*Examples Illustrating Expansion Procedures for the Navier-Stokes Equations**

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Communicated by C. C. LIN

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* This research was carried out with the support of the Office of Naval Research under Contract N6 ONR-244 T.O. VIII, Project No. 041-156. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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1. INTRODUCTION

This paper contains a study of approximation methods used in the theory of viscous incompressible flow. The general viewpoint adopted is that most of these approximation methods (such as Prandtl boundary layer theory, Stokes and Oseen approximations) are implicitly based on the assumption that exact solutions of the Navier-Stokes equations have asymptotic expansions of certain forms. For a critical study of an approximation method the exact form of the relevant expansion should be given explicitly. In the present paper a systematic and detailed investigation is made of the expansions of the solutions of the Navier-Stokes equations for a specially restricted class of flow problems. However, the solutions given, approximate or exact, have the main purpose of verifying and exemplifying general ideas about approximation methods. To this end often several different methods are used to arrive at the same answers. The application of the general ideas to more general flow problems is obvious in many cases; it is discussed in some detail in Section 6.

In Section 2 the basic flow problem studied is introduced: An infinitely long circular cylinder is immersed in a viscous incompressible fluid and everything is initially at rest. At the initial instant ($t = 0$) the cylinder starts to move in the direction of its axis with a constant speed U . Also at $t = 0$ the radius of the cylinder starts to increase proportionally to t^n ; $0 \leq n \leq 1$ (t = time). This problem is thus a generalization of Rayleigh's problem of the suddenly started infinite plane and may be referred to as the sliding-expanding cylinder problem. Due to the simple geometry the radial velocity component is easily found explicitly and the complete Navier-Stokes equations reduce to a single equation for the axial velocity component u . The partial differential equation satisfied by u is a linear second-order equation of two independent variables, radius r and time t ; it has variable coefficients and is of parabolic type. In fact for $n = 0$, the case of the non-expanding cylinder, this equation reduces to the classical heat equation with axial symmetry for which the solution is well known. For $n = \frac{1}{2}$ it can be reduced by similarity to an ordinary differential equation which is easily solved. These exact solutions of the Navier-Stokes equations are presented in Section 4. Various asymptotic representations are derived by expanding the exact solutions in terms of certain variables or parameters. A basic parameter of viscous flow is the (overall) Reynolds number $\mathcal{R}_* = UL/\nu$, where L = characteristic length, ν = kinematic viscosity. For $n \neq 1$ a characteristic length and hence an overall Reynolds number can be defined for the sliding-expanding cylinder (the exceptional case $n = 1$ requires a separate discussion).

The different expansions that occur are conveniently defined in terms of different limit processes. The basic definitions of variables and limit process are presented in Section 3. For example the Prandtl variables are defined there

(Eq. (3-16)) as certain dimensionless forms of r and t . The Prandtl limit of $u^* = u/U$ is then defined as the limit of u^* as Reynolds number ($\mathcal{R}_e$) tends to infinity while the Prandtl variables are kept fixed. The repeated application of the limit process is also defined, and it yields an asymptotic expansion of u^* in Prandtl variables. The Prandtl limit is related to boundary layer theory, and subsequent terms of the expansion are related to corrections to boundary layer theory. Another set of dimensionless variables in r, t , the Euler variables, are also of importance in obtaining approximations for high Reynolds number flow. The Euler limit and the Euler expansion are defined. The Euler limit gives the inviscid flow and the expansion includes corrections due to displacement effects of the boundary layer. In finding an expansion of the exact solution for high Reynolds numbers which is valid in as large a domain as possible one needs in general a combination of the Euler and the Prandtl expansions. A precise definition of this double expansion as well as the relation ("matching conditions") between its two parts can very conveniently be expressed with the aid of the limit processes (*cf.* [2]). Similarly, for low Reynolds numbers, important limit processes and expansions can be defined. These are called Stokes and Oseen expansions.

The four different expansions just referred to occur in the study of flow past infinite bodies. For semi-infinite or infinite bodies an additional set of variables and limits are of importance. In the present case they are defined as follows: $\sqrt{\nu t}$ is a variable which has the dimensions of a length and contains the kinematic viscosity ν . Thus $r/\sqrt{\nu t}$ is a dimensionless variable which is denoted by ξ . The value of ξ at the surface of the cylinder is denoted by η . Thus, η measures the ratio of a geometrical length, the cylinder radius, to a "viscous" length at the time t . η has the nature of a Reynolds number and will be called the cross-sectional Reynolds number. Similarity properties of the special problem treated here lead in a natural way to the introduction of ξ and η as shown in Section 3. If these variables are used the overall Reynolds number $\mathcal{R}_e$ may be completely eliminated from the problem. In a certain sense η takes its place; $\mathcal{R}_e \rightarrow \infty$ may correspond to $\eta \rightarrow \infty$ *etc.*

The ideas introduced in Section 3 are used in Section 4 to obtain expansions of the exact solutions of the cases $n = 0$ and $n = \frac{1}{2}$. In Section 5, certain forms of the expansion of the solution are assumed for $0 \leq n \leq 1$ and these expansions are substituted directly into the partial differential equations and the boundary conditions. Recursive methods for finding the successive terms are developed and the first few terms are determined explicitly. The domain of uniform validity of the various expansions is studied. It is seen from Sections 4 and 5 that, for the special problem studied, the expansions in the high Reynolds number case are identical whether $\mathcal{R}_e$ or η is considered large. However, in the low Reynolds number case it turns out that the original Oseen and Stokes expansions in terms of $\mathcal{R}_e$ are inferior (that is, valid in a smaller domain) to the corresponding expansion in terms of η .

A summary and discussion of the results obtained and of possible generalizations is given in Section 6. First it is proved that expansions which are uniformly

valid for all (r, t) cannot be obtained except for $n = \frac{1}{2}$. This result follows directly from the similarity discussed in Section 3. The same situation occurs for stationary viscous flow past infinite wedges. Then general high Reynolds number expansions are discussed, and it is shown how, with simple generalizations, the expansion procedures apply to flow past bodies without separation. The problem of low Re is, at present, less well understood and is therefore treated in considerably more detail. A general discussion of the nature of the Stokes and Oseen expansions and their relationship to one another is presented. After this some special examples are considered in more detail. In particular the problem of a sliding-expanding cylinder where the expansion rate is not a power of time is studied in order to clarify the role of η and Re . Finally, the application of these ideas to the problem of flow past semi-infinite bodies is indicated.

The work contained in this paper is part of a general study of expansion procedures being conducted at GALCIT under sponsorship of ONR. It is related to previous work by LATTI [1] and KAPLUN [2] on high Reynolds number expansions and current work by KAPLUN [3] on low Reynolds number expansions. Some of the basic ideas used are due to KAPLUN and his help is gratefully acknowledged. The solution in the special case $n = 0$ has been discussed previously by BATCHELOR [4]. Variables equivalent to ξ and η have been used previously by SEBAN & BOND [5] and PROBSTEN [6] in the study of corrections to the boundary layer on semi-infinite bodies due to transverse curvature; STEWARTSON [7] also used these variables in his study of the viscous layer on the cylindrical afterbody far downstream on a body of revolution. It is hoped that the work of STEWARTSON and that of MARK [8] on flow past a paraboloid at high and low Reynolds numbers can be clarified in terms of the general results presented here.

For questions of mathematical analysis the authors have found [9] and [10] very helpful references. Whenever possible their notation has been used. We also have had the privilege of receiving information and advice directly from A. ERDELYI, the director of the Bateman Manuscript Project, which authored [9] and [10]. However, the present paper does not aim at mathematical completeness and rigor. Emphasis has instead been placed on developing and illustrating certain general ideas.

2. FORMULATION OF THE PROBLEM

In this section the mathematical problem will be formulated, some notation introduced, and the solution given for the radial velocity.

Consider an infinitely long circular cylinder of radius R , whose axis is taken to be the x -axis, immersed in an incompressible viscous fluid. For $t < 0$ the cylinder and the fluid are at rest. At $t = 0$ the cylinder starts sliding in the positive x -direction with constant velocity U . Also at $t = 0$ the radius of the cylinder starts varying as a function of time: $R = R(t)$. We shall use the following notation:

r = radial distance from x -axis

u = x -component of velocity

v = r -component of velocity

p = pressure

ν = kinematic viscosity

ρ_∞ = density = const.

From the symmetries of the problem it follows that the Navier-Stokes equations reduce to the following simple form:

$$u_t + vu_r = \nu \left(u_{rr} + \frac{1}{r} u_r \right) \quad (2-1a)$$

$$(rv)_r = 0 \quad (2-1b)$$

$$v_t + vv_r = -\frac{1}{\rho_\infty} p_r + \nu \left(v_{rr} + \frac{1}{r} v_r - \frac{v}{r^2} \right) \quad (2-1c)$$

The initial and boundary conditions are:

$$p = p_0, \quad u = v = 0 \quad \text{for} \quad t < 0 \quad (2-2a)$$

$$u = U, \quad v = \frac{dR}{dt} \quad \text{for} \quad t \geq 0, \quad r = R(t) \quad (2-2b)$$

$$u = v = 0 \quad \text{for all} \quad t, \quad r = \infty \quad (2-2c)$$

The subscript “ w ” will be used to denote conditions at the wall $r = R$.

The solution for v is immediately found from the continuity equation (2-1b) and the boundary conditions. It is

$$v = \frac{v_w R}{r} = \frac{1}{2r} \frac{dR^2}{dt} \quad (2-3)$$

where $v_w = dR/dt$. In principle the solution for the pressure may be obtained by integrating Eq. (2-1c). In the following discussion attention will be focused on the equation for u (Eq. (2-1a)).

Only the following special case will be considered in detail

$$R = At^n; \quad A = \text{Const.}, \quad 0 \leq n \leq 1 \quad (2-4)$$

For $n \neq 1$ a quantity L of dimension length may be formed from A and U such that

$$\frac{R}{L} = \left(\frac{Ut}{L} \right)^n \quad (2-5)$$

Multiplying the right-hand side by a non-dimensional number β does not lead to a more general case. By redefining the length the resulting equation may be brought back to the original form. For $n = 1$ no length may be formed from A and U . It will then be convenient to introduce an arbitrary length L (independent of ν). This length, and the Reynolds number based thereupon, will disappear in the final formulas. For $n = 1$ one obtains, however, a more general expression by multiplying the right-hand side of Eq. (2-5) by β :

$$R = \beta Ut \quad (2-6)$$

The case of $\beta \neq 1$ will be treated separately and unless otherwise stated we shall assume $\beta = 1$.

If R is given by Eq. (2-5) it follows from Eq. (2-3) that

$$\frac{v}{U} = \frac{nL}{r} \left(\frac{Ut}{L} \right)^{2n-1} \quad (2-7)$$

Finally we note that in the present case the *Oseen equation* for u is

$$u_t = \nu \left(u_{rrr} + \frac{1}{r} u_r \right) \quad (2-8)$$

The *Stokes equation* is

$$0 = u_{rr} + \frac{1}{r} u_r \quad (2-9)$$

and the *Euler equation* is

$$u_t + vu_r = 0 \quad (2-10)$$

3. LIMIT PROCESSES AND EXPANSIONS

In this section various non-dimensional variables will be defined and the equations of Section 2 expressed in these variables. For each set of variables a certain limit process will be introduced. Each such limit process will lead to an asymptotic expansion of a solution of Eqs. (2-1a, b). One of the main questions to be discussed in this paper is to what extent such asymptotic expansions represent valid approximations. A brief discussion of similarity of the problem will be given in connection with the introduction of similarity variables. In general, however, the application and justification for the concepts introduced here will be postponed to subsequent sections.

A fundamental parameter is the Reynolds number, more specifically the overall Reynolds number, defined by

$$\mathcal{R}_e = \frac{UL}{\nu} \quad (3-1)$$

In the latter part of this section it will be shown how, in a certain sense, $\mathcal{R}_e$ may be eliminated from the problem. Nevertheless this parameter will play a fundamental role in the discussion.

For the simplest non-dimensional form of independent variables the following notation will be used

$$u^* = \frac{u}{U}, \quad v^* = \frac{v}{U} \quad (3-2)$$

Similarly v_w^* will denote v_w/U .

There are two simple ways of making the independent variables non-dimensional: r and Ut may be referred either to the body length L or to the viscous length, ν/U .

The following notation will be used:

$$r^* = \frac{r}{L} \quad t^* = \frac{Ut}{L} \quad (3-3)$$

$$\tilde{r} = \frac{rU}{\nu} = \mathcal{R}_e r^* \quad \tilde{t} = \frac{U^2 t}{\nu} = \mathcal{R}_e t^* \quad (3-4)$$

Similarly, $R^* = R/L$ etc.

In addition the Prandtl limit process will make use of a special non-dimensional form of v and r (cf. Eq. (3-16)) and the Stokes limit a special form of r (cf. Eq. (3-6)). Furthermore, in the discussion of cross sectional Reynolds numbers (cf. (3-30) ff.) the time dependent viscous length $\sqrt{\nu t}$ will be used as a reference length.

The various limit processes and the corresponding variables will now be introduced.

3.1. Low Reynolds numbers. The discussion of the low Reynolds number case will incorporate some ideas introduced by S. KAPLUN (*cf.* [3]). Two sets of variables, Oseen variables and Stokes variables, will be used.

$$\text{Oseen variables: } \tilde{r}, \quad \tilde{t}, \quad u^*, \quad v^* \quad (3-5)$$

$$\text{Stokes variables: } r^+, \quad \tilde{t}, \quad u^*, \quad v^* \quad (3-6a)$$

where

$$r^+ = \frac{\tilde{r}}{\text{Re}^{1-n}} \quad (3-6b)$$

Introducing Oseen variables into the equations of motion (2-1a, b) and the boundary conditions (2-2b, c) one obtains (the initial condition $u = 0$ will in general not be written out explicitly).

$$u_i^* + v^* u_r^* = u_{r\tilde{r}}^* + \frac{1}{\tilde{r}} u_r^* \quad (3-7a)$$

$$v^* = \frac{n(\tilde{t})^{2n-1} \text{Re}^{2-n}}{\tilde{r}} \quad (3-7b)$$

$$u^* = 1 \text{ at } \tilde{r} = \tilde{R} = (\tilde{t})^n \text{Re}^{1-n} \quad (3-8a)$$

$$u^* = 0 \text{ at } \tilde{r} = \infty \quad (3-8b)$$

The same equations in Stokes variables are

$$\text{Re}^{2(1-n)} u_i^* + \text{Re}^{1-n} v^* u_{r^+}^* = u_{r^+ r^+}^* + \frac{1}{r^+} u_{r^+}^* \quad (3-9a)$$

$$v^* = \frac{n(\tilde{t})^{2n-1} \text{Re}^{1-n}}{r^+} \quad (3-9b)$$

$$u^* = 1 \text{ at } r^+ = R^+ = (\tilde{t})^n \quad (3-10a)$$

$$u^* = 0 \text{ at } r^+ = \infty \quad (3-10b)$$

Let f denote a variable quantity such as u^* or v^* . The *Oseen limit* of f is then defined as

$$\lim_0 f = \lim f \text{ as } \text{Re} \rightarrow 0, \quad (\tilde{r}, \tilde{t}) \text{ fixed} \quad (3-11)$$

The *Stokes limit* is defined as

$$\lim_s f = \lim f \text{ as } Re \rightarrow 0, \quad (r^+, \bar{t}) \text{ fixed} \quad (3-12)$$

Expansion procedures associated with these two limit processes depend on a sequence of $\epsilon_j = \epsilon_j(Re)$ which are functions of Re only such that

$$\epsilon_0 = 1, \quad \lim_{Re \rightarrow 0} \frac{\epsilon_{j+1}}{\epsilon_j} = 0 \quad (3-13)$$

The appropriate choice of such a sequence $\epsilon_j(Re)$ will be discussed in the following sections. It will be seen that such a choice is not unique. When the sequence ϵ_j is given each limit leads to an asymptotic expansion of f in the following manner. Apply the Oseen limit to f . This yields $\lim_o f = f_0$. One then forms $\lim_o [(f - f_0)/\epsilon_1] = f_1$, $\lim_o [(f - f_0 - \epsilon_1 f_1)/\epsilon_2] = f_2$, etc. This associates with f a formal series which is the result of the repeated application of the Oseen limit process:

$$f \rightarrow \sum_{j=0} \epsilon_j(Re) f_j(\tilde{r}, \bar{t}) \quad (3-14)$$

By construction this is an asymptotic expansion of f . It will be referred to as the Oseen expansion. Similarly one obtains the Stokes expansion of f by a repeated application of the Stokes limit. Of course, nothing can be said *a priori* about the problem, to what extent these expansions are uniformly valid in $\tilde{r}$ and $\bar{t}$. It is one of the main purposes of this paper to investigate such problems.

