*Remarks on Transforms and Boundary Value Problems*¹

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Introduction. The notion of “transform” is one of the most useful notions [1]*, [2] which can be used in applied mathematics, especially for the solutions of some boundary value problems. On the other hand these boundary value problems may be studied by the technique of the fundamental solution† and Green’s function. Looking more closely at the connection between these two mathematical techniques shows that to every problem which can be solved by the Green’s function technique there corresponds a theory of a special transform. It seems that this idea could be developed into a general and simple theory of such transforms. The following paper is an attempt to show briefly how such a theory may be developed starting with a simple class of equations. The main conclusion is that results which are generally given for the classical special functions—especially Bessel functions—can be simply proved and extended to the solutions of a class of differential equations. This work was undertaken initially in order to see if the method of transforms, at least in the simplest cases, could be applied to “mixed” problems. This problem is discussed in Section 4.

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* Numbers in brackets refer to the bibliography at the end of this paper.

† The word “elementary solution” is more usual in the French terminology.

1. Generalization of the sine Fourier transform. The basic differential equation* in this paper is

$$u_{zz} + k(z)u_{xx} = 0 \quad (1)$$

where u is the dependent variable, x and z the independent variables, $k(z)$ a positive continuously differentiable function of z in the interval $(0, \infty)$. The problem is to find a solution of (1) in the domain $x > 0, z > a$, where a is a positive constant, which satisfies the following boundary conditions:

$$\begin{aligned} x = 0, \quad z > a \quad u &= f(z) \\ z = a, \quad x > 0 \quad u &= 0. \end{aligned} \quad (2)$$

$f(z)$ is a given function such that $f(a) = 0$. Of course, $u(x, z)$ must be regular at infinity. One method which can be used in order to solve this problem is to represent this solution by a Laplace integral. Consider the ordinary differential equation

$$u_{zz}^* + \beta^2 k(z)u^* = 0, \quad (3)$$

and denote by $S_a(z, \beta)$ the solution of (3) which satisfies the following boundary conditions:

$$S_a(a, \beta) = 0 \quad \frac{\partial}{\partial z} S_a(a, \beta) = 1. \quad (4)$$

We can try to find the solution of the problem in the form

$$u(x, z) = \int_0^\infty \beta \varphi(\beta) S_a(z, \beta) \exp(-\beta x) d\beta; \quad (5)$$

the function on the right hand side is a solution of (3) and satisfies the second condition (2). Thus the problem is to choose $\varphi(\beta)$ so as to satisfy the first condition (2), *i.e.*

$$f(z) = \int_0^\infty \beta \varphi(\beta) S_a(z, \beta) d\beta. \quad (6)$$

The formula giving $\varphi(\beta)$ as a functional of $f(z)$ will be the inverse transform of the transform (6). Before proceeding let us remark that if $h(x)$ is the value of the z derivative of the solution along $z = a, x > 0$, then

$$h(x) = \int_0^\infty \beta \varphi(\beta) e^{-\beta x} d\beta; \quad (7)$$

* Such an equation is very important in theoretical fluid mechanics. It can be interpreted, for instance, as the equation of the stream function in the hodograph plane of an inviscid, two-dimensional, flow of a compressible fluid, where x is the angle of the velocity vector with a fixed direction, z is a function of the magnitude of the velocity, and $k(z)$ is a function which depends on the equation of state [4].

hence $h(x)$ is the Laplace transform of $\beta\varphi(\beta)$. If one can relate h and f by another method, one may expect to solve the equation (6).

It has been shown [4] how to construct the Green's function of such a problem by using a Fourier transform, the Fourier transform defined for summable functions by

$$\begin{aligned} U(\alpha) &= \mathbf{f}u = \int_{-\infty}^{+\infty} u(x, z) \exp(-2i\pi\alpha x) dx \\ u &= \mathbf{f}^{-1}U = \int_{-\infty}^{+\infty} U(\alpha) \exp(2i\pi\alpha x) d\alpha. \end{aligned} \quad (8)$$

Let us summarize the principal conclusions. A fundamental solution of (1) singular at $x = 0, z = z_0$ may be defined as $\mathbf{f}^{-1}(E(z, z_0))$, where E is itself a fundamental solution of

$$U_{zz} - 4\pi^2 \alpha^2 k(z)U = 0. \quad (9)$$

More precisely E is a solution of (9) except for $z = z_0$. If we set

$$E = E_1 \quad \text{when } z > z_0, \quad E = E_2 \quad \text{when } 0 < z < z_0,$$

E_1 and E_2 are solutions of (9) which satisfy

$$E_1(z_0, z_0) = E_2(z_0, z_0), \quad \frac{\partial E_1}{\partial z}(z_0, z_0) - \frac{\partial E_2}{\partial z}(z_0, z_0) = +1 \quad (10)$$

for every α . For the present problem, E_1 and E_2 must be defined in the following way. Let $K_2(z, \alpha)$ as the solution of (9) defined by the initial conditions

$$K_2(a, \alpha) = 0, \quad \frac{\partial K_2}{\partial z}(a, \alpha) = 1. \quad (11)$$

$K_2(z, \alpha)$ is an entire function in α according to the Poincaré theorem; define also $K_1(z, \alpha)$ as the solution of (9) which satisfies the following condition

$$K_1(a, \alpha) = -1, \quad \lim_{z \rightarrow \infty} K_1(z, \alpha) = 0, \quad \text{for } \alpha \text{ real and positive;} \quad (12)$$

the existence and the properties of such a solution will not be discussed here. We shall assume that as a function of α , $K_1(z, \alpha)$ is analytic in the right half plane $\Re(\alpha) \geq 0$ except perhaps at the origin $\alpha = 0$.

