On a Class of Partial Differential Equations

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1. INTRODUCTION

1.1. The problem considered in the present paper. One of the efficient tools in the study of the harmonic equation $\Delta\psi = 0$ is the theorem that the harmonic polynomials

$$\operatorname{Re}(x + iy)^n, \quad \operatorname{Im}(x + iy)^n, \quad n = 0, 1, \dots,$$

represent an infinite set of independent particular solutions of this equation. It is natural in passing to other partial differential equations in two variables to determine an infinite set of independent particular solutions which can be characterized in a simple manner, *e.g.*, by the fact that each of these solutions satisfies an ordinary differential equation, and to draw from this fact various conclusions.

In the present paper we shall consider a class of partial differential equations possessing particular solutions which satisfy not only the partial differential equation itself but also ordinary differential equations with algebraic coefficients. The existence and the form of these ordinary differential equations will be investigated in a later paper. In this way we can use the theory of ordinary differential equations as an efficient tool when investigating the properties of partial differential equations. The construction of the desired particular solutions of the partial differential equations will be done by the aid of BERGMAN's operators (see definition below).

Obviously if instead of a linear space of functions one considers an algebra (*i.e.*, a set of functions such that addition and multiplication reproduces the functions belonging to the set) one obtains various new relations and one can gain a deeper insight into the laws governing the class under consideration. The basic idea of the BERGMAN theory of operators consists in establishing different

possibilities of mapping the set of solutions of a linear partial differential equation on the algebra of functions of complex variables. It is of great interest that as BERGMAN showed we can choose the operator $\mathbf{P}(f)$ so that the mapping

$$\psi = \mathbf{P}(f)$$

which at first is defined in the small can be continued in the large and that many properties of the analytic function f are preserved. In this way we see that many results in the theory of functions of a complex variable lead to results not only for harmonic functions but also for solutions of other partial differential equations. BERGMAN's procedure shows further that for various purposes different operators are particularly useful. The choice of the special form of the operator for investigating a problem is therefore of basic importance.

There exists an interesting class of operators denoted as "Bergman's operators of the third kind" where the generating function $E(z, z^*, t)$ (see definition below) possesses the property that it satisfies ordinary differential equations considered as a function of z as well as a function of t .

1.2. Content of the present paper. We shall restrict ourselves to a certain subclass of these operators of the third kind which we call "*operators of exponential type*" (definition below); it may be noted however that the whole theory can be developed also for a much more general type of operators and corresponding partial differential equations.

The purpose of the present paper is to determine partial differential equations of second order which possess solutions generated by "*generating functions of exponential type*" (definition below). We shall see that it is possible to determine the whole class of partial differential equations with this property. Particular solutions of this class of partial differential equations satisfy also an ordinary differential equation with algebraic coefficients, as we shall prove in a later paper; we shall then be able to investigate relations between the singularities of the coefficients of the partial differential equation and those of the coefficients of the corresponding ordinary differential equation.

1.3. Notation. We start from the given partial differential equation

$$(1.1) \quad \Delta\psi + \alpha(x, y) \frac{\partial\psi}{\partial x} + \beta(x, y) \frac{\partial\psi}{\partial y} + \gamma(x, y)\psi = 0,$$

where the coefficients α , β , and γ are analytic functions of two real variables x and y . We continue these functions to complex values of the variables. We write

$$z_1 = x + iX, \quad z_2 = y + iY.$$

For our purpose it is then convenient to introduce the new variables

$$z = z_1 + iz_2, \quad z^* = z_1 - iz_2.$$

z and z^* are independent for complex values of z_1 and z_2 . If however z_1 and z_2 are real, *i.e.*, $X = Y = 0$, then

$$z^* = \bar{z}$$

where $\bar{z}$ denotes the number conjugate complex to z . Introducing the well known notations used by WIRTINGER [1]*, BERGMAN [2 *etc.*], KNESER [3] and others

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial z_1} - i \frac{\partial}{\partial z_2} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial z_1} + i \frac{\partial}{\partial z_2} \right),$$

we write the differential equation (1.1) in the form

$$(1.2) \quad w_{zz^*} + a(z, z^*)w_z + b(z, z^*)w_{z^*} + c(z, z^*)w = 0,$$

where the indices denote partial derivatives. The coefficients of (1.2) are analytic functions; they can have singularities.