3.2. High Reynolds numbers. In the study of solutions for high Reynolds numbers two sets of variables will be used, Euler variables and Prandtl variables.

$$\text{Euler Variables:} \quad r^*, \quad t^*, \quad u^*, \quad v^* \quad (3-15)$$

$$\text{Prandtl Variables:} \quad \tilde{r} \quad \bar{t}, \quad u^*, \quad \bar{v} \quad (3-16a)$$

where

$$\tilde{r} = \frac{(r - R)\sqrt{U}}{\sqrt{\nu L}} = \sqrt{Re}(r^* - R^*) \quad (3-16b)$$

$$\bar{v} = \frac{(v - v_w)\sqrt{L}}{\sqrt{\nu U}} = \sqrt{Re}(v^* - v_w^*) \quad (3-16c)$$

The Prandtl variables as defined above are the variables usually employed in boundary layer theory.

Using Euler variables one obtains from Eq. (2-1a, b) and (2-2b, c)

$$\Re(u_{i*}^* + v^* u_{r*}^*) = u_{r*}^* + \frac{1}{r^*} u_{r*}^* \quad (3-17a)$$

$$v^* = \frac{n(t^*)^{2n-1}}{r^*} \quad (3-17b)$$

$$u^* = 1 \text{ at } r^* = R^* = (t^*)^n \quad (3-18a)$$

$$u^* = 0 \text{ at } r^* = \infty \quad (3-18b)$$

In Prandtl variables the equations and boundary conditions are:

$$u_{i*}^* + \bar{v} u_{\bar{r}}^* = u_{\bar{r}}^* + \frac{1}{\sqrt{\Re_e}} \frac{1}{R^* + \frac{\bar{r}}{\sqrt{\Re_e}}} u_{\bar{r}}^* \quad (3-19a)$$

where $R^* = (t^*)^n$ and the t^* derivative is to be taken with $\bar{r}$ fixed.

$$\bar{v} = - \frac{n\bar{r}(t^*)^{n-1}}{\frac{\bar{r}}{\sqrt{\Re_e}} + R^*} \quad (3-19b)$$

$$u^* = 1 \text{ at } \bar{r} = 0 \quad (3-20a)$$

$$u^* = 0 \text{ at } \bar{r} = \infty \quad (3-20b)$$

The *Euler limit process* is defined by

$$\lim_E f = \lim f \text{ as } \Re_e \rightarrow \infty, \quad (r^*, t^*) \text{ fixed} \quad (3-21)$$

The *Prandtl limit process* is defined by

$$\lim_P f = \lim f \text{ as } \Re_e \rightarrow \infty, \quad (\bar{r}, t^*) \text{ fixed} \quad (3-22)$$

Again the repeated application of a limit process may be defined. In the high Reynolds number case the sequence ϵ_j is defined by

$$\epsilon_j = (\Re_e)^{-(j/2)} \quad (3-23)$$

As an example, by repeated application of the Prandtl limit process to $\bar{v}$ one obtains from Eq. (3-19b)

$$\bar{v} \rightarrow \sum_{j=0}^{\infty} (\Re_e)^{-j/2} V_j(\bar{r}, t^*) \quad (3-24a)$$

where

$$V_j = (-1)^{j+1} n(\bar{r})^{j+1} (t^*)^{-(jn+1)} \quad (3-24b)$$

The power series in Eq. (3-24a) is actually a convergent representation of $\bar{v}$ for $\bar{r} < \sqrt{\mathcal{R}_e}(t^*)^n$.

3.3. Elimination of Overall Reynolds Number $\mathcal{R}_e$; Cross-sectional Reynolds Numbers. In the preceding paragraphs various forms of the basic equations have been introduced, suitable for obtaining solutions at low and high Reynolds numbers, respectively. We shall now show how, except for $n = \frac{1}{2}$, the Reynolds number $\mathcal{R}_e$ can be eliminated altogether. In its stead will appear an instantaneous Reynolds number $\frac{1}{2}R(t)/\sqrt{\nu t}$, which measures the ratio of the cylinder radius at any time to a characteristic viscous length at the same instant. For want of a better name we shall call this quantity the *cross-sectional Reynolds number*.

It is easily seen that the equations and boundary conditions in Oseen form (Eqs. (3-7) and (3-8)) are invariant under the mapping

$$\bar{t} \rightarrow a^2 \bar{t}, \quad \bar{r} \rightarrow a \bar{r}, \quad \mathcal{R}_e \rightarrow a^{1-2n/1-n} \mathcal{R}_e \quad (3-25)$$

where a is any constant. Assuming uniqueness of the solution it then follows that one may replace the three quantities $\bar{t}$, $\bar{r}$, $\mathcal{R}_e$ by two combinations of these, which are invariant under the mapping (3-25).

For $n \neq \frac{1}{2}$ one may introduce

$$\tau = \bar{t} \mathcal{R}_e^{2(1-n)/2n-1} = t^* \mathcal{R}_e^{1/2n-1} \quad (3-26a)$$

$$\rho = \bar{r} \mathcal{R}_e^{1-n/2n-1} = r^* \mathcal{R}_e^{n/2n-1} \quad (3-26b)$$

This reduces the problem for any given Reynolds number to an equivalent problem for $\mathcal{R}_e = 1$. With this choice of variables Eqs. (3-7) and (3-8) become

$$u_\tau^* + \frac{n}{\rho} \tau^{2n-1} u_\rho^* = u_{\rho\rho}^* + \frac{1}{\rho} u_\rho^* \quad (3-27)$$

$$u^* = 1 \quad \text{for} \quad \rho = \tau^n \quad (3-28a)$$

$$u^* = 0 \quad \text{for} \quad \rho = \infty \quad (3-28b)$$

Note that for $n = 1$, $\tau = \bar{t}$ and $\rho = \bar{r}$. This is consistent with the fact that no length may be defined and hence the introduction of a Reynolds number is spurious. However, in the conical case ($n = 1$), we assume R to have the form

$$R = \beta U t \quad (2-6)$$

where β is an arbitrary constant. In this case β can be eliminated by introducing

$$\tau = \beta^2 \bar{t} \quad (3-29a)$$

$$\rho = \beta \bar{r} \quad (3-29b)$$

In these variables Eqs. (3-27) and (3-28) are valid with $n = 1$. Thus for $n = 1$ the problem with β arbitrary may be reduced to the case of $\beta = 1$. In this sense β plays a role equivalent to that of the Reynolds number $\mathcal{R}_e$. The details of this will be shown later.

Now, other sets of variables in addition to (τ, ρ) are invariant under the mapping (3-14). A particularly advantageous set which includes the case $n = \frac{1}{2}$ is

$$\xi = \frac{r}{2\sqrt{\nu t}} = \frac{\tilde{r}}{2\sqrt{\tilde{t}}} \quad (3-30a)$$

$$\begin{aligned} \eta &= \frac{R}{2\sqrt{\nu t}} = \frac{\tilde{R}}{2\sqrt{\tilde{t}}} = \frac{1}{2} \mathcal{R}_e^{1-n}(\tilde{t})^{n-\frac{1}{2}} \\ &= \frac{1}{2} \mathcal{R}_e^{\frac{1}{2}}(t^*)^{n-\frac{1}{2}} \end{aligned} \quad (3-30b)$$

In cases when it is justified to talk about a boundary layer, $\sqrt{\nu t}$ is proportional to the thickness of this boundary layer; this viscous length also is a significant quantity in other cases. The coordinate η thus measures the ratio between a body dimension and a viscous length and will be referred to as the *cross-sectional Reynolds number*. The significance of η has been pointed out by various authors, e.g. SEBAN & BOND [5] and PROBSTEN [6]. The variables introduced by Eq. (3-24) are for $n = 0$, equivalent to variables used in [5] and by STEWARTSON in [7]. Note that the parabolic case $n = \frac{1}{2}$ is exceptional. In this case $\eta = \frac{1}{2}\sqrt{\mathcal{R}_e}$ and is not a variable properly speaking. For $n = \frac{1}{2}$ the cross-sectional Reynolds number η is simply a function of the overall Reynolds number. For $0 \leq \eta < \frac{1}{2}$, η decreases with time at a rate depending on the overall Reynolds number. For $\frac{1}{2} < \eta < 1$, η increases with time. The conical case $n = 1$ is again exceptional in that $\mathcal{R}_e$ does not occur in the definition of η (cf. comment above). When $n = 1$ and $R = \beta U t$ we eliminate β as follows:

$$\xi = \frac{\rho}{2\sqrt{\tau}} = \frac{\tilde{r}}{2\sqrt{\tilde{t}}} \quad (3-31a)$$

$$\eta = \frac{1}{2}\sqrt{\tau} = \frac{1}{2}\beta\sqrt{\tilde{t}} \quad (3-31b)$$

Because of the invariance mentioned previously, introduction of the new variables defined by Eqs. (3-30) or (3-31) into Eqs. (3-7) and (3-8) yields equations and boundary conditions with $\mathcal{R}_e$ (or β) eliminated:

$$u_{\xi\xi}^* + \left[2\xi + \frac{1}{\xi} (1 - 4n\eta^2) \right] u_{\xi}^* + (2 - 4n)\eta u_{\eta}^* = 0 \quad (3-32)$$

$$u^* = 1 \quad \text{for} \quad \xi = \eta \quad (3-33a)$$

$$u^* = 1 \quad \text{for} \quad \xi = \infty \quad (3-33b)$$

The domain of variation of ξ and η is $\eta \geq 0$, $\eta \leq \xi \leq \infty$. The equations above are valid for $0 \leq n \leq 1$. Note that for $n = \frac{1}{2}$, *i.e.* $\eta = \frac{1}{2}\sqrt{\mathfrak{R}_e}$, the last term has a factor zero and Eq. (3-32) reduces to an ordinary differential equation of first order in u_ξ^* .

For the understanding of the limiting cases of Eq. (3-32) it is of importance to realize the physical meaning of its various terms. The individual terms of the equation can be identified as follows:

$$u_{\xi\xi}^* \rightarrow \text{viscous stress term} \sim \nu u_{rr} \quad (3-34a)$$

$$\frac{1}{\xi} u_\xi^* \rightarrow \text{viscous stress curvature term} \sim \frac{\nu}{r} u_r \quad (3-34b)$$

$$-2\xi u_\xi^* - (2 - 4n)\eta u_\eta^* \rightarrow \text{acceleration term} \sim u_t \quad (3-34c)$$

$$\frac{4n\eta^2}{\xi} u_\xi^* \rightarrow \text{transport term} \sim \nu u_r \quad (3-34d)$$

Since the Reynolds number is eliminated (except for $n = \frac{1}{2}$) there are no limit processes and expansions which correspond to $\mathfrak{R}_e$ tending to zero and infinity. However, one may consider expansions for η small or large. η then takes formally the role of the parameter $\mathfrak{R}_e$ and the variables $\tilde{r}$, $\tilde{t}$ (or r^* , t^* *etc.*) are replaced by a single variable ξ . Such expansions will be called *coordinate-type expansions* as distinguished from *parameter-type expansions*, *i.e.* the expansions based on $\mathfrak{R}_e$ as discussed in the first half of this section.

ξ and η were formed with the aid of the Oseen variables (Eq. (3-30)). Later on we shall have occasion to use the corresponding Stokes form

$$\xi^+ = \frac{\xi}{\eta} = \frac{\tilde{r}}{\tilde{R}} = \frac{r^+}{R^+} \quad (3-35)$$

with η defined as before, and the Prandtl form

$$\bar{\xi} = \frac{\tilde{r}}{2\sqrt{\tilde{t}^*}} = \frac{\tilde{r} - \tilde{R}}{2\sqrt{\tilde{t}}} = \xi - \eta \quad (3-36a)$$

which may be combined with η or η^* defined by

$$\eta^* = \frac{\eta}{\sqrt{\mathfrak{R}_e}} \quad (3-36b)$$

3.4. Skin friction coefficient. The viscous stress at the wall (skin friction) is often one of the main quantities to be determined in a problem of viscous flow. In the present case the value of the skin friction will often be used as a convenient test in comparing two approximations with each other or an approximation with

an exact solution. For future reference the expression for the skin friction coefficient C_f will be now given in terms of the various variables introduced above:

$$C_f = \frac{-\mu \left(\frac{\partial u}{\partial r} \right)_w}{\frac{1}{2} \rho_\infty U^2} \quad (3-37a)$$

or

$$\begin{aligned} C_f &= \frac{2}{\mathcal{R}_e} \left(-\frac{\partial u^*}{\partial r^*} \right)_{r^*=R^*} = 2 \left(-\frac{\partial u^*}{\partial \tilde{r}} \right)_{\tilde{r}=\tilde{R}} = \frac{2}{\sqrt{\mathcal{R}_e}} \left(-\frac{\partial u^*}{\partial \bar{r}} \right)_{\bar{r}=0} \\ &= \frac{1}{\sqrt{\tilde{t}}} \left(-\frac{\partial u^*}{\partial \xi} \right)_{\xi=\eta} = 2 \left(-\frac{\partial u^*}{\partial r^+} \right)_{r^+=R^+} \mathcal{R}_e^{n-1} \end{aligned} \quad (3-37b)$$

4. EXACT SOLUTIONS AND THEIR EXPANSIONS, $n = 0$ AND $n = \frac{1}{2}$

In this section the exact solutions for the special cases $n = 0$ and $n = \frac{1}{2}$ will be presented. The various limit processes introduced in Section 3 will be applied to u^* and the resulting expansions discussed. These two cases, $n = 0$ and $n = \frac{1}{2}$, will then be used as check points in the later sections where the corresponding expansions are obtained for $0 \leq n \leq 1$ by solving the appropriate approximate equations.

4.1. $n = 0$. Cylinder of Constant Radius. For $n = 0$ the radius $R = \text{const.} = L$. The radial velocity v is zero. Hence the equation for u is the Oseen equation (2-8). The present problem is then mathematically identical with the problem of finding the temperature distribution in a solid medium surrounding a circular cylinder whose temperature has been raised instantaneously to a fixed value. The solution is easily obtained by standard methods (*cf.* [11] and [4]). For our purposes it is convenient to represent the solution as an inverse Laplace transform:

$$\frac{u(r, t)}{U} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma t} \frac{K_0\left(\sqrt{\frac{\sigma}{\nu}} r\right)}{K_0\left(\sqrt{\frac{\sigma}{\nu}} L\right)} \frac{d\sigma}{\sigma} \quad (4-1)$$

The contour of integration is to be taken to the right of the imaginary axis. $K_0(z)$ is the modified Bessel function of the second kind. We note for reference the following expansions: For small values of its argument $K_0(z)$ may be expressed conveniently by

$$K_0(z) = -\sum_{m=0}^{\infty} \frac{z^{2m}}{2^{2m}(m!)^2} \log \frac{\gamma_m z}{2} \quad (4-2)$$

where

$$\gamma_m = e^{-\psi(m+1)}, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

in particular

$$\gamma_0 = e^{\gamma} \doteq 1.781, \quad \gamma = -\psi(1) = \text{Euler's constant} = .577$$

Thus for small z

$$K_0(z) = -\log \frac{\gamma_0 z}{2} + O(z^2 \log z) \quad (4-2b)$$

$$K'_0(z) = -\frac{1}{z} + O(z \log z) \quad (4-2c)$$

For large values of the argument an asymptotic expansion is

$$K_0(z) \cong \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 - \frac{1}{8z} + \dots \right] \quad (4-3)$$

4.1.1. *Oseen Expansion.* Expressed in Oseen variables (Eq. (3-5)) the solution (4-1) reads

$$u^* = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{K_0(\sqrt{\sigma} \tilde{r})}{K_0(\sqrt{\sigma} \mathcal{R}_e)} \frac{d\sigma}{\sigma} \quad (4-4)$$

and the skin friction coefficient is (cf. Eq. (3-37))

$$C_f = \frac{-2}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{K'_0(\sqrt{\sigma} \mathcal{R}_e)}{K_0(\sqrt{\sigma} \mathcal{R}_e)} \frac{d\sigma}{\sigma} \quad (4-5)$$

The range of the variables is $\mathcal{R}_e \leq \tilde{r} < \infty$, $\tilde{t} > 0$.

The Oseen expansion for u^* will be found by a repeated application of the Oseen limit. A suitable sequence of functions $\epsilon_j(\mathcal{R}_e)$ must be determined at the same time (cf. Eq. (3-14) and the discussion preceding it). As a preparation for finding the Oseen expansion we use Eq. (4-2) to rewrite Eqs. (4-4) and (4-5) for small $\mathcal{R}_e$ as

$$u^* = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{K_0(\sqrt{\sigma} \tilde{r})}{\left(-\log \frac{\sqrt{\sigma} \gamma_0 \mathcal{R}_e}{2} \right)} \frac{d\sigma}{\sigma} + O\left(\frac{\mathcal{R}_e^2}{\log \mathcal{R}_e} \right) \quad (4-6)$$

and

$$\mathcal{R}_e C_f = \frac{1}{2\pi i} \frac{1}{2} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{1}{\left(-\log \frac{\sqrt{\sigma} \gamma_0 \mathcal{R}_e}{2} \right)} \frac{d\sigma}{\sigma} + O(\mathcal{R}_e^2) \quad (4-7)$$

In the above integrals the contour is taken to the right of the imaginary axis but to the left of pole at $\sigma = 4/(\gamma_0^2 \mathcal{R}_e^2)$, which was introduced by the approximation. This is so because, as will be explained later, the approximation is not uniform with respect to time. It cannot be required that the initial conditions be satisfied, but only that the solution not grow exponentially with time.