Now E_1 and E_2 are given by

$$E_1 = K_1(z, |\alpha|)K_2(z_0, \alpha), \quad E_2 = K_1(z_0, |\alpha|)K_2(z, \alpha), \quad (13)$$

and this choice is a correct one, because the conditions (10) are satisfied (for the second, note that the Wronskian $K_1' K_2 - K_2' K_1$, which is independent of z , is

equal to one for $z = a$). Finally the Green's function, $G(x, z; x_0, z_0)$, of the Dirichlet problem considered is given by

$$G(x, z; x_0, z_0) = \mathbf{f}^{-1}[E(z)\{\exp(-2i\pi\alpha x_0) - \exp(2i\pi\alpha x_0)\}]. \quad (14)$$

This Green's function gives the solution of the problem immediately in the form

$$u(x_0, z_0) = - \int_0^\infty k(z)f(z) \frac{\partial G}{\partial x}(0, z; x_0, z_0) dz. \quad (15)$$

In other words the doublet at the point $z = z_1$ on the z axis ($z_1 > a$) which vanishes along $z = a$ is given by

$$d(x, z; z_1) = -k(z_1) \frac{\partial G}{\partial x}(0, z_1, x, z)$$

(it can be said also that this doublet corresponds to the case in which $f(z) = \delta_{z_1}$, δ_{z_1} being the Dirac mass at $z = z_1$). Therefore

$$d(x, z; z_1) = \int_{-\infty}^{+\infty} k(z_1)E((z_1, z)2i\pi\alpha[\exp(2i\pi\alpha x) - \exp(-2i\pi\alpha x)] d\alpha, \quad (16)$$

and $h^*(x, z_1)$, the value of the z derivative of d for $z = a$, ($h^*(x, z_1)$ is the value of $h(x)$ for $f(z) = \delta_{z_1}$) is given by

$$h^*(x, z_1) = 2 \int_0^\infty k(z_1)K_1(z_1, \alpha)2i\pi\alpha [\exp(2i\pi\alpha x) - \exp(-2i\pi\alpha x)] d\alpha. \quad (17)$$

What we have to do is to compare (7) and (17) (in (7), h will be replaced by h^* , $\varphi(\beta)$ by $\varphi^*(B, z_1)$). Starting with (17), we divide the integral on the right hand side into two parts, each of them involving only one exponential. In the first of the two parts set $2\pi\alpha = i\beta$, in the second $2\pi\alpha = -i\beta$; here we use the fact that $K_1(\alpha, z_1)$ is analytic in α for $\Re(\alpha) \geq 0$. Thus (17) can be rewritten

$$h^*(x, z_1) = \frac{1}{i\pi} \int_0^\infty \beta \exp(-\beta x) k(z_1) \left\{ K_1\left(z_1, \frac{i\beta}{2\pi}\right) - K_1\left(z_1, -\frac{i\beta}{2\pi}\right) \right\} d\beta. \quad (18)$$

As a result, taking account of the symmetry properties of K_1 , we get

$$\varphi^*(\beta, z_1) = \frac{2k(z_1)}{\pi} \Im \left\{ K_1\left(z_1, \frac{i\beta}{2\pi}\right) \right\} = \frac{2k(z_1)}{\pi} \Im \{ \bar{K}_1(z_1, \beta) \}.$$

$\bar{K}_1(z, \beta)$ is the solution of (3) equal to -1 for $z = a$, which for values of β of argument $-\pi/2$ tends towards zero when $z \rightarrow +\infty$. $\Im \{ \bar{K}_1(z, \beta) \}$ is a solution of (3) which is zero for $z = a$; thus this solution as a function of z is proportional to $S_a(z, \beta)$, and so it is possible to write

$$\varphi^*(\beta, z_1) = \frac{2\omega(\beta)}{\pi} k(z_1) S_a(z_1, \beta) \quad (19)$$

with

$$\omega(\beta) = \Im \left\{ \frac{\partial \bar{K}_1}{\partial z}(a, \beta) \right\}. \quad (20)$$

By a classical argument, which is in fact nothing else than the superposition principle, we have the following

Theorem 1. *The transform defined by*

$$f(z) = \int_0^\infty \beta \varphi(\beta) S_a(z, \beta) d\beta \quad (6)$$

admits the following inverse formula

$$\varphi(\beta) = \frac{2\omega(\beta)}{\pi} \int_a^\infty k(z) f(z) S_a(z, \beta) dz. \quad (21)$$

The precise conditions required for the validity of this theorem will not be discussed here. Let us recall that $\bar{K}_1(z, \beta)$ defined above must be an analytic function of β for $\Im(\beta) < 0$. The conditions for $f(z)$ and $\varphi(\beta)$ in order to have convergent integrals are similar to the classical conditions for the Fourier and Hankel transforms.

Applications. A verification of the previous formulae can be obtained by considering the Laplace equation with $a = 0$. In this case $k(z) = 1$, $S_0(z, \beta) = \beta^{-1} \sin \beta z$, $\bar{K}_1(z, \beta) = -\exp(-i\beta z)$, $\omega(\beta) = \beta$. Thus we obtain the sine Fourier transform. But this cannot be considered as a new proof of the formulae of Fourier transform because the Fourier transform was used in order to establish the theorem. Now let us consider the case

$$k(z) = z^{1/n-2}; \quad (22)$$

the corresponding equation is an Euler-Poisson equation. In the special case $n = \frac{1}{3}$ it may be reduced to the Tricomi equation. The corresponding equation (3) is solved by introducing a new variable s defined by $s = 2nz^{\frac{1}{3}n}$; two independent solutions are $s^n J_n(\beta s)$, $s^n J_{-n}(\beta s)$ where J_n is the classical Bessel function. Thus in this case, taking first $a = 0$, we get

