*On Representation of Stream Functions of Subsonic and Supersonic Flows of Compressible Fluids**

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Introduction. One of the aims of the modern theory of partial differential equations is the unification of methods. In particular, it seems of great interest to develop the theory of compressible fluids sufficiently that at least in the subsonic case we can use various procedures which were proved to be successful in the case of incompressible fluids, and to derive explicit formulas for stream functions of various types. This particular problem seems to be of comparative importance today: the development of modern computational machinery will permit in the not too distant future evaluation of rather complicated formulas.

The present paper consists of two parts. In §§1–4 we consider flows of compressible fluids in the pseudo-logarithmic plane, *i.e.*, a plane whose cartesian coordinates are λ and θ . λ is a function of the speed (see (1.3a)) and θ is the angle which the velocity vector forms with the positive x -direction of the physical plane. The differential equations of motion for the stream function in the case of subsonic flows assume a very simple form (see (1.2b) or (1.4b)). To a flow which in the physical plane is bounded by segments of straight lines and segments of free boundaries corresponds in the pseudo-logarithmic plane a flow bounded by segments of lines which are parallel to the axes λ and θ . The main complication arising in this case is the fact that in the pseudo-logarithmic plane the stream functions of flows of this type are defined, in general, on a Riemann surface. In §§1–4 we consider flows of this type and define operators transforming analytic functions of a complex variable into stream functions (defined in the pseudo-logarithmic plane) yielding flow patterns of the desired type in the physical plane.

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In the second part of the paper, §5, we discuss the problem of determining bounds for the stream function $\psi(H, \theta)$, where H is a function of the Mach number. We assume that the coefficient $l(H)$ in equations (1.2a) and (1.2b) satisfies the inequality $|a_1^2 H^M| \leq |l(H)| \leq |a_2^2 H^M|$. If the value of the stream function $\psi(0, \theta)$ and its derivatives $\partial\psi(H, \theta)/\partial H|_{H=0}$ are given and we assume that ψ is an analytic function of H and θ , then we derive bounds for ψ depending on $a_1, a_2, \psi(0, \theta)$ and $\partial\psi(H, \theta)/\partial H|_{H=0}$. In doing so we use the fact that certain functions connected with ψ are *monotone* functionals of $l(H)$.

1. The equations of motion in the hodograph and in the pseudo-logarithmic planes of flow patterns bounded by segments of straight lines and segments of free boundaries. By using either the Schwarz-Christoffel formula or the kernel function one obtains explicit formulas for various flow patterns of incompressible fluids. It is possible, in certain instances, to generalize the second representations to the case of adiabatic subsonic flows of compressible fluids.

As has been shown by MOLENBROEK [18]* and CHAPLYGIN [13], it is preferable in many instances to consider flows of compressible fluids in the hodograph plane, whose coordinates are components of the velocity vector.

We assume that the pressure p and the density ρ are connected by the relation $p = \sigma\rho^k + \beta$, where σ, k, β are constants, and we introduce instead of the speed v the variable

$$(1.1) \quad H = \int_{v_1}^v \frac{\rho}{v} dv = \int_{v_1}^v \left[1 - \frac{k-1}{2} \left(\frac{v}{a_0} \right)^2 \right]^{1/(k-1)} \frac{dv}{v},$$

where v_1 denotes the speed corresponding to $M = 1$ and a_0 is a conveniently chosen constant. Then the equations connecting the potential (ϕ) and the stream-function (ψ) assume the form

$$(1.2a) \quad \phi_\theta = \psi_H,$$

$$(1.2b) \quad \phi_H = -l(H)\psi_\theta, \quad l(H) \equiv \frac{1 - M^2}{\rho^2}.$$

Here

$$(1.3) \quad M = v/[a_0^2 - \frac{1}{2}(k-1)v^2]^{\frac{1}{2}}$$

is the Mach number, θ is the angle which the velocity vector forms with positive direction of the x -axis, and v is the speed. In the *subsonic* case it is useful to consider the flow in the pseudo-logarithmic plane,† whose cartesian co-ordinates are θ and

* Numbers in brackets refer to the bibliography, p. 905.

† λ and θ are connected in a simple way with the velocity components (coordinates of the hodograph plane).

$$(1.3a) \quad \lambda = \frac{1}{2} \log \left[\frac{1 - (1 - M^2)^{\frac{1}{2}}}{1 + (1 - M^2)^{\frac{1}{2}}} \left(\frac{1 + h(1 - M^2)^{\frac{1}{2}}}{1 - h(1 - M^2)^{\frac{1}{2}}} \right)^{1/h} \right], \quad \text{where}$$

$$h = \left(\frac{k - 1}{k + 1} \right)^{\frac{1}{2}}.$$

Then the equations for the potential function ϕ and for the stream function ψ assume the form

$$(1.4a) \quad \phi_{\lambda\lambda} + \phi_{\theta\theta} - 4N(\lambda)\phi_{\lambda} = 0,$$

$$(1.4b) \quad \psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N(\lambda)\psi_{\lambda} = 0,$$

$$N = -\frac{(k+1)}{8} \frac{M^4}{(1-M^2)^{\frac{3}{2}}}, \quad \phi_{\lambda\lambda} \equiv \frac{\partial^2 \phi}{\partial \lambda^2}, \dots$$

If we introduce the *modified stream and potential functions* $\phi^* = \tilde{H}\phi$, $\psi^* = \psi/\tilde{H}$, where

$$(1.5) \quad \tilde{H} = [1 - M^2]^{-1} [1 + (k-1)M^2/2]^{-1/2(k-1)},$$

the system (1.2) is reduced to

$$(1.6a) \quad \phi_{\lambda\lambda}^* + \phi_{\theta\theta}^* + 4P(\lambda)\phi^* = 0,$$

$$(1.6b) \quad \psi_{\lambda\lambda}^* + \psi_{\theta\theta}^* + 4F(\lambda)\psi^* = 0,$$

(see [4] page 463 *ff.*). The equation (1.6b) has no eigenfunctions, so that the first boundary value problem for equation (1.6b) always has a solution. (See [9] p. 146 *ff.*).

In the present paper we shall show that using conveniently chosen orthogonal functions one obtains explicit formulas for flow patterns of certain types. See Figs. 1a, 2a, and 3a.

Remark 1.1. In what follows we shall often derive formulas for the *modified* stream functions, ψ^* . Multiplying by $\tilde{H}$ the expressions obtained, we get the corresponding stream functions, ψ .

2. Hodographs and pseudographs of flow patterns bounded by segments of straight lines and free boundaries. Let us consider a flow pattern in a tube (see Fig. 1a, where the upper half of the flow is indicated) which in the physical plane is bounded by segments of straight lines, shown full, and segments of free boundaries, shown dashed. Its hodograph will be a domain bounded by segments of straight lines and segments of circles whose center is the origin (see Fig. 1b), and its pseudograph (the image in the pseudo-logarithmic plane, *i.e.*, in the λ , θ -plane) will be a domain bounded by segments of straight lines [$\lambda = \text{const.}$] and [$\theta = \text{const.}$] (see Fig. 1c). Indeed, to a segment of a straight line which

forms an angle θ_k with the positive x -direction corresponds in the pseudo-logarithmic plane a segment of the line $[\theta = \theta_k, \lambda_k \leq \lambda \leq \lambda_{k+1}]$, while to a segment of a free boundary corresponds $[\theta_{k-1} \leq \theta \leq \theta_k, \lambda = \lambda_k = \text{const.}]$. Flows of this type will be denoted as flows of the type $\mathcal{S}$.

