Periodic Solutions of Singularly Perturbed Systems

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1. Here a periodic solution for the system

$$(1.0) \quad \frac{dx}{dt} = f(t, x, y, \epsilon) \quad \epsilon \frac{dy}{dt} = g(t, x, y, \epsilon)$$

will be found for small $|\epsilon|$, $\epsilon \neq 0$, when the degenerate system

$$(1.1) \quad \frac{dx}{dt} = f(t, x, y, 0) \quad g(t, x, y, 0) = 0$$

has a periodic solution. It is assumed that x and f are vectors with n_0 components, and y and g are vectors with n_1 components. The components of x are x_i , $i = 1, \dots, n_0$, and similarly for f , y and g . The norm of a vector x is denoted by $|x|$ and is $\sum |x_j|$.

In what follows it will be convenient to use the notation f_x, f_y , etc. to represent matrices. Thus f_y is the matrix with element in the i^{th} row and j^{th} column given by $\partial f_i(t, x, y, \epsilon) / \partial y_j$. The norm of a matrix A is denoted by $|A|$ and is the sum of the magnitudes of all the elements of A .

Let the system (1.1) have a solution $x = \theta(t)$, $y = \chi(t)$ for $0 \leq t \leq T$. Let f , g , $\theta(t)$ and χ all be real and periodic in t of period T . Let f , f_x , f_y , g , g_x and g_y all exist as continuous functions of (t, x, y, ϵ) for $|x - \theta(t)| + |y - \chi(t)| + |\epsilon|$ small and $0 \leq t \leq T$.

If (1.1) is assumed to have a solution depending on some initial value s as a parameter then differentiation with respect to s gives

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial x}{\partial s} \right) &= f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} \\ g_x \frac{\partial x}{\partial s} + g_y \frac{\partial y}{\partial s} &= 0 \end{aligned}$$

which is the first variation of (1.1). If the second equation is solved for $\partial y/\partial s$ and the result substituted in the first there follows

$$\frac{d}{dt} \left(\frac{\partial x}{\partial s} \right) = (f_x - f_y g_y^{-1} g_x) \frac{\partial x}{\partial s}$$

Let

$$(1.2) \quad A(t) = f_x(t) - f_y(t) g_y^{-1}(t) g_x(t)$$

where throughout $f_x(t)$ will mean $f_x(t, \theta(t), \chi(t), 0)$ and similarly for $f_y(t)$, $g_x(t)$ and $g_y(t)$. Then $\xi' = A(t)\xi$ is the first variation of (1.1) with respect to the solution (θ, χ) and as will be seen is of importance in going from (1.1) to (1.0).

It will be convenient to assume the matrix $U(t) = g_y^{-1}(t)g_x(t)$ to be of class $C_1(t)$ so that it has a continuous derivative. [Without this assumption the proof would be more complicated.]

A fundamental hypothesis will be that $g_y(t)$ has m characteristic roots with negative real part and n with positive real part where $m + n = n_1$. Indeed let there exist a real non-singular matrix $P(t)$ of class C_1 on $0 \leq t \leq T$ and of period T such that

$$(1.3) \quad P^{-1}(t)g_y(t)P(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix}$$

where $B(t)$ is a matrix of m rows and columns with characteristic roots having negative real parts and $C(t)$ is a matrix of n rows and columns with characteristic roots having positive real parts.

Theorem. *Let $P(t)$ exist as described above and let the variational equation*

$$(1.4) \quad \frac{d\xi}{dt} = A(t)\xi$$

have no solution of period T except $\xi = 0$. Then for small $|\epsilon|$ the system (1.0) has a unique solution $x = p(t, \epsilon)$ $y = q(t, \epsilon)$ periodic of period T such that

$$|p(t, \epsilon) - \theta(t)| + |q(t, \epsilon) - \chi(t)| \rightarrow 0$$

uniformly in t as $\epsilon \rightarrow 0$.

In case f and g do not depend explicitly on t then the theorem is modified in a familiar way to give the existence of p and q of period $T + \tau$ where $\tau \rightarrow 0$ as $\epsilon \rightarrow 0$. (In this case (1.4) has one solution of period T .) For the case where $n_1 = 1$ so that y and g are scalars see [1].