We note that the expression (1.2) is in general equivalent to a system of two partial differential equations for the real and the imaginary part of the function $w(z, z^*)$. These two equations are completely independent and then both of the same form if and only if

$$b = \bar{a} \quad c = \bar{c}.$$

Using the operator

$$\mathbf{R}(w) = \frac{1}{2}[w(z, z^*) + \bar{w}(z^*, z)],$$

which for real values of z_1 and z_2 is the operator Re (“*take the real part*”), we obtain again a solution of (1.2) which is real for real z_1 and z_2 . As is well known, the partial differential equation (1.2) can be always transformed in such a manner that the term containing the derivative with respect to z does not appear. If we set

$$w = u \exp \left[- \int_0^{z^*} a(z, \zeta^*) d\zeta^* \right],$$

then we obtain from (1.2) the partial differential equation

$$(1.3) \quad L(u) \equiv u_{zz^*} + B(z, z^*)u_{z^*} + C(z, z^*)u = 0,$$

where

$$u_{zz^*} \equiv \frac{\partial^2 u}{\partial z \partial z^*}, \text{ etc.}$$

* Numbers in brackets refer to the bibliography at the end of this paper.

which we consider instead of (1.2) without any loss of generality. The coefficients of this equation are related to those of (1.2) by the following simple formulas:

$$(1.4) \quad B = b - \int_0^{z^*} a_z(z, \zeta^*) d\zeta^*, \quad C = c - a_z - ab.$$

For the sake of simplicity let us consider the equation (1.3) in our following investigations.

2. OPERATORS OF EXPONENTIAL TYPE

2.1. Bergman's operators. There exist operators transforming analytic functions of a complex variable into solutions of a given partial differential equation. These operators can be written in different forms. The choice of the operator and also the choice of its form depends on the purpose. In order to investigate particular solutions $u(z, z^*)$ of the partial differential equation (1.3) we represent them using BERGMAN's operators [4-7].

Definition 1. An operator of the form

$$(2.1) \quad u(z, z^*) = \mathbf{P}(f, O) \equiv \int_{-1}^1 E(z, z^*, t) f \left\{ \frac{z}{2} (1 - t^2) \right\} \frac{dt}{\sqrt{1 - t^2}}$$

is called a Bergman operator. The function E is denoted as the generating function of this operator; the function f is denoted as the associated function of $u(z, z^*)$.

E is independent of the special choice of f and depends on the coefficients of the partial differential equation (1.3) only. In the expression $\mathbf{P}(f, O)$ the second argument denotes the point of reference on which the integral operator depends. In the following we shall assume that the point of reference is the origin and we shall omit this second argument of $\mathbf{P}$.

By substituting (2.1) into (1.3) one can prove that E satisfies the partial differential equation

$$(2.2) \quad (1 - t^2)E_{z^*t} - E_{z^*}/t + 2zt L(E) = 0,$$

where L is defined by (1.3). The associated function f is an analytic function of a complex variable regular at the origin.

The functions E and f can have singularities. We assume that this is the case in general for the functions occurring in the present investigation. We consider all these functions in the neighborhood of a point where they are all regular. We can always find such a point in the neighborhood of the origin. Then analytic continuation permits us to extend the relations valid in the small also into the large.

Of course one can find other representations of solutions of a given partial differential equation; various authors have considered different operators and their representation. If one takes a Bergman operator then there exist again various possibilities for the special choice of the generating function E of this operator,

since every solution of (2.2) satisfying certain conditions can be used as a generating function of the operator (2.1) (see BERGMAN & SCHIFFER [2], BERGMAN, [4–7], EICHLER [8, 9] *etc.*). The choice depends on the purpose.

2.2. The idea of the operators (2.1). From (2.1) we see that the operator $\mathbf{P}(f)$ transforms analytic functions $f(z)$ into solutions $u(z, z^*)$ of partial differential equations. The idea of the operators developed by BERGMAN can therefore be considered as a natural generalization of the connection between functions of one complex variable and the solutions of Laplace's equation,

$$(2.3) \quad \Delta\psi \equiv 4u_{zz^*} = 0.$$

In the case (2.3) the simplest operator which transforms analytic functions into (complex) harmonic functions is given by $\mathbf{P}(f) = \mathbf{I}(f) \equiv f$ (identity). Then using the operation $\mathbf{R}(\mathbf{P}(f))$ (see Section 1.3), we obtain a real solution if z_1 and z_2 are real. The fact that for general equations (1.3) we take operators involving integrations causes difficulties which are to be overcome in the general theory.

The theorems of function theory can be formulated as theorems about harmonic functions. If one is interested in extending this correspondence to the solutions of partial differential equations (1.3), then Bergman's operator of the first kind is advantageous, *i.e.*, an operator (2.1) with a generating function of the form (see BERGMAN [10])

$$E = \exp \left[- \int_0^{z^*} a(z, \zeta^*) d\zeta^* \right] \left\{ 1 + \sum_{n=1}^{\infty} Q_n(z, z^*) t^n \right\}, \quad Q_n(z, 0) = 0.$$

2.3. The definition of the operators of exponential type. As already indicated the choice of the operator depends on the purpose. For the present investigations we take operators (2.1) which possess the property that if the associated function f of (2.1) is a polynomial the functions $u(z, z^*)$ generated by these operators satisfy also an ordinary differential equation with algebraic coefficients in addition to the given partial differential equation (1.3).

