 f'_1)}{(f'_2 g'_1 - g'_2 f'_1)^2}$$

and

$$\tau_1 = \frac{\partial}{\partial Z_2} \left[\frac{k}{f'_1 g'_2 - f'_2 g'_1} \right] + \frac{h(g''_2 g'_1 - g''_1 g'_2)}{(f'_2 g'_1 - g'_2 f'_1)^2}. \quad (10.8)$$

With the notation (9.7), equations (10.8) may be re-written

$$\tau_2 = \frac{\partial}{\partial Z_1} \left(\frac{h}{\Delta} \right) + \frac{k}{\Delta^2} (f''_1 f'_2 - f''_2 f'_1)$$

and

$$\tau_1 = \frac{\partial}{\partial Z_2} \left(\frac{k}{\Delta} \right) + \frac{h}{\Delta^2} (g_2'' g_1' - g_1'' g_2'). \quad (10.9)$$

The second of these equations may be re-written as

$$\frac{h}{\Delta} = \frac{\Delta}{g_2'' g_1' - g_1'' g_2'} \left[\tau_1 - \frac{\partial}{\partial Z_2} \left(\frac{k}{\Delta} \right) \right]. \quad (10.10)$$

Substituting from (10.10) for h/Δ in the first of equations (10.9), we obtain

$$\begin{aligned} \tau_2 &= \frac{1}{g_2'' g_1' - g_1'' g_2'} \frac{\partial}{\partial Z_1} \left\{ \Delta \left[\tau_1 - \frac{\partial}{\partial Z_2} \left(\frac{k}{\Delta} \right) \right] \right\} + \frac{k}{\Delta^2} (f_1'' f_2' - f_2'' f_1') \\ &= \frac{1}{g_2'' g_1' - g_1'' g_2'} \frac{\partial}{\partial Z_1} (\Delta \tau_1). \end{aligned} \quad (10.11)$$

In a similar manner, we can obtain, by substituting for k/Δ from the first of equations (10.9) in the second,

$$\tau_1 = \frac{1}{f_1'' f_2' - f_2'' f_1'} \frac{\partial}{\partial Z_2} (\Delta \tau_2). \quad (10.12)$$

FIG. 5

11. Edge Traction Specified on a Closed Curve. Let us consider a plane sheet of the ideal fabric bounded by the closed curve S , as shown in Fig. 5. The curve is considered to be made up of a number of arcs for each of which the slope of the tangent to the curve changes continuously. At the junctions of these arcs, the slope of the tangent may or may not change discontinuously.

We assume that the sheet is deformed by edge tractions lying in the plane of S and that it remains plane in the deformed state. We take as our reference sys-

tem a rectangular Cartesian coordinate system (x_1, x_2) , the axes of which bisect the angles between the cords, and assume that the components (ξ, η) , in this coordinate system, of the edge tractions, measured per unit length of edge in the undeformed state of the sheet, are specified on S . Let us suppose that the distance measured along S of an arbitrary point P from a fixed point Q is s in the undeformed state of the sheet and s' in the deformed state of the sheet.

Let (l_1, m_1) be the direction-cosines of the outward drawn normal to S at P in the undeformed state and let (l'_1, m'_1) be the direction-cosines of the outward drawn normal to S at P in the deformed state. Let (l_2, m_2) and (l'_2, m'_2) be the direction-cosines of an element of length of S at P in the undeformed and deformed states respectively. Then,

$$l_1 = m_2, \quad m_1 = -l_2$$

and

$$l'_1 = m'_2, \quad m'_1 = -l'_2. \quad (11.1)$$

If (ξ', η') are the components of the edge traction, per unit length of edge measured in the deformed state, then

$$\xi' = l'_1 t_{11} + m'_1 t_{12}$$

and

$$\eta' = l'_1 t_{12} + m'_1 t_{22}, \quad (11.2)$$

where t_{11} , t_{22} and t_{12} are the components of stress in the coordinate system (x_1, x_2) . With (11.1), we obtain

$$\xi' = m'_2 t_{11} - l'_2 t_{12}$$

and

$$\eta' = m'_2 t_{12} - l'_2 t_{22}. \quad (11.3)$$

We note that

$$\xi' = \frac{ds}{ds'} \xi \quad \text{and} \quad \eta' = \frac{ds}{ds'} \eta. \quad (11.4)$$

Also,

$$l'_2 = \frac{ds}{ds'} \left(\frac{\partial x_1}{\partial X_1} l_2 + \frac{\partial x_1}{\partial X_2} m_2 \right)$$

and

$$m'_2 = \frac{ds}{ds'} \left(\frac{\partial x_2}{\partial X_1} l_2 + \frac{\partial x_2}{\partial X_2} m_2 \right). \quad (11.5)$$