$$S_0(z, \beta) = 2^{-n} \Gamma(n+1) (n\beta)^{-2n} (\beta s)^n J_n(\beta s),$$

provided that n is positive. $\bar{K}_1(\beta, s)$ is expressed as a Hankel function,

$$\bar{K}_1(\beta, z) = i 2^{-n} \Gamma(1-n) (\beta s)^n \sin(\pi n) H_n^{(2)}(\beta s),$$

and

$$\omega(\beta) = \pi n^{2n+1} [\Gamma(n+1)]^{-2} \beta^{2n}.$$

If we put

$$f(z) = s^n F(s), \quad \Phi(\beta) = 2^n (2n)^{-2n} \Gamma(n+1) \beta^{-n} \varphi(\beta)$$

in the formulae (6), (21) we obtain the classical Hankel transform

$$\int_0^\infty \beta \Phi(\beta) J_n(\beta s) = F(s), \quad \Phi(\beta) = \int_0^\infty s J_n(\beta s) F(s) ds \quad (23)$$

which are valid for $n > 0$.

Now for the same value of $k(z)$ given by (22) consider the case $a > 0$ and define σ as the value of s which corresponds to $z = a$. Using an elementary property of the Wronskian, one can write

$$S_a(z, \beta) = \frac{\pi(\sigma s)^n}{2(2n)^{2n-1} \sin(\pi n)} \{J_n(\beta s) J_{-n}(\beta \sigma) - J_n(\beta \sigma) J_{-n}(\beta s)\},$$

and similarly

$$\bar{K}_1(\beta, z) = - \left(\frac{s}{\sigma} \right)^n \frac{H_n^{(2)}(\beta s)}{H_n^{(2)}(\beta \sigma)}.$$

The corresponding value of $\omega(\beta)$ is given by

$$\omega(\beta) = \frac{2 \sin^2(\pi n)}{\pi \sigma^{2n}} \frac{(2n)^{2n-1}}{\rho_n(\beta \sigma)},$$

where $\rho_n(z)$ is defined by

$$\rho_n(z) = J_n^2(z) + J_{-n}^2(z) - 2J_n(z)J_{-n}(z) \cos(\pi n). \quad (24)$$

To obtain the formulae in a convenient form, let

$$f(z) = s^n F(s), \quad \Phi(\beta) = \frac{\pi \sigma^n}{2(2n)^{2n-1}} \varphi(\beta).$$

Then the following formulae are obtained:

$$\int_0^\infty (\sin \pi n)^{-1} \{J_n(\beta s) J_{-n}(\beta \sigma) - J_n(\beta \sigma) J_{-n}(\beta s)\} \beta \Phi(\beta) d\beta = F(s), \quad (25)$$

$$\int_0^\infty (\sin \pi n)^{-1} \{J_n(\beta s) J_{-n}(\beta \sigma) - J_n(\beta \sigma) J_{-n}(\beta s)\} s F(s) ds = \rho_n(\beta \sigma) \sin^{-2} \pi n \Phi(\beta)$$

which is one of the possible generalizations of the Hankel transform [1], [5]. The last example, which to the best of the author's knowledge is new, is related to the equation (3) when

$$k(z) = \exp(2z) - 1. \quad (26)$$

The corresponding equation is closely related to an approximation in transonic flow theory initially proposed by S. TOMOTIKA and K. TAMADA. The correspond-

ing equation (3) can also be solved in terms of Bessel functions; with the substitution $t = \exp(z)$, (3) becomes

$$t^2 u_{tt}^* + tu_t^* + \beta^2(t^2 - 1)u^* = 0, \quad (27)$$

and two independent solutions are $J_\beta(\beta t)$ and $J_{-\beta}(\beta t)$. Let t_0 be defined by $t_0 = \exp(a)$. After a few easy calculations, it is found that

$$S_a(z, \beta) = (2 \sin \pi \beta)^{-1} \pi \{J_\beta(\beta t)J_{-\beta}(\beta t_0) - J_{-\beta}(\beta t)J_\beta(\beta t_0)\}$$

$$\bar{K}_1(\beta, z) = -\frac{H_\beta^2(\beta t)}{H_\beta^{(2)}(\beta t_0)}, \quad \omega(\beta) = \frac{2 \sin^2 \pi \beta}{\pi \rho_\beta(\beta t_0)},$$

where $\rho_n(z)$ is defined by (24). Thus the following pair of formulae are obtained

$$\begin{aligned} \int_0^\infty \beta \Phi(\beta) [J_\beta(\beta t)J_{-\beta}(\beta t_0) - J_{-\beta}(\beta t)J_\beta(\beta t_0)] d\beta &= f(t) \\ \int_{t_0}^\infty f(t) \left[t - \frac{1}{t} \right] [J_\beta(\beta t)J_{-\beta}(\beta t_0) - J_{-\beta}(\beta t)J_\beta(\beta t_0)] dt &= \rho_\beta(\beta t_0) \Phi(\beta). \end{aligned} \quad (28)$$

2. Generalization of the fourier cosine transform. Going back to the equation (1), let us consider the following problem: Find a solution u defined in $x > 0$, $z > a$ which satisfies the following boundary conditions:

$$\begin{aligned} x = 0, \quad z > 0, \quad u &= f(z), \\ z = a, \quad x > 0, \quad u_x &= 0. \end{aligned} \quad (29)$$

A method similar to the previous one may be applied. First of all, if $D_a(z, \beta)$ is the solution of (3) which satisfies

$$D_a(a, \beta) = 1, \quad \frac{\partial D_a}{\partial z}(a, \beta) = 0, \quad (30)$$

then it is possible to represent the solution by

$$u(x, z) = \int_0^\infty \beta \varphi(\beta) D_a(z, \beta) e^{-\beta x} d\beta. \quad (31)$$