Remark 2.1. We note that if on a segment of the line $\lambda = \lambda_k = \text{const.}$, $\psi(\lambda, \theta) = c_0 = \text{const.}$, then $\psi^*(\lambda, \theta) = \psi(\lambda_k, \theta)/\tilde{H}(\lambda_k)$ will be equal to $[c_0/\tilde{H}(\lambda_k)] = \text{const.}$ *

At the images of the points corresponding to $y = \infty$ and $y = -\infty$, the flow will have a sink and a source, respectively. At these points the stream function

FIGURES 1a, 1b, 1c

has singularities described in [4], page 473 (see also §3 of the present paper). On the other hand, if the stream function ψ in the pseudo-logarithmic plane is of the form described above, it will represent in the physical plane a flow in a tube around a domain bounded by segments of free boundaries and segments of straight lines. Of course, in the general case the image of flow patterns in the pseudo-logarithmic plane will be multiply covered, but we consider at first (see §3) the case where the pseudograph is univalent (schlicht).

Remark 2.2. It should be stressed that starting from an arbitrary domain in the hodograph (or in the pseudo-logarithmic) plane may yield an image in the

* Tables of M , $\tilde{H}$, $[d\tilde{H}(\lambda)/d\lambda]$ as functions of λ are published in [8] p. 197.

physical plane which is not necessarily univalent and consequently has no physical significance.

FIGURES 2a, 2b, 2c

We shall now give some examples of flow patterns of the type *s* described above. Since our examples are flows which are symmetric with respect to the x -axis, we consider only the upper half plane. However, the method discussed here can be applied also in cases of flows which are not symmetric with respect to the x -axis.

a

b

c

FIGURES 3a, 3b, 3c

In the case where the angles formed by $[\infty C_1$ and $\widehat{D_1 E_1}]$ and by $[\widehat{D_1 E_1}$ and $\widehat{J_1 F_1}]$ are smaller than π (see Fig. 1a) the limiting process in which the segments $\widehat{C_1 D_1}$ and $\widehat{E_1 J_1}$ shrink to points is such that the velocity remains finite. We then obtain the situation indicated in Figures 2a, 2b, and 2c.

Other types of flow patterns in which we are interested are those which are around a finite body, as indicated in Fig. 3a. In this case at the point J_3 which corresponds to infinity, we have a doublet. Again, if the angles between $\widehat{B_1 C_1}$ and $\widehat{D_1 E_1}$ etc. are smaller than π we can allow the limiting case in which segments of free boundaries shrink to points. It is of considerable interest to consider also the limiting case when the number of the sides tends to infinity, while the lengths of each individual segment tends to zero, so that in the limit we obtain a curved line.

3. An analytic representation of stream functions of flows of the type S. In this section we shall derive a representation of the stream function whose pseudograph is of the form indicated in Fig. 1c.* At the points A_3 and B_3 where the flow has a source and sink, the stream function has logarithmic singularities. The stream functions

$$(3.1) \quad \psi^{(L,1)}(\lambda, \theta, \lambda_A, \theta_A) \quad \text{and} \quad \psi^{(L,1)}(\lambda, \theta, \lambda_B, \theta_B),$$

representing these singularities are described in [4] p. 473; (λ_A, θ_A) and (λ_B, θ_B) are the coordinates of the points A_3 and B_3 , respectively. Therefore the stream function of the flow is given by

$$(3.2) \quad \psi(\lambda, \theta) = \psi^{(0)}(\lambda, \theta) + \frac{1}{2}\psi^{(L,1)}(\lambda, \theta; \lambda_A, \theta_A) - \psi^{(L,1)}(\lambda, \theta; \lambda_B, \theta_B)$$

where $\dagger \psi^{(0)}(\lambda, \theta)$ is a solution of (1.4b) which is regular in the domain B (bounded by $b = A_3 B_3 C_3 D_3 E_3 J_3 F_3 A_3$). It is chosen in the way we shall now describe.

The function

$$(3.3) \quad T(\lambda, \theta) \equiv C[\frac{1}{2}\psi^{(L,1)}(\lambda, \theta; \lambda_A, \theta_A) - \psi^{(L,1)}(\lambda, \theta; \lambda_B, \theta_B)]$$

(where C is a normalizing constant) converges to 0 when we approach A_3 and B_3 along $B_3 A_3$, and converges to 1 when we approach the same points moving along $C_3 B_3$ and $A_3 F_3$, respectively.

The function $h(s)$ (defined on the boundary $b = [\lambda = \tilde{\lambda}(s), \theta = \tilde{\theta}(s), 0 \leq s \leq 2\pi]$ of B) which assumes on $B_3 A_3$ the values $-T(\lambda, \theta)$ and on $B_3 C_3 D_3 E_3 J_3 F_3$ the values $-T(\lambda, \theta) + 1$ will be continuous on the whole boundary curve b of B .

We determine now that solution $C\psi^{(0)}(\lambda, \theta)$ of the differential equation (1.4b)

* The pseudographs considered in the present section are assumed to be univalent.

In connection with the considerations of this present section see also [5].

$\dagger$ The existence of the function $\psi^{(0)}$ can easily be proved, *e.g.*, using the theory of the integral equations and the fact that equation (1.4b) has no eigenvalues.

which is regular in B and assumes on b the values $h(s)$. According to [1, 9] this function can be represented in B in the form

$$(3.4) \quad \psi^{(0)}(\lambda, \theta) = \sum_{\nu=1}^{\infty} \psi_{\nu}(\lambda, \theta) \int_b h^*(\lambda_1, \theta_1) \frac{\partial \psi_{\nu}^*(\lambda_1, \theta_1)}{\partial n_1} ds_1, \quad h^* = \frac{h}{\tilde{H}},$$

where $\{\psi_{\nu}^*\}$, $\nu = 1, 2, \dots$, is a set of orthonormal solutions* complete for the class of solutions $\psi = \tilde{H}\psi^*$, see (1.5), which are regular in B and for which

$$(3.5) \quad D(\psi^*, \psi^*) = \iint_B \left[\left(\frac{\partial \psi^*}{\partial \lambda} \right)^2 + \left(\frac{\partial \psi^*}{\partial \theta} \right)^2 - 4F\psi^{*2} \right] d\lambda d\theta < \infty.$$

In [2, 3, 4 and 9 p. 275] are discussed different methods of obtaining a set $\{\psi_{\nu}\}$, $\nu = 1, 2, \dots$, of particular solutions. For large classes of functions we can orthogonalize the CHAPLYGIN [13] solutions or solutions introduced in [2, 3, 11, 12].

Remark 3.1. In the special case of a hodograph indicated in Fig. 1b, when the points B_2, C_2, D_2 coincide with the origin O and the angle E_2OA_2 equals $\pi/2$, instead of (3.2) we have $\psi(\lambda, \theta) = \psi^{(0)}(\gamma, \theta) + c \sin \theta$, $c = \text{const}$.

4. Stream functions of flow patterns whose pseudograph is located on a Riemann surface. We proceed now to the case where the pseudograph is multiply covered. We shall consider a pseudograph of the form indicated in Fig. 3c, which is located on a two-sheeted Riemann surface with a branch point of the second order at J_3 . A pseudograph of this kind yields flow patterns whose upper half is indicated in Fig. 3a. The stream function $\psi(\lambda, \theta)$ can be written in this case in the form

$$(4.1) \quad \psi = \psi^{(1,1)} + \psi_0$$

where ψ_0 is a regular solution of (1.4b) defined on the Riemann surface R and having boundary values such that ψ assumes constant values on the boundary of the domain, see Fig. 3c. $\psi^{(1,1)}$ is a solution of (1.4b) possessing a singularity of first order, so that $\psi^{(1,1)}$ represents the stream function of a doublet (see (4.10)). As before we operate with the modified stream function $\psi^* = \psi/\tilde{H}$, and we denote $\psi^{*(1,1)} = \psi^{(1,1)}/\tilde{H}$ and $\psi_0^* = \psi_0/\tilde{H}$. See (1.6b).