The expansion of u^* will be discussed first. Eq. (4-6) may be further expanded as

$$\begin{aligned} u^* &\cong \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} K_0(\sqrt{\sigma} \tilde{r}) \left[\sum_{j=1}^{\infty} \epsilon^j (\log \sqrt{\gamma_0} \sigma)^{j-1} \right] \frac{d\sigma}{\sigma} \\ &= \sum_{j=1}^{\infty} \epsilon^j u_j(\tilde{r}, \tilde{t}) \end{aligned} \quad (4-8a)$$

where

$$\epsilon = \epsilon(\mathfrak{R}_e) = \frac{-1}{\log \frac{\sqrt{\gamma_0} \mathfrak{R}_e}{2}} \quad (4-8b)$$

and the functions u_j are defined by the above equality. Using known inversion integrals [10] and applying the convolution theorem when necessary one obtains

$$u_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} K_0(\sqrt{\sigma} \tilde{r}) \frac{d\sigma}{\sigma} = \frac{1}{2} E_1(\xi^2) \quad (4-9a)$$

$$\begin{aligned} u_2 &= \frac{1}{2\pi i} \frac{1}{2} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} K_0(\sqrt{\sigma} \tilde{r}) \log \gamma_0 \sigma \frac{d\sigma}{\sigma} \\ &= -\frac{1}{4} [E_1(\xi^2) \log \tilde{t} + G_1(\xi^2)] \end{aligned} \quad (4-9b)$$

Here $\xi = \tilde{r}/(2\sqrt{\tilde{t}})$ (Eq. (3-30a)). The function E_1 is related to the exponential integral Ei by

$$E_1(z) = -\text{Ei}(-z) = \int_z^{\infty} e^{-s} \frac{ds}{s} \quad (4-10a)$$

$$= -\log \gamma_0 z + e^{-z} \sum_{m=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m} \right) \frac{z^m}{m!} \quad (4-10b)$$

For large values of z an asymptotic expansion is

$$E_1(z) \cong \frac{e^{-z}}{z} \left(1 - \frac{1!}{z} + \frac{2!}{z^2} - \cdots \right) \quad (4-11)$$

$G_1(z)$ is defined by

$$G_1(z) = \int_z^{\infty} \log \left(1 - \frac{z}{s} \right) e^{-s} \frac{ds}{s} = \frac{-1}{\pi i} \int_{-i\infty}^{i\infty} e^{\sigma} K_0(2\sqrt{\sigma} z) \log \gamma_0 \sigma \frac{d\sigma}{\sigma} \quad (4-12)$$

For small values of z

$$G_1(z) = -\frac{1}{6}\pi^2 - z \log z + O(z) \quad (4-13a)$$

and for z large

$$G_1(z) \cong -\frac{e^{-z}}{z} \log \gamma_0 z + \dots \quad (4-13b)$$

(The function $G_1(z)$ does not appear to be simply related to known functions. The notation " G_1 " used here is not standard.)

We shall now study the behavior of the expansion (4-8a) near the wall. Since $\tilde{r} = \tilde{R} = \mathcal{R}_\epsilon$ at the wall and we consider the limit of $\mathcal{R}_\epsilon$ small an expansion valid near the wall means an expansion valid for $\tilde{r}$ small. We find for the functions u_j defined by Eq. (4-8a)

$$u_j(\tilde{r}, \tilde{t}) = -\log \frac{\sqrt{\gamma_0} \tilde{r}}{2} A_{j-1}(\tilde{t}) - A_j(\tilde{t}) + O(\tilde{r}^2 \log \tilde{r}) \quad (4-14a)$$

where

$$A_j(\tilde{t}) = \frac{1}{2\pi i} \frac{1}{2^j} \int_{-i\infty}^{i\infty} e^{i(\log \gamma_0 \sigma)^j} \frac{d\sigma}{\sigma} \quad (4-14b)$$

With the aid of Hankel's contour integral representation of $1/\Gamma$ (cf. [9], Vol. 1, p. 14) one may show that

$$A_j(\tilde{t}) = \frac{1}{2^j} \left\{ \left(\frac{d}{dz} \right)^j \left[\frac{e^{-z \log(\tilde{t}/\gamma_0)}}{\Gamma(1-z)} \right] \right\}_{z=0} \quad (4-15)$$

In particular

$$A_0 = 1 \quad (4-16a)$$

$$A_1 = -\frac{1}{2} \log \tilde{t} \quad (4-16b)$$

$$A_2 = \frac{1}{4} \left(\log^2 \tilde{t} - \frac{\pi^2}{6} \right) \quad (4-16c)$$

The above three functions may be obtained from Eq. (4-14b) with the aid of the table of inverse Laplace transforms (cf. [10], Vol. 1, p. 251). (Note that only positive values of $\tilde{t}$ are considered. Strictly speaking A_0 is of course the Heaviside function which is zero for negative $\tilde{t}$.) Subsequent A_j may be determined by straightforward calculations using the values of the derivatives of $\Gamma(z)$ for $z = 1$. These values have been tabulated in [12].

Near the wall one then finds (from Eqs. (4-14a) or (4-9))

$$\begin{aligned} u^* \cong \epsilon(\mathcal{R}_\epsilon) \left[-\log \frac{\sqrt{\gamma_0} \tilde{r}}{2} + \frac{1}{2} \log \tilde{t} \right] \\ + \epsilon^2(\mathcal{R}_\epsilon) \left[\frac{1}{2} \left(\log \frac{\sqrt{\gamma_0} \tilde{r}}{2} \right) \log \tilde{t} - \frac{1}{4} \left(\log^2 \tilde{t} - \frac{\pi^2}{6} \right) \right] + \dots \end{aligned} \quad (4-17)$$

The results obtained will now be summarized and discussed. If one defines

$$\epsilon_j(\mathcal{R}_\epsilon) = [\epsilon(\mathcal{R}_\epsilon)]^j, \quad \epsilon(\mathcal{R}_\epsilon) = \frac{-1}{\log \frac{\sqrt{\gamma_0} \mathcal{R}_\epsilon}{2}} \quad (4-18)$$

then the repeated application of the Oseen limit yields an Oseen expansion of the form

$$u^* \cong \sum_{j=0}^{\infty} \epsilon_j(\mathcal{R}_\epsilon) u_j(\tilde{r}, \tilde{t}) \quad (4-19)$$

The Oseen limit of u^* is zero. In our notation

$$\lim_{\mathcal{O}} u^* = u_0 = 0 \quad (4-20)$$

The remaining functions u_j are defined by Eq. (4-8a) and u_1 and u_2 are explicitly given by Eqs. (4-9a, b).

As for the validity of the expansion (4-19) we first note that the occurrence of the function $\log \tilde{t}$ in the functions u_j prevents the expansion from being uniformly valid near $\tilde{t} = 0$ or near $\tilde{t} = \infty$. It will be seen later that the non-uniformity for small $\tilde{t}$ is essential. On the other hand the non-uniformity at $\tilde{t} = \infty$ is due to an inadequate choice of variables. This inadequacy is indicated by the fact that the similarity property of u^* (*cf.* Eqs. (3-25) *ff.*) is lost in the expansion (4-19). It will be shown later in the discussion of coordinate-type expansions that similarity may be restored by a very simple substitution. The resulting expansion will have its validity extended to large $\tilde{t}$.

We shall now restrict ourselves to a fixed time interval $t_0 \leq \tilde{t} \leq t_1$ where t_0 and t_1 are only subject to the restriction $0 < t_0 < t_1 < \infty$. In studying the uniformity in $\tilde{r}$ we first observe that the Oseen limit is non-uniform at the wall, *i.e.* at $\tilde{r} = 0$ since in the Oseen limit $\tilde{R}$ becomes zero. At the wall $u^* = 1$ and hence remains so in the limit. The Oseen limit, *i.e.* $u_0 = 0$, is then an approximation which is uniformly valid for $r_0 \leq \tilde{r} < \infty$ where r_0 is any number > 0 . The approximation is however not uniformly valid at the wall. Intuitively this may be understood as follows: One may think of the Oseen limit as letting L tend to zero while r , t , U and ν are fixed. Since the circumference is $2\pi L$ the ability of the cylinder to create a flow disturbance decreases with L . In the limit no disturbance can be created (*cf.* Section 6.3) at a fixed point not on the cylinder. At the surface of the cylinder, on the other hand, u must always have the value U .

The non-uniformity in $\tilde{r}$ just discussed may be remedied by including the term ϵu_1 . Eq. (4-17) shows that the error at the wall is then of order ϵ . More generally from Eq. (4-14a) it is seen that $u_j(\tilde{r}, \tilde{t})$ is of order $1/\epsilon$ for $\tilde{r} = \mathcal{R}_\epsilon$. From this it follows that the m^{th} partial sum of the expansion (4-19) is valid to order ϵ^{m-1} .

More precisely

$$\lim_{\epsilon \rightarrow 0} \frac{u^* - \sum_{j=1}^l \epsilon^j u_j}{\epsilon^{l-1}} = 0 \quad (4-21)$$

uniformly for $\Re \epsilon \leq \tilde{r} < \infty$, $0 < t_0 \leq \tilde{t} \leq t_1 < \infty$. Eq. (4-21) is the key statement regarding the validity of the Oseen expansion (4-19).

In order to obtain an Oseen expansion which is uniformly valid in the sense of Eq. (4-21) one may choose the function $\epsilon_j(\Re \epsilon)$ (Eq. (4-18)) in many different ways. The only essential requirement is that for $\Re \epsilon$ small

$$\epsilon_j(\Re \epsilon) = O\left(\frac{1}{\log^j \Re \epsilon}\right) \quad (4-22)$$

One may for example replace the definition of ϵ in Eq. (4-18) by

$$\epsilon^* = \frac{-1}{\log \Re \epsilon + C}, \quad C = \text{arbitrary constant} \quad (4-23)$$

Since ϵ^* may be expanded in powers of ϵ and vice versa, one obtains the expansion of u^* in powers of ϵ^* by rearranging the terms in the expansion in powers of ϵ . Note furthermore that Eq. (4-23) implies

$$\Re \epsilon = e^{-(1/\epsilon^*) - C} \quad (4-24)$$

From this it follows that $\Re \epsilon$ (or, more generally, $\Re \epsilon^l \log^k \Re \epsilon$, $l > 0$, k arbitrary) is transcendentally small with respect to ϵ , i.e. is $o(\epsilon^m)$ for any m . Terms of order $\Re \epsilon \log \Re \epsilon$ may then be neglected in the Oseen expansion for u^* . This justifies the substitution of the approximate integral of Eq. (4-6) for the exact integral of Eq. (4-4). If, however, one starts with the integral of Eq. (4-6) (or some other function which has the expansion (4-19)) one may of course add a new series of terms which are transcendentally small as $\Re \epsilon \rightarrow 0$ in order to improve the approximation for finite $\Re \epsilon$.

It follows of course that even if the series (4-19) is convergent it cannot represent the exact value of u^* . Moreover, it is useless as a numerical approximation unless $\Re \epsilon$ is very small. Note that the "small" parameter ϵ becomes infinitely large for $\Re \epsilon = 2/\sqrt{\gamma_0} \doteq 1.5$.

Finally, it follows that if

$$\bar{\epsilon} = \frac{-1}{f(\Re \epsilon)}, \quad f(\Re \epsilon) = \log \frac{\sqrt{\gamma_0} \Re \epsilon}{2} + O(\Re \epsilon \log^b \Re \epsilon) \quad (4-25)$$

then each function $\bar{u}_j$ in the expansion $u^* \cong \sum \bar{\epsilon}^j \bar{u}_j$ may be taken identical with the corresponding u_j in the expansion (4-19).

4.1.2. *Skin Friction.* Eq. (4-14a) yields

$$-\left(\frac{\partial u_j}{\partial \tilde{r}}\right)_{\tilde{r}=\mathcal{R}_e} = \frac{1}{\mathcal{R}_e} A_{j-1}(\tilde{t}) + O(\mathcal{R}_e \log \mathcal{R}_e) \quad (4-26)$$

Hence (cf. Eq. (3-37))

$$\begin{aligned} \frac{1}{2} \tilde{R} C_f &= \frac{1}{2} \mathcal{R}_e C_f \cong \sum_{j=1}^{\infty} \epsilon^j A_{j-1}(\tilde{t}) + O(\mathcal{R}_e^2) \\ &= \left[\epsilon - \frac{\epsilon^2}{2} \log \tilde{t} + \frac{\epsilon^3}{4} \left(\log^2 \tilde{t} - \frac{\pi^2}{6} \right) + \dots \right] \end{aligned} \quad (4-27)$$

One may also find the coefficients of this expansion by starting from Eq. (4-5). This will be discussed later.

4.1.3. *Stokes Expansion.* For $n = 0$ the Stokes variable r^+ is equal to r^* (cf. Eq. (3-6)). Hence written in Stokes variables the integral of Eq. (4-1) is

$$u^*(r^*, \tilde{t}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{K_0(\sqrt{\sigma \mathcal{R}_e} r^*)}{K_0(\sqrt{\sigma \mathcal{R}_e})} \frac{d\sigma}{\sigma} \quad (4-28)$$

We shall obtain the Stokes expansion by a repeated application of the Stokes limit to the above integral, thereby using the same $\epsilon_i(\mathcal{R}_e)$ as in the Oseen case (Eq. (4-18)). Since within transcendental terms

$$\frac{K_0(\sqrt{\sigma \mathcal{R}_e} r^*)}{K_0(\sqrt{\sigma \mathcal{R}_e})} = 1 - \log r^* \sum_{j=1}^{\infty} \epsilon^j (\log \sqrt{\lambda_0 \sigma})^{j-1} \quad (4-29)$$

the Stokes expansion is (cf. Eqs. (4-14b) and (4-15))

$$\begin{aligned} u^* &\cong 1 - \log r^* \sum_{j=1}^{\infty} \epsilon^j A_{j-1}(\tilde{t}) \\ &= 1 - \log r^* \left[\epsilon - \frac{\epsilon^2}{2} \log \tilde{t} + \frac{\epsilon^3}{4} \left(\log^2 \tilde{t} - \frac{\pi^2}{6} \right) + \dots \right] \end{aligned} \quad (4-30)$$

Any partial sum of this expansion satisfies the Stokes equation (Eq. (2-9)) as well as the boundary conditions at the wall. The initial condition and the condition at $r^* = \infty$ are not satisfied. The dependence on $\tilde{t}$ is the same as for the Oseen expansion, *i.e.* the Stokes expansion diverges as $\tilde{t}$ tends to zero or infinity. In addition it diverges as r^* tends to infinity. Near the wall the Oseen and Stokes expansions agree. If one inserts the value $\mathcal{R}_e r^*$ for $\tilde{r}$ in the Oseen expansion, using for u_j the approximations given by Eq. (4-14a) one obtains the Stokes expansion (4-30). In particular the Stokes expansion gives the same formula for the skin friction as the Oseen expansion (Eq. (4-27)).

The Stokes limit of u^* is the first term of Eq. (4-30), namely unity. To understand this result one may think of the Stokes limit as a limit where r , t , $R = L$ and U are fixed while ν tends to infinity. The viscous influence of the wall velocity

on a fixed point not on the wall then becomes increasingly stronger and in the limit the velocity u at the fixed point tends to U . On the other hand the convergence to the limit is not uniform in r (or t). The Stokes limit $\lim_s u^* = 1$ is then an approximation which, in r , is uniformly valid near the wall, *i.e.* for $1 = R^* \leq r^* \leq r_1 < \infty$. In the Oseen limit the corresponding non-uniformity occurred at the wall. Note that in the latter case the non-uniformity could be remedied by adding more terms of the Oseen expansion. On the other hand all partial sums of Eq. (4-30) diverge at infinity.

The relation between Stokes and Oseen flow shows a certain resemblance to the relation between boundary layer flow and external flow in the high Reynolds number case. The Oseen limit represents an outer limit (external flow) which is non-uniform at the wall. A solution valid near the wall may be obtained by stretching the Oseen coordinate $\tilde{r}$ by dividing by the small parameter $\mathcal{R}_e$, *i.e.* by introducing Stokes coordinates. The Stokes limit represents an inner limit (boundary layer flow) which gives solutions near the wall. Two significant differences between the two cases should, however, be emphasized: (1) In the low Reynolds number case the inner limit may be dispensed with, *i.e.* the Oseen expansion alone gives a solution which is uniformly valid to any order if sufficiently many terms are used. (2) The matching conditions between inner and outer solutions are different in the two cases.