The function defined by the right hand side is a solution of (1) which satisfies the second condition (29). In order to solve the problem, $\varphi(\beta)$ must be determined when $f(z)$ is given, i.e. $\varphi(\beta)$ is a solution of the integral equation

$$f(z) = \int_0^\infty \beta \varphi(\beta) D_a(z, \beta) d\beta. \quad (32)$$

But if $l(x) = u(x, a)$ is the value of the solution for $z = a, x > 0$, then

$$l(x) = \int_0^\infty \beta \varphi(\beta) e^{-\beta x} d\beta. \quad (33)$$

Now a direct solution may be given using what may still be called the Green's function of the problem, *i.e.* a fundamental solution in $x > 0, z > a$ which vanishes for $x = 0, z > a$ and whose normal derivative vanishes for $z = a, x > 0$. Let us start with a fundamental solution of (9), which satisfies the conditions (10). E_1 and E_2 will be given by (compare with (13))

$$E_1 = K_3(z, |\alpha|) K_4(z_0, \alpha), \quad E_2 = K_3(z_0, |\alpha|) K_4(z, \alpha),$$

where K_3 and K_4 are the solutions of (9) defined by the following properties:

$$K_4(a, \alpha) = 1, \quad \frac{\partial K_4}{\partial z}(a, \alpha) = 0.$$

Thus K_4 is an entire function of α according to the Poincaré Theorem

$$\frac{\partial K_3}{\partial z}(a, \alpha) = 1, \quad \lim_{z \rightarrow \infty} K_3(z, \alpha) = 0 \text{ for } \alpha \text{ real and positive.} \quad (34)$$

The existence and properties of K_3 will not be discussed here. It will be assumed in the following that $K_3(z, \alpha)$ is an analytic function of α for $\Re(\alpha) > 0$. Observe that the Wronskian $K_3' K_4 - K_3 K_4'$ is equal to 1, which shows that (10) is satisfied. Now it is possible to write the Green's function of our problem as

$$G(x, z; x_0, z_0) = f^{-1} E(z, z_0) \{ \exp(-2i\pi\alpha x_0) - \exp(2i\pi\alpha x) \}. \quad (32)$$

Now the associated doublet which corresponds to the case $f(z) = \delta_{z_1}$ is again given by (16), and the value $l^*(x, z_1)$ of this doublet for $z = a$ is

$$l^*(x, z_1) = 2 \int_0^\infty k(z_1) K_3(z_1, \alpha) 2i\pi\alpha [\exp(2i\pi\alpha x) - \exp(-2i\pi\alpha)] d\alpha.$$

Using the same transformation, previously employed for passing from (17) to (18), we can express $l^*(x, z_1)$ by the right hand side of (18) when K_1 is replaced by K_3 . If we denote by $\varphi^*(\beta, z_1)$ the value of $\varphi(\beta)$ in (31) which corresponds to this doublet, and if we compare the expression of l^* with the formula (31), we obtain

$$\varphi^*(\beta, z_1) = \frac{2k(z_1)}{\pi} \Im K_3 \left(z_1, \frac{i\beta}{2\pi} \right) = \frac{2k(z_1)}{\pi} \Im \bar{K}_3(z_1, \beta).$$

But $\bar{K}_3(z, \beta)$ is the solution of (3) such that $\frac{\partial \bar{K}_3}{\partial z}(a, \beta) = 1$; for values of β of ar-

gument $-\pi/2$ this solution tends towards zero when $z \rightarrow \infty$. Thus $\Im \bar{K}_3(\beta, z)$, which is also a solution of (3), is proportional to $D_a(z, \beta)$, and thus

$$\varphi^*(\beta, z) = \frac{2k(z)}{\pi} \bar{\omega}(\beta) D_a(z, \beta), \quad (33)$$

where

$$\bar{\omega}(\beta) = \Im \{ \bar{K}_3(a, \beta) \}. \quad (34)$$

Applying the superposition principle we get the

Theorem 2. *The transform defined by*

$$f(z) = \int_0^\infty \beta \varphi(\beta) D_a(z, \beta) d\beta$$

admits the following inverse formula (35)

$$\varphi(\beta) = \frac{2\bar{\omega}(\beta)}{\pi} \int_a^\infty k(z) f(z) D_a(z, \beta) dz.$$

Applications. The classical Fourier cosine transform will be used in order to check the formulae (35). In the case $a = 0$

$$D_0(z, \beta) = \cos \beta z, \quad \bar{K}_3(z, \beta) = i\beta^{-1} \exp(-i\beta z), \quad \bar{\omega}(\beta) = \beta^{-1},$$

and (35) gives the classical formulae.

Now for the equation defined by (22) and $a = 0$ (recall that $s = 2nz^{\frac{1}{2}n}$)

$$D_0(z, \beta) = 2^{-n} \Gamma(1-n)(\beta s)^n J_{-n}(\beta s)$$

provided that n is less than one. Moreover,

$$\bar{K}_3(z, \beta) = \frac{i \sin(\pi n) 2^n \Gamma(n+1)(\beta s)^n H_n^{(2)}(\beta s)}{(2n\beta)^{2n} \exp(in\pi)}$$

and as a result

$$\bar{\omega}(\beta) = \pi \beta^{-2n} n^{1-2n} [\Gamma(1-n)]^{-2}.$$

The final formulae obtained from this example are identical to (23), but now the range of validity is extended to $n > -1$.