By a conformal transformation

$$(4.2) \quad \zeta = \zeta(z) \equiv [(z - \alpha)/(1 - \alpha)]^{\frac{1}{2}}$$

the two-sheeted Riemann surface with branch points at $z = \alpha$ and $z = \infty$ is transformed into the single-sheeted ζ -plane. The points $z = \alpha, 1, \infty$ are mapped into $\zeta = 0, 1, \infty$. The solutions ψ^* of the differential equation

$$(4.3) \quad \frac{\partial^2 \psi^*}{\partial z \partial \bar{z}} + F \left(\frac{z + \bar{z}}{2} \right) \psi^* = 0, \quad z = \lambda + i\theta, \quad \bar{z} = \lambda - i\theta,$$

* I.e., solutions for which $D(\psi_{\nu}^*, \psi_{\mu}^*) = \delta_{\nu\mu}$. See [9], p. 259 and (3.5).

go over into the solutions of

$$(4.4) \quad \frac{\partial^2 \chi^*}{\partial \zeta \partial \bar{\zeta}} + T \chi^* = 0, \quad \zeta = \tau + i\mu, \quad \bar{\zeta} = \tau - i\mu,$$

where

$$\chi^*(\zeta, \bar{\zeta}) \equiv \psi^*[z(\zeta), \bar{z}(\bar{\zeta})],$$

$$(4.4a) \quad T = \frac{dz}{d\zeta} \frac{d\bar{z}}{d\bar{\zeta}} F \left[\frac{z(\zeta) + \bar{z}(\bar{\zeta})}{2} \right]$$

$$(4.4b) \quad = [4 | 1 - \alpha |^2 \zeta \bar{\zeta}] F \left[\frac{1}{2} (\alpha + \bar{\alpha} + (1 - \alpha)\zeta^2 + (1 - \bar{\alpha})\bar{\zeta}^2) \right].$$

In the case of pseudographs located on Riemann surfaces we are interested in two questions:

I. The construction of singularities located at the branch point, in this case at $z = \alpha$; and

II. The determination of a set of functions which is complete for the class of regular functions defined on the Riemann surface.

I. One obtains the desired singularity $\psi^{(1,1)}$ by starting from the considerations of [4, p. 473 ff.], namely, using the procedure indicated there and in §3. We obtain a function

$$(4.5) \quad \psi^{(1,1)}(\zeta, \bar{\zeta}; \zeta_0, \bar{\zeta}_0) = \frac{1}{2} A [\log(\zeta - \zeta_0) + \log(\bar{\zeta} - \bar{\zeta}_0)] + B, \quad \zeta_0 = \tau_0 + i\mu_0,$$

possessing a logarithmic singularity at the point (τ_0, μ_0) of the ζ -plane. Here

$$(4.6) \quad A = \tilde{H} p_1^*(1, \zeta_0) \equiv \tilde{H} p_1^*(1, \zeta_0; \zeta, \bar{\zeta}),$$

where $p_1^*(g, \zeta_0)$ is the integral operator (of the first kind) transforming analytic functions of a complex variable into (complex) solutions of (4.4).

$$(4.7) \quad \begin{aligned} p_1^*(g, \zeta_0) \equiv & g(\zeta) - \int_{\zeta_0}^{\zeta} \int_{\bar{\zeta}_0}^{\bar{\zeta}} T(\zeta_1, \bar{\zeta}_1) g(\zeta_1) d\zeta_1 d\bar{\zeta}_1 \\ & - \int_{\zeta_0}^{\zeta} \int_{\bar{\zeta}_0}^{\bar{\zeta}} T(\zeta_1, \bar{\zeta}_1) \int_{\zeta_0}^{\zeta_1} \int_{\bar{\zeta}_0}^{\bar{\zeta}_1} G(\zeta_2, \bar{\zeta}_2) d\zeta_2 d\bar{\zeta}_2 d\zeta_1 d\bar{\zeta}_1 + \dots \end{aligned}$$

Remark 4.1. In defining the integral operator* $p_1^*(g, \zeta_0)$ we denote ζ_0 as the *reference point of the integral operator*. We assume that this reference point is

* In [6 p. 39 and papers mentioned there] the integral operator in this form has been considered for univalent domains. It can be easily generalized to the case of Riemann surfaces; see the following.

fixed during all our further considerations.

$\tilde{H}$ is given by (1.5) and

$$(4.8) \quad B = \tilde{H} \left[\int_{\zeta_0}^{\zeta} \int_{\bar{\zeta}_0}^{\bar{\zeta}} G(\zeta_1, \bar{\zeta}_1) d\zeta_1 d\bar{\zeta}_1 - \int_{\zeta_0}^{\zeta} \int_{\bar{\zeta}_0}^{\bar{\zeta}} T(\zeta_1, \bar{\zeta}_1) \int_{\zeta_0}^{\zeta_1} \int_{\bar{\zeta}_0}^{\bar{\zeta}_1} G(\zeta_2, \bar{\zeta}_2) d\zeta_2 d\bar{\zeta}_2 d\zeta_1 d\bar{\zeta}_1 + \cdots \right],$$

where

$$(4.9) \quad G = -\frac{1}{\bar{\zeta} - \bar{\zeta}_0} \frac{\partial(p_1^*(1, \zeta_0))}{\partial\zeta} - \frac{1}{\zeta - \zeta_0} \frac{\partial(p_1^*(1, \zeta_0))}{\partial\bar{\zeta}}.$$

Differentiating (4.5) with respect to τ_0 , we obtain a function with a singularity which is infinite of the first order at the point (τ_0, μ_0) , namely

$$(4.10) \quad \psi^{(1,1)}(\zeta, \bar{\zeta}; \zeta_0, \bar{\zeta}_0) = -\frac{1}{2} A[(\zeta - \zeta_0)^{-1} + (\bar{\zeta} - \bar{\zeta}_0)^{-1}] + \frac{1}{2} A_{\tau_0} [\log(\zeta - \zeta_0) + \log(\bar{\zeta} - \bar{\zeta}_0)] + B_{\tau_0}, \quad A_{\tau_0} = \frac{\partial A}{\partial \tau_0}.$$

If we substitute for $\zeta - \zeta_0$ and $\bar{\zeta} - \bar{\zeta}_0$ the functions $(z - z_0)^{\frac{1}{2}}$ and $(\bar{z} - \bar{z}_0)^{\frac{1}{2}}$, respectively, we obtain the singularity of order $\frac{1}{2}$ at the point (λ_0, θ_0) in the λ, θ -plane.

We proceed now to the problem II, *i.e.*, to determining a set of functions which are complete for regular functions defined in a simply connected, bounded domain B , lying on the Riemann surface R .

For our purposes it is convenient to continue our function $\psi^*(\lambda, \theta)$ to *complex* values of the arguments. In the following we write λ_1 and θ_1 instead of λ and θ . In accordance with (4.3) we write

$$(4.11) \quad z = (\lambda_1 + i\lambda_2) + i(\theta_1 + i\theta_2), \quad z^* = (\lambda_1 + i\lambda_2) - i(\theta_1 + i\theta_2).$$

When z and $\bar{z}$ are replaced by z and z^* the equation (4.3) assumes the form

$$(4.12) \quad \frac{\partial^2 \psi^*}{\partial z \partial z^*} + \hat{F} \left(\frac{z + z^*}{2} \right) \psi^* = 0, \quad \psi^* \equiv \psi^*(z, z^*).$$

$\hat{F}$ is obtained from F by substituting z^* for $\bar{z}$.

Let $\dagger R(z, \zeta)$ be the Riemann surface where z and ζ are connected by the relation

$$(4.13) \quad \zeta^n + a_1(z)\zeta^{n-1} + \cdots + a_n(z) = 0$$

$\dagger$ The following discussion we shall carry out in the case of general Riemann surface, since replacing the equation (4.2) by (4.13) causes no complications.

and $a_j(z)$, $z = 1, 2, \dots, n$, are functions which are regular in a domain D_1 of the Riemann surface.