Definition 2. An operator (1.3) whose generating function has the form

$$(2.4) \quad E = e^Q, \quad Q(z, z^*, t) = \sum_{\mu=0}^m q_{\mu}(z, z^*) t^{\mu},$$

is called an operator of exponential type.

We shall prove in Theorem 1 that these operators arise in a large and interesting class of partial differential equations (1.3). The functions generated by these operators when applied to the associated functions $f(z) = z^n$, $n = 0, 1, \dots$, satisfy also an ordinary differential equation as we shall investigate in detail in a later paper. In the present paper we shall determine all partial differential equations (1.3) possessing the property that their particular solutions can be generated by operators of exponential type.

2.4. Generalizations. It may be noted that the theory developed in the present paper can be generalized and holds then also for other classes of partial differential equations (1.3) and corresponding operators. This generalization can be investigated using the following result of BERGMAN [5]:

If

(a) the generating function E of (2.1) satisfies differential equations with respect to z and t ,

$$\sum_{\nu=0}^n R_{\nu} \frac{\partial^{\nu} E}{\partial z^{\nu}} = 0, \quad R_n = 1, \quad \frac{\partial E}{\partial t} + \sum_{\nu=0}^{n-1} Q_{\nu} \frac{\partial^{\nu} E}{\partial z^{\nu}} = 0,$$

where R_{ν} and Q_{ν} are polynomials in t and algebraic functions in z and z^* ;

(b) the associated function $f(z)$ satisfies an ordinary differential equation

$$\frac{d^k f}{du^k} + \frac{1}{V} \sum_{\nu=0}^{k-1} P_{\nu} \frac{d^{\nu} f}{du^{\nu}} = 0, \quad V(0) \neq 0,$$

where $u = z(1 - t^2)/2$ and P_{ν} and V are polynomials in u ;

(c) there exists no relation of the form

$$\sum_{l=0}^{n-1} \sum_{\nu=0}^{k-1} \frac{\partial^{\nu} f(u)}{\partial z^{\nu}} \frac{\partial^l E}{\partial z^l} S^{(\nu l)}(z, z^*, t) = 0,$$

where $S^{(\nu l)}$ are algebraic functions of z and z^* and polynomials in t , then (2.1) satisfies a linear ordinary differential equation with respect to z .

The operators with such a generating function are denoted as *Bergman's operators of the third kind*.

3. PARTIAL DIFFERENTIAL EQUATIONS CORRESPONDING TO OPERATORS OF EXPONENTIAL TYPE

As we mentioned in the section before, it is possible to determine all partial differential equations (1.3) whose solutions can be generated by an operator (2.4) of exponential type. Of course a special choice of the generating function E of the operators (2.1) imposes certain restrictions with respect to the coefficients B and C of (1.3); Theorem 1 gives a full knowledge of these restrictions caused by (2.4).

3.1. The trivial case. We denote the following case as the "trivial case of the partial differential equation (1.3)" (with respect to our considerations): If $C = 0$ then (1.3) reduces to

$$(3.1) \quad u_{zz^*} + B(z, z^*)u_{z^*} = 0$$

which is an ordinary differential equation for the function u_{z^*} . If $C = 0$ and in addition the generating function E of (2.1) is independent of z^* then $u(z, z^*)$ is a trivial solution of (3.1). All other cases are denoted as "non trivial cases".

3.2. Special operators of exponential type. Partial differential equations corresponding to the very special case

$$(3.2) \quad Q(z, z^*, t) = q_r(z, z^*)t^r + q_s(z, z^*)t^s$$

of (2.4) were considered by BERGMAN [4] and NIELSEN & RAMSEY [11, 12]. These authors obtained the surprising result that only for sufficiently small values of r and s is the corresponding partial differential equation (1.3) non trivial; in detail [12], *the only values of r and s in (3.2) which do not correspond to trivial cases of (1.3) are*

$$(3.2') \quad \begin{array}{ll} (a) \ r = 1 & s = 0 \\ (b) \ r = 2 & (\alpha) \ s = 0 \quad \text{or} \quad (\beta) \ s = 1 \\ (c) \ r = 3 & s = 1 \\ (d) \ r = 4 & s = 2. \end{array}$$

In order to detect the essential properties of the results of NIELSEN [12] and to get a better understanding we write them in a more advantageous manner. The coefficients B and C of the partial differential equation (1.3) and the coefficients of (3.2) corresponding to (3.2') are