In the next example the same equation is considered, but a is taken different from zero and σ is the corresponding value of s ($\sigma = 2na^{\frac{1}{2}n}$). Then

$$D_a(z, \beta) = \left(\frac{s}{\sigma}\right)^n \frac{\pi \beta \sigma}{2 \sin(\pi n)} [J_{-n}(\beta s) J_{n-1}(\beta \sigma) + J_n(\beta s) J_{1-n}(\beta \sigma)]$$

$$\bar{K}_3(z, \beta) = \frac{(s\sigma)^n}{(2n)^{2n-1} \beta \sigma} \frac{J_n(\beta s) \exp(in\pi) - J_{-n}(\beta s)}{J_{n-1}(\beta \sigma) \exp(in\pi) + J_{1-n}(\beta \sigma)},$$

and then

$$\bar{\omega}(\beta) = \frac{2\sigma^{2n} \sin^2(n\pi)}{(2n)^{2n-1} \pi(\beta\sigma)^2 \pi_n(\beta\sigma)},$$

where

$$\pi_n(z) = J_{n-1}^2(z) + J_{1-n}^2(z) + 2J_{1-n}(z)J_{n-1}(z) \cos(n\pi).$$

Then after some obvious simplifications, the corresponding transform formulae are obtained:

$$\begin{aligned} \int_0^\infty \beta \Phi(\beta) [J_{-n}(\beta s) J_{n-1}(\beta \sigma) + J_n(\beta s) J_{1-n}(\beta \sigma)] d\beta &= F(s) \\ \pi_n(\beta \sigma) \Phi(\beta) &= \int_0^\infty [J_{-n}(\beta s) J_{n-1}(\beta \sigma) + J_n(\beta s) J_{1-n}(\beta \sigma)] s F(s) ds. \end{aligned} \quad (36)$$

The last example corresponds to the value of $k(z)$ given by (26), when the equation (3) takes the form (27). The following formulae are easily obtained, if t_0 is the number, greater than one, defined by $t_0 = \exp(a)$:

$$D_a(z, \beta) = \frac{\pi \beta t_0}{2 \sin \pi \beta} [J'_\beta(\beta t_0) J_{-\beta}(\beta t) - J_\beta(\beta t) J'_{-\beta}(\beta t_0)],$$

where J' is used to denote the ordinary derivative of the Bessel function, and

$$\bar{K}_s(z, \beta) = \frac{H_\beta^{(2)}(\beta t)}{\beta t_0 H_\beta^{(2)}(\beta t_0)} = \frac{J_\beta(\beta t) \exp(i\pi\beta) - J_{-\beta}(\beta t)}{\beta t_0 (J_\beta(\beta t_0) \exp(i\pi\beta) - J_{-\beta}(\beta t_0))}.$$

Thus

$$\bar{\omega}(\beta) = \frac{2 \sin^2 \pi \beta}{\pi(\beta t_0)^2 q_\beta(\beta t_0)},$$

where

$$q_n(z) = J_n'^2(z) + J_{-n}^2(z) - 2J_n'(z)J_{-n}'(z) \cos(n\pi).$$

The final transform formulae can be written

$$\begin{aligned} \int_0^\infty \beta [J'_\beta(\beta t_0) J_{-\beta}(\beta t) - J_\beta(\beta t) J'_{-\beta}(\beta t_0)] \Phi(\beta) d\beta &= f(t) \\ q_\beta(\beta t_0) \Phi(\beta) &= \int_{t_0}^\infty [J'_\beta(\beta t_0) J_{-\beta}(\beta t) - J_\beta(\beta t) J'_{-\beta}(\beta t_0)] \left(t - \frac{1}{t}\right) f(t) dt. \end{aligned} \quad (37)$$

3. Possibility of obtaining the previous formulae from problems related to an equation of the hyperbolic type. The main interest of the preceding formulae is

to allow one to write the solutions of special boundary value problems in a very simple manner. But these formulae may be applied to different problems. It will be shown now how these formulae can be used in order to solve some special boundary value problems related to an equation of the hyperbolic type. More precisely, it will be proved that the same formulae may be derived by a similar method from the solutions of particular problems.

Consider the equation

$$u_{zz} - k(z)u_{xx} = 0 \quad (38)$$

where $k(z)$ is a positive, continuously differentiable function of z in the interval $(0, +\infty)$. The solution of (38) is to be defined in $x > 0, z > a$ with the following data

$$\begin{aligned} u(x, a) &= 0 & \text{for } x > 0 \\ u(0, z) &= 0, \quad u_x(0, z) = f(z) & \text{for } z > a. \end{aligned} \quad (39)$$

The problem is "well set," since the data along Oz define a Cauchy problem and the solution in the remaining part of the domain is the solution of a classical problem (data on a line and an adjacent characteristic); but the solution generally will admit discontinuous derivatives along the characteristics which separate the two sub-domains. Obviously any product of a solution of (3) by $\cos \beta x$ or $\sin \beta x$ will be a solution of (38). Thus if $S_a(z, \beta)$ is the solution of (3) and previously defined by (4), it is possible to write the solution we are looking for in the form

$$u(x, z) = \int_0^\infty \varphi(\beta) S_a(z, \beta) \sin(\beta x) d\beta. \quad (40)$$

Moreover, all the conditions (39) will be satisfied if $\varphi(\beta)$ is chosen in such a way that

$$f(z) = \int_0^\infty \beta \varphi(\beta) S_a(z, \beta) d\beta. \quad (41)$$

This is precisely the condition (6), and thus (21) gives the solution of this integral equation. But it will be instructive to solve the problem directly. Observe first that the solution of (38) defined by the right hand side of (40) has a normal derivative $m(x)$ along $z = u, x > 0$ given by

$$m(x) = \int_0^\infty \varphi(\beta) \sin(\beta x) d\beta. \quad (42)$$

Now, by using the method of [3] or [4] the Green's function of this problem can be formed. To obtain a fundamental solution of (38) we have to consider the differential equation

$$U_{zz} + 4\pi^2 \alpha^2 k(z)U = 0 \quad (43)$$

and its two solutions $X_1(z, \alpha)$ and $X_2(z, \alpha)$ defined by

$$X_2(a, \alpha) = 0, \quad \frac{\partial X_2}{\partial z}(a, \alpha) = 1$$

$$X_1(a, \alpha) = -1, \quad \lim_{z \rightarrow \infty} X_1(z, \alpha) = 0 \quad \text{for} \quad \arg(\alpha) = \tfrac{1}{2}\pi.$$