Let D_1^+ be a domain which is obtained from D_1 by reflection on the real λ -axis and let $R^*(z^*, \zeta^*)$ be the Riemann surface

$$(4.14) \quad \zeta^{*n} + \bar{a}_1(z^*)\zeta^{*n-1} + \dots + \bar{a}_n(z^*) = 0, \quad z^* \in D_1^+.$$

Here, if $a_j(z) = \sum_{k=0}^{n_j} A_{jk}(z - z_0)^k$, $z_0 \in D_1$, then[†]

$$\bar{a}_j(z^*) = \sum_{k=0}^{n_j} \bar{A}_{jk}(z^* - \bar{z}_0)^k, \quad z^* \in D_1^+.$$

The function $\chi^*(\zeta, \zeta^*) \equiv \psi^*[z(\zeta), z^*(\zeta^*)]$ satisfies the equation (4.4), where T is given by (4.4a) with ζ replaced by ζ^* and z by z^* .

In [6, p. 39], it has been shown that the integral operator of the first kind $p_1^*(g, \zeta_0)$ can be written in the form (4.7), where $g(\zeta(z))$ was an arbitrary function regular in a simply connected domain $\bar{D} \subset D_1$. D lies completely in $\text{Re } z < 0$.

Remark 4.2. The value of the integral operator $p_1^*(g, \zeta_0)$ with the point of reference $\zeta = \zeta_0$, is independent of the choice of the auxiliary variables ζ, ζ^* . Indeed

$$(4.15) \quad T(\zeta_k, \zeta_k^*) d\zeta_k d\zeta_k^* = F(z_k(\zeta_k), z_k^*(\zeta_k^*)) \frac{dz_k(\zeta_k)}{d\zeta_k} \frac{dz_k^*(\zeta_k^*)}{d\zeta_k^*} d\zeta_k d\zeta_k^* \\ = F(z_k, z_k^*) dz_k dz_k^*.$$

Therefore we can speak of the integral operator defined on the Riemann surface and denote it as

$$p_1^*(\bar{g}(z), z_0), \quad \bar{g}(z) = g(z(\zeta)), \quad z_0 = z(\zeta_0).$$

By the relation (4.7) the integral operator $p_1^*(\bar{g}, z_0)$ of the first kind is defined also in the case where $\bar{g}$ is a *multivalued* function of z . Every real solution of the equation at a point z_0 , $\text{Re } z_0 < 0$, can be continued to *complex* values of λ and θ , and can be represented in a sufficiently small neighborhood of z_0 in the form

$$(4.16) \quad \Psi^*(z, z^*) = \frac{1}{2}[p_1^*(\bar{g}, z_0) + \bar{p}_1^*(\bar{g}, z_0^*)].$$

Let D be a simply connected domain located on R in the part $[\lambda_1 < 0]$. Let

$$(4.17) \quad \Psi^*(z, z^*) = \frac{1}{2}[p_1^*(\bar{g}, z_0) + \bar{p}_1^*(\bar{g}, z_0^*)] \equiv \chi^*(\zeta, \zeta^*)$$

be a solution of (4.4). Then $\Psi^*(\lambda_1, \theta_1)$ continued to complex values $\lambda = \lambda_1 + i\lambda_2$, $\theta = \theta_1 + i\theta_2$ is defined in the domain $[z \in D_1, z^* \in D_2^*]$, where D_2^* is obtained from D_1 by reflecting it along the real axis.

If δ denotes a quantity which exceeds the length of the paths of integration

[†] See Remark 4.1.

$\widehat{z_0 z}$ and $\widehat{z_0 z^*}$, γ the maximum of $|\tilde{g}(z_1)|$ and $|\tilde{g}(z_1^*)|$ when z_1 and z_1^* vary along the above mentioned paths, then there exists a constant $M_1 < \infty$ such that*

$$(4.18) \quad \frac{1}{2} [|p_1^*(\tilde{g}(z), z_0)| + |\bar{p}_1^*(\tilde{g}(z_0^*), \bar{z}_0)|] \\ \leq \left[1 + \frac{M_1^2 \delta^2}{(1!)^2} + \frac{M_1^4 \delta^4}{(2!)^2} + \frac{M_1^6 \delta^6}{(3!)^2} + \dots \right] < \infty.$$

We obtain the inequality (4.18) immediately by taking into account that if $\operatorname{Re} z < 0$, and $\operatorname{Re} z^* < 0$, then $\operatorname{Re} (z + z^*) < 0$ and therefore $F(\frac{1}{2}(z + z^*))$ is regular along the (closed) paths $\widehat{z_0 z}$ and $\widehat{z_0 z^*}$. There exists, therefore, a constant $M_1 < \infty$ so that along these paths we have

$$(4.19) \quad F\left(\frac{z + z^*}{2}\right) \leq M_1.$$

If then we majorize in (4.7) $\tilde{g}$, F by γ and M_1 , respectively, we obtain the bound (4.18).

5. Initial value problem in the case of subsonic and supersonic flows. One of the problems of the theory of compressible fluids is the question of developing new procedures permitting us to obtain various results about the behavior of flows of mixed type. In the present section we consider certain flows of this kind. One of the new tools which we use in this case consists in introducing a conveniently chosen set of particular solutions of the compressibility equation and exploiting their properties. As indicated in §1 if instead of λ we use the variable H (see (1.1)) the compressibility equation assumes the form**

$$(5.1) \quad S(\psi) = l(H) \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial H^2} = 0, \quad l(H) = \frac{1 - M^2}{\rho^2}.$$

Remark 5.1. The variable λ (used in §§1-4) and H are connected by the relation

$$(5.2) \quad -\lambda = \int_0^{-H} [l(\tau)]^{\frac{1}{2}} d\tau.$$

We note that in the subsonic region $H < 0$, $M < 1$ while in the supersonic region $H > 0$, $M > 1$. Further $\lambda \equiv \lambda(H)$ is smaller than 0 in the subsonic region, while in the supersonic region $\lambda \equiv i\Lambda$ becomes purely imaginary and $\Lambda > 0$. (See (1.1), (1.3), (1.3a)). Also, $l(H)$ is positive in the subsonic region and negative in the supersonic region.

In [2 and 3] two sets of particular solutions of the system (1.2a), (1.2b) have been introduced.† These particular solutions $\phi_{n,k}(H, \theta)$ and $\psi_{n,k}(H, \theta)$, $k = 1, 2$,

* If $\tilde{g}(z) = \Sigma a_\nu (z - z_0)^\nu$, then $\tilde{g}(z^*) = \Sigma \bar{a}_\nu (z^* - \bar{z}_0)$.

** In the following $l(H)$ is assumed to be continuous.

† These solutions were introduced independently by BERS & GELBART [11, 12].

$n = 0, 1, 2, \dots$ satisfying the system (1.2a), (1.2b), have been written in the form:

$$(5.3) \quad \begin{aligned} \phi_{n,1}(H, \theta) + i\psi_{n,1}(H, \theta) &= (\theta + iH)^{[n]} \\ &\equiv \left[\theta^n - 2! \binom{n}{2} \theta^{n-2} p_2(H) + 4! \binom{n}{4} \theta^{n-4} p_4(H) \dots \right] \\ &\quad + i \left[1! \binom{n}{1} \theta^{n-1} s_1(H) - 3! \binom{n}{3} \theta^{n-3} s_3(H) + \dots \right], \end{aligned}$$

$$(5.4) \quad \begin{aligned} \phi_{n,2}(H, \theta) + i\psi_{n,2}(H, \theta) &= i \odot (\theta + iH)^{\{n\}} \\ &\equiv - \left[1! \binom{n}{1} \theta^{n-1} p_1(H) - 3! \binom{n}{3} \theta^{n-3} p_3(H) + \dots \right] \\ &\quad + i \left[\theta^n - 2! \binom{n}{2} \theta^{n-2} s_2(H) + 4! \binom{n}{4} \theta^{n-4} s_4(H) - \dots \right], \end{aligned}$$

where

$$(5.5) \quad p_m(H) = \int_{H_0}^H l(H_m) \int_{H_0}^{H_m} \int_{H_0}^{H_{m-1}} l(H_{m-2}) \int_{H_0}^{H_{m-2}} \dots dH_1 dH_2 \dots dH_m,$$

$$p_0 = 1, \quad m = 1, 2, \dots$$

$$s_m(H) = \int_{H_0}^H \int_{H_0}^{H_m} l(H_{m-1}) \int_{H_0}^{H_{m-1}} \int_{H_0}^{H_{m-2}} l(H_{m-3}) \dots dH_1 dH_2 \dots dH_m,$$

$$(5.6) \quad s_0 = 1, \quad m = 1, 2, \dots$$

$$s_1 = (H - H_0) \equiv \int_{H_0}^H dH_1.$$

For* $H = H_0$, the functions $\phi_{n,1}$ and $\psi_{n,2}$ become θ^n , and their derivatives with respect to H vanish, while $\psi_{n,1}$ and $\phi_{n,2}$ vanish and their derivatives become $n\theta^{n-1}$ and $-nl(H_0)\theta^{n-1}$, respectively.