Throughout this paper $\omega(\epsilon)$ will represent a continuous function of ϵ for $0 \leq \epsilon \leq \epsilon_0$ for some $\epsilon_0 > 0$ and $\omega(0) = 0$. However in each formula $\omega(\epsilon)$ may be different so that $\omega(\epsilon)$ is really a way of representing a quantity that tends to zero as $\epsilon \rightarrow 0$. It will also be assumed in what follows that $\epsilon > 0$. The case $\epsilon < 0$ is handled similarly.

2. Lemma. Let $Q(t)$ be an $N \times N$ continuous matrix for $a \leq t \leq b$ and let the real parts of its characteristic roots all be less than -2μ on $a \leq t \leq b$ for some $\mu > 0$. Let $\phi(t, s, \epsilon)$ be the solution of

$$\frac{dX}{dt} = \frac{1}{\epsilon} Q(t)X \quad X(s) = E$$

on $a \leq t \leq b$ for s on $a \leq s \leq b$ where E is the $N \times N$ unit matrix. Then there exists a constant K such that

$$(2.0) \quad |\phi(t, s, \epsilon)| \leq K e^{-\mu(t-s)/\epsilon}$$

for $a \leq s \leq t \leq b$.

(This lemma can be used to effect a considerable simplification of the proof of [2].)

For $a \leq s \leq t \leq b$ it has been shown in Lemma 2.4 of [2] that

$$(2.1) \quad |e^{Q(t)(t-s)/\epsilon}| \leq K e^{-2\mu(t-s)/\epsilon}$$

(The constant K may be different in each formula below.)

Proof of Lemma. For any fixed τ

$$\frac{d\phi}{dt} = \frac{1}{\epsilon} Q(\tau)\phi + \frac{1}{\epsilon} [Q(t) - Q(\tau)]\phi$$

or using the variation of constants formula

$$\phi(t, s, \epsilon) = e^{Q(\tau)(t-s)/\epsilon} + \frac{1}{\epsilon} \int_s^t e^{Q(\tau)(t-\lambda)/\epsilon} [Q(\lambda) - Q(\tau)]\phi(\lambda, s, \epsilon) d\lambda$$

for all t, τ, s on $[a, b]$. In particular this holds with $\tau = t$ so that

$$(2.2) \quad \phi(t, s, \epsilon) = e^{Q(t)(t-s)/\epsilon} + \frac{1}{\epsilon} \int_s^t e^{Q(t)(t-\lambda)/\epsilon} [Q(\lambda) - Q(t)]\phi(\lambda, s, \epsilon) d\lambda$$

For $s \leq t$ let

$$(2.3) \quad |\phi(t, s, \epsilon)| e^{\mu(t-s)/\epsilon} = F(t, s, \epsilon)$$

Then using (2.1) in (2.2)

$$F \leq K e^{-\mu(t-s)/\epsilon} + \frac{K}{\epsilon} e^{\mu(t-s)/\epsilon} \int_s^t e^{-2\mu(t-\lambda)/\epsilon} |Q(\lambda) - Q(t)| e^{-\mu(\lambda-s)/\epsilon} F(\lambda, s, \epsilon) d\lambda.$$

Or

$$F \leq K + \frac{K}{\epsilon} \int_s^t e^{-\mu(t-\lambda)/\epsilon} |Q(\lambda) - Q(t)| F(\lambda, s, \epsilon) d\lambda$$

For fixed ϵ let $\max F = M$ where the maximum is taken over $a \leq s \leq t \leq b$. Then

$$(2.4) \quad M \leq K + \frac{KM}{\epsilon} J$$

where

$$(2.5) \quad J = \max \int_s^t e^{-\mu(t-\lambda)/\epsilon} |Q(\lambda) - Q(t)| d\lambda$$

Taking $\int_s^t = \int_s^{t-\epsilon^{\frac{1}{2}}} + \int_{t-\epsilon^{\frac{1}{2}}}^t$ above and using the continuity of $Q(\lambda) - Q(t)$ in the second integrand it follows there that $|Q(\lambda) - Q(t)| \leq \omega(\epsilon)$ so that

$$(2.6) \quad J \leq Ke^{-\mu/\epsilon^{\frac{1}{2}}} + \epsilon\omega(\epsilon)$$

Thus in (2.4)

$$M \leq K + K \left[\frac{1}{\epsilon} e^{-\mu/\epsilon^{\frac{1}{2}}} + \omega(\epsilon) \right] M$$

Choosing ϵ small enough so that

$$K \left[\frac{1}{\epsilon} e^{-\mu/\epsilon^{\frac{1}{2}}} + \omega(\epsilon) \right] < \frac{1}{2}$$

there follows $M \leq 2K$. In (2.3) this gives

$$|\phi(t, s, \epsilon)| \leq Ke^{-\mu(t-s)/\epsilon}$$

which is (2.0).