The fact that the outer (Oseen) solution agrees with the inner (Stokes) solution near the wall may be taken to mean that the two solutions are properly matched. In terms of limits one may express this fact by stating that a repeated application of the Stokes limit to the Oseen expansion gives the Stokes expansion. This is so because the Oseen expansion actually represents u^* uniformly in r^* . The converse cannot be expected to hold since the Stokes expansion does not contain the flow at large distances. However, the following restricted relation, whose significance will be discussed in Sections 5 and 6, is easily verified:

$$\lim_0 (1 - \epsilon \log r^+) = \lim_0 u^* = 0 \quad (4-31)$$

4.1.4. *Euler Expansion.* Turning now to the case of high Reynolds numbers we first consider Eq. (4-1) in Euler variables:

$$u^* = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma t^*} \frac{K_0(\sqrt{\sigma \mathcal{R}_e} r^*)}{K_0(\sqrt{\sigma \mathcal{R}_e})} \frac{d\sigma}{\sigma} \quad (4-32)$$

It is then seen, using the asymptotic formula (4-3), that for $r^* > R^* = 1$

$$\lim_E u^* = 0 \quad (4-33)$$

The approach to zero is exponential as $\mathcal{R}_e \rightarrow \infty$, so that in fact a repeated application of the Euler limit process (with $\epsilon_i = \mathcal{R}_e^{-(i/2)}$) gives zero. As was the case for the Oseen limit, the convergence of the Euler limit is non-uniform at the cylinder where $u^* = 1$.

4.1.5. *Prandtl Expansion.* In Prandtl variables Eq. (4-1) reads

$$u^* = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma t^*} \frac{K_0[\sqrt{\sigma}(\bar{r} + \sqrt{\Re \epsilon})]}{K_0[\sqrt{\sigma \Re \epsilon}]} \frac{d\sigma}{\sigma} \quad (4-34)$$

The corresponding skin friction formula is

$$\sqrt{\Re \epsilon} C_t = \frac{-1}{\pi i} \int_{-i\infty}^{i\infty} e^{\sigma t^*} \frac{K_0'(\sqrt{\sigma \Re \epsilon})}{K_0(\sqrt{\sigma \Re \epsilon})} \frac{d\sigma}{\sigma} \quad (4-35)$$

Using the asymptotic expansion of $K_0(z)$ for large z (Eq. (4-3)) one then finds the Prandtl expansion

$$u^* = \operatorname{erfc} \bar{\xi} - \epsilon \frac{\bar{r}}{2} \operatorname{erfc} \bar{\xi} + \epsilon^2 \frac{\bar{r}^2}{8} \left(\frac{1}{\sqrt{\pi} \bar{\xi}} e^{-\bar{\xi}^2} + 2 \operatorname{erfc} \bar{\xi} \right) + O(\epsilon^3) \quad (4-36)$$

where $\bar{\xi} = \bar{r}/(2\sqrt{t^*})$ and $\epsilon = \Re \epsilon^{-1/2}$.

The corresponding expansion for the skin friction is obtained either directly by expanding Eq. (4-35) or by differentiating Eq. (4-36). It is

$$\begin{aligned} C_t &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma t^*} \left[2\epsilon + \frac{\epsilon^2}{\sqrt{\sigma}} + O(\epsilon^3) \right] \frac{d\sigma}{\sigma} \\ &= \frac{2\epsilon}{\sqrt{\pi t^*}} + \epsilon^2 - \frac{\epsilon^3 \sqrt{t^*}}{2\sqrt{\pi}} + O(\epsilon^3) \end{aligned} \quad (4-37a)$$

or

$$\sqrt{\frac{U^2 t}{\nu}} C_t = \frac{2}{\sqrt{\pi}} + \frac{\sqrt{\nu t}}{L} - \frac{1}{2\sqrt{\pi}} \frac{\nu t}{L^2} + O(\nu^{3/2}) \quad (4-37b)$$

It follows from Eq. (4-36) that

$$\lim_{\mathbb{P}} u^* = \operatorname{erfc} \xi \quad (4-38)$$

This term represents the conventional boundary layer solution to our problem and is identical with the Rayleigh solution for the impulsively started sliding plane. It represents the first order inner solution for high Reynolds numbers. It is uniformly valid to order unity in the entire domain $0 \leq \bar{r} < \infty$ (or $R \leq r < \infty$). Its uniform validity in time is restricted to a semi-infinite time interval $0 \leq t^* \leq t_1 < \infty$. The corresponding outer solution, *i.e.* the external flow $\lim_{\mathbb{P}} u^*$, is zero. The higher order terms in Eq. (4-36) correct the boundary layer theory for transverse curvature (*cf.* [6]). Since the corresponding terms of the outer solution (the Euler expansion) are all zero the Prandtl expansion gives the flow in the entire r -domain. If all terms including the ϵ^m term are included one obtains an expansion that is uniformly valid of order ϵ^m in this domain. Note that the fact that the second application of the Euler limit is zero means in ordinary

language that there is no flow due to displacement thickness, a fact which is intuitively obvious in this example. Similarly the higher order corrections to the boundary layer have no displacement effects.

It can be seen, especially from the formulas for the skin friction, that the smaller the value of t^* is, the fewer terms of the Prandtl expansion are needed to get a certain accuracy. In particular, boundary layer theory is a very good approximation right at the start of the motion. This is consistent with the fact that for very small times the viscous layer is very thin compared to the transverse radius of curvature (*i.e.* L) and hence the viscous stress due to curvature $\mu v_r/r$ may be neglected compared to μv_{rr} . Successive corrections to boundary layer theory consider parts of the term $\mu v_r/r$ (*cf.* Section 5.2). However, for very large times the whole idea of the Prandtl expansion fails completely since L is constant while the thickness of the viscous layer grows indefinitely. By the same reasoning one would expect the Oseen expansion to be good when the viscous layer is thick but to fail for small values of time. Unfortunately, as was seen above, the Oseen expansion fails also for large values of time. To study the reason for this one has to put the above arguments about the thickness of the viscous layer on a more exact basis. This is done by introducing the coordinates ξ and η defined by Eqs. (3-30). η measures the ratio of L and the thickness of the viscous layer as represented by $\sqrt{\nu t}$, whereas ξ is proportional to the ratio of the radius of curvature r at an arbitrary point and of $\sqrt{\nu t}$. We shall now study the coordinate-type expansions for small and large η analogous to the parameter type expansions for small and large $\mathcal{R}_e$.

4.1.6. *Coordinate-type Expansions.* Expressed in $\xi = \frac{1}{2}r/\sqrt{\nu t}$ and $\eta = \frac{1}{2}L/\sqrt{\nu t}$ Eq. (4-1) is

$$u^*(\xi, \eta) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma} \frac{K_0(2\sqrt{\sigma\xi})}{K_0(2\sqrt{\sigma\eta})} \frac{d\sigma}{\sigma} \quad (4-39)$$

The corresponding skin friction formula is

$$\sqrt{i} C_t = \frac{-2}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma} \frac{K'_0(2\sqrt{\sigma\xi})}{K_0(2\sqrt{\sigma\eta})} \frac{d\sigma}{\sigma} \quad (4-40)$$

We shall obtain an expansion of the form

$$u^* \cong \sum_{j=0}^{\infty} \epsilon_j(\eta) u_j(\xi) \quad (4-41)$$

valid for small χ . The $\epsilon_j(\eta)$ above play the role of the $\epsilon_j(\mathcal{R}_e)$ in Eq. (4-19) and will of course be required to be of successively smaller magnitude, *i.e.*

$$\epsilon_0(\eta) = 1, \quad \lim_{\eta \rightarrow 0} \frac{\epsilon_{j+1}(\eta)}{\epsilon_j(\eta)} = 0 \quad (4-42)$$

A comparison shows immediately that the problem of expanding Eq. (4-39)

for small η is formally identical with the problem of expanding Eq. (4-4) for small $\Re_e$ if one correlates the quantities in the latter with the former as follows:

$$\tilde{t} \rightarrow 1, \quad \Re_e \rightarrow 2\eta, \quad \tilde{r} \rightarrow 2\xi \quad (4-43)$$

One then finds

$$\epsilon_j(\eta) = [\epsilon(\eta)]^j, \quad \epsilon(\eta) = \frac{-1}{\log \sqrt{\gamma_0} \eta} \quad (4-44)$$

$$u_0(\xi) = 0 \quad (4-45a)$$

and for $j \geq 1$

$$u_j(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^\sigma K_0(2\sqrt{\sigma}\xi) (\log \sqrt{\gamma_0} \sigma)^{j-1} \frac{d\sigma}{\sigma} \quad (4-45b)$$

In particular

$$u_1(\xi) = \frac{1}{2} E_1(\xi^2) \quad (4-46a)$$

and

$$u_2(\xi) = -\frac{1}{4} G_1(\xi^2) \quad (4-46b)$$

Near the wall, *i.e.* for small values of ξ

$$u^* = - \sum_{j=1}^{\infty} \epsilon^j (a_{j-1} \log \sqrt{\gamma_0} \xi + a_j) + O(\xi^2 \log \xi) \quad (4-47a)$$

where

$$a_j = A_j(1) = \frac{1}{2\pi i} \frac{1}{2^j} \int_{-i\infty}^{i\infty} e^\sigma (\log \gamma_0 \sigma)^j \frac{d\sigma}{\sigma} \quad (4-47b)$$

Special values of the a_j are

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\frac{\pi^2}{24} \quad (4-48)$$

Since $\log \sqrt{\gamma_0} \xi$ is $-1/\epsilon$ at the boundary one concludes that in general the m^{th} partial sum of the expansion (4-41) is uniformly valid to order ϵ^{m-1} in the following sense (*cf.* Eq. (4-21))

$$\lim_{\eta \rightarrow 0} \frac{u^* - \sum_{j=1}^l \epsilon_j u_j(\xi)}{\epsilon^{l-1}} = 0 \quad (4-49)$$

uniformly in the whole range of ξ , *i.e.* $\eta \leq \xi < \infty$.

While the formal methods of derivation of the expansion (4-41) exactly parallel those used in deriving the Oseen expansion (4-19), the results are different. In

fact the former expansion is an improvement over the latter. If ξ and η are expressed in physical coordinates one sees that the expansion (4-41) is valid for large values of $\tilde{t}$. It is even valid for an arbitrary Reynolds number provided $\tilde{t}$ is sufficiently large since η is small whenever $\tilde{t} \gg \mathcal{R}_e$. On the other hand the approximation fails eventually as $\tilde{t}$ approaches zero no matter how small $\mathcal{R}_e$ is. Note that the two expansions agree for $\tilde{t} = 1$. The correspondence expressed by Eq. (4-43) is a redefinition of the expansion (4-19) by similarity, *i.e.* this expansion is taken to be valid for $\tilde{t} = 1$ and then redefined for any other value of $\tilde{t}$ by the similarity laws discussed in Section 3.

The failure of the Oseen expansion for large values of $\tilde{t}$ can now be explained from the following formula

$$\epsilon(\eta) = \frac{-1}{\log \sqrt{\gamma_0} \eta} = \frac{\epsilon(\mathcal{R}_e)}{1 + \epsilon(\mathcal{R}_e) \log \tilde{t}} \quad (4-50a)$$

or

$$\epsilon(\eta) = \epsilon(\mathcal{R}_e) - [\epsilon(\mathcal{R}_e)]^2 \log \tilde{t} + \dots \quad (4-50b)$$

If Eq. (4-50b) is used in the η -expansion (4-41) one obtains the Oseen expansion (4-19). However, the derivation of Eq. (4-50b) from Eq. (4-50a) is valid only for $\epsilon(\mathcal{R}_e) \log \tilde{t} < 1$. This is the reason that the Oseen expansion fails for large values of $\tilde{t}$.

It has thus been shown that the cross-sectional Reynolds number η is a more significant quantity than the overall Reynolds number $\mathcal{R}_e$. The introduction of ξ and η as coordinates, which was prompted both by physical reasoning and by similarity considerations (*cf.* Section 3), has thus been justified.

The comments above will of course apply equally well to the skin friction formula which will now be discussed.

4.1.7. *Skin friction.* From Eqs. (3-37) and (4-47) one finds

$$\eta \sqrt{\tilde{t}} C_f = \frac{1}{2} \tilde{R} C_f = \frac{1}{2} \mathcal{R}_e C_f \cong \sum_{j=1}^{\infty} [\epsilon(\eta)]^j a_{j-1} = \epsilon - \frac{\epsilon^3 \pi^2}{24} + \dots \quad (4-51)$$

Note now that the skin-friction formula given by Eq. (4-7) has the correct similarity. (4-7) can be rewritten as

$$\mathcal{R}_e C_f = \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma(4\tilde{t}/(\gamma_0^2 \mathcal{R}_e^2))} \frac{1}{\log \sigma} \frac{d\sigma}{\sigma} + O(\mathcal{R}_e) \quad (4-52)$$

where the contour is along the imaginary axis indented at $\sigma = 0$. Using the table of inverse Laplace Transforms [10, Vol. 1, p. 251] and accounting for the pole at $\sigma = 1$ Eq. (4-52) is

$$\mathcal{R}_e C_f = e^{(4\tilde{t}/(\gamma_0^2 \mathcal{R}_e^2))} - \nu \left(\frac{4\tilde{t}}{\gamma_0^2 \mathcal{R}_e^2} \right) + O(\mathcal{R}_e) \quad (4-53)$$

The special function $\nu(z)$ introduced is discussed in [9, Vol. 3] and here we note only

$$\begin{aligned}\nu(z) &= \int_0^\infty \frac{s^z}{\Gamma(s+1)} ds = e^{-z} - \int_0^\infty e^{-zs} \frac{ds}{s(\pi^2 + \log^2 s)} \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{sz} \frac{ds}{s \log s} + e^z\end{aligned}\quad (4-54)$$

The expansion for $\Re_e \rightarrow 0$ of Eq. (4-52) is related to the asymptotic behavior of $\nu(z) - e^z$ as $z \rightarrow \infty$. The contour integral in Eq. (4-54) representing this has been studied in [13] where the asymptotic validity as $z \rightarrow \infty$ of the following expansion is proved

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{sz} \frac{ds}{s \log s} = -\frac{1}{\log z} \sum_{l=0}^{\infty} \frac{(-1)^l}{(\log z)^l} \left[\left(\frac{d}{dz} \right)^l \frac{1}{\Gamma(z)} \right]_{z=1} + o(e^{-\alpha z}), \alpha > 0 \quad (4-55)$$

Applying this expansion formula to Eq. (4-52) one obtains an expansion equivalent to Eq. (4-51), using, however, a different definition of $\epsilon(\Re_e)$ (cf. Eqs. (4-44), (4-47) and the discussion after Eq. (4-22)).

4.1.8. *Stokes-type Expansion in ξ^+ and η .* We shall now derive an expansion which bears the same relation to the expansion in ξ and η as the Stokes expansion does to the Oseen expansion. We first rewrite the exact formula for u^* in terms of ξ^+ and η , where (cf. Eq. (3-35))

$$\xi^+ = \frac{\xi}{\eta} \quad (4-56)$$

The problem of expanding the resulting integral for η small and ξ^+ fixed is formally identical with the problem of deriving the Stokes expansion, if one makes the following correlation (cf. Eq. (4.43))

$$\bar{t} \rightarrow 1, \quad \Re_e \rightarrow 2\eta, \quad r^* \rightarrow \xi^+ \quad (4-57)$$

Hence we find from Eqs. (4-30) and (4-37)

$$u^* \cong 1 - \log \xi^+ \sum_{j=1}^{\infty} [\epsilon(\eta)]^j a_{j-1} \quad (4-58)$$

The comments on this expansion are obvious from the previous discussions on Stokes expansion.

4.1.9. *Large values of η .* Finally, we shall expand Eq. 4-39 for large values of η . Since $\bar{\xi} = \frac{1}{2}\bar{r}/\sqrt{t^*} = \xi - \eta$ and $\bar{r}/\sqrt{\Re_e} = \bar{\xi}/\eta$ it follows that the Prandtl expansion (Eq. (4-36)) is of the form

$$u^* \cong \sum_{j=0}^{\infty} \frac{1}{\eta^j} u_j(\bar{\xi}) \quad (4-59)$$

and hence is already expressed in the similarity variables ξ and η (or $\bar{\xi}$ and η). The expansion of Eq. (4-39) for large η has then already been found. Hence for large Reynolds numbers the Re_e -expansion and the η -expansion are formally identical.

4.2. $n = \frac{1}{2}$. Cylinder Expanding at Parabolic Rate.