The introduction of the functions $\psi_{n,1}$ and $\psi_{n,2}$ yields interesting formulas for the representation of a function $\psi(H, \theta)$ in terms of values of ψ and $\partial\psi/\partial H$ for $H = H_0$ and $-\theta_1 \leq \theta \leq \theta_1$.

In particular it is of interest that certain results in theory of analytic functions of a complex variable yield results for the above mentioned equations in the subsonic as well as the supersonic case provided that the initial data on the line $H = H_0$ are analytic functions of the variable θ .

Suppose that for $-\theta_1 \leq \theta \leq \theta_1$ we have

$$(5.7) \quad \psi(H_0, \theta) = \sum_{n=0}^{\infty} C_n \theta^n \equiv f(\theta), \quad \left. \frac{\partial\psi(H, \theta)}{\partial H} \right|_{H=H_0} = \sum_{n=0}^{\infty} nD_n \theta^{n-1} \equiv g^{(1)}(\theta),$$

* The index m indicates the number of integrations involved in the right-hand sides of (5.5) and (5.6).

where the series $\sum C_n \theta^n$ and $\sum D_n \theta^n$ converge uniformly and absolutely for $-\theta_1 \leq \theta \leq \theta_1$. Suppose also that $|l(H)| \leq c^2$, $0 < c < \infty$ when $|H - H_0| \leq H_1$. Then in the domain $|\theta| + c|H - H_0| \leq \theta_1$

$$(5.8) \quad \begin{aligned} \psi(H, \theta) = & [f(\theta) - f^{(2)}(\theta)s_2(H) + f^{(4)}(\theta)s_4(H) - \dots] \\ & + [g^{(1)}(\theta)s_1(H) - g^{(3)}(\theta)s_3(H) + g^{(5)}(\theta)s_5(H) - \dots] \end{aligned}$$

is the (analytic) solution of (5.1) satisfying (5.7). Here $f^{(1)} \equiv df/d\theta$, $f^{(2)} \equiv d^2f/d\theta^2$, etc.*

Proof. Since $s_0(H_0) = 1$ and $s_1(H) = (H - H_0)$ it is easy to check that the initial conditions are satisfied. If $|l(H)| \leq c^2$ in the interval $|H - H_0| \leq H_1$, then

$$(5.9) \quad |p_m(H)| \leq \frac{\delta_m}{m!} c^m |H - H_0|^m, \quad |s_m(H)| \leq \frac{\delta_m^{-1}}{m!} c^m |H - H_0|^m$$

where $\delta_m = c$ if m is odd and $= 1$ if m is even, and therefore

$$(5.10) \quad \begin{aligned} & |\theta^n - n(n-1)\theta^{n-2}s_2(H) + n(n-1)(n-2)(n-3)\theta^{n-4}s_4(H) + \dots| \\ & \leq |\theta|^n + \binom{n}{2} |\theta|^{n-2} c^2 |H - H_0|^2 + \dots \\ & = \frac{(|\theta| + c|H - H_0|)^n + (|\theta| - c|H - H_0|)^n}{2} \leq (|\theta| + c|H - H_0|)^n. \end{aligned}$$

In a similar way

$$(5.11) \quad \begin{aligned} & |n\theta^{n-1}s_1(H) - n(n-1)(n-2)\theta^{n-3}s_3(H) + \dots| \\ & \leq \binom{n}{1} |\theta|^{n-1} |H - H_0| + \binom{n}{3} |\theta|^{n-3} c^2 |H - H_0|^3 + \dots \\ & \leq \frac{1}{2} [(|\theta| + c|H - H_0|)^n - (|\theta| - c|H - H_0|)^n] c^{-1} \\ & \leq (|\theta| + c|H - H_0|)^n c^{-1}. \end{aligned}$$

From the convergence of the series (5.7) it follows that the series

$$(5.12) \quad \sum_{n=0}^{\infty} |C_n| [|\theta| + c|H - H_0|]^n \quad \text{and} \quad \sum_{n=0}^{\infty} |D_n| [|\theta| + c|H - H_0|]^n c^{-1}$$

which majorize the series in the square brackets of (5.8), converge uniformly if

$$(5.13) \quad |\theta| + c|H - H_0| \leq \theta_1.$$

The question of dependence of solutions of (5.1) upon the variation of $l(H)$ has been considered in various papers (see BERS [10], FRANKL [14], GELBART [15, 16], GERMAIN & BADER [17, 18], and PROTTER [20]). In particular BERS [10] showed that the solutions ϕ and ψ depend continuously upon the variations of $l(H)$. It is of interest that a more detailed analysis shows that certain special

* The upper index (n) denotes the n^{th} derivative. Compare also [7], p. 448.

solutions ψ of (5.1) are even monotone functions of $l(H)$. This yields stronger bounds for the stream function $\psi(H, \theta)$ in terms of the values of ψ and its normal derivative $\partial\psi/\partial H$ on the line $H = H_0$, in particular for $H_0 = 0$.

Let $\psi(H, \theta)$ be a solution of (5.1) which is an analytic function of H and θ in the domain under consideration.

Suppose further that in the supersonic region $[0 \leq H \leq H_1]$ the inequality

$$(5.14) \quad a_1^2 H^M \leq (-l(H)) \leq a_2^2 H^M, \quad 0 < a_1 < a_2,$$

holds [see (5.1)], and on the line $H = 0$ the stream function and its derivative satisfy the initial conditions

$$(5.15) \quad \psi(0, \theta) = \sum_{n=0}^N C_n \theta^n \equiv f(\theta), \quad \left. \frac{\partial\psi(H, \theta)}{\partial H} \right|_{H=0} = 0;$$

then in $[0 \leq H \leq H_1]$, $\psi(H, \theta)$ is bounded by the limits* (5.24). If in the subsonic region $[0 \geq H \geq H_1]$ the coefficient $l(H)$ satisfies

$$(5.16) \quad a_2^2 H^M \leq -l(H) \leq a_1^2 H^M, \quad 0 < a_1 < a_2,$$

and M is odd then in $[0 \geq H \geq H_1]$, $\psi(H, \theta)$ satisfies the same inequalities (5.24).

Under the same conditions if (5.15) is replaced by

$$(5.17) \quad \psi(0, \theta) = 0, \quad \left. \frac{\partial\psi(H, \theta)}{\partial H} \right|_{H=0} = \sum_{n=1}^N n D_n \theta^{n-1} \equiv g^{(1)}(\theta) = \frac{dg(\theta)}{d\theta}.$$

the inequality (5.24) has to be replaced for $H > 0$ by (5.32) and for $H < 0$ and an odd M by (5.52), respectively.†

Remark 5.2. It can be shown that similar inequalities hold for the potential function ϕ .

Proof. We begin with deriving the inequality for the special case $f(\theta) = \theta^n$, $g^{(1)}(\theta) = 0$.