Using (2.1) and (2.0) in appraising the integral on the right of (2.2) there follows incidentally

$$(2.7) \quad |\phi(t, s, \epsilon) - e^{Q(t)(t-s)/\epsilon}| \leq \frac{K}{\epsilon} J e^{-\mu(t-s)/\epsilon}$$

where J is given by (2.5). Using (2.6) in (2.7) proves

$$(2.8) \quad |\phi(t, s, \epsilon) - e^{Q(t)(t-s)/\epsilon}| \leq \omega(\epsilon) e^{-\mu(t-s)/\epsilon}$$

for $a \leq s \leq t \leq b$. However this result will not be required here.

3. It would seem natural now to make the change of variables $\xi = x - \theta(t)$ and $\eta = y - \chi(t)$ in (1.0) where (θ, χ) is the solution of (1.1) as already indicated. In (1.0) this would lead to

$$(3.0) \quad \begin{aligned} \frac{d\xi}{dt} &= f_x(t)\xi + f_y(t)\eta + \Delta_1 \\ \epsilon \frac{d\eta}{dt} &= g_x(t)\xi + g_y(t)\eta + \Delta_2 \end{aligned}$$

where $f_x(t)$ denotes $f_x(t, \theta, \chi, 0)$ etc. and where Δ_1 and Δ_2 are of less significance in $|\xi|$ and $|\eta|$ than the linear terms which appear explicitly on the right side of (3.0).

Actually there is a great advantage to removing the term $g_x(t)\xi$ from the right side of the second equation of (3.0) and this can be done quite easily by setting $w = \eta + U(t)\xi$, where $U = g_y^{-1}(t)g_x(t)$. Indeed it is convenient to eliminate η entirely from consideration and simply to make the change of variable

$$(3.1) \quad \xi = x - \theta(t) \quad w = y - \chi(t) + U(t)\xi$$

In (1.0) this becomes

$$(3.1) \quad \begin{aligned} \frac{d\xi}{dt} &= f(t, \xi + \theta, w + \chi - U\xi, \epsilon) - f(t, \theta, \chi, 0) \\ \epsilon \frac{dw}{dt} &= g(t, \xi + \theta, w + \chi - U\xi, \epsilon) - g(t, \theta, \chi, 0) \\ &\quad - \epsilon \frac{d\chi}{dt} + \epsilon \frac{dU}{dt} \xi + \epsilon U \frac{d\xi}{dt} \end{aligned}$$

Using the first equation of (3.1) to replace $d\xi/dt$ on the right of the second equation it follows that (3.1) can be written as

$$(3.2) \quad \begin{aligned} \frac{d\xi}{dt} &= (f_x(t) - f_y(t)U)\xi + f_y(t)w + F(t, \xi, w, \epsilon) \\ \epsilon \frac{dw}{dt} &= g_y(t)w + G(t, \xi, w, \epsilon) \end{aligned}$$

where $F = f(t, \xi + \theta, w + \chi - U\xi, \epsilon) - f(t, \theta, \chi, 0) - (f_x - f_y U)\xi - f_y w$ and similarly for G . (The analogue of (3.1) could be used in [2] to simplify the proofs there very considerably.) It follows from the mean value theorem that given any $\delta > 0$ there exists an ϵ_1 such that

$$(3.3) \quad |F(t, \tilde{\xi}, \tilde{w}, \epsilon) - F(t, \xi, w, \epsilon)| \leq \delta(|\tilde{\xi} - \xi| + |\tilde{w} - w|)$$

for $|\epsilon|$, $|\xi|$, $|\tilde{\xi}|$, $|w|$ and $|\tilde{w}| < \epsilon_1$ and a similar inequality holds for G . Also

$$(3.4) \quad |F(t, 0, 0, \epsilon)| \leq \omega(\epsilon)$$

and similarly for G . Note also that F_ξ , F_w , G_ξ and G_w exist and are continuous.