4.2.1. *Exact Solution.* The second case where an exact solution may be obtained is that of $n = \frac{1}{2}$. In this case the radius grows parabolically ($\sim \sqrt{t}$), i.e. at the same rate as the viscous thickness $\sqrt{\nu t}$. The cross-sectional Reynolds number is then independent of the time:

$$\eta = \frac{1}{2} \sqrt{\text{Re}_e} \quad (4-60)$$

Eq. (3-32) then reduces to the ordinary differential equation

$$\frac{d^2 u^*}{d\xi^2} + \left[2\xi + \frac{1}{\xi} (1 - 2\eta^2) \right] \frac{du^*}{d\xi} = 0 \quad (4-61)$$

with the boundary conditions

$$u^* = 1 \quad \text{for} \quad \xi = \eta \quad (4-62a)$$

$$u^* = 0 \quad \text{for} \quad \xi = \infty \quad (4-62b)$$

Since the above equation is of first order in the derivative $du^*/d\xi$ it is easily solved. The solution is

$$u^*(\xi, \eta) = \frac{g(\xi, \eta)}{g(\eta, \eta)} \quad (4-63a)$$

where

$$g(\xi, \eta) = \int_{\xi}^{\infty} e^{-\sigma^2} \sigma^{(2\eta^2-1)} d\sigma = \frac{1}{2} \Gamma(\eta^2, \xi^2) \quad (4-63b)$$

The function $\Gamma(x, y)$ is the incomplete gamma function (cf. [9, Vol. 2, Ch. IX]). The tabulated values (cf. [14]) of this function are useful in computing an exact skin friction curve. However, all the expansions of the exact solution will be obtained below directly from the integral in Eq. (4-63b).

The formula for the skin friction is

$$\sqrt{i}C_f = \frac{\eta^{(2\eta^2-1)} e^{-\eta^2}}{g(\eta, \eta)} \quad (4-64)$$

For special values of η the integral in Eq. (4-48b) may be evaluated in terms of elementary functions:

For $\eta^2 = \frac{1}{2}$, $\mathcal{R}_e = 2$ one obtains

$$u^* = \frac{\operatorname{erfc} \xi}{\operatorname{erfc} \frac{1}{\sqrt{2}}} \quad (4-65a)$$

$$\sqrt{i}C_t = \frac{2}{\pi} \frac{1}{\operatorname{erfc} \frac{1}{\sqrt{2}}} \frac{1}{\sqrt{e}} \quad (4-65b)$$

For $\eta^2 = 1$, $\mathcal{R}_e = 4$ one obtains

$$u^* = e^{-(\xi^2-1)} \quad (4-66a)$$

$$\sqrt{i}C_t = 2 \quad (4-66b)$$

Similar expressions may be obtained whenever $\eta^2 = \frac{1}{2}m$, m a positive integer.

Since η depends on $\mathcal{R}_e$ only and not on i , the parameter expansions and the coordinate expansions are identical. We shall now derive expansions valid for small or large $\mathcal{R}_e$, *i.e.* small or large η .

4.2.2. *Oseen Expansion*, $n = \frac{1}{2}$. For small values of η the function $g(\xi, \eta)$ may be written

$$\begin{aligned} g(\xi, \eta) &= \int_{\xi}^{\infty} e^{-\sigma^2} \frac{d\sigma}{\sigma} \{1 + 2\eta^2 \log \sigma + O(\eta^4)\} \\ &= \frac{1}{2} E_1(\xi^2) + \frac{1}{2}\eta^2 G_2(\xi^2) + O(\eta^4) \end{aligned} \quad (4-67)$$

The function E_1 was discussed above (Eqs. 4-1 and 4-11). For the second integral we introduce the temporary notation:

$$G_2(z) = \int_z^{\infty} e^{-s} \frac{\log s}{s} ds = E_1(z) \log z + \int_z^{\infty} \frac{E_1(s)}{s} ds \quad (4-68a)$$

For small z the dominant terms of G_2 are

$$G_2(z) = -\frac{1}{2} \log^2 \gamma_0 z + (\log \gamma_0) \log z + O(1) \quad (4-68b)$$

Inserting Eq. (4-52) into Eq. (4-48) one obtains

$$u^* = \frac{E_1(\xi^2)}{E_1(\eta^2)} - \frac{\eta^2}{E_1^2(\eta^2)} \{G_2(\eta^2)E_1(\xi^2) - G_2(\xi^2)E_1(\eta^2)\} + O(\eta^4) \quad (4-69)$$

The corresponding skin friction formula is

$$\sqrt{i}C_t = \frac{2e^{-\eta^2}}{\eta} \left[\frac{1}{E_1(\eta^2)} - \frac{\eta^2}{E_1^2(\eta^2)} \{G_2(\eta^2) - E_1(\eta^2) \log \eta^2\} \right] \quad (4-70)$$

To obtain an Oseen expansion we note that since $\xi = \frac{1}{2}\tilde{r}/\sqrt{i}$ and $\eta = \frac{1}{2}\sqrt{\mathcal{R}_e}$

the Oseen limit in the present case is a limit where ξ is fixed and η tends to zero. Defining

$$\epsilon = - \frac{2}{\log (\gamma_0 \mathcal{R}_e/4)} \quad (4-71)$$

one finds from Eq. (4-10a) and the above equations

$$u^* = \epsilon \frac{E_1(\xi^2)}{2} + \eta^2 \frac{E_1(\xi^2)}{2} + O\left(\frac{\eta^2}{\log^2 \eta^2}\right) \quad (4-72)$$

and

$$\eta \sqrt{i} C_i = \frac{1}{2} \tilde{R} C_i = \epsilon + \eta^2 + O\left(\frac{\eta^2}{\log^2 \eta^2}\right) \quad (4-73)$$

The first term in Eq. (4-72) corresponds to the entire Oseen expansion (4-19) for $n = 0$ and also to the expansion (4-41). In the present case all u_j , $j > 1$ vanish identically. This is of course true only with the special choice of the multiplicative constant in the argument of $\log$ in Eq. (4-71) (cf. Eqs. (4-22) ff). The "infinite" Oseen series in powers of ϵ has thus been summed and it is meaningful to include further terms that are transcendently small in ϵ . As is seen from Eq. (4-72) the first transcendental term is of order η^2 , i.e. of order $\mathcal{R}_e$. An equivalent definition of ϵ , in the sense of Eq. (4-25) would be

$$\epsilon = \frac{1}{E_1(\eta^2)} \quad (4-74)$$

While this definition is suitable for the present case, it would, however, be difficult to find an analogous definition for a general value of n .

4.2.3. *Stokes Expansion.* For $n = \frac{1}{2}$

$$r^+ = \frac{\tilde{r}}{\sqrt{\mathcal{R}_e}} \quad (4-75a)$$

and

$$\xi^+ = \frac{\xi}{\eta} = \frac{r^+}{\sqrt{i}} \quad (4-75b)$$

(cf. Eqs. (3-6) and (3-35)). In the Stokes limit $\mathcal{R}_e$ tends to zero while $\tilde{r}$ and r^+ are fixed. The same limit is obtained by letting η tend to zero with ξ^+ fixed. One then finds the Stokes expansion

$$\begin{aligned} u^* &= 1 - \epsilon (\log r^+ - \log \sqrt{i}) + O(\eta^2) \\ &= 1 - \epsilon \log \xi^+ + O(\eta^2) \end{aligned} \quad (4-76)$$

and

$$\frac{1}{2}\tilde{R}C_t = \epsilon + O(\eta^2) \quad (4-77)$$

This expansion is analogous to both the Stokes expansion (4-30) and the expansion (4-58) for the case $n = 0$. The comments made in the case $n = 0$ apply in the present case also with the following two additional remarks. For $n = \frac{1}{2}$ the coefficients of ϵ^j are identically zero for $j > 1$. Furthermore Eq. (4-76) is valid for large t^+ near the cylinder since there $r^+ = O(\sqrt{t})$.

4.2.4. *Euler and Prandtl Expansions.* Repeated applications of the Euler limit give zero. The Prandtl limit is obtained by letting $\mathcal{R}_e$ tend to infinity while $\bar{r}$ and t^* are fixed. The same limit is obtained if η tends to infinity while $\bar{\xi} = \xi - \eta$ is fixed (cf. Eq. (3-36)). One then finds

$$u^* = \operatorname{erfc}(\sqrt{2}\bar{\xi}) + \epsilon \frac{2}{3} \sqrt{\frac{2}{\pi}} [\operatorname{erfc}(\sqrt{2}\bar{\xi}) - (1 - \bar{\xi}^2)e^{-2\bar{\xi}^2}] + O(\epsilon^2) \quad (4-78a)$$

where

$$\epsilon = \mathcal{R}_e^{-\frac{1}{2}} = \frac{1}{2\eta} \quad (4-78b)$$

and

$$\sqrt{t}C_t = \sqrt{2} \frac{2}{\sqrt{\pi}} + \epsilon \frac{8}{3\pi} + O(\epsilon^2) \quad (4-79)$$

4.2.5. *Note on Uniformity.* The Oseen and Prandtl expansions are uniformly valid for the entire range of ξ ($\eta \leq \xi < \infty$) for small and large η respectively. Since η depends on $\mathcal{R}_e$ only, the expansions are valid for the entire (r, t) -domain for small and large $\mathcal{R}_e$ respectively. The non-uniformities encountered for small or large t in the case $n = 0$ do not occur in the present case.

4.2.6. *Skin-friction Curves.* $n = \frac{1}{2}$

The variation of the skin-friction coefficient, as represented by $\sqrt{t}C_t$, with the Reynolds number is shown in Fig. 1 for the exact solution and various approximations.

The curve labeled as the exact solution has been computed from Eq. (4-64) using Pearson's table of the incomplete gamma function [14]. The curve is drawn for the range of values available in Pearson's tables. The special exact solutions Eq. (4-65b) and Eq. (4-66b) have been used as check points.

The boundary layer solution in the figure is the first term of Eq. (4-79) and the curve of boundary layer plus first correction is found from both terms in Eq. (4-79). The second correction to boundary layer theory has also been computed but it turns out to be negligible over the range of $\mathcal{R}_e$ considered.

The curve representing the Oseen solution is computed from Eq. (4-73) with ϵ defined by Eq. (4-71). A different choice of ϵ , as for example given by Eq.

(4-74), would give a skin-friction curve which decreases as η increases but is still far away from the exact solution if $\eta > \frac{1}{4}$.

FIG. 1 - VARIATION OF SKIN-FRICTION COEFFICIENT WITH REYNOLDS NUMBER; $n = 1/2$, EXACT SOLUTION AND VARIOUS APPROXIMATIONS

The most striking feature of these curves is the extremely good agreement of the corrected boundary layer theory with the exact solution over the range $1 \leq \eta \leq \infty$ or $1 \leq Re \leq \infty$. For example at $\eta = 2$ the exact solution gives

$\sqrt{i}C_f = 2.00$ (*cf.* Eq. (4-66b)) while the boundary layer theory plus the first correction gives $\sqrt{i}C_f = 2.02$. The Oseen solution on the other hand seems to be a valid approximation only for η or $\mathcal{R}_e < \frac{1}{4}$. It is difficult to relate the Reynolds number for this special problem to the corresponding entity for steady flow. However, it seems from this example that the first correction to boundary layer theory would predict the skin-friction (in separationless flow) down to much lower Reynolds numbers than generally imagined, say $\mathcal{R}_e = 10$ or even 5.

5. EXPANSIONS DERIVED DIRECTLY FROM THE EQUATIONS, $0 \leq n \leq 1$

In the preceding section expansions were derived from the existing exact solutions, *i.e.* in the cases $n = 0$ and $n = \frac{1}{2}$. We shall now derive approximate solutions for $0 \leq n \leq 1$ by assuming that the solutions may be represented by expansions of suitable form. Differential equations for the various terms of these expansions will be derived. Recursive methods for solving these equations will be developed and the solutions given explicitly for some of the leading terms. In each case the region of uniform validity of the expansion will be discussed. For $n = 0$ and $n = \frac{1}{2}$ the expansions derived directly from the equations may be checked against the expansions derived from the exact solutions.

5.1. Expansions for Low Reynolds Numbers. The discussion of the case $n = 0$ in Section 4 showed that in the low Reynolds number case one may consider either a parameter-type expansion (small $\mathcal{R}_e$) or a coordinate-type expansion (small η). The latter type was shown to be an improvement over the former. Both types of expansion will be discussed below for general values of n . It will be seen that their relation to each other is very much the same as in the case $n = 0$. In principle, the parameter-type expansion could then have been avoided altogether. It is, however, included for various reasons. The method used in deriving it is of certain interest. In particular, its underlying ideas may be applied to studying flow past finite bodies in which case coordinates such as ξ and η lose their significance. Furthermore the method used may be modified so as to lead to a coordinate type expansion. As pointed out in the Introduction the aim of the present paper is to discuss various methods with a view to their generality. Actually, the reasoning starting with Eq. (5-45) contains a derivation of the final results, which is complete in itself.

5.1.1. Parameter-type Expansions for Low $\mathcal{R}_e$. We shall first derive an *Oseen expansion* for $0 \leq n < 1$ (*cf.* the discussion preceding Eq. (3-14)). Such an expansion will have the form

$$u^*(\tilde{r}, \tilde{t}, \mathcal{R}_e) \cong \sum_{j=0}^{\infty} \epsilon_j(\mathcal{R}_e) u_j(\tilde{r}, \tilde{t}) \quad (5-1)$$

The form of the function $\epsilon_j(\mathcal{R}_e)$ will have to be determined in the course of the derivation. At present we shall only require the ϵ_j to be a decreasing sequence (*cf.* Eq. (3-13)). The expansion (5-1) should be an asymptotic expansion that is uniformly valid in as large a domain as possible. It will later be seen that the requirement of uniformity will take the following form (*cf.* Eq. (4-21))

$$\lim_{\mathcal{R}_e \rightarrow 0} \frac{u^* - \sum_{j=1}^l \epsilon_j u_j}{\epsilon_{l-1}} = 0 \quad (5-2)$$

uniformly for $\tilde{R} \leq \tilde{r} < \infty$, $t_0 \leq \tilde{t} \leq t_1$ where t_0 and t_1 are any numbers such that $0 < t_0 < t_1 < \infty$. This means that uniformity in time, or even in a half-infinite time interval cannot be achieved. Note furthermore that in order to have uniformity of order ϵ_{l-1} it is necessary to include the term $\epsilon_l u_l$. For example, it may be seen as in Section 4 that the Oseen limit is zero:

$$u_0 = \lim_O u^* = 0 \quad (5-3)$$

However, this approximation is not uniformly valid at $\tilde{r} = \tilde{R}$. To obtain a uniform approximation of order ϵ_0 (= unity) it is necessary to include the term $\epsilon_1 u_1$.

If the expansion (5-1) is inserted into Eq. (3-7a) and the expression for v^* (Eq. (3-7b)) is used one obtains the result that for any ϵ_j the transport term $v^* u_r^*$ will contain the factor $\mathcal{R}\epsilon^{2(1-n)} \epsilon_j$. Hence this factor is either equal to ϵ_k for some $k > j$ or transcendental with respect to all ϵ_j . At any rate u_1 and possibly some u_j with $j > 1$ will satisfy the Oseen equation (cf. Eq. (2-8))

$$u_{j\tilde{t}} = u_{j\tilde{r}\tilde{r}} + \frac{1}{\tilde{r}} u_{j\tilde{r}} \quad (5-4)$$

In fact it will be shown below that $\mathcal{R}\epsilon^{2(1-n)} \epsilon_j$ is always transcendental with respect to the ϵ_j so that Eq. (5-4) is valid for all u_j . If properly interpreted this statement is true even for $n = \frac{1}{2}$ as discussed in Section 4.2.

The boundary condition at $\tilde{r} = \tilde{R}$ is

$$\sum_{j=1}^{\infty} \epsilon_j (\mathcal{R}\epsilon) u_{j(\mathcal{R}\epsilon^{1-n} \tilde{t}^n, \tilde{t})} = 1 \quad (5-5)$$

In order to achieve the uniformity expressed in Eq. (5-2), Eq. (5-5) is interpreted to mean

$$\lim_{\mathcal{R}\epsilon \rightarrow 0} \frac{\sum_{j=1}^l \epsilon_j (\mathcal{R}\epsilon) u_{j(\mathcal{R}\epsilon^{1-n} \tilde{t}^n, \tilde{t})} - 1}{\epsilon_{l-1}(\mathcal{R}\epsilon)} = 0 \quad (5-6a)$$

The convergence need not be uniform in $\tilde{t}$. It is required that each u_j satisfy the condition at infinity and the initial condition

$$u_j(\infty, \tilde{t}) = 0 \text{ all } \tilde{t} \quad (5-6b)$$

$$u_j(\tilde{r}, \tilde{t}) = 0, \quad \tilde{t} < 0 \quad (5-6c)$$

The Oseen equation (5-4) is simply the heat equation with axial symmetry. Its general solution satisfying conditions (5-6b) and (5-6c) may be written in terms of an arbitrary function φ_j as

$$u_j = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} K_0(\sqrt{\sigma} \tilde{r}) \varphi_j(\sigma) \frac{d\sigma}{\sigma} \quad (5-7)$$

The φ_j have to be determined from Eq. (5-6a). In the Oseen limit $\tilde{R} = \mathcal{R}_e^{1-n} \tilde{t}^n$ tends to zero. Hence it is sufficient to use an approximation to the integral in Eq. (5-7) valid for $\tilde{r}$ near zero when the boundary condition Eq. (5-6a) is imposed. For small values of $\tilde{r}$ one obtains

$$\begin{aligned} u_j &= \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \left[\log \frac{\sqrt{\gamma_0} \tilde{r}}{2} + \frac{1}{2} \log \gamma_0 \sigma + O(\tilde{r}^2 \log \tilde{r}) \right] \varphi_j(\sigma) \frac{d\sigma}{\sigma} \\ &= -\Phi_j(\tilde{t}) \log \frac{\sqrt{\gamma_0} \tilde{r}}{2} - \psi_j(\tilde{t}) + O(\tilde{r}^2 \log \tilde{r}) \end{aligned} \quad (5-8)$$

where Φ_j and ψ_j are defined as the inverse Laplace transforms

$$\Phi_j(\tilde{t}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \varphi_j(\sigma) \frac{d\sigma}{\sigma} \quad (5-9a)$$

$$\psi_j(\tilde{t}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{1}{2} (\log \gamma_0 \sigma) \varphi_j(\sigma) \frac{d\sigma}{\sigma} \quad (5-9b)$$

ψ_j and Φ_j are thus related functions.