We consider the expression (see (5.8))

$$(5.18) \quad \psi(H, \theta) = \sum_{x=0}^{\infty} (-1)^x f^{(2x)}(\theta) s_{2x}(H), \quad H > 0, \quad f^{(0)}(\theta) \equiv f(\theta),$$

which becomes

$$(5.19) \quad \begin{aligned} & \theta^n S_{1n}(H, \theta; -l(H)), \text{ where} \\ & S_{1n}(H, \theta; -l(H)) = 1 + \frac{n(n-1)}{\theta^2} \int_0^H \int_0^{H_2} (-l(H_1)) dH_1 dH_2 \\ & + \frac{n(n-1)(n-2)(n-3)}{\theta^4} \int_0^H \int_0^{H_4} (-l(H_3)) \int_0^{H_3} \int_0^{H_2} \\ & \quad \cdot (-l(H_1)) dH_1 dH_2 dH_3 dH_4 + \dots \end{aligned}$$

* $F(a, b, c; x)$ denotes the hypergeometric series.

† Similar relations hold also if the initial data are given on a line $H = H_0 \neq 0$. In this case we replace H by $H' = H - H_0$ and $l(H)$ by $l'(H) = l(H) - l(H_0)$.

$S_{1n}(\mathbf{H}, \theta; -l(\mathbf{H}))$ are monotone functionals of $(-l(\mathbf{H}))$, because in the supersonic region ($\mathbf{H} > 0$) we have $-l(\mathbf{H}) > 0$. In particular, if $(-l(\mathbf{H}))$ satisfies the inequality

$$(5.20) \quad a_1^2 \mathbf{H}^M \leq (-l(\mathbf{H})) \leq a_2^2 \mathbf{H}^M, \quad 0 \leq \mathbf{H} \leq \mathbf{H}_1, \quad 0 < a_1 < a_2 < \infty,$$

then

$$(5.21) \quad S_{1n}(\mathbf{H}, \theta; a_1^2 \mathbf{H}^M) \leq S_{1n}(\mathbf{H}, \theta; -l(\mathbf{H})) \leq S_{1n}(\mathbf{H}, \theta; a_2^2 \mathbf{H}^M).$$

But for $-l(\mathbf{H}) \Rightarrow a^2 \mathbf{H}^M$ and m even (see (5.6))

$$(5.22) \quad \begin{aligned} s_m(\mathbf{H}) &= \int_0^{\mathbf{H}} \int_0^{\mathbf{H}_m} (-a^2 \mathbf{H}_{m-1}^M) \cdots \int_0^{\mathbf{H}_3} \int_0^{\mathbf{H}_2} (-a^2 \mathbf{H}_1^M) d\mathbf{H}_1 \cdots d\mathbf{H}_m \\ &= \frac{(-1)^{m/2} a^m \mathbf{H}^{\frac{1}{2}m(M+2)}}{\prod_{s=1}^{m/2} [(sM + 2s - 1)(sM + 2s)]} = \frac{(-1)^{m/2} a^m \mathbf{H}^{\frac{1}{2}m(M+2)}}{(M + 2)^m \left(\frac{m}{2}\right)! \left[\frac{M+1}{M+2}\right]_{m/2}}, \end{aligned}$$

where $[a]_j = a(a+1) \cdots (a+j-1)$. Consequently,

$$(5.23) \quad \begin{aligned} S_{1n}(\mathbf{H}, \theta; a^2 \mathbf{H}^M) &= 1 + \frac{4 \left(-\frac{n}{2}\right) \left(\frac{-n+1}{2}\right) a^2 \mathbf{H}^{M+2}}{(M+2)^2 1 \cdot \left(\frac{M+1}{M+2}\right) \theta^2} \\ &+ \frac{4^2 \left(-\frac{n}{2}\right) \left(-\frac{n}{2} + 1\right) \left(\frac{-n+1}{2}\right) \left(\frac{-n+1}{2} + 1\right) a^4 \mathbf{H}^{2(M+2)}}{(M+2)^4 1 \cdot 2 \left(\frac{M+1}{M+2}\right) \left(\frac{M+1}{M+2} + 1\right) \theta^4} + \cdots \\ &= F\left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; \frac{4a^2 \mathbf{H}^{M+2}}{(M+2)^2 \theta^2}\right). \end{aligned}$$

Therefore,* for the case $f(\theta) = \sum_{n=0}^N C_n \theta^n$, $g(\theta) = 0$,

$$(5.24) \quad \begin{aligned} &\sum_{\mu=1}^2 (-1)^{\mu+1} \sum_{n=0}^N ((-1)^{\mu-1} C_n \theta^n)^+ F\left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; \frac{4a_\mu^2 |\mathbf{H}|^{M+2}}{(M+2)^2 \theta^2}\right) \\ &\leq \sum_{n=0}^N C_n \theta^n S_{1n}(\mathbf{H}, \theta, -l(\mathbf{H})) \equiv \psi(\mathbf{H}, \theta) \\ &\leq \sum_{\mu=1}^2 (-1)^\mu \sum_{n=0}^N ((-1)^\mu C_n \theta^n)^+ F\left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; \frac{4a_\mu^2 |\mathbf{H}|^{M+2}}{(M+2)^2 \theta^2}\right). \end{aligned}$$

Here

$$(5.25) \quad A^+ = A \quad \text{if} \quad A \geq 0, \quad A^+ = 0 \quad \text{if} \quad A \leq 0.$$

* In the supersonic case $|\mathbf{H}| = \mathbf{H}$, but in connection with later considerations concerning the subsonic case we here replace $\mathbf{H}$ by $|\mathbf{H}|$.

We proceed now to consider the series (see (5.8))

$$(5.26) \quad \psi(H, \theta) = \sum_{k=0}^{\infty} (-1)^k g^{(2k+1)}(\theta) s_{2k+1}(H), \quad H > 0,$$

where $f(\theta) = 0, \quad g(\theta) = \theta^n$.

It becomes equal to

$$(5.27) \quad \begin{aligned} & [n\theta^{n-1} s_1(H) - n(n-1)(n-2)\theta^{n-3} s_3(H) \\ & + n(n-1)(n-2)(n-3)(n-4)\theta^{n-5} s_5(H) - \dots] \\ & = \left[n\theta^{n-1} \int_0^H dH_1 + n(n-1)(n-2)\theta^{n-3} \int_0^H \int_0^{H_1} (-l(H_2)) \int_0^{H_2} dH_1 dH_2 dH_3 \right. \\ & + n(n-1)(n-2)(n-3)(n-4)\theta^{n-5} \int_0^H \int_0^{H_1} (-l(H_4)) \int_0^{H_4} \int_0^{H_3} \\ & \quad \left. \cdot (-l(H_2)) \int_0^{H_2} dH_1 dH_2 dH_3 dH_4 dH_5 + \dots \right] \\ & \equiv n\theta^{n-1} S_{2n}[H, \theta; -l(H)], \end{aligned}$$

where

$$(5.28) \quad \begin{aligned} & S_{2n}[H, \theta; -l(H)] \\ & = \int_0^H dH_1 + \frac{(n-1)(n-2)}{\theta^2} \int_0^H \int_0^{H_1} (-l(H_2)) \int_0^{H_2} dH_1 dH_2 dH_3 \\ & + \frac{(n-1)(n-2)(n-3)(n-4)}{\theta^4} \int_0^H \int_0^{H_1} (-l(H_4)) \int_0^{H_4} \int_0^{H_3} \\ & \quad \cdot (-l(H_2)) \int_0^{H_2} dH_1 dH_2 dH_3 dH_4 dH_5 + \dots \end{aligned}$$

If $(-l(H_2))$ satisfies the inequality (5.20) we obtain

$$(5.29) \quad S_{2n}(H, \theta; a_1 H^M) \leq S_{2n}(H, \theta; -l(H)) \leq S_{2n}(H, \theta; a_2 H^M).$$

If m is odd and $-l(H) = a^2 H^M$ we easily prove by induction that

$$(5.30) \quad \begin{aligned} s_m(H) &= - \int_0^H \int_0^{H_1} a^2 H_2^M s_{m-2}(H_2) dH_2 dH_1 \\ &= (-1)^{\frac{1}{2}(m-1)} \int_0^H \int_0^{H_1} a^2 H^M \frac{a^{m-3} H^{\frac{1}{2}(m-3)M+m-2}}{(2+M)^{m-3} \left(\frac{m-3}{2}\right)! \left(\frac{M+3}{M+2}\right)_{(m-3/2)}} dH_2 dH_1 \end{aligned}$$

$$= \frac{(-1)^{\frac{1}{2}(m-1)} a^{m-1} H^{\frac{1}{2}(m-1)M+m}}{(2+M)^{m-1} \left(\frac{m-1}{2}\right)! \left(\frac{M+3}{M+2}\right)^{\frac{1}{2}(m-1)}}$$