	Coefficients of (1.3)		Coefficients of (3.2)
	B	C	
(3.3) (a)	ρ_1	ρ_2	$q_0 = q_0(z), \quad q_1 = \sqrt{z}a_0(z^*)$
(b, α)	$\rho_1 + \rho_3$	ρ_4	$q_0 = q_0(z), \quad q_2 = d_1(z^*)z$
(b, β)	ρ_3	ρ_2	$q_1 = \sqrt{z}a_0(z^*), \quad q_2 = d_1 z$
(c)	0	ρ_2	$q_1 = \sqrt{z}\{a_0(z^*) + a_1 z\}, \quad q_3 = -\frac{2}{3}a_1 z^{\frac{3}{2}}$
(d)	ρ_3	ρ_4	$q_2 = d_1(z^*)z + d_2 z^2, \quad q_4 = -\frac{1}{2}d_2 z^2$

where

$$\rho_1 = -\frac{\partial q_0}{\partial z}, \quad \rho_2 = -\frac{q_1}{2z} \frac{\partial q_1}{\partial z^*}, \quad \rho_3 = -\frac{q_2}{z}, \quad \rho_4 = -\frac{1}{2z} \frac{\partial q_2}{\partial z^*}.$$

In these expressions the quantities a_r and d_r which are written without argument are constants. Of course the cases (a)–(d) are independent of each other; for sake of simplicity we used the same notations in all cases. It can be proved by direct calculation that (3.3) is equivalent to the Theorem 1 of the paper by NIELSEN & RAMSEY [11]. The inner reason for this interesting result cannot be seen from this special case but will appear in a clear manner from Theorem 1 of the present paper.

3.3. The general case of (2.4). The investigation of the general case (2.4) with a polynomial $Q(z, z^*, t)$ containing all powers of t seems to be somewhat difficult

because of the fact that the partial differential equation (2.2) satisfied by E has a not very simple form. Nevertheless it is possible to attack this problem using certain conveniently chosen symbols and proceeding stepwise. In this way we shall see that the connection between B and C and the coefficients q_μ of Q also in the general case is not more complicated than in the special case (3.3). In addition we shall find simple expressions relating the coefficients $q_3, q_4, \dots, q_m$ of (2.4) with q_1 and q_2 in such a manner that the odd coefficients depend on q_1 only and the even coefficients depend on q_2 only. Moreover it is possible to determine all partial differential equations (1.3) corresponding to operators (2.4) of exponential type. The following result holds:

Theorem 1. (a) *If the coefficients B and C of the partial differential equation (1.3) can be represented in the form*

$$(3.4) \quad B = -\frac{\partial q_0}{\partial z} - \frac{q_2}{z},$$

$$(3.5) \quad C = -\frac{q_1}{2z} \frac{\partial q_1}{\partial z^*},$$

where

$$(3.6) \quad q_0 = q_0(z),$$

$$(3.7) \quad q_1 = \sum_{\nu=0}^{\sigma(m)} a_\nu z^{\nu+\frac{1}{2}}, \quad a_0 = a_0(z^*),$$

$$a_\nu = \text{const. } (\nu = 1, 2, \dots, \sigma(m)), \quad \sigma(m) = \left[\frac{m-1}{2} \right],$$

$$(3.8) \quad q_2 = \sum_{\nu=1}^{\tau(m)} d_\nu z^\nu, \quad d_\nu = \text{const. } (\nu = 1, 2, \dots, \tau(m)), \tau(m) = \left[\frac{m}{2} \right],$$

then the solutions $u(z, z^*)$ of (1.3) can be generated by the generating function (2.4) of exponential type, and the other coefficients q_μ , $2 < \mu \leq m$, of $Q(z, z^*, t)$ are given by the following expressions:

$$(3.9) \quad q_{2\mu+1} = \frac{(-2)^\mu}{3 \cdot 5 \cdots (2\mu+1)} \sum_{\nu=\mu}^{\sigma(m)} \nu(\nu-1) \cdots (\nu-\mu+1) a_\nu z^{\nu+\frac{1}{2}},$$

$$\mu = 1, 2, \dots, \sigma(m),$$

$$(3.10) \quad q_{2\mu} = -\frac{(-2)^\mu}{2 \cdot 4 \cdots 2\mu} \sum_{\nu=\mu}^{\tau(m)} (\nu-1)(\nu-2) \cdots (\nu-\mu+1) d_\nu z^\nu,$$

$$\mu = 2, 3, \dots, \tau(m).$$

(b) *The same holds if B can be represented in the form (3.4) and*

$$(3.11) \quad C = -\frac{1}{2z} \frac{\partial q_2}{\partial z^*},$$

where q_2 is of the form (3.8); however, in this case

$$q_1 = 0$$

(and therefore according to (3.9))

$$q_{2\mu+1} = 0, \quad \mu = 1, 2, \dots, \sigma(m),$$

while d_1 can be a function of z^* .