The boundary condition (5-5) then takes the form

$$\begin{aligned} \epsilon_1(\mathcal{R}_e) &\left[-\Phi_1(\tilde{t}) \log \frac{\sqrt{\gamma_0} \mathcal{R}_e^{1-n}}{2} - \Phi_1(\tilde{t}) \log \tilde{t}^n - \psi_1(\tilde{t}) + O(\mathcal{R}_e^{2(1-n)} \log \mathcal{R}_e) \right] \\ &+ \epsilon_2(\mathcal{R}_e) \left[-\Phi_2(\tilde{t}) \log \frac{\sqrt{\gamma_0} \mathcal{R}_e^{1-n}}{2} - \Phi_2(\tilde{t}) \log \tilde{t}^n - \psi_2(\tilde{t}) + O(\mathcal{R}_e^{2(1-n)} \log \mathcal{R}_e) \right] \\ &+ \epsilon_3(\mathcal{R}_e) [\dots] + \dots \cong 1 \end{aligned} \quad (5-10)$$

It is seen that this equation is satisfied to order unity (Eq. (5-6a) with $l = 1$) if one chooses

$$\epsilon_1(\mathcal{R}_e) = \frac{-1}{\log \frac{\sqrt{\gamma_0} \mathcal{R}_e^{1-n}}{2}} = \epsilon \quad (5-11)$$

and

$$\Phi_1(\tilde{t}) = 1 \quad \text{for } \tilde{t} > 0 \quad (5-12)$$

(more precisely, $\Phi_1(\tilde{t})$ is the Heaviside function $H(\tilde{t})$ which vanishes for $\tilde{t} < 0$). The choice of ϵ_1 is of course not unique. Alternate choices of ϵ_1 may be found as in Section 4 (*cf.* the discussion of Eqs. (4-23) *ff.*) The determination of $\Phi_1(\tilde{t})$ implies

$$\varphi_1(\sigma) = 1 \quad (5-13)$$

Hence the general solution (5-7) becomes

$$u_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} \frac{K_0(\sqrt{\sigma} \tilde{r})}{\sigma} d\sigma = \frac{1}{2} E_1(\xi^2) \quad (5-14)$$

where $\xi = \frac{1}{2} \tilde{r} / \sqrt{\tilde{t}}$.

Similarly, it is seen that Eq. (5-10) is satisfied to order ϵ_1 (Eq. (5-6a) with $l = 2$) if one chooses

$$\epsilon_2(\mathcal{R}\epsilon) = \epsilon^2 \quad (5-15)$$

and

$$\Phi_2(\tilde{t}) = \Phi_1(\tilde{t}) \log \tilde{t}^n + \psi_1(\tilde{t}) \quad (5-16)$$

This implies

$$\varphi_2(\sigma) = (\tfrac{1}{2} - n) \log \gamma_0 \sigma \quad (5-17)$$

and

$$u_2 = \frac{2n-1}{4} [E_1(\xi^2) \log \tilde{t} + G_1(\xi^2)] \quad (5-18)$$

where G_1 is the function defined by Eq. (4-12). Continuing this procedure one finds

$$\epsilon_j = \epsilon^j \quad (5-19)$$

and

$$\Phi_j(\tilde{t}) = \Phi_{j-1}(\tilde{t}) \log \tilde{t}^n + \psi_{j-1}(\tilde{t}) \quad (5-20)$$

from which the φ_j and the u_j may be determined successively.

A consequence of Eq. (5-19) is that $\mathcal{R}\epsilon^{2(1-n)}\epsilon_j$ is always transcendentally small and that all u_j obey the Oseen equation. Thus even if the series in Eq. (5-1) is summed to $j = \infty$ the result cannot represent the exact solution (see below).

The uniformity requirement (Eq. (5-6a)) can now be discussed. The presence of the terms containing $\log \tilde{t}$ in Eq. (5-10) prevents uniformity near $\tilde{t} = 0$ or ∞ . Furthermore for $\tilde{r}$ near $\tilde{R}$, $u_1 = O(\log \mathcal{R}\epsilon) = O(1/\epsilon_1)$. This is the reason why $\epsilon_1 u_1$ is valid only to order unity and not to order ϵ_1 . Similar remarks apply for $j > 1$. The leading term in the expansion (5-1) is actually $\epsilon_0 u_0 = 0$. This term is uniformly valid for $\tilde{r} > \tilde{R}$ but not at $\tilde{r} = \tilde{R}$ where $u^* = 1$, *i.e.* the convergence of the Oseen limit is non-uniform at the boundary. However, as was seen this non-uniformity can be remedied by the simple expedient of including more terms of the expansion.

The following expansion has thus been obtained

$$u^* = \epsilon \frac{E_1(\xi^2)}{2} - \frac{\epsilon^2}{4} [(\log \tilde{t}^{1-2n}) E_1(\xi^2) + (1 - 2n) G_1(\xi^2)] + \dots \quad (5-21)$$

where $\epsilon = -1/\log \frac{1}{2}\sqrt{\gamma_0} \mathcal{R}_e^{1-n}$. The corresponding skin friction formula is

$$\tilde{R}C_f = \mathcal{R}_e^{1-n} \tilde{t}^n C_f = 2\epsilon - \epsilon^2 \log \tilde{t}^{1-2n} + \dots \quad (5-22)$$

For $n = 0$ and $n = \frac{1}{2}$ Eqs. (5-21) and (5-22) specialize to the formulas previously derived from the exact solutions (Eqs. (4-8), (4-9), (4-27), and (4-72), (4-73)).

We shall now derive a *Stokes solution* for $0 \leq n < 1$. If Stokes variables are used, it follows from Eq. (3-9) that the leading terms of an expansion for low $\mathcal{R}_e$ must satisfy the Stokes equation (2-9). The general solution of this equation satisfying the boundary condition at the cylinder is

$$u^* = 1 + A(\mathcal{R}_e, \tilde{t}) \log \frac{r^+}{R^+} \quad (5-23)$$

where A is an arbitrary function.

To determine A we impose the matching condition that the Oseen limit of Stokes solution is equal to the Oseen limit of the exact solution, namely zero (cf. Eqs. (5-3) and (4-31)). Since $R^+ = \tilde{t}^n$ and $r^+ = \tilde{r} \mathcal{R}_e^{n-1}$ one finds that

$$A(\mathcal{R}_e, \tilde{t}) = \frac{1}{\log \mathcal{R}_e^{1-n}} + O\left(\frac{1}{\log^2 \mathcal{R}_e}\right) \quad (5-24)$$

Hence the Stokes solution, that is, the beginning of the Stokes expansion, is

$$u^* = 1 + \frac{1}{\log \mathcal{R}_e^{1-n}} \log \frac{r^+}{R^+} + O\left(\frac{1}{\log^2 \mathcal{R}_e}\right) \quad (5-25)$$

The corresponding skin friction is

$$C_f = \frac{-2}{\log \mathcal{R}_e^{1-n}} \frac{\mathcal{R}_e^{n-1}}{R^+} + O\left(\frac{1}{\log^2 \mathcal{R}_e}\right) = \frac{-2}{\log \mathcal{R}_e^{1-n}} \frac{1}{\tilde{R}} + O\left(\frac{1}{\log^2 \mathcal{R}_e}\right) \quad (5-26)$$

The comments made about the Stokes solution in Section 4 apply also to the general case. Thus near the wall the Stokes solution agrees with the leading term of the Oseen solution. In particular it gives correctly the leading term of the skin friction (cf. Eqs. (5-22) and (5-26)).

5.1.2. Parameter-type Expansion for Small β , $n = 1$. As was remarked in Section 2 no length L and hence no overall Reynolds number $\mathcal{R}_e$ may be defined in this case. It can also be seen that many of the arguments above lose their meaning when $n = 1$. However, as pointed out, for $n = 1$ the most general form of R is

$$R = At \quad \text{or} \quad \tilde{R} = \beta \tilde{t} \quad (2-6)$$

Here β may not be put equal to unity without loss of generality. A small value of β means a slow rate of expansion. It is then to be expected that for $n = 1$ β plays the same role as $\mathcal{R}_e$ does for $n < 1$. For small β one should then be able to deduce an expansion similar to the expansion for low $\mathcal{R}_e$ discussed above.

It may be seen immediately that the reasoning above is valid for small β if one replaces n by 1 and $\mathcal{R}\epsilon^{1-n}$ by β . One then obtains from Eq. (5-21) the *Oseen expansion*

$$u^* = \frac{\epsilon}{2} E_1(\xi^2) + \frac{\epsilon^2}{4} [E_1(\xi^2) \log \bar{t} + G_1(\xi^2)] + \dots \quad (5-27a)$$

where

$$\epsilon = \frac{-1}{\log \frac{\sqrt{\gamma_0} \beta}{2}} \quad (5-27b)$$

The corresponding skin friction is

$$\bar{R}C_t = \beta \bar{t} C_t = 2\epsilon + \epsilon^2 \log \bar{t} + \epsilon^3(\quad) + \dots \quad (5-27c)$$

The comments on the uniformity of the expansion are the same as in the case $n < 1$.

A *Stokes solution* for β small may be obtained by solving the Stokes equation, neglecting the boundary condition at infinity. This gives

$$u^* = 1 + A(\beta, \bar{t}) \log \frac{r^+}{R^+} \quad (5-27d)$$

where $r^+ = \bar{r}/\beta$ and A is an arbitrary function. Applying the same matching conditions as for Eq. (5-23) one obtains

$$u^* = 1 + \frac{1}{\log \beta} \log \frac{r^+}{R^+} + O\left(\frac{1}{\log^2 \beta}\right) \quad (5-28)$$

and

$$\bar{R}C_t = \frac{-2}{\log \beta} + O\left(\frac{1}{\log^2 \beta}\right) \quad (5-29)$$

The relation to the Oseen solution is the same as for $n < 1$.

5.1.3. Coordinate-type Expansions for Small η . The expansions derived so far are non-uniform for both small and large values of time if $n \neq \frac{1}{2}$. We note also that the partial sums of the expansions do not have the correct similarity property (*cf.* the discussion after Eq. (3-25)). These two facts are connected. By introducing the similarity variables ξ and η one may obtain expansions that are uniformly valid for large times ($0 \leq n \leq \frac{1}{2}$) or small times ($\frac{1}{2} \leq n \leq 1$). This result is consistent with the intuitive interpretation of the cross sectional Reynolds number η (*cf.* the discussion preceding Eq. (4-39)). For $\frac{1}{2} \leq n \leq 1$ the body grows at a slower rate than the viscous length for small times so that $\eta = \frac{1}{2}R/\sqrt{\nu t}$ is small for small times, *etc.* Coordinate-type expansions valid for small η will be derived below by two different methods. The first method is a revision

of the method of Laplace transformation used above. The second method is based on the equation in ξ, η variables (3-32).

In order to combine the *method of Laplace transformation* with similarity considerations we proceed as follows: Each term in the series in Eq. (5-1) was shown to satisfy the Oseen equation (5-4). Hence the sum of the series will satisfy the same equation, or at least it follows that there exists a function $U^*(\tilde{r}, \tilde{t}; \mathcal{R}_e)$ which satisfies the Oseen equation and which has the same asymptotic expansion as u^* . Then

$$U^*(\tilde{r}, \tilde{t}; \mathcal{R}_e) \cong \sum_{j=1}^{\infty} \epsilon_j(\mathcal{R}_e) u_j(\tilde{r}, \tilde{t}) \quad (5-30)$$

where the functions on the right hand side are found as above. Furthermore, we may require that $U^*(\tilde{r}, \tilde{t}; \mathcal{R}_e)$ has the similarity defined by Eq. (3-25). The statement made above about U^* are correct if, for instance, one defines U^* to be the solution of the Oseen equation which satisfies the exact boundary condition. This is, however, not the only possible choice since this solution contains transcendently small terms, which are irrelevant for Eq. (5-30) (*cf.* the case $n = 0$ discussed in Section 4).

Since U^* satisfies the Oseen equation it may be represented as

$$U^*(\tilde{r}, \tilde{t}; \mathcal{R}_e) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma \tilde{t}} K_0(\sqrt{\sigma} \tilde{r}) \Phi(\sigma, \mathcal{R}_e) \frac{d\sigma}{\sigma} \quad (5-31)$$

Furthermore, if U^* has the correct similarity it is invariant under the mapping described by Eq. (3-25) (extended to include the mapping $\sigma \rightarrow \sigma/a^2$). From this it follows that

$$\Phi(\sigma, \mathcal{R}_e) = \Phi(\sigma^{\frac{1}{2}-n} \mathcal{R}_e^{1-n}) \quad (5-32)$$

Now the variables in the integral can be combined further to introduce the similarity coordinates of Eq. (3-30). Replacing σ by $\sigma/\tilde{t}$ gives

$$U^*(\tilde{r}, \tilde{t}; \mathcal{R}_e) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma} K_0(\sqrt{\sigma} 2\xi) \Phi(\sigma^{\frac{1}{2}-n} 2\eta) \frac{d\sigma}{\sigma} \quad (5-33)$$

An integral equation for Φ will be derived from the fact that U^* satisfies the boundary conditions

$$U^* = 1 \quad \text{at} \quad \xi = \eta \quad (5-34)$$

This last equation is to be understood in a sense equivalent to Eq. (5-6a) with $\mathcal{R}_e$ replaced by η . Proceeding as before, we replace Eq. (5-33) by an expression valid for small values of ξ .

$$U^* = \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma} (\log \sqrt{\gamma_0} \xi + \frac{1}{2} \log \gamma_0 \sigma) \Phi(\sigma^{\frac{1}{2}-n} 2\eta) + O(\xi^2 \log \xi) \quad (5-35)$$

Using the boundary condition (5-34) and leaving out transcendently small terms one then obtains the following integral equation for Φ

$$1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma} \left[\frac{1}{\epsilon} - \frac{1}{2} \log(\gamma_0 \sigma) \right] \Phi(\sigma^{\frac{1}{2}-n} 2\eta) \frac{d\sigma}{\sigma} \quad (5-36a)$$

where

$$\epsilon = \epsilon(\eta) = \frac{-1}{\log \sqrt{\gamma_0 \eta}} \quad (5-36b)$$

A solution valid for the cases $n = 0$ and $n = \frac{1}{2}$ is (*cf.* Eqs. (4-6), (4-43), and (4-48) for $j = 1$)

$$\Phi = \frac{-1}{\log [\sqrt{\gamma_0 \eta} (\gamma_0 \sigma)^{\frac{1}{2}-n}]} \quad (5-37)$$

It is natural to assume for the general case that Φ is of the following form

$$\Phi(\sigma^{\frac{1}{2}-n} 2\eta) = \sum_{j=1}^{\infty} \frac{b_j}{\{\log [\sqrt{\gamma_0 \eta} (\gamma_0 \sigma)^{\frac{1}{2}-n}]\}^j} \quad (5-38a)$$

which, introducing the ϵ of Eq. (5-36b), may be written

$$\Phi(\sigma^{\frac{1}{2}-n} 2\eta) = \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^j b_j}{[1 - \epsilon(\frac{1}{2} - n) \log(\gamma_0 \sigma)]^j} \quad (5-38b)$$

To determine the coefficients ϵ we first rearrange the above series in powers of ϵ :

$$\Phi(\sigma^{\frac{1}{2}-n} 2\eta) = \sum_{m=1}^{\infty} (-1)^m \Gamma(m) \epsilon^m \sum_{k=0}^{m-1} (-1)^k \frac{(\frac{1}{2} - n)^k \log^k(\gamma_0 \sigma) b_{m-k}}{\Gamma(k+1) \Gamma(m-k)} \quad (5-39)$$

Substituting Eq. (5-39) into Eq. (5-36a) introduces the known integrals described by Eq. (4-47b) and (4-14b). The integral equation (5-36a) then yields a recursive system of equations for the b_j . The first equation is obtained by equating the term independent of ϵ to unity. The remaining equations are found by equating the coefficients of each ϵ^m to zero. One then finds (*cf.* Eqs. (4-47b) and (4-48))

$$b_{m+1} = \Gamma(m) \sum_{k=0}^{m-1} (-1)^k \frac{a_{k+1} b_{m-k} (1 - 2n)^k}{\Gamma(k+2) \Gamma(m-k)} [(1 - 2n)m - (k+1)], \quad (5-40a)$$

$$m = 1, 2, \dots$$

In particular

$$b_1 = -1, \quad b_2 = 0, \quad b_3 = \frac{\pi^2}{12} n(1 - 2n) \quad (5-40b)$$

Thus Φ can be expanded in powers of $\epsilon(\eta)$ as

$$\Phi(\sigma^{\frac{1}{2}-n} 2\eta) = \epsilon + \epsilon^2 \left(\frac{1}{2} - n\right) \log \gamma_0 \sigma - \epsilon^3 [b_3 - (\frac{1}{2} - n)^2 \log^2 \gamma_0 \sigma] \quad (5-41)$$

Inserting the value for Φ into Eq. (5-35) one obtains the expansion

$$u^* \cong U^* \cong \sum_{j=1}^{\infty} \epsilon_j(\eta) u_j(\xi) = \frac{\epsilon E_1(\xi^2)}{2} - \frac{\epsilon^2}{4} (1 - 2n) G_1(\xi^2) + O(\epsilon^3) \quad (5-42a)$$

where

$$\epsilon_j(\eta) = \epsilon^j, \quad \epsilon = \frac{-1}{\log \sqrt{\gamma_0} \eta} \quad (5-42b)$$

The function $G_1(z)$ was described by Eqs. (4-12) and (4-13). In the case $n = 0$ the expansion (5-42) reduces to its previously found value (Eq. (4-46)). It follows from the method of derivation that each $u_j(\xi)$ may be expressed as a simple combination of functions occurring in the case $n = 0$. For $n = \frac{1}{2}$, $u_j(\xi) = 0$ for $j > 1$, so that the result agrees with Eq. (4-72).