In the special case where $-l(H) = a^2 H^M$ we obtain

$$\begin{aligned}
 n\theta^{n-1} S_{2n}[H, \theta; a^2 H^M] &= n\theta^{n-1} H + n(n-1)(n-2)\theta^{n-3} \frac{a^2 H^{M+3}}{(M+2)(M+3)} \\
 &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)\theta^{n-5} a^4 H^{2M+5}}{(M+2)(M+3)(2M+4)(2M+5)} + \dots \\
 (5.31) \quad &= n\theta^{n-1} H \left[1 + \frac{4 \left(\frac{-n+1}{2}\right) \left(-\frac{n}{2} + 1\right) a^2 H^{M+2}}{(M+2)^2 1 \cdot \left(\frac{M+3}{M+2}\right) \theta^2} \right. \\
 &\quad \left. + \frac{4^2 \left(\frac{-n+1}{2}\right) \left(\frac{-n+1}{2} + 1\right) \left(-\frac{n}{2} + 1\right) \left(-\frac{n}{2} + 2\right) a^4 H^{2(M+2)}}{(M+2)^4 1 \cdot 2 \cdot \left(\frac{M+3}{M+2}\right) \left(\frac{M+3}{M+2} + 1\right) \theta^4} + \dots \right] \\
 &= n\theta^{n-1} HF \left(\frac{-n+1}{2}, -\frac{n}{2} + 1, \frac{M+3}{M+2}; \frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2} \right).
 \end{aligned}$$

Therefore, in analogy to (5.24), when $f(\theta) = 0$, $g(\theta) = \sum_{n=1}^N D_n \theta^n$,

$$\begin{aligned}
 &\sum_{n=1}^N \sum_{\mu=1}^2 (-1)^{\mu+1} ((-1)^{\mu+1} n D_n \theta^{n-1} H)^+ \\
 &\quad \cdot F \left(\frac{-n+1}{2}, -\frac{n}{2} + 1, \frac{M+3}{M+2}, \frac{4a_\mu H^{M+2}}{(M+2)^2 \theta^2} \right) \\
 (5.32) \quad &\leq \sum_{n=0}^N n D_n \theta^{n-1} S_{2n}(H, \theta; -l(H)) \equiv \psi(H, \theta) \\
 &\leq \sum_{n=1}^N \sum_{\mu=1}^2 (-1)^{\mu+1} ((-1)^{\mu+1} n D_n \theta^{n-1} H)^+ \\
 &\quad \cdot F \left(\frac{-n+1}{2}, -\frac{n}{2} + 1, \frac{M+3}{M+2}, \frac{4a_{3-\mu} H^{M+2}}{(M+2)^2 \theta^2} \right).
 \end{aligned}$$

We proceed now to the subsonic case, *i.e.*, to the case where $H < 0$. In this case we have to distinguish two possibilities, M odd and M even. We assume at first that M is odd and that $l(H)$ satisfies the inequality

$$(5.33) \quad a_2^2 H^M \leq -l(H) \leq a_1^2 H^M, \quad 0 < a_1^2 < a_2^2 < \infty.$$

Now $l(H) > 0$, and therefore according to (5.19), for $f(\theta) = \theta^n$, $g(\theta) = 0$ we obtain

$$(5.34) \quad S_{1n}(H, \theta; -l(H)) = P_{1n}(H, \theta; -l(H)) - P_{2n}(H, \theta; -l(H)),$$

where

$$(5.35) \quad \begin{aligned} \theta^n P_{1n}(H, \theta; -l(H)) &= \sum_{k=0}^{\infty} f^{(4k)}(\theta) s_{4k}(H), \\ \theta^n P_{2n}(H, \theta; -l(H)) &= \sum_{k=0}^{\infty} f^{(4k+2)}(\theta) s_{4k+2}(H). \end{aligned}$$

Now

$$(5.36) \quad \begin{aligned} P_{1n}(H, \theta; a^2 H^M) &= 1 + \frac{n(n-1)(n-2)(n-3)}{\theta^4} \int_0^H \int_0^{H_1} a^2 H_3^M \\ &\quad \cdot \int_0^{H_3} \int_0^{H_2} a^2 H_1^M dH_1 dH_2 dH_3 dH_4 + \dots \\ &= 1 + \frac{4^2}{(M+2)^4} \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}+1\right)\left(\frac{-n+1}{2}\right)\left(\frac{-n+1}{2}+1\right) a^4 H^{2(M+2)}}{2! \theta^4 \left(\frac{M+1}{M+2}\right)\left(\frac{M+1}{M+2}+1\right)} \\ &\quad + \dots \\ &= \frac{1}{2} \left[\sum_{\mu=1}^2 F\left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}, (-1)^{3-\mu} \frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2}\right) \right]. \end{aligned}$$

In a similar way we show that

$$(5.37) \quad \begin{aligned} P_{2n}(H, \theta; a^2 H^M) &= \frac{n(n-1)}{\theta^2} \int_0^H \int_0^{H_1} a^2 H_1^M dH_1 dH_2 \\ &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{\theta^6} \\ &\quad \cdot \int_0^H \int_0^{H_6} a^2 H_5^M \int_0^{H_5} \int_0^{H_4} a^2 H_3^M \int_0^{H_3} \int_0^{H_2} a^2 H_1^M dH_1 dH_2 dH_3 dH_4 dH_5 dH_6 \\ &= \frac{n(n-1)a^2 H^{M+2}}{(M+2)^2 1! [(M+1)/(M+2)]_1 \theta^2} \\ &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)a^6 H^{3(M+2)}}{(M+2)^6 3! [(M+1)/(M+2)]_3 \theta^6} + \dots \end{aligned}$$

$$= \frac{1}{2} \left[\sum_{\mu=1}^2 (-1)^{3-\mu} F \left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; (-1)^{3-\mu} \frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2} \right) \right]$$

If M is an odd number, the series (5.19) satisfies the inequality:

$$(5.38) \quad \begin{aligned} P_{1n}(H, \theta; a_1^2 H^M) - P_{2n}(H, \theta; a_1^2 H^M) &\leq S_{1n}(H, \theta; -l(H)) \\ &\leq P_{1n}(H, \theta; a_2^2 H^M) - P_{2n}(H, \theta; a_2^2 H^M). \end{aligned}$$

Indeed, in this case $P_{1n}(H, \theta; a^2 H^M)$ is positive, while $P_{2n}(H, \theta; a^2 H^M)$ is negative (since $H < 0$) and therefore, for $a_2^2 H^M \leq -l(H) \leq a_1^2 H^M$,

$$(5.39) \quad P_{1n}(H, \theta; a_1^2 H^M) \leq P_{1n}(H, \theta; -l(H)) \leq P_{1n}(H, \theta; a_2^2 H^M),$$

$$(5.40) \quad P_{2n}(H, \theta; a_2^2 H^M) \leq P_{2n}(H, \theta; -l(H)) \leq P_{2n}(H, \theta; a_1^2 H^M),$$

so that

$$(5.41) \quad S_{1n}(H, \theta; a_1^2 H^M) \leq S_{1n}(H, \theta; -l(H)) \leq S_{1n}(H, \theta; a_2^2 H^M).$$

According to (5.34), (5.36), and (5.37)

$$(5.42) \quad \begin{aligned} S_{1n}(H, \theta; a^2 H^M) &= P_{1n}(H, \theta; a^2 H^M) - P_{2n}(H, \theta; a^2 H^M) \\ &= \frac{1}{2} \left[\sum_{\mu=1}^2 F \left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; (-1)^{3-\mu} \frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2} \right) \right. \\ &\quad \left. - \sum_{\mu=1}^2 (-1)^{3-\mu} F \left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; (-1)^{3-\mu} \frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2} \right) \right] \\ &= F \left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; -\frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2} \right). \end{aligned}$$

Therefore, $\psi(H, \theta) \equiv \sum_{n=0}^N C_n \theta^n S_{1n}(H, \theta; -l(H))$ again satisfies the inequality (5.24).