(c) *Excluding trivial cases of (1.3) (see Section 1.3) no further types of the coefficients B and C are possible than those indicated in (a) and (b) if the generating function E of the operator (2.1) is a function (2.4) of exponential type.*

3.4. The proof of Theorem 1. Let the generating function E of the operator (2.1) which generates particular solutions $u(z, z^*)$ of the partial differential equation (1.3) be a function (2.4) of exponential type. Then we can express the derivatives of E as a product of the function E itself and certain polynomials in t with coefficients depending on z and z^* . We obtain

$$E_t = Q_t E, \quad E_{z^*} = Q_{z^*} E,$$

$$E_{tz^*} = (Q_{tz^*} + Q_t Q_{z^*}) E,$$

$$E_{zz^*} = (Q_{zz^*} + Q_z Q_{z^*}) E.$$

E must satisfy the partial differential equation (2.2). Substituting the expressions for the derivatives of E into (2.2) we obtain

$$(2.2') \quad \left\{ (1 - t^2)(Q_{tz^*} + Q_t Q_{z^*}) - \frac{Q_{z^*}}{t} + 2zt[Q_{zz^*} + Q_z Q_{z^*} + BQ_{z^*} + C] \right\} E = 0.$$

The common factor E drops out. The remaining expression is a polynomial in t with coefficients depending on z and z^* . In order that (2.2') be satisfied, the sums of the coefficients of every power t^σ , $\sigma = -1, 0, \dots, 2m+1$, must be zero. In this manner we obtain a system of $2m+3$ linear partial differential equations. Solving this system, we shall find conditions for the coefficients B and C of the partial differential equation (1.3) and also for the coefficients g_μ of the polynomial $Q(z, z^*, t)$ of the generating function (2.4). Each of these equations corresponds to a certain power of t ; we denote by $\langle \sigma \rangle$ the equation corresponding to the power t^σ . The lowest and the highest of these equations are the simplest ones. From the equation $\langle -1 \rangle$ we obtain

$$q_{0,z^*} = 0,$$

where the index behind the comma denotes the partial derivative with respect to z^* . This expression is equivalent to the equation (3.6). The equation $\langle 0 \rangle$ is satisfied identically. Using the equations $\langle -1 \rangle$ and $\langle 1 \rangle$, we find

$$(3.12) \quad C = -\frac{1}{2z} \{q_1 q_{1,z^*} + q_{2,z^*}\}.$$

We shall later obtain the relations (3.5) and (3.11) from this expression. The other $2m$ equations to be solved can be written in the form

$$(3.13) \quad \begin{aligned} & \sigma q_{\sigma+1,z^*} + (2zB - \sigma + 1)q_{\sigma-1,z^*} + 2zq_{\sigma-1,zz^*} \\ & + \sum_{\nu=0}^{2m-1} \{2zq_{\sigma-m-1+\nu,z} - (\sigma - m - 1 + \nu)q_{\sigma-m-1+\nu} \\ & + (\sigma - m + 1 + \nu)q_{\sigma-m+1+\nu}\} q_{m-\nu,z^*} = 0, \end{aligned}$$

where $\sigma = 2, 3, \dots, 2m+1$ and $q_\mu = 0$ for $\mu > m$.

We shall now prove that

$$q_{m,z^*} = 0.$$

In order to do this we shall show that the assumption

$$(3.14) \quad q_{m,z^*} \neq 0$$

leads to a contradiction. For this purpose let us consider at first the equations $\langle 2m+1 \rangle, \langle 2m \rangle, \dots, \langle m+2 \rangle$ stepwise and in the order indicated here. In all these equations the terms in the first line of (3.13) do not occur because these terms correspond to powers of t from 1 to $m+1$. For our purpose it is convenient to introduce the symbols

$$(3.15) \quad G_\alpha = 2zq_{\alpha-m-1,z} - (\alpha - m - 1)q_{\alpha-m-1} + (\alpha - m + 1)q_{\alpha-m+1}$$

where $\alpha = m+2, m+3, \dots, 2m+1$ and $q_\mu = 0$ for $\mu > m$.

Using these symbols the equations $\langle 2m+1 \rangle, \langle 2m \rangle, \dots, \langle m+2 \rangle$ can be written in the form

$$(3.16) \quad \sum_{\nu=0}^{2m-1} G_{\sigma+\nu} q_{m-\nu,z^*} = 0, \quad \sigma = m+2, m+3, \dots, 2m+1.$$