Let

$$(3.5) \quad w = P(t) \begin{pmatrix} u \\ v \end{pmatrix}$$

where $P(t)$ satisfies (1.3) and u and v are vectors with m and n components respectively where as already indicated $m + n = n_1$. Let

$$P = (P_1 P_2)$$

where $P_1(t)$ has m columns and $P_2(t)$ has n columns. Using (3.5) in (3.2) and recalling that $f_x - f_y U = A$ leads to

$$\begin{aligned} \frac{d\xi}{dt} &= A(t)\xi + A_1(t)u + A_2(t)v + H(t, \xi, u, v, \epsilon) \\ (3.6) \quad \epsilon \frac{du}{dt} &= B(t)u + R(t, \xi, u, v, \epsilon) \\ \epsilon \frac{dv}{dt} &= C(t)v + S(t, \xi, u, v, \epsilon) \end{aligned}$$

where $A_1 = f_y P_1$, $A_2 = f_y P_2$ and B and C are given by (1.3). Furthermore it is clear that A , A_1 , A_2 , B and C have period T in t as do H , R , and S . From (3.3) and (3.4) it follows easily that for any given $\delta > 0$ there exists an $\epsilon_2 > 0$ such that

$$(3.7) \quad |H(t, \tilde{\xi}, \tilde{u}, \tilde{v}, \epsilon) - H(t, \xi, u, v, \epsilon)| \leq \delta(|\tilde{\xi} - \xi| + |\tilde{u} - u| + |\tilde{v} - v|)$$

if $|\epsilon|$, $|\tilde{\xi}|$, $|\xi|$, $|\tilde{u}|$, $|u|$, $|\tilde{v}|$, and $|v| < \epsilon_2$ and similar inequalities hold for R and S . Also

$$(3.8) \quad |H(t, 0, 0, 0, \epsilon)| \leq \omega(\epsilon)$$

and similarly for R and S . Moreover since F_ξ , F_u , G_ξ and G_u exist and are continuous so are H_ξ , H_u , H_v , R_ξ etc.

4. To prove that (3.6) has a periodic solution a successive approximations procedure will be used. Let $\phi(t, s, \epsilon)$ be the solution of

$$\epsilon \frac{du}{dt} = B(t)u \quad u(s) = E$$

for $0 \leq s \leq t \leq T$. By the lemma of §2 there exist a K_1 and $\sigma > 0$ such that

$$(4.0) \quad |\phi(t, s, \epsilon)| \leq K_1 e^{-\sigma(t-s)/\epsilon} \quad s \leq t$$

and similarly for $\epsilon(dv/dt) = C(t)v$ there is a $\psi(t, s, \epsilon)$ such that

$$(4.1) \quad |\psi(t, s, \epsilon)| \leq K_2 e^{-\sigma(t-s)/\epsilon} \quad t \leq s$$

where the constant σ may be decreased in (4.0) if necessary for (4.1) also to be valid. Note K_1 and $K_2 > 1$ and it can be assumed $\sigma < 1$.

Because (1.4) has no solution of period T except $\xi = 0$ it follows that there

exists a Green's function $\Phi(t, s)$ such that if $b(t)$ is a continuous function of t on $0 \leq t \leq T$ then

$$\frac{d\xi}{dt} = A(t)\xi + b(t)$$

has as its unique solution of period T

$$\int_0^T \Phi(t, s)b(s) ds \quad 0 \leq t \leq T$$

Moreover $\Phi(t, s)$ is bounded and is continuous except at $t = s$.

Then (variation of constants formula) a solution of

$$(4.2) \quad \underline{\xi}(t) = \int_0^T \Phi(t, s)[A_1(s)u(s) + A_2(s)v(s) + H(s, \underline{\xi}(s), u(s), v(s), \epsilon)] ds$$

$$(4.3) \quad u(t) = \phi(t, 0, \epsilon)[E - \phi(T, 0, \epsilon)]^{-1} \frac{1}{\epsilon} \int_0^T \phi(T, s, \epsilon)R(s, \underline{\xi}, u, v, \epsilon) ds \\ + \frac{1}{\epsilon} \int_0^t \phi(t, s, \epsilon)R(s, \underline{\xi}(s), u(s), v(s), \epsilon) ds$$

$$(4.4) \quad v(t) = -\psi(t, T, \epsilon)[E - \psi(0, T, \epsilon)]^{-1} \frac{1}{\epsilon} \int_0^T \psi(0, s, \epsilon)S ds \\ - \frac{1}{\epsilon} \int_t^T \psi(t, s, \epsilon)S(s, \underline{\xi}(s), u(s), v(s), \epsilon) ds$$

is a periodic solution of (3.6). Conversely a periodic solution of (3.6) of period T is a solution of (4.2)–(4.4). For small ϵ the existence of the inverse matrices in (4.3) and (4.4) is an easy consequence of (4.0) and (4.1).