Remark 5.3. If M is even*, then the situation changes somewhat. In this case P_{1n} and P_{2n} are sums of even powers of H , and instead of (5.38) we obtain

$$(5.43) \quad \begin{aligned} P_{1n}(H, \theta; a_1^2 H^M) - P_{2n}(H, \theta; a_2^2 H^M) &\leq S_{1n}(H, \theta; -l(H)) \\ &\leq P_{1n}(H, \theta; a_2^2 H^M) - P_{2n}(H, \theta; a_1^2 H^M). \end{aligned}$$

Hence for $f(\theta) = C_n \theta^n$, $g(\theta) = 0$ we obtain

$$(5.43) \quad \begin{aligned} C_n \theta^n S_{1n}(H, \theta, -l(H)) &\leq \frac{1}{2} \sum_{\sigma=1}^2 \sum_{\mu=1}^2 (-1)^{\mu+1} ((-1)^{\mu+1} C_n \theta^n)^+ \\ &\cdot \left[F \left(-\frac{n}{2}, \frac{-n+1}{2}, \frac{M+1}{M+2}; (-1)^{\sigma+1} \frac{4a_{3-\mu}^2 H^{M+2}}{(M+2)^2 \theta^2} \right) \right] \end{aligned}$$

* Let us stress that this case does not occur in the theory of compressible fluids.

$$+ (-1)^{\sigma} F \left(-\frac{n}{2}, -\frac{n+1}{2}, \frac{M+1}{M+2}; (-1)^{\sigma+1} \frac{4a_{\mu}^2 H^{M+2}}{(M+2)^2 \theta^2} \right) \Big].$$

A similar expression can be obtained for the lower bound.

We proceed now to the case where $H < 0$ and M is odd.

In analogy to (5.34) (see (5.3) and (5.6)) we write when $f(\theta) = 0$, $g(\theta) = \theta^n$,

$$(5.44) \quad n\theta^{n-1} HP_{3n}(H, \theta; -l(H)) = g^{(1)}(\theta)s_1(H) + g^{(5)}(\theta)s_5(H) + \dots,$$

$$(5.45) \quad n\theta^{n-1} HP_{4n}(H, \theta; -l(H)) = g^{(3)}(\theta)s_3(H) + g^{(7)}(\theta)s_7(H) + \dots$$

where

$$(5.46) \quad \begin{aligned} P_{3n}(H, \theta; -l(H)) = & -\frac{1}{H} \left[\int_H^0 dH_1 + \frac{(n-1)(n-2)(n-3)(n-4)}{\theta^4} \right. \\ & \cdot \int_H^0 \int_{H_5}^0 l(H) \int_{H_4}^0 \int_{H_3}^0 l(H_2) \int_{H_2}^0 dH_1 dH_2 dH_3 dH_4 dH_5 + \dots \Big] \end{aligned}$$

and

$$(5.47) \quad \begin{aligned} P_{4n}(H, \theta; -l(H)) \\ = -\frac{1}{H} \left[\frac{(n-1)(n-2)}{\theta^2} \int_H^0 \int_{H_3}^0 l(H) \int_{H_2}^0 dH_1 dH_2 dH_3 + \dots \right]. \end{aligned}$$

If $l(H) > 0$, i.e., for $H < 0$ and M odd $P_{\kappa n}(H, \theta; -l(H))$, $\kappa = 3, 4$ are increasing functionals of $l(H)$. In analogy to (5.34), from (5.28), (5.46) and (5.47) we have

$$(5.48) \quad H^{-1} S_{2n}(H, \theta; -l(H)) = P_{3n}(H, \theta; -l(H)) - P_{4n}(H, \theta; -l(H)).$$

We obtain now (for an odd M)

$$(5.49) \quad \begin{aligned} \sum_{\nu=1}^2 (-1)^{\nu+1} P_{2+\nu, n}(H, \theta; a_{\nu} H^M) \\ \leq H^{-1} S_{2n}(H, \theta; -l(H)) \leq \sum_{\nu=1}^2 (-1)^{\nu+1} P_{2+\nu, n}(H, \theta; a_{3-\nu} H^M). \end{aligned}$$

Repeating the same considerations as in (5.36) and (5.37) we obtain

$$(5.50) \quad \begin{aligned} n\theta^{n-1} HP_{3n}(H, \theta; a^2 H^M) \\ = n\theta^{n-1} H + \frac{n(n-1)(n-2)(n-3)(n-4)\theta^{n-5} a^4 H^{2(M+2)+1}}{(M+2)^4 2! \left(\frac{M+3}{M+2} \right)_2} + \dots \\ = \frac{n\theta^{n-1} H}{2} \sum_{\mu=1}^2 F \left(-\frac{n+1}{2}, -\frac{n}{2} + 1, \frac{M+3}{M+2}; (-1)^{3-\mu} \frac{4a^2 H^{M+2}}{(M+2)^2 \theta^2} \right) \end{aligned}$$

and

$$\begin{aligned}
 n\theta^{n-1} \text{HP}_{4n}(\text{H}, \theta); a^2 \text{H}^M) &= \frac{n(n-1)(n-2)\theta^{n-3} a^2 \text{H}^{(M+2)+1}}{(M+2)^2 1! [(M+3)/(M+2)]_1} \\
 &+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)\theta^{n-7} a^6 \text{H}^{3(M+2)+1}}{(M+2)^6 3! [(M+3)/(M+2)]_3} + \dots \\
 (5.51) \quad &= \frac{n\theta^{n-1} \text{H}}{2} \left[\sum_{\mu=1}^2 (-1)^{3-\mu} F\left(\frac{-n+1}{2}, -\frac{n}{2}+1, \frac{M+3}{M+2}; \right. \right. \\
 &\quad \left. \left. (-1)^{3-\mu} \frac{4a^2 \text{H}^{M+2}}{(M+2)^2 \theta^2} \right) \right].
 \end{aligned}$$

Therefore, for $f(\theta) = 0$, $g^{(1)}(\theta) = \sum_{n=1}^N n D_n \theta^{n-1}$, it follows from (5.49), (5.50) and (5.51) that

$$\begin{aligned}
 \sum_{n=1}^N D_n n\theta^{n-1} \text{H} [\text{H}^{-1} S_{2n}(\text{H}, \theta; -l(\text{H}))] &\equiv \psi(\text{H}, \theta) \\
 (5.52) \quad &\leq \sum_{n=1}^N \left[\frac{1}{2} (D_n n\theta^{n-1} \text{H})^+ \sum_{\nu=1}^2 \sum_{\mu=1}^2 (-1)^{3\mu-\nu\mu} F\left(\frac{-n+1}{2}, -\frac{n}{2}+1, \frac{M+3}{M+2}; \right. \right. \\
 &\quad \left. \left. (-1)^{3-\mu} \frac{a_{3-\nu}^2 \text{H}^{M+2}}{(M+2)^2 \theta^2} \right) - \frac{1}{2} (-D_n n\theta^{n-1} \text{H})^+ \sum_{\nu=1}^2 \sum_{\mu=1}^2 (-1)^{3\mu-\nu\mu} \right. \\
 &\quad \left. \cdot F\left(\frac{-n+1}{2}, -\frac{n}{2}+1, \frac{M+3}{M+2}; (-1)^{3-\mu} \frac{a_{\nu}^2 \text{H}^{M+2}}{(M+2)^2 \theta^2} \right) \right].
 \end{aligned}$$

Replacing $a_{3-\nu}^2$ by a_{ν}^2 , $\nu = 1, 2$, we obtain the lower bound.

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