If $\sigma = 2m+1$ we obtain from (3.16)

$$G_{2m+1} q_{m,z^*} = 0.$$

If we assume that (3.14) holds, then

$$(3.17) \quad G_{2m+1} \equiv 2zq_{m,z} - mq_m = 0.$$

Integrating this equation with respect to z we find

$$(3.17') \quad q_m = z^{\frac{1}{2}m} k_m(z^*),$$

where $k_m(z^*)$ is an arbitrary regular function of z^* which enters in consequence of this integration. Using (3.17), we find from the equation $\langle 2m \rangle$

$$G_{2m} q_{m,z^*} = 0,$$

which under the assumption (3.14) can be integrated with respect to z , yielding

$$(3.18) \quad q_{m-1} = z^{(m-1)/2} k_{m-1}(z^*).$$

We proceed successively in this manner and solve the equations $\langle 2m-1 \rangle$, $\langle 2m-2 \rangle$, $\dots$, $\langle m+2 \rangle$. In each step of this procedure the equation to be solved can be simplified using the equations already solved. In this way we obtain from the equation $\langle \sigma \rangle$ the relations

$$(3.19) \quad G_\sigma = 0, \quad \sigma = 2m-1, 2m-2, \dots, m+2.$$

The last of these expressions is of the form

$$G_{m+2} \equiv 2zq_{1,z} - q_1 + 3q_3 = 0.$$

The relations (3.19) are equivalent to the existence of the differential recurrence formulas

$$(3.20) \quad q_{\lambda,z} - \frac{\lambda}{2z} q_\lambda + \frac{\lambda+2}{2z} q_{\lambda+2} = 0, \quad \lambda = 1, 2, \dots, m-2.$$

These linear differential equations can be integrated in the form

$$(3.21) \quad q_\lambda = -\frac{\lambda+2}{2} z^{\lambda/2} \int_0^z \zeta^{-(\lambda+2)/2} q_{\lambda+2}(\zeta, z^*) d\zeta.$$

For abbreviation we set

$$(3.22) \quad H = 2q_2 + 2zq_{0,z} + 2zB.$$

We consider now the next three equations $\langle m+1 \rangle$, $\langle m \rangle$, and $\langle m-1 \rangle$, from which we shall find that our assumption (3.14) does not hold. By (3.22), the equation $\langle m+1 \rangle$ can be written in the form

$$(3.23) \quad Hq_{m,z^*} - mq_{m,z^*} + 2zq_{m,zz^*} + \sum_{\nu=1}^m G_{m+1+\nu} q_{m-\nu,z^*} = 0.$$

Differentiating (3.17) with respect to z^* , we find

$$(3.17'') \quad 2zq_{m,zz^*} = mq_{m,z^*}.$$

Assuming (3.14) holds, we obtain from (3.19), (3.17'') and (3.23)

$$(3.24) \quad H = 0$$

(see (3.22)). Using (3.19) and (3.24), we write the equation $\langle m \rangle$ in the form

$$(3.25) \quad q_1 q_{m,z^*} - (m-1) q_{m-1,z^*} + 2zq_{m-1,zz^*} = 0.$$

Differentiating the relation

$$G_{2m} = 0$$

(see (3.15)) with respect to z^* , we obtain

$$2zq_{m-1,zz^*} = (m-1)q_{m-1,z^*};$$

hence we have from (3.25)

$$(3.26) \quad q_1 q_{m,z^*} = 0,$$

or, because of our assumption (3.14),

$$(3.27) \quad q_1 = 0.$$

Finally, using (3.16), (3.24) and (3.27), we obtain from the equation $\langle m-1 \rangle$ the relation

$$q_{m,z^*} = 0.$$

This is a contradiction to our assumption (3.14). Proceeding in similar manner we find analogous contradictions from which we derive the relations

$$(3.28) \quad q_{\mu,z^*} = 0, \quad \mu = m-1, \quad m-2, \dots, 4, 3,$$

stepwise in the above indicated order; the contradiction of the assumption $q_{\mu,z^*} \neq 0$ is always found from the equation $\langle \mu-1 \rangle$. However, the differential recurrence formulas (3.20) which we proved under the assumption (3.14) remain valid; because of (3.28) we obtain them from the equations $\langle m+1 \rangle$, $\langle m \rangle$, $\dots$, $\langle 4 \rangle$, proceeding stepwise in this order. In (3.17') and (3.18) the quantities k_m and k_{m-1} must be independent of z^* , i.e., they are constants, in consequence of (3.28). The relation (3.27) was proved under the assumption (3.14), which does not hold. Therefore we must also consider the case $q_1 \neq 0$. Also in this case (3.28) is valid, and using the equation $\langle 2 \rangle$ we find

$$q_{2,z^*} = 0.$$

Hence we obtain from (3.12) the result

$$(3.29) \quad C = \begin{cases} -\frac{q_1 q_{1,z^*}}{2z} & (\text{if } q_1 \neq 0) \\ -\frac{q_{2,z^*}}{2z} & (\text{if } q_1 = 0), \end{cases}$$