Substituting from (11.4) and (11.5) in (11.3), we obtain

$$\xi = \left(\frac{\partial x_2}{\partial X_1} l_2 + \frac{\partial x_2}{\partial X_2} m_2 \right) t_{11} - \left(\frac{\partial x_1}{\partial X_1} l_2 + \frac{\partial x_1}{\partial X_2} m_2 \right) t_{12}$$

and

$$\eta = \left(\frac{\partial x_2}{\partial X_1} l_2 + \frac{\partial x_2}{\partial X_2} m_2 \right) t_{12} - \left(\frac{\partial x_1}{\partial X_1} l_2 + \frac{\partial x_1}{\partial X_2} m_2 \right) t_{22}. \quad (11.6)$$

Introducing the expressions (8.7), for t_{11} , t_{22} and t_{12} into (11.6), we obtain

$$\begin{aligned} \xi = \frac{\sin 2\alpha}{d(f'_2 g'_1 - f'_1 g'_2)} & \left[\left(\frac{\partial x_2}{\partial X_1} l_2 + \frac{\partial x_2}{\partial X_2} m_2 \right) (\tau_1 f_1'^2 + \tau_2 g_1'^2) \right. \\ & \left. - \left(\frac{\partial x_1}{\partial X_1} l_2 + \frac{\partial x_1}{\partial X_2} m_2 \right) (\tau_1 f'_1 f'_2 + \tau_2 g'_1 g'_2) \right] \end{aligned}$$

and

$$\begin{aligned} \eta = \frac{\sin 2\alpha}{d(f'_2 g'_1 - f'_1 g'_2)} & \left[\left(\frac{\partial x_2}{\partial X_1} l_2 + \frac{\partial x_2}{\partial X_2} m_2 \right) (\tau_1 f'_1 f'_2 + \tau_2 g'_1 g'_2) \right. \\ & \left. - \left(\frac{\partial x_1}{\partial X_1} l_2 + \frac{\partial x_1}{\partial X_2} m_2 \right) (\tau_1 f_2'^2 + \tau_2 g_2'^2) \right]. \quad (11.7) \end{aligned}$$

From equations (5.1), (5.2), (4.8) and (4.9), we obtain

$$\begin{aligned} \frac{\partial x_1}{\partial X_1} &= \frac{1}{2 \cos \alpha} (f'_1 + g'_1), & \frac{\partial x_1}{\partial X_2} &= \frac{1}{2 \sin \alpha} (f'_1 - g'_1), \\ \frac{\partial x_2}{\partial X_1} &= \frac{1}{2 \cos \alpha} (f'_2 + g'_2) & \text{and} & \frac{\partial x_2}{\partial X_2} = \frac{1}{2 \sin \alpha} (f'_2 - g'_2). \end{aligned} \quad (11.8)$$

Introducing the expressions (11.8) into (11.7), we obtain

$$\xi = \frac{1}{d} [\tau_1 f'_1 (m_2 \cos \alpha - l_2 \sin \alpha) + \tau_2 g'_1 (m_2 \cos \alpha + l_2 \sin \alpha)]$$

and

$$\eta = \frac{1}{d} [\tau_1 f'_2 (m_2 \cos \alpha - l_2 \sin \alpha) + \tau_2 g'_2 (m_2 \cos \alpha + l_2 \sin \alpha)]. \quad (11.9)$$

From equations (10.2) and (10.3) and equations (11.9), we obtain

$$\xi = \frac{1}{d} \left[\frac{\partial \varphi}{\partial Z_2} (m_2 \cos \alpha - l_2 \sin \alpha) - \frac{\partial \varphi}{\partial Z_1} (m_2 \cos \alpha + l_2 \sin \alpha) \right]$$

and

$$\eta = \frac{1}{d} \left[\frac{\partial \psi}{\partial Z_2} (m_2 \cos \alpha - l_2 \sin \alpha) - \frac{\partial \psi}{\partial Z_1} (m_2 \cos \alpha + l_2 \sin \alpha) \right]. \quad (11.10)$$

Now,

$$\begin{aligned} \frac{\partial \varphi}{\partial s} &= \frac{\partial \varphi}{\partial X_1} l_2 + \frac{\partial \varphi}{\partial X_2} m_2 \\ &= \frac{\partial \varphi}{\partial Z_1} \left(l_2 \frac{\partial Z_1}{\partial X_1} + m_2 \frac{\partial Z_1}{\partial X_2} \right) + \frac{\partial \varphi}{\partial Z_2} \left(l_2 \frac{\partial Z_2}{\partial X_1} + m_2 \frac{\partial Z_2}{\partial X_2} \right). \end{aligned} \quad (11.11)$$