One fundamental solution $E(z, z_0)$ of (43) is given by

$$E = E_1 = X_1(z, \alpha)X_2(z_0, \alpha) \quad \text{for} \quad z > z_0,$$

$$E = E_2 = X_1(z_0, \alpha)X_2(z, \alpha) \quad \text{for} \quad a < z < z_0,$$

and the conditions (10) are satisfied because the Wronskian $X_1' X_2 - X_1 X_2' = 1$. Thus according to the results of [3] or [4],

$$\mathbf{f}^{-1}(E(z, z_0) \exp(-2i\pi\alpha x_0))$$

is the corresponding Green's function of the problem because it is zero for $z = a$; on the other hand, according to the choice of X_1 , if $X_1(z, \alpha)$ is assumed to be analytic for $\Im m(\alpha) \geq 0$, the function is different from zero only in the region bounded by the two arcs of characteristics starting from x_0, z_0 and drawn for $x < x_0$. As previously, the case $f(z) = \delta_{z_1}$ may be considered first. (The solution is a distribution solution.) The corresponding solution is written by application of the Green's formula

$$d(x, z; z_1) = -k(z_1) \int_{-\infty}^{+\infty} E(z_1, z) \exp(-2i\pi\alpha x) d\alpha,$$

and its normal derivative $m^*(x, z_1)$ along $z = a, x > 0$ is

$$m^*(x, z_1) = -k(z_1) \int_{-\infty}^{+\infty} X_1(z_1, \alpha) \exp(-2i\pi\alpha x) d\alpha.$$

Using once more the properties of $X_1(z, \alpha)$, by application of Cauchy's theorem we can rewrite this formula in a more convenient form:

$$\begin{aligned} m^*(x, z_1) &= -2ik(z_1) \int_{-\infty}^{+\infty} X_1(z_1, \alpha) \sin 2\pi\alpha x d\alpha \\ &= -4k(z_1) \int_0^{\infty} \operatorname{Im}(X_1) \sin(2\pi\alpha x) d\alpha \\ &= -\frac{2}{\pi} k(z_1) \int_0^{\infty} \operatorname{Im}(\bar{X}_1(z_1, \beta)) \sin \beta x d\beta \end{aligned} \tag{44}$$

if

$$\bar{X}_1(z, \beta) = X_1(z, 2\pi\alpha).$$

$\bar{X}_1(z, \beta)$ is a solution of (3) which assumes the value -1 for $z = a$. Therefore

$$\Im\{\bar{X}_1(z, \beta)\} = -\omega(\beta)S_a(z, \beta) \quad (45)$$

where

$$\bar{\omega}(\beta) = -\frac{\partial}{\partial z} \Im\{\bar{X}_1(a, \beta)\}. \quad (46)$$

Now writing (42) for this special case with $m^*(x, z_1)$ and $\varphi^*(\beta, z_1)$ instead of $m(x)$ and $\varphi(\beta)$, respectively, then comparing with (44) and (45), we obtain

$$\varphi^*(\beta, z_1) = \frac{2}{\pi} \omega(\beta)k(z_1)S_a(z, \beta). \quad (47)$$

This result must be compared with (19), and the only thing required in order to prove the identity of the two treatments is to show that $\omega(\beta)$ defined by (20) and (46) is the same function. This is a direct consequence of the fact that

$$\bar{X}_1(z, \beta) = \bar{K}_1(z, \beta \exp(-i\pi))$$

and that $\bar{X}_1(z, \beta)$ and $\bar{K}_1(z, \beta)$, for purely imaginary values of β , are real. Of course, it would be possible to derive similarly the result of the second section by considering the solution of (38) defined in $x > 0, z > a$, by the following boundary data:

$$\begin{aligned} \frac{\partial u}{\partial z}(x, a) &= 0 & \text{for } x > 0 \\ u(0, z) &= 0, \quad \frac{\partial u}{\partial x}(0, z) = f(z) & \text{for } z > a. \end{aligned}$$

4. Remarks on a special formula associated with a problem related to an equation of the mixed type. In [5] the following result is proved: if

$$f(t) = \int_0^\infty \beta \Phi(\beta) J_\beta(\beta t) d\beta \quad (48)$$

then

$$\Phi(\beta) = \int_0^\infty \left(t - \frac{1}{t}\right) f(t) J_\beta(\beta t) dt. \quad (49)$$

The same result is reproduced in [1]. It is presented as a generalized Hankel Transform, and the result is formulated in the same terms used for the classical transforms (Fourier or Hankel). It will be pointed out now that this result has not at all the same significance; in fact, the converse is generally untrue, *i.e.* if $f(t)$ is given and $\Phi(\beta)$ computed by (49), it is not at all certain that (48) is satisfied.

If (48) and (49) are compared with (28) and (37), it appears that the formulae are related to a problem concerning an equation of the mixed type for which the solution is to be found in a region which involves, at the same time, points in which the equation is of elliptic type and points in which the equation is of hyperbolic type. *A priori*, this makes the formulae very interesting. To be more precise consider, for instance, the equation

$$u_{zz} + [1 - \exp(-2z)]u_{xx} = 0, \quad (50)$$

which has been considered previously in [4] and which is of special interest for certain transonic flow problems. It is possible to use the method of separation of variables and to derive solutions of (50) of the type $U(z) \exp(i\beta x)$. If we introduce $t = \exp(-z)$, then U as a function of t must be a solution of

$$t^2 U_{tt} + tU_t - \beta^2(1 - t^2)U = 0, \quad (51)$$

which is in fact, with obvious changes in notations, nothing but (27). In order to interpret the pair of formulae (48), (49), the following solution of (50) may be considered

$$u(x, z) = \int_0^\infty \beta \Phi(\beta) J_\beta(\beta t) \cos \beta x \, d\beta. \quad (52)$$

This shows that this transform is related to some problem involving the value of t such that $t > 0$, *i.e.* the whole range of values of z . Thus the mixed type of equation will be involved in the analysis.