which is equivalent to (3.5) and (3.11) respectively. The relation (3.24), which was proved under the assumption (3.14), remains valid if we assume (3.28): In the case $q_1 = 0$ we find (3.24) from the equation $\langle 3 \rangle$; an exception occurs if $q_{2,z^*} = 0$. But in this exceptional case we see from (3.29) that $C = 0$; furthermore in this case E and therefore also u is a function of z alone; that means this case is a trivial case of (1.3) (see Section 3.1). In the case $q_1 \neq 0$ the relation (3.24) is a consequence of $\langle 2 \rangle$; an analogous exceptional case occurs if $q_{1,z^*} = 0$. In this exceptional case from (3.29) follows $C = 0$, and E is a function of z alone;

hence we have again a trivial case of (1.3). (3.24) is equivalent to (3.4). So far, we have proved the relations (3.4)–(3.6), and (3.11). We proceed to the proof of formulas (3.7)–(3.10). This can be done by using the recurrence formulas (3.20). Starting from (3.17') if m is odd and from (3.18) if m is even and using (3.20), we find that q_1 must be of the form (3.7). In this part of the proof we must take into account that the integration “constants” occurring in (3.21) must be chosen in such a manner that (3.28) holds. Consequently a_0 can be a function of z^* , but all other coefficients a_ν , $\nu = 1, 2, \dots, \sigma(m)$, of (3.7) must be constant. The recurrence relations (3.20) can be written in the form

$$(3.30) \quad q_{\lambda+2} = \frac{\lambda}{\lambda+2} q_\lambda - \frac{2z}{\lambda+2} q_{\lambda,z}$$

and then be used for proving (3.9); starting from (3.7) this can be done stepwise. In an analogous manner we find that q_2 must be of the form (3.8); in order to prove this we start from (3.17') or (3.18) respectively. Taking into account (3.28) and using (3.20) we obtain (3.8). Starting from (3.8) and using (3.30), we can prove that (3.10) holds. This completes the proof of Theorem 1. In particular it includes also the proof of the statement (c), because we excluded all possibilities for B and C of (1.3) other than those indicated in (a) and (b).

3.5. Remarks about Theorem 1. From Theorem 1 we see that in the case of the generating function (2.4) of exponential type the coefficients B and C of (1.3) depend on the coefficients q_0 , q_1 , and q_2 of (2.4) only. The odd coefficients q_3 , q_5 , $\dots$ depend on q_1 only, while the even coefficients q_4 , q_6 , $\dots$ depend on q_2 only. Of course the choice of m in (2.4) has a considerable influence on the form of each of the coefficients which can be seen from (3.7)–(3.10).

It should be noted that it is possible to express each coefficient q_μ of (2.4) as a linear combination of all coefficients with indices larger than μ . Using (3.17'), (3.18), and (3.21) we find the following relations

$$(3.31) \quad \begin{aligned} q_{m-2\sigma} &= z^{(m-2\sigma)/2} k_{m-2\sigma} \\ &- \sum_{\nu=1}^{\sigma} \frac{1}{\nu! 2^\nu} (m-2\sigma+2)(m-2\sigma+4) \cdots (m-2\sigma+2\nu) q_{m-2\sigma+2\nu}, \\ \sigma &= 1, 2, \dots, \left[\frac{m-1}{2} \right], \end{aligned}$$

and

$$(3.32) \quad \begin{aligned} q_{m-2\sigma-1} &= z^{(m-2\sigma-1)/2} k_{m-2\sigma-1} \\ &- \sum_{\nu=1}^{\sigma} \frac{1}{\nu! 2^\nu} (m-2\sigma+1)(m-2\sigma+3) \cdots (m-2\sigma+2\nu-1) q_{m-2\sigma+2\nu-1}, \\ \sigma &= 1, 2, \dots, \left[\frac{m-2}{2} \right]. \end{aligned}$$

These expressions are related to (3.9) and (3.10); the quantity $k_1 = k_1(z^*)$ corresponds to $a_0(z^*)$ of (3.9), and the other quantities k_λ occurring in (3.31) and (3.32) are constants corresponding to the coefficients a_ν and d_ν of (3.9) and (3.10).

By use of the relations (1.4) between the coefficients of the partial differential equations (1.2) and (1.3), Theorem 1 can easily be formulated also for the partial differential equation (1.2). From (3.5) we obtain the following necessary condition: In order that the coefficient C of the partial differential equation (1.3) has the form indicated in the case (a) of Theorem 1, there must exist an analytic function $\alpha(z)$ such that

$$(3.33) \quad \frac{\partial^2}{\partial z \partial z^*} \left\{ \sqrt{\int_0^{z^*} C d\zeta^*} + \alpha(z) \right\} = 0$$

holds.

From Theorem 1 we obtain a full explanation of the fact that if the generating function E is of the special form (3.2) the only non trivial cases are those given by (3.3). In particular, we see from (3.9) and (3.10) the reason why higher values of the exponents r and s in (3.2) always correspond to trivial cases of the partial differential equation (1.3). In consequence of (3.9) and (3.10) certain other coefficients q_μ of (2.4) must vanish if some of these coefficients are zero.