With equations (4.8) and (4.9), we can rewrite equation (11.11) as

$$\frac{\partial \varphi}{\partial s} = \frac{1}{\sin 2\alpha} \left[\frac{\partial \varphi}{\partial Z_1} (l_2 \sin \alpha + m_2 \cos \alpha) + \frac{\partial \varphi}{\partial Z_2} (l_2 \sin \alpha - m_2 \cos \alpha) \right]. \quad (11.12)$$

In a similar way, we obtain

$$\frac{\partial \psi}{\partial s} = \frac{1}{\sin 2\alpha} \left[\frac{\partial \psi}{\partial Z_1} (l_2 \sin \alpha + m_2 \cos \alpha) + \frac{\partial \psi}{\partial Z_2} (l_2 \sin \alpha - m_2 \cos \alpha) \right]. \quad (11.13)$$

From (11.12), (11.13) and (11.10), we have

$$\xi = - \frac{\sin 2\alpha}{d} \frac{\partial \varphi}{\partial s} \quad \text{and} \quad \eta = - \frac{\sin 2\alpha}{d} \frac{\partial \psi}{\partial s}. \quad (11.14)$$

From (11.14) we can obtain φ and ψ at all points of the curve S apart from additive constants. By fixing the values of φ and ψ at some point of S , we can determine φ and ψ completely at each point of S .

Let (Z_1, Z_2) be a point R in the region bounded by S . Let the cords $Z_1 = \text{const.}$ and $Z_2 = \text{const.}$ passing through R intersect the curve S in the points A, B, C and D , which have coordinates (Z_1, Z'_2) , (Z_1, Z''_2) , (Z'_1, Z_2) and (Z''_1, Z_2) respectively.

The values of φ and ψ are completely determined at A, B, C and D . From equations (10.6), we have

$$\left. \begin{aligned} [f'_2(Z'_1)g'_1(Z_2) - f'_1(Z'_1)g'_2(Z_2)]\varphi(Z'_1, Z_2) &= h(Z'_1)g'_1(Z_2) - k(Z_2)f'_1(Z'_1), \\ [f'_2(Z'_1)g'_1(Z_2) - f'_1(Z'_1)g'_2(Z_2)]\psi(Z'_1, Z_2) &= h(Z'_1)g'_2(Z_2) - k(Z_2)f'_1(Z'_1), \end{aligned} \right\} \quad (11.15)$$

$$\left. \begin{aligned} [f'_2(Z''_1)g'_1(Z_2) - f'_1(Z''_1)g'_2(Z_2)]\varphi(Z''_1, Z_2) &= h(Z''_1)g'_1(Z_2) - k(Z_2)f'_1(Z''_1), \\ [f'_2(Z''_1)g'_1(Z_2) - f'_1(Z''_1)g'_2(Z_2)]\psi(Z''_1, Z_2) &= h(Z''_1)g'_2(Z_2) - k(Z_2)f'_2(Z''_1), \end{aligned} \right\} \quad (11.16)$$

$$\left. \begin{aligned} [f'_2(Z_1)g'_1(Z'_2) - f'_1(Z_1)g'_2(Z'_2)]\varphi(Z_1, Z'_2) &= h(Z_1)g'_1(Z'_2) - k(Z'_2)f'_1(Z_1), \\ [f'_2(Z_1)g'_1(Z'_2) - f'_1(Z_1)g'_2(Z'_2)]\psi(Z_1, Z'_2) &= h(Z_1)g'_2(Z'_2) - k(Z'_2)f'_2(Z_1), \end{aligned} \right\} \quad (11.17)$$

$$\left. \begin{aligned} [f'_2(Z_1)g'_1(Z''_2) - f'_1(Z_1)g'_2(Z''_2)]\varphi(Z_1, Z''_2) &= h(Z_1)g'_1(Z''_2) - k(Z''_2)f'_1(Z_1), \\ [f'_2(Z_1)g'_1(Z''_2) - f'_1(Z_1)g'_2(Z''_2)]\psi(Z_1, Z''_2) &= h(Z_1)g'_2(Z''_2) - k(Z''_2)f'_2(Z_1). \end{aligned} \right\} \quad (11.18)$$

Multiplying the first of equations (11.15) by $g'_2(Z_2)$ and the second by $g'_1(Z_2)$ and subtracting them, we obtain

$$k(Z_2) = g'_2(Z_2)\varphi(Z'_1, Z_2) - g'_1(Z_2)\psi(Z'_1, Z_2). \quad (11.19)$$