It can be seen that the expansion (5-42) is obtained by assuming the expansion (5-21) to be valid for $\bar{i} = 1$ and redefining it by similarity for all other values of $\bar{i}$. Note that in the expansion (5-21) each term satisfies the Oseen equation but only the total sum has the proper similarity. On the other hand, each term of the expansion (5-42) has the proper similarity whereas only the total sum satisfies the Oseen equation.

The expansion (5-42) is uniformly valid in the sense of Eq. (4-49) and the comments following that equation apply. Note only that η small means $\bar{i}$ large for $0 \leq n < \frac{1}{2}$ and $\bar{i}$ small for $\frac{1}{2} < n \leq 1$.

It can easily be checked that Eq. (5-42) is still valid for the case $n = 1$, where of course ξ and η have to be defined by Eq. (3-31).

Near the wall one finds, neglecting terms of order $\xi \log \xi$,

$$U^* = -\epsilon \log \sqrt{\gamma_0} \xi + \epsilon^2 \frac{(1 - 2n)\pi^2}{24} + \epsilon^3 \left[(\log \sqrt{\gamma_0} \xi) \frac{(1 - 2n)(1 + 2n)\pi^2}{24} + \text{const.} \right] + \dots \quad (5-43)$$

The skin friction coefficient is then given by (cf. Eq. (3-37))

$$\frac{1}{2} \tilde{R} C_f = \eta \sqrt{i} C_f = \epsilon - \epsilon^3 \frac{(1 - 2n)(1 + 2n)\pi^2}{24} + \dots \quad (5-44)$$

A coordinate-type expansion of u^* has thus been found, using the method of Laplace transformation. An alternate method for finding the same expansion will now be discussed, namely a *method based on Eq. (3-32)*. We shall work di-

rectly with the similarity variables ξ and η and assume that u^* has an asymptotic expansion of the form (cf. Eq. (5-42))

$$u^* \cong \sum_{j=0}^{\infty} \epsilon_j(\eta) u_j(\xi) \quad (5-45)$$

valid for small values of η . The functions ϵ_j and u_j will now be determined by insertion of the expansion (5-45) into Eq. (3-32). *A priori* we shall only make the standard assumptions that the $\epsilon_j(\eta)$ are successively smaller in the sense of Eq. (4-42) and that the expansion be uniform in the sense of Eq. (4-49).

It can easily be verified that $u_0 = 0$. Inserting the expansion (5-45) into Eq. (3-32) one then finds that u_1 satisfies the equation

$$u_{1\xi\xi} + \left(2\xi + \frac{1}{\xi}\right) u_{1\xi} = 0 \quad (5-46)$$

The general solution of this equation, satisfying $u_1(\infty) = 0$ is within a multiplicative constant

$$u_1 = \frac{1}{2} E_1(\xi^2) \quad (5-47)$$

The multiplicative constant can be absorbed into the definition of $\epsilon_1(\eta)$. Using the boundary condition at $\xi = \eta$ which now reads (cf. Eq. (4-49))

$$\lim_{\eta \rightarrow 0} [1 - \epsilon_1(\eta) u_1(\eta)] = 0 \quad (5-48)$$

one finds with the aid of (4-10b) that a possible choice of ϵ_1 is

$$\epsilon_1 = \frac{-1}{\log \sqrt{\gamma_0 \eta}} = \epsilon \quad (5-49)$$

This formula agrees with Eq. (5-42b). As usual the factor in the argument of log is not uniquely determined. It has been chosen in such a way that the boundary condition (5-48) is satisfied up to transcendental order. It was seen before that for $n = \frac{1}{2}$ a consequence of this choice is that the u_j , $j > 1$, may be put equal to zero within transcendental terms. For $n \neq \frac{1}{2}$ it will also lead to a simplification of the higher order u_j .

Since

$$\eta \frac{d\epsilon_1}{d\eta} = \epsilon_1^2 = \epsilon^2 \quad (5-50)$$

it follows by insertion of the expansion into the equation that $\epsilon_2(\eta) = \epsilon^2$, and by induction

$$\epsilon_j = \epsilon^j \quad (5-51a)$$

$$\eta \frac{d\epsilon_j}{d\eta} = j\epsilon_{j+1} \quad (5-51b)$$

The u_j then satisfy the recursive equations

$$u_{j\xi\xi} + \left(2\xi + \frac{1}{\xi}\right) u_{j\xi} = 2(1-j)(1-2n)u_{j-1} \quad (5-52a)$$

with the boundary conditions

$$u_j(\infty) = 0 \quad (5-52b)$$

and

$$\lim_{\eta \rightarrow 0} \frac{\sum_{j=2}^l \epsilon_j u_j(\eta)}{\epsilon_{l-1}} = 0 \quad (5-52c)$$

The last condition follows from Eq. (4-49) and that fact that $\epsilon u_l(\eta) = 1$ within transcendentally small terms.

Note that the transport term $-(4\eta^2/\xi)u_\xi$ is transcendentally small, and hence plays no role in the equations (5-52a). It thus follows that the expansion (5-45) actually only represents a solution of the Oseen equation, expressed in ξ and η , which satisfies the boundary condition up to transcendental order.

Since u_1 is known the u_j may be found from Eq. (5-52) by successive integrations. However, actually the case of general n may be reduced to the case of $n = 0$. The number n occurs in Eq. (5-52a) only in the factor $(1 - 2n)$ on the right hand side and it does not occur in the boundary conditions. This fact leads to the following recursive scheme:

First we note (using a self-explanatory notation) that according to Eq. (5-47)

$$u_1(\xi; n) = u_1(\xi; 0) = \frac{1}{2}E_1(\xi^2) \quad (5-53)$$

From Eq. (4-46b) it then follows that a particular solution of Eq. (5-52) for $j = 2$ is

$$u_2(\xi; n) = (1 - 2n)u_2(\xi; 0) = \frac{(2n - 1)}{2} \frac{G_1(\xi^2)}{4} \quad (5-54)$$

At the wall one then finds from Eqs. (4-47a) and (4-48) that within transcendental terms

$$\epsilon^2 u_2(\eta; n) = -\epsilon^2 a_2(1 - 2n) = \epsilon^2 \frac{(1 - 2n)\pi^2}{24} \quad (5-55)$$

Thus the function $u_2(\xi; n)$ defined above satisfies the boundary condition (5-52c). In the next step, however, one finds that while $(1 - 2n)^2 u_3(\xi, 0)$ is a particular solution of Eq. (5-52a) for $j = 3$ it does not satisfy the correct boundary condition. The error of order ϵ^2 has to be canceled by the addition of a solution of the homogeneous part of Eq. (5-52a), *i.e.* of Eq. (5-46). One finds that the proper solution is

$$u_3(\xi; n) = (1 - 2n)^2 u_3(\xi, 0) + 2n(1 - 2n)a_2 u_1(\xi, 0) \quad (5-56)$$

For η small one then finds from Eq. (4-47a)

$$\begin{aligned}\epsilon^3 u_3(\eta; n) &= \epsilon^2 a_2(1 - 2n)^2 - \epsilon^3 a_3(1 - 2n)^2 + \epsilon^2 2n(1 - 2n)a_2 \\ &= \epsilon^2 a_2(1 - 2n) - \epsilon^3 a_3(1 - 2n)^2\end{aligned}\quad (5-57)$$

A comparison with Eq. (5-55) then shows that the boundary condition (5-52c) is satisfied.

The above examples show how the recursive method may be continued indefinitely. The results agree with the previous recursive method that was used in deriving Eq. (5-42). Note that for $n = \frac{1}{2}$ the right-hand side of Eq. (5-52a) is zero so that all u_j are zero for $j > 1$.

5.1.4. *Stokes-type Solution with Similarity.* If one introduces the Stokes form of the similarity variables (Eq. (3-35)) into Eq. (3-32) and then lets η tend to zero while $\xi^+ = \xi/\eta$ is fixed one obtains the equation

$$u_{\xi^+\xi^+}^* + \frac{1}{\xi^+} u_{\xi^+}^* = 0 \quad (5-58)$$

This equation contains only the viscous stress term and is thus the same as Eq. (2-9). The general solution of this equation satisfying the boundary condition at the wall is

$$u^* = A(\eta) \log \xi^+ + 1 \quad (5-59)$$

To determine the function $A(\eta)$ we use the matching condition that the limit of u^* as η tends to zero while ξ is fixed should be zero. This gives the Stokes-type solution

$$u^* = \frac{\log \xi^+}{\log \eta} + 1 + O\left(\frac{1}{\log^2 \eta}\right) = \frac{\log \xi}{\log \eta} + O\left(\frac{1}{\log^2 \eta}\right) \quad (5-60)$$

This solution agrees with Eq. (4-58) for $n = 0$ and Eq. (4-76) for $n = \frac{1}{2}$. For further comments on the method, see the discussion following Eq. (4-30). Eq. (5-60) bears the same relation to the expansion (5-45) as the Stokes solution (5-28) to the Oseen expansion (5-27a).

5.2. Expansions for High Reynolds Numbers. In this section the form of the expansion of u^* valid for large values of $\mathcal{R}_e$ will be derived. An iterative method for computing the successive terms will be given and the first two terms computed explicitly. The first term is the solution according to boundary layer theory; the second term represents a correction to boundary layer theory. The form of the expansion will make it obvious that it is also an expansion for high values of η . Thus for high Reynolds numbers the results are the same whether one considers the overall Reynolds number or the cross-sectional Reynolds number.

For a class of problems similar to the one studied here (see Section 6.2 for

further discussion) one may expect that a flow quantity A , such as u^* or v^* , has an asymptotic expansion of the form

$$A = A_L^{(m)} + A_T^{(m)} + o(\epsilon^m) \quad (5-61a)$$

$$A_L^{(m)} = \sum_{j=0}^m \epsilon^j B_j(r^*, t^*) \quad (5-61b)$$

$$A_T^{(m)} = \sum_{j=0}^m \epsilon^j C_j(\bar{r}, t^*) \quad (5-61c)$$

where

$$\epsilon = \frac{1}{\sqrt{\text{Re}}}$$

$A_L^{(m)}$, which may be called the longitudinal component, is obtained from A by repeated application of the outer limit process, *i.e.* of the Euler limit. This may yield discontinuities at the wall, $r^* = R^*$, in which case each B_j is redefined by continuity

$$B_j(R^*, t^*) = \lim_{r^* \downarrow R^*} B_j(r^*, t^*) \quad (5-62)$$

$A_T^{(m)}$, the transversal component is obtained by a repeated application of the inner limit, *i.e.* the Prandtl limit, to the difference $A - A_L^{(m)}$. Eq. (5-61a) should be valid in a region which includes the whole range of r . However, it is not in general uniform in t^* . The longitudinal component itself is not in general uniformly valid near the boundary; that is the reason for adding the transversal component. For regions not including the boundary the transversal component is transcendentally small, *i.e.* smaller than any power of ϵ . This follows from the fact that any term which is of order ϵ^k off the boundary has been included in the longitudinal part $A_L^{(m)}$ for $m \geq k$.

In more familiar and intuitive terms B_0 is the external flow and ϵB_1 is the flow due to displacement thickness. The term C_0 is closely related to the boundary layer approximation. To see this we first observe that

$$\begin{aligned} \lim_P [B_0(r^*, t^*) + \epsilon B_1(r^*, t^*) + \dots] \\ = \lim_P [B_0(R^* + \epsilon \bar{r}, t^*) + \epsilon B_1(R^* + \epsilon \bar{r}, t^*) + \dots] \\ = B_0(R^*, t^*) \end{aligned} \quad (5-63)$$

Hence

$$C_0 = \lim_P (A - A_L^{(m)}) = (\lim_P A) - B_0(R^*, t^*) \quad (5-64)$$

where $\lim_P A$ is the boundary layer approximation. Similarly C_1 is a correction to the boundary layer solution, $\epsilon^2 B_2$ is the corresponding correction to the flow due to displacement thickness etc.

In order to find the functions B_j and C_j for a given flow problem, one replaces the various quantities A occurring in the equations by expansions of the form described by Eq. (5-61). By rewriting these equations in Euler and Prandtl variables, respectively, and by repeated application of the corresponding limit processes, one obtains successively equations for the various B_j and C_j . The equations and the corresponding boundary conditions show the order in which the B_j and C_j have to be determined.

The expansion form given above actually has a greater generality than is required for the present problem. First we note that v^* (cf. Eq. (3-17b)) is its own Euler limit. Hence v^* has only a longitudinal component, and this component has only one term. Secondly, u^* has only a transversal component. While this statement is intuitively quite plausible, especially for B_0 and B_1 , it is of interest to see the mechanism by which it may be derived from the equations. If the expansion (5-61) is inserted into the equation for u^* (Eq. (3-17a)) one sees that the contribution of the transversal component will vanish whenever the Euler limit is applied. This is so because the Euler limit is to be applied only off the boundary where the transversal component is transcendentally small. Since the right hand side of the equation for u^* is of higher order than the left hand side one can prove by induction that it will always vanish. The B_j will then satisfy the Euler equation (Eq. (2-10))

$$\frac{\partial B_j}{\partial t^*} + v^* \frac{\partial B_j}{\partial r^*} = 0 \quad (5-65)$$

Using the formula for v^* (Eq. (3-17b)) one sees that the B_j must be constant along the characteristic lines

$$(r^*)^2 = (t^*)^{2n} + \text{constant} \quad (5-66)$$

The conditions $u^*(r^*, 0) = u^*(\infty, t^*) = 0$, which should be applied individually to the B_j and the C_j , then imply that the B_j are everywhere zero. Note that the condition $u_w^* = 1$ should not be imposed on the longitudinal component. The general significance of the argument just given will be discussed in Section 6.2. Here we only remark that it fails if compressibility is taken into account.