To prove that (4.2)–(4.4) has a solution replace $\underline{\xi}$, u and v on the right by $\underline{\xi}_n$, u_n and v_n and $\underline{\xi}$, u , v on the left by $\underline{\xi}_{n+1}$, u_{n+1} , v_{n+1} . Let the largest $\omega(\epsilon)$ appearing in (3.8) and its analogues for R and S be $\omega_0(\epsilon)$. Let $\xi_0 = u_0 = v_0 = 0$. Then using (4.0) and (4.1) it follows, since

$$(4.5) \quad |\Phi(t, s)|(1 + |A_1| + |A_2|) < K_3$$

for some $K_3 > 1$, that for small ϵ

$$(4.6) \quad |\underline{\xi}_1(t)| \leq K_3 T \omega_0(\epsilon) \\ |u_1(t)| \leq \frac{3}{\sigma} K_1^2 \omega_0(\epsilon) \\ |v_1(t)| \leq \frac{3}{\sigma} K_2^2 \omega_0(\epsilon)$$

With the aid of (3.7) and (4.6), if $|\xi_{n-1}|$, $|\xi_n|$, $|u_{n-1}|$, $|u_n|$, $|v_{n-1}|$ and $|v_n| < \epsilon_2$ then

$$(4.7) \quad |\xi_{n+1} - \xi_n| \leq K_3 T [2 \max(|u_n - u_{n-1}| + |v_n - v_{n-1}|) + \delta \max|\xi_n - \xi_{n-1}|]$$

$$(4.8) \quad |u_{n+1} - u_n| \leq \frac{3}{\sigma} K_1^2 \delta \max[|u_n - u_{n-1}| + |v_n - v_{n-1}| + |\xi_n - \xi_{n-1}|]$$

$$(4.9) \quad |v_{n+1} - v_n| \leq \frac{3}{\sigma} K_2^2 \delta \max[|u_n - u_{n-1}| + |v_n - v_{n-1}| + |\xi_n - \xi_{n-1}|]$$

Let $a = \min(1, (8K_3T)^{-1})$. If δ is chosen so that

$$(4.10) \quad \frac{3}{\sigma} (K_1^2 + K_2^2) \delta + \delta K_3 T < \frac{a}{10}$$

and if

$$D_n = \max(a|\xi_{n+1} - \xi_n| + |u_{n+1} - u_n| + |v_{n+1} - v_n|)$$

then (4.7)–(4.10) gives

$$(4.11) \quad D_n \leq \frac{1}{4}D_{n-1} + \frac{1}{10}D_{n-1} \leq \frac{1}{2}D_{n-1}$$

This assures the uniform convergence of ξ_n , u_n , v_n if these quantities remain less than ϵ_2 in magnitude. But since $\xi_0 = u_0 = v_0 = 0$

$$a|\xi_m| + |u_n| + |v_n| \leq a|\xi_1| + |u_1| + |v_1| + D_1 + \cdots + D_{n-1}$$

Using (4.11)

$$\begin{aligned} a|\xi_n| + |u_n| + |v_n| &\leq (a|\xi_1| + |u_1| + |v_1|)(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \\ &= 2(a|\xi_1| + |u_1| + |v_1|) \end{aligned}$$

Thus (4.6) shows that if $|\epsilon|$ is small enough all $|\xi_n|$, $|u_n|$, $|v_n| < \epsilon_2$.

Since ξ_n , u_n , and v_n have period T so do their limit functions ξ , u , v which are solutions of (4.2)–(4.4). It will next be shown that the solution is unique. If not let there be two solutions of (4.2)–(4.4) ξ , u , v and $\tilde{\xi}$, $\tilde{u}$, $\tilde{v}$.

Using (4.2)–(4.4) it follows easily that if

$$D = \max(a|\xi - \tilde{\xi}| + |\tilde{u} - u| + |\tilde{v} - v|)$$

then $D \leq \frac{1}{2}D$ much as in (4.11). Thus $D = 0$. This completes the proof.

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