In a similar manner we obtain from equations (11.16)

$$k(Z_2) = g'_2(Z_2)\varphi(Z''_1, Z_2) - g'_1(Z_2)\psi(Z''_1, Z_2). \quad (11.20)$$

From (11.19) and (11.20), we obtain

$$\frac{g'_2(Z_2)}{g'_1(Z_2)} = \frac{\psi(Z'_1, Z_2) - \psi(Z''_1, Z_2)}{\varphi(Z'_1, Z_2) - \varphi(Z''_1, Z_2)}$$

and

$$k(Z_2) = \frac{\psi(Z'_1, Z_2)\varphi(Z''_1, Z_2) - \varphi(Z'_1, Z_2)\psi(Z''_1, Z_2)}{\varphi(Z'_1, Z_2) - \varphi(Z''_1, Z_2)} g'_1(Z_2). \quad (11.21)$$

In an analogous manner, we obtain, from equations (11.17) and (11.18),

$$\frac{f'_2(Z_1)}{f'_1(Z_1)} = \frac{\psi(Z_1, Z'_2) - \psi(Z_1, Z''_2)}{\varphi(Z_1, Z'_2) - \varphi(Z_1, Z''_2)}$$

and

$$h(Z_1) = \frac{\varphi(Z_1, Z''_2)\psi(Z_1, Z'_2) - \psi(Z_1, Z''_2)\varphi(Z_1, Z'_2)}{\varphi(Z_1, Z'_2) - \varphi(Z_1, Z''_2)} f'_1(Z_1). \quad (11.22)$$

Thus, from equations (10.6), φ and ψ are uniquely determined throughout the sheet and from equations (10.2) and (10.3), $f'_1 \tau_1$, $f'_2 \tau_1$, $g'_1 \tau_2$ and $g'_2 \tau_2$ are uniquely determined throughout the sheet.

FIG. 6

Since f'_2/f'_1 is determined throughout the sheet and $f_1'^2 + f_2'^2 = 1$, we can determine f'_1 and f'_2 throughout the sheet apart from an ambiguity which allows the sign of both of them to be changed simultaneously at any point. Again,

since g_2/g_1 is determined throughout the sheet and $g_1'^2 + g_2'^2 = 1$, we can determine g_1' and g_2' throughout the sheet apart from an ambiguity which allows the signs of both of them to be changed simultaneously at any point. It follows from the fact that

$$\frac{1}{\sin 2\alpha} (g_1' f_2' - f_1' g_2') > 0$$

at each point that if the signs of f_1' and f_2' are changed at any point, then the signs of g_1' and g_2' must also be changed. Thus, f_1', f_2', g_1' and g_2' are uniquely determined at each point of the sheet apart from a simultaneous change of sign of all of them. Since, from equations (10.2), $f_1' \tau_1$ and $g_1' \tau_2$ are uniquely determined at each point of the sheet, $f_1', f_2', g_1', g_2', \tau_1$ and τ_2 are uniquely determined at each point of the sheet apart from the fact that the signs of all of them can be simultaneously changed. We can readily show that if the signs of these quantities are changed at one point of the sheet, they must be changed at all points of the sheet. For let them be changed at the point L at (Z_1', Z_2') , as shown in Fig. 6. Let M and N be the points (Z_1', Z_2'') and (Z_1'', Z_2'') respectively. We see that if the signs of $f_1', f_2', g_1', g_2', \tau_1$ and τ_2 are changed at L , since the values of f_1' and f_2' at L and M are the same, we must change the signs of all of the quantities at M and, since the values of g_1' and g_2' at M and N are the same, we must change the signs of all of the quantities at N . Since it is possible to reach any point of the sheet from any other by passing alternately along cords of the families $Z_1 = \text{const.}$ and $Z_2 = \text{const.}$, if we change the signs of $f_1', f_2', g_1', g_2', \tau_1$ and τ_2 at one point of the sheet, we must change them at all points of the sheet.

Thus, $f_1', f_2', g_1', g_2', \tau_1$ and τ_2 are uniquely determined apart from the possibility that the signs of all of them may be simultaneously changed at *all* points of the sheet. Such a change of sign corresponds to a rotation of the sheet through 180° about the normal to it, *before* the edge tractions are applied to it.

Since f_1', f_2', g_1' and g_2' are known throughout the sheet, we can of course determine, from the formulae

$$x_1 = f_1 + g_1 \quad \text{and} \quad x_2 = f_2 + g_2,$$

the position of each point of the sheet in its deformed state, if any specified point of the sheet is assumed not to move in the deformation.