3.6. Examples. The partial differential equation

$$(3.34) \quad \frac{1}{4}(\Delta\psi + \psi) \equiv u_{zz^*} + \frac{1}{4}u = 0$$

belongs to the class of equations (1.3) whose solutions can be generated by generating functions (2.4) of exponential type. In this case we have

$$B = 0, \quad C = \frac{1}{4}.$$

Therefore we obtain from Theorem 1, case (a), that the simplest corresponding generating function is of the form

$$(3.35) \quad E = e^{t\sqrt{zz^*}t},$$

in agreement with results of BERGMAN [4]. But in addition some other generating functions are possible; from $B = 0$ and (3.4) we obtain

$$q_0 = - \int_0^z \frac{q_2}{\zeta} d\zeta,$$

where q_2 must be a polynomial in ζ of the form (3.8); therefore we have

$$q_0 = d_0 - \sum_{\nu=1}^{r(m)} \frac{d_\nu}{\nu} z^\nu,$$

where d_0 denotes an integration constant. If we chose $q_1 = 0$ then we obtain from (3.11)

$$q_2 = -\frac{z}{2}(z^* + \varphi(z)),$$

where $\varphi(z)$ is a polynomial in z with constant coefficients. Further examples corresponding to Theorem 1 are given in (3.3), and other more complicated examples can be obtained easily.

In the simplest case, Laplace's equation,

$$(3.36) \quad \Delta\psi \equiv 4u_{zz^*} = 0,$$

(if $q_1 \neq 0$) we conclude from (3.5) that q_1 must be a function of z alone. Since in this case all other coefficients q_μ of (2.4) are also functions of z alone (including q_2) the operator (2.1) transforms the associated function $f(z)$ again into an analytic function of one complex variable. The same holds also in the case (b) of Theorem 1; if $q_1 = 0$ then we obtain from (3.11) that q_2 is a function of z alone. The simplest generating function compatible with (3.36) is $E = 1$, but we see from Theorem 1 that also more complicated generating functions (2.4) with an arbitrary value of m are possible.

3.7. Relations to special functions. It should be noted that the integral (2.1) with the generating function (3.35) and the associated functions

$$f(z) = z^n, \quad n = 0, 1, \dots,$$

leads to a well known integral representation for the functions

$$(3.37) \quad \sqrt{\pi} \Gamma(n + \frac{1}{2}) \left(\frac{z}{z^*}\right)^{n/2} J_n(\sqrt{zz^*}),$$

where J_n are Bessel functions (see [4] and [13]). The functions (3.37) satisfy ordinary differential equations with respect to z_1 whose coefficients are algebraic functions of z_1 and z_2 . This is a very special case of the connections between partial and ordinary differential equations which we shall consider in a later paper.

The present theory is therefore of interest in connection with the investigation of special functions. By using operators of different forms, various properties and relations between special functions can be obtained. An example in this connection was investigated by HENRICI [14], who considered the partial differential equation (3.34) and generalizations of it using different operators (2.1). See also EICHLER [9].

3.8. The partial differential equation (1.2) and operators of exponential type. We consider finally the partial differential equation (1.2). The case where (1.2) is equivalent to two partial differential equations for the functions $\text{Re}(w)$ and

$\text{Im}(w)$ which are completely independent of each other is of special interest. For this case we obtain from Theorem 1 the following

Theorem 2. *If the partial differential equation (1.2) possesses constant coefficients*

$$a, \quad b = \bar{a}, \quad c = \bar{c},$$

then this equation is equivalent to a system of two completely independent partial differential equations for the real part $\text{Re}(w)$ and the imaginary part $\text{Im}(w)$ of the function $w(z, z^)$. The solutions of this equation can be generated in the form (2.1) by a generating function (2.4) of exponential type, and in (2.4) $m = 1$ can be, and in (3.7)*

$$a_0 = 2i\sqrt{\{c - |a|^2\}z^*}.$$

Proof. According to (1.4) we have in general

$$B = b - \int_0^{z^*} a_z d\zeta^*, \quad C = c - a_z - ab.$$

Hence we have in our case

$$B = \bar{a}, \quad C = c - |a|^2.$$

From Theorem 1 (see (3.4)) we obtain

$$B = -q_{0,z} - \frac{q_2}{z}$$

using the case (a). If we set

$$q_0 = - \int_0^z \bar{a} d\zeta = -\bar{a}z + \text{const.}, \quad q_2 = 0,$$

then (3.4) is satisfied. From (3.5) we obtain by integration with respect to z^* the expression

$$-4z \left(\int_0^{z^*} (c - |a|^2) d\zeta^* + \varphi(z) \right) = q_1^2.$$

In order that q_1 be of the form (3.7) the function $\varphi(z)$ which enters in consequence of the integration must vanish, and we obtain the expression for a_0 given above.

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