Let us first investigate the properties of the solution defined by (52). $J_\beta(\beta t)$ is a solution of (51) which tends toward zero when t tends toward zero, that is to say, when z increases indefinitely. Thus u as defined by (52) is a regular solution at infinity in the elliptic half plane. Along the z axis, $u_x(0, z) = 0$, and according to (48) $u(0, z) = f(t)$. In order to determine whether a formula like (49) is valid whatever be $f(t)$, we have to see if it is possible to find a solution of (50), regular at infinity in the elliptic half plane, which satisfies the following data:

$$u_x(0, z) = 0, \quad u(0, z) = f(z) \quad \text{for } -\infty < z < +\infty. \quad (53)$$

Along $x = 0$, $-\infty < z < 0$, these are Cauchy data and so define the solution for each x such that $0 < x < h(z)$ if $x = h(z)$ defines the equation of the characteristic Γ passing through the origin drawn for $x > 0$, $z < 0$. Now u is known on $x = 0$, $0 < z < \infty$ and on Γ . Thus u satisfies Tricomi data, and these data are sufficient in order to define u in the rest of the domain $x > 0$. As a result, it is seen that for an arbitrary $f(z)$, the condition $u_x(0, z) = 0$ for $0 < z < +\infty$ will not be satisfied. Thus it appears likely that, for an arbitrary $f(z)$, no $\Phi(\beta)$ can be found which checks (48).

A more precise analysis using the same ideas developed in the previous sections would allow one to prove the formula (49) when it is valid.

Put $\beta = 2\pi\alpha$ —in order to be able to use the Fourier transform defined by (8)—

and consider the fundamental solution which is defined in the whole x, z plane and which has an "orientation" opposite to the x axis [4]. Such a fundamental solution, singular in x_0, z_0 ($x_0 > 0$) is given by $\mathbf{f}^{-1}\{E(z, z_0) \exp(-2i\pi\alpha x_0)\}$ where

$$E(z, z_0) = \begin{cases} -\frac{\pi i}{2} J_{|\beta|}(|\beta t|) H_{\beta}^{(1)}(\beta t_0) & \text{for } z > z_0 \\ -\frac{\pi i}{2} J_{|\beta|}(|\beta t_0|) H_{\beta}^{(1)}(\beta t) & \text{for } z < z_0. \end{cases} \quad (54)$$

The Fourier integral will be taken along the real axis of the α plane. For β negative $H_{\beta}^{(1)}(\beta t)$ is defined by analytic continuation in the half plane $\text{Im}(\beta) > 0$. The normal derivative of this solution along $x = 0$ is given by $\mathbf{f}^{-1}\{2i\pi\alpha E(z, z_0) \exp(-2i\pi\alpha x_0)\}$. Now we apply the Green's formula relative to (50) in the domain $x > 0$ to this fundamental solution and to a solution $u(x, z)$ defined in $x > 0$, regular at infinity in the elliptic half plane, which satisfies the boundary conditions (53) along the z axis; [for instance, such a solution is defined by (52) if $f(z)$ is given by (48)]. It can be shown that no contribution arises from infinity either in the elliptic half plane because of the regularity of u at infinity or in the hyperbolic half plane because of the "orientation" of the fundamental solution. Accordingly, the Green's formula allows one to express $u(x, z)$ in terms of the data along $0z$, that is to say, in terms of $f(z)$. The result can be written

$$\begin{aligned} u(x_0, z_0) = & \int_0^{t_0} f(t) \left(t - \frac{1}{t}\right) dt \int_{-\infty}^{+\infty} \pi^2 \alpha J_{|\beta|}(|\beta t|) H_{\beta}^{(1)}(\beta t_0) \exp(-2i\pi\alpha x_0) d\alpha \\ & + \int_{t_0}^{\infty} f(t) \left(t - \frac{1}{t}\right) dt \int_{-\infty}^{+\infty} \pi^2 \alpha J_{|\beta|}(|\beta t_0|) H_{\beta}^{(1)}(\beta t) \exp(-2i\pi\alpha x_0) d\alpha. \end{aligned} \quad (55)$$

Now, using the property

$$H_{\beta}^{(1)}(\beta \exp(i\pi)t) = -H_{\beta}^{(2)}(\beta t),$$

which allows one to obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \pi^2 \alpha J_{|\beta|}(|\beta t|) H_{\beta}^{(1)}(\beta t_0) \exp(-2i\pi\alpha x_0) d\alpha \\ & = \frac{1}{2} \int_0^{\infty} \beta J_{\beta}(\beta t) J_{\beta}(\beta t_0) \cos(\beta x_0) d\beta + \frac{1}{2} \int_0^{\infty} \beta J_{\beta}(\beta t) Y_{\beta}(\beta t_0) \sin(\beta x_0) d\beta \end{aligned}$$

and the similar formula obtained by exchanging t and t_0 , we can rewrite (55) as

$$\begin{aligned} u(x_0, z_0) = & \frac{1}{2} \int_0^{\infty} f(t) \left(t - \frac{1}{t}\right) dt \int_0^{\infty} \beta J_{\beta}(\beta t) J_{\beta}(\beta t_0) \cos(\beta x_0) d\beta \\ & + \frac{1}{2} \int_0^{\infty} f(t) \left(t - \frac{1}{t}\right) dt \int_0^{\infty} \beta \sin(\beta x_0) d\beta \begin{cases} J_{\beta}(\beta t) Y_{\beta}(\beta t_0) & \text{for } t < t_0 \\ J_{\beta}(\beta t_0) Y_{\beta}(\beta t) & \text{for } t > t_0. \end{cases} \end{aligned} \quad (56)$$

The term in the right hand side written in the second line is easy to interpret* as a distribution of doublets along the z axis. As a function of x_0 and z_0 , it represents an odd function in x_0 and is a solution of (50) except on the z axes. Across this axis, this function admits a jump equal to $f(t_0)$, but its first derivative in x_0 is continuous. On the other hand, the first term in the right hand side of (56) represents a continuously differentiable solution of (50) and its normal derivative along Oz is zero. Now for $x_0 \geq 0$, $u(x_0, z_0)$, according to the assumptions, satisfies the boundary conditions along the z axis. Thus for $x_0 < 0$ the function defined by the right hand side of (56) is a solution of (50) which vanishes when x_0 tends towards zero and whose normal derivative is zero along the z axis. Thus this solution is identically zero for $x_0 < 0$, and accordingly for $x_0 > 0$ (56) may be written

$$u(x_0, z_0) = \int_0^\infty f(t) \left(t - \frac{1}{t} \right) dt \int_0^\infty \beta J_\beta(\beta t) J_\beta(\beta t_0) \cos(\beta x_0) d\beta. \quad (57)$$

But it must be kept in mind that this passage from (56) to (57) is *valid only for this special choice of $f(t)$* which is the value for $x = 0$ of a solution u of (50) whose normal derivative is zero on the z axis. Now changing the order of integrations in (57) and comparing with the formula (50) in which x and z are replaced by x_0 and z_0 , it is quite clear that the function $\Phi(\beta)$ which corresponds to this solution u is given by (49). In conclusion considering a function $\Phi(\beta)$, if we compute $f(t)$ by (48), $\Phi(\beta)$ can be expressed in terms of $f(t)$ by means of (49). But the proof which was given here shows very clearly that the converse is not true. The pair of formulae (48), (49) cannot be interpreted as a transform, and it is very unlikely that this nice result can be of use for the study of any mixed boundary value problem. This is why, despite the fact that similar formulae can be written by starting with an equation of mixed type different from (50), it does not seem worthwhile to proceed in such a direction. If any transform formulae may be associated with an equation of mixed type, it seems that they must be related to a "well set" problem. Unfortunately, it seems unlikely that formulae as simple as those found in the previous sections may be found for the Tricomi problem. In order to support this feeling, let us consider briefly the Tricomi equation,

* The function $\mathfrak{F}^{-1} E(t, t_0)$ where $E(t, t_0)$ is defined by

$$E(t, t_0) = \begin{cases} \frac{\pi}{2} J_\beta(\beta t) Y(\beta t_0) & \text{for } t < t_0 \\ \frac{\pi}{2} J_\beta(\beta t_0) Y(\beta t) & \text{for } t > t_0, \end{cases}$$

is a fundamental solution of (50) at the singular point in $x_0 = 0$, $z_0 = -\log t_0$, defined in the whole x, z plane. The associated doublet can be obtained by the method used in the previous sections. It is defined by

$$-\frac{k(z)}{2} \int_0^\infty \beta \sin(\beta x_0) E(t, t_0) d\beta.$$

$zu_{xx} + u_{zz} = 0$, and the Tricomi problem defined in the domain lying on the right of the contour $x = 0, z > 0$ and $3x = 2(-z)^{\frac{1}{3}}, z < 0$, when the solution must take prescribed values on $0z$, be regular at infinity in the elliptic part, and vanish on the characteristic at the infinity. An expression for the Green's function of this problem was given recently [6]. Let y be defined by $y = \frac{2}{3}(-z)^{\frac{1}{3}}$ in order to apply the ideas of the previous section and look for the solution in the form

$$u(x, z) = \int_0^\infty \varphi(\beta) K_{\frac{1}{3}}(\beta y) y^{\frac{1}{3}} e^{-\beta x} dx. \quad (58)$$

This solution satisfies all the requirements at infinity and $\varphi(\beta)$ must be chosen in order that u take the prescribed value for $z > 0, x = 0$. Thus using $s = \frac{2}{3}z^{\frac{1}{3}}$ for $z > 0, K_{\frac{1}{3}}(\beta y)$ for $z > 0$ is to be replaced by $3^{\frac{1}{3}} 2^{-1}(J_{\frac{1}{3}}(\beta s) + J_{-\frac{1}{3}}(\beta s))$; the problem reduces to solving the equation

$$3^{\frac{1}{3}} s^{\frac{1}{3}} \int_0^\infty \varphi(\beta) [J_{\frac{1}{3}}(\beta s) + J_{-\frac{1}{3}}(\beta s)] = 2f(s), \quad (59)$$

where $f(s)$ is a known function. It is known that for sufficiently regular f , this equation has a solution; theoretically it is possible to find the inverse formula of (59) by considering, as previously, (58) for $y = 0$ and using the direct solution of [6]. But it does not seem that the result may be expressed in a simple manner in terms of the known special functions.

Conclusion. This paper was an attempt to study the relations between transforms and simple boundary value problems. Some general formulae (Theorems I and II) show in fact that the Fourier integral theorem, the Fourier-Dini integral theorem and many of the consequences which can be derived from them can easily be generalized. The same formulae can be applied as well to elliptic as to hyperbolic problems. It does not seem to be possible to obtain correspondingly simple results for the case of problems related to an equation of mixed type. As an application of the approach, the interpretation of a result, called a generalized Hankel Transform, was made precise. Of course, it will be easy to extend the method and the results to the case of finite transforms [2].

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