We may thus assume that u^* is purely transversal and hence of the form

$$u^* \cong \sum_{j=0}^{\infty} \epsilon^j u_j(\bar{r}, t^*) \quad (5-67)$$

To find the equations for the u_j one starts with the equation for u^* in Prandtl variables (Eq. (3-19a)). Into this equation one inserts the expansion for u^* given above, the expansion for $\bar{v}$ given by Eq. (3-24) and the expansion

$$\frac{1}{\sqrt{\mathcal{R}_e}} \frac{1}{R^* + \frac{\bar{r}}{\sqrt{\mathcal{R}_e}}} = \frac{\epsilon}{R^*} \left(1 - \epsilon \frac{\bar{r}}{R^*} + \epsilon^2 \left(\frac{\bar{r}}{R^*} \right)^2 - \dots \right) \quad (5-68)$$

By collecting terms of the same order, which is equivalent to a repeated application of the Prandtl limit process, one then obtains:

$$\frac{\partial u_0}{\partial t^*} + v_0 \frac{\partial u_0}{\partial \bar{r}} - \frac{\partial^2 u_0}{\partial \bar{r}^2} = 0 \quad (5-69a)$$

$$\frac{\partial u_1}{\partial t^*} + v_0 \frac{\partial u_1}{\partial \bar{r}} - \frac{\partial^2 u_1}{\partial \bar{r}^2} = \left(\frac{1}{R^*} - v_1 \right) \frac{\partial u_0}{\partial \bar{r}} \quad (5-69b)$$

$$\frac{\partial u_2}{\partial t^*} + v_0 \frac{\partial u_2}{\partial \bar{r}} - \frac{\partial^2 u_2}{\partial \bar{r}^2} = \left(\frac{1}{R^*} - v_1 \right) \frac{\partial u_1}{\partial \bar{r}} - \left(\frac{\bar{r}}{(R^*)^2} + v_2 \right) \frac{\partial u_0}{\partial \bar{r}} \quad (5-69c)$$

etc., where $R^* = (t^*)^n$ and

$$v_0 = -\frac{n\bar{r}}{t^*}, \quad v_1 = \frac{n(\bar{r})^2}{(t^*)^{n+1}}, \quad v_2 = -\frac{n(\bar{r})^3}{(t^*)^{2n+1}} \quad (5-69d)$$

The boundary and initial conditions are:

$$u_0 = 1 \quad u_j = 0 \quad (j > 0) \quad \text{for} \quad \bar{r} = 0 \quad (5-70a)$$

$$u_j = 0 \quad \text{for} \quad t^* = 0 \quad \text{or} \quad \bar{r} = \infty \quad (5-70b)$$

Note that since the longitudinal part is zero the transversal part has to carry all the boundary conditions. Actually, its first term satisfies all the boundary conditions and the successive u_j satisfy zero boundary conditions.

The equation for u_0 , which of course is identical with Prandtl's boundary layer equation, is easily solved. The equation as well as its boundary conditions is invariant under the mapping

$$t^* \rightarrow a^2 t^* \quad \bar{r} \rightarrow a\bar{r} \quad a = \text{const.} \quad (5-71)$$

Hence one may put

$$u_0 = g_0(\bar{\xi}) = g_0(\bar{\xi}; n) \quad (5-72)$$

where

$$\bar{\xi} = \frac{\bar{r}}{2\sqrt{t^*}} = \frac{r - R}{2\sqrt{\nu t}} = \xi - \eta$$

g_0 then satisfies the equation

$$g_0'' + 2b^2 \bar{\xi} g_0' = 0 \quad b^2 = 1 + 2n \quad (5-73a)$$

with boundary conditions

$$g_0(0) = 1 \quad g_0(\infty) = 0 \quad (5-73b)$$

The solution is

$$g_0(\bar{\xi}; n) = \text{erfc}(b\bar{\xi}) \quad (5-74)$$

The equation for u_1 gives a first correction for transverse curvature (*cf.* [6]). It is not invariant under the mapping expressed by Eq. (5-71) (except for $n = \frac{1}{2}$). However, since the boundary conditions are homogeneous for $j > 1$ one has the added freedom of changing u_1 in the mapping. It is then easily seen that Eq. (5-69b) is invariant under the mapping

$$t^* = a^2 t^*, \quad \bar{r} \rightarrow a\bar{r}, \quad u_1 \rightarrow a^{1-2n} u_1 \quad (5-75)$$

Thus one may put

$$u_1(\bar{r}, t^*) = (t^*)^{\frac{1}{2}-n} g_1(\bar{\xi}; n) \quad (5-76)$$

Eq. (5-69b) then reduces to

$$g_1'' + 2(1 + 2n)\bar{\xi}g_1' + 2(2n - 1)g_1 = 2(4n\bar{\xi}^2 - 1)g_0' \quad (5-77)$$

where

$$g_0' = -\frac{2b}{\sqrt{\pi}} e^{-b^2\bar{\xi}^2}.$$

A particular solution g_{1p} is easily found

$$g_{1p} = \frac{\sqrt{1 + 2n}}{\sqrt{\pi}(1 + n)} (2n\bar{\xi}^2 - 1)e^{-b^2\bar{\xi}^2} \quad (5-78)$$

To this has to be added a solution of the homogeneous equation corresponding to Eq. (5-77) so that the sum satisfies the correct boundary conditions. For the cases $n = 0$ and $n = \frac{1}{2}$ the general solution to the homogeneous equation may be found by elementary methods. These solutions will, however, be derived below as special cases of the solutions for a general value of n , which may be derived as follows:

Referring to [9, Vol. 1, p. 250] it is seen that the general solution to the homogeneous part of Eq. (5-77) is expressed by two confluent hypergeometric functions for which we introduce notations and note some properties

$$g_1^{(1)}(\bar{\xi}; n) = \Phi\left(\frac{1}{2}\frac{2n-1}{2n+1}; \frac{1}{2}; -(2n+1)\bar{\xi}^2\right) \quad (5-79a)$$

$$= 1 - (2n-1)\bar{\xi}^2 + \dots \quad (5-79b)$$

$$\cong \frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1}{2n+1}\right)} [(2n+1)\bar{\xi}^2]^{-\frac{1}{2}(2n-1)/2n+1} \left[1 + O\left(\frac{1}{|\bar{\xi}|^2}\right)\right] \quad (5-79c)$$

$$g_1^{(2)}(\bar{\xi}; n) = \sqrt{1 + 2n} \bar{\xi} \Phi\left(\frac{2n}{2n+1}; \frac{3}{2}; -(2n+1)\bar{\xi}^2\right) \quad (5-80a)$$

$$= \sqrt{1 + 2n} \bar{\xi} \left\{ 1 - \frac{2n}{3} \bar{\xi}^2 + \dots \right\} \quad (5-80b)$$

$$\cong \frac{\Gamma(\frac{3}{2})}{\Gamma\left(\frac{n + \frac{3}{2}}{2n + 1}\right)} [(2n + 1)\bar{\xi}^2]^{-\frac{1}{2}(2n-1)/2n+1} \left[1 + O\left(\frac{1}{|\bar{\xi}|^2}\right) \right] \quad (5-80c)$$

It may be remarked that $g_1^{(1)}$ and $g_1^{(2)}$ can also be expressed as parabolic cylinder functions.

The complete solution to Eq. (5-77) is thus

$$g_1(\bar{\xi}; n) = g_{1P}(\bar{\xi}; n) + \frac{\sqrt{1 + 2n}}{\sqrt{\pi}(1 + n)} \{A g_1^{(1)}(\bar{\xi}; n) + B g_1^{(2)}(\bar{\xi}; n)\} \quad (5-81)$$

The conditions as $\bar{\xi} \rightarrow \infty$ and on the boundary $\bar{\xi} = 0$ provide the two relations for determining A and B . For $0 < n < \frac{1}{2}$, the asymptotic formulas show that both $g_1^{(1)}$ and $g_1^{(2)}$ become large as $\bar{r}$ and $\bar{\xi} \rightarrow \infty$. To insure that $g_1(\bar{\xi}; n)$ and $u^* \rightarrow 0$ as $\bar{r} \rightarrow \infty$ we choose

$$A \frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1}{2n + 1}\right)} + B \frac{\Gamma(\frac{3}{2})}{\Gamma\left(\frac{n + \frac{3}{2}}{2n + 1}\right)} = 0 \quad (5-82)$$

This same relationship holds also for $\frac{1}{2} < n < 1$. In this case $g_1^{(1)}$ and $g_2^{(2)}$ tend to zero as $\bar{r}$ and $\bar{\xi} \rightarrow \infty$. However the rate of approach to zero is important because of the factor $(t^*)^{\frac{1}{2}-n}$ in Eq. (5-76). Actually, the initial conditions ($t^* \rightarrow 0$, $\bar{\xi} \rightarrow \infty$) are not satisfied on u_1 unless Eq. (5-82) is taken to hold. Since $g_{1P}(0; n) = -\sqrt{1 + 2n}/[\sqrt{\pi}(1 + n)]$

$$A = 1 \quad (5-83)$$

to make $g_1(0; n) = 0$. Thus the complete solution of Eq. (5-69b) is

$$\begin{aligned} u_1(\bar{r}, t^*) = (t^*)^{\frac{1}{2}-n} \frac{\sqrt{1 + 2n}}{\sqrt{\pi}(1 + n)} & \left\{ (2n\bar{\xi}^2 - 1)e^{-(2n+1)\bar{\xi}^2} \right. \\ & + \Phi\left(\frac{1}{2} \frac{2n - 1}{2n + 1}; \frac{1}{2}; -(2n + 1)\bar{\xi}^2\right) \\ & \left. - 2\sqrt{1 + 2n} \frac{\Gamma\left(\frac{n + \frac{3}{2}}{2n + 1}\right)}{\Gamma\left(\frac{1}{2n + 1}\right)} \bar{\xi} \Phi\left(\frac{2n}{2n + 1}; \frac{3}{2}; -(2n + 1)\bar{\xi}^2\right) \right\} \end{aligned} \quad (5-84)$$

When n is either 0 or $\frac{1}{2}$ the functions $g_1^{(1)}$ and $g_1^{(2)}$ may be expressed in terms of error functions and exponentials (cf. [9, Vol. 1, p. 266 and Vol. 2, p. 133]). Eq. (5-84) then reduces to the functions previously derived by expanding the exact solutions (cf. Eqs. (4-36) and (4-78)).

From Eqs. (5-74), (5-79), (5-80) and (5-84) one finds the following formula for the skin friction coefficient

$$\sqrt{t} C_f = \frac{2\sqrt{1+2n}}{\sqrt{\pi}} + \frac{1}{2\eta} \frac{2(1+2n)}{\sqrt{\pi}(1+n)} \frac{\Gamma\left(\frac{n+\frac{3}{2}}{2n+1}\right)}{\Gamma\left(\frac{1}{2n+1}\right)} \quad (5-85)$$

This formula agrees with Eqs. (4-37) and (4-79) for $n = 0$ and $\frac{1}{2}$ respectively.

About the higher order terms in the expansion, only a few brief remarks will be made. By the same similarity conditions as used for u_1 it may be seen that u_j is of the form:

$$u_j = (t^*)^{(\frac{1}{2}-n)j} g_j(\xi) \quad (5-86)$$

so that the expansion for u^* Eq. (5-67) has the form:

$$u^* = \sum_{j=0}^{\infty} [\epsilon(t^*)^{(\frac{1}{2}-n)}]^j g_j(\xi), \quad \epsilon = \frac{1}{\sqrt{\mathcal{R}_e}} \quad (5-87a)$$

or (cf. Eq. (3-30b))

$$u^* = \sum_{j=0}^{\infty} \frac{1}{(2\eta)^j} g_j(\xi - \eta) \quad (5-87b)$$

This expansion, which is the result of application of boundary layer theory and higher order corrections thereto, shows the following: The exponents of t^* are positive, zero and negative for $n < \frac{1}{2}$, $n = \frac{1}{2}$ and $n > \frac{1}{2}$, respectively. One may therefore expect the expansion to be uniformly valid only for a limited domain if $n \neq \frac{1}{2}$. For $n < \frac{1}{2}$ it is valid for small values of time and for $n > \frac{1}{2}$ it is valid for large values of time, and for $n = \frac{1}{2}$ it is uniformly valid. This is consistent with the fact that for $n < \frac{1}{2}$ the cross-sectional Reynolds number η is large for small values of time whereas for $n > \frac{1}{2}$ it is large for large values of time. This bears out the intuitive idea that boundary layer theory applies for large values of the cross-sectional Reynolds number. Eq. (5-87) also shows that the expansion obtained for large values of $\mathcal{R}_e$ is simultaneously an expansion for large values of η . The equations for the g_j could also have been found by assuming u^* to have an expansion of the form given by (5-87b) and by inserting this expansion into Eq. (3-32). For the case $n = 1$ Eq. (3-32) is the same equation in (ξ, η) redefined by Eq. (3-31). Hence the previous expansion and skin-friction formula carry over unchanged when expressed in (ξ, η) .

6. DISCUSSION

The significance of the results of this paper with respect to more general problems will be discussed in Sections 6.2 and 6.3 which treat the high and low Reynolds number cases respectively. Details of the calculations for some special cases of more general flows will be presented in Section 6.4. Before embarking on this general discussion the question of similarity, as occurs in our special problem for $n \neq \frac{1}{2}$ and related problems, is taken up in Section 6.1. Section 6.5 contains concluding remarks with respect to generalization and applications.

6.1. Non-uniformity and Similarity. The form of the expansions obtained in the preceding sections indicate that none of these expansions are uniformly valid for all time ($0 \leq t \leq \infty$) except if $n = \frac{1}{2}$. It actually follows from the similarity of the problem that such a uniformly valid expansion cannot be obtained. This statement is intuitively rather obvious. A formal proof will, however, be given below in order to obtain a more exact formulation. To fix the ideas we shall consider the case of high Reynolds numbers. The generality of the proof will be obvious.

Let u^* be the exact solutions. It follows from the similarity discussed following Eq. (3-25) that

$$u^* = fn(\rho, \tau) \quad (6-1a)$$

$$\rho = \tilde{r}\mathcal{R}_e^{1-n/2n-1}, \quad \tau = \tilde{t}\mathcal{R}_e^{2(1-n)/2n-1} \quad (6-1b)$$

(or see Eq. (3-29) if $n = 1$).

The similarity of Eq. (6-1) implies that given any point $(\tilde{r}_1, \tilde{t}_1)$ in the domain ($\tilde{R} \leq \tilde{r} \leq \infty, 0 \leq \tilde{t} \leq \infty$) and any two Reynolds numbers $\mathcal{R}_{e1}$ and $\mathcal{R}_{e2}$ a second point $(\tilde{r}_2, \tilde{t}_2)$ in the domain can be found such that

$$u^*(\tilde{r}_1, \tilde{t}_1; \mathcal{R}_{e1}) = u^*(\tilde{r}_2, \tilde{t}_2; \mathcal{R}_{e2}) \quad (6-2)$$

Now assume that w^* is an approximation to u^* valid to order unity uniformly in $(\tilde{r}, \tilde{t})$. Formally this means that given any $\delta > 0$ there exists a number $N(\delta)$ depending only on δ such that

$$|u^* - w^*| < \delta \quad \text{if} \quad \mathcal{R}_e > N(\delta) \quad (6-3)$$

for all $\tilde{r}, \tilde{t}$ in the domain.

Now it can be proved that if w^* has the same similarity properties as u^* then w^* is identical with u^* , that is, w^* is the exact solution. To prove this it is sufficient to show that, given any $\delta > 0$,

$$|u^* - w^*| < \delta \quad \text{for all} \quad \mathcal{R}_e \quad (6-4)$$

and all $\tilde{r}, \tilde{t}$ in the domain. Now let $\mathcal{R}_{e1}, r_1, t_1$ be arbitrary values in the domain considered and δ_1 a positive number. Choose a $\mathcal{R}_{e2} > N(\delta_1)$ according to Eq. (6-3) and r_2 and t_2 according to the similarity laws. Eq. (6-3) then implies that the inequality (6-4) holds for $\delta_1, \mathcal{R}_{e2}, r_2, t_2$. If now w^* has the same similarity as u^* (Eqs. (6-1) and (6-2)) it then follows that the inequality (6-4) also holds for the arbitrarily chosen $\delta_1, \mathcal{R}_{e1}, r_1, t_1$. Hence the inequality (6-4) is true in general.

If one removes the restriction that the approximation w^* have the same similarity as the exact solution one may construct trivial counter examples for which Eq. (6-4) does not hold. However, such examples are of no importance. When one tries to obtain approximate solutions one usually gains by utilizing the similarity properties of the problem.

Thus, putting aside approximations without similarity, it follows that for $n \neq \frac{1}{2}$ there are only two cases possible *a priori*. The first case is one of exceptional luck in which the first approximation for high $\mathcal{R}_e$ is uniformly valid and hence, by the proof just given, is the exact solution for all $\mathcal{R}_e$. The second and more usual case is the one in which the first approximation is valid only in a restricted domain and to a certain order.

The latter case is the one that occurs in our special problem for $n \neq \frac{1}{2}$. Another problem to which the same reasoning applies is that of steady viscous flow past a conical object (wedge, cone, semi-infinite flat plate). In these problems ν can be regarded as a parameter. It can be eliminated from both the boundary conditions and the equations by introducing the coordinates $\tilde{x} = Ux/\nu$, etc. Flows with different values of ν are similar. Some of these "conical" flows are cases of exceptional luck, namely flow normal to an infinite plane and Oseen flow past a semi-infinite flat plate. In these cases the boundary layer approximation in suitable coordinates is identical with the exact solution (*cf.* [2]). In other cases the expansion in $\sqrt{\nu}$ does not break off with the first term and hence is not uniformly valid. For example, in the case of viscous flow past a wedge of half-angle $\neq \frac{1}{2}\pi$ it is not possible to find an expansion for small ν such that the leading term is valid uniformly to order unity. In this problem, as is well known, the non-uniformity occurs in a region near the leading edge.

The fact that $\mathcal{R}_e$ may be eliminated in the present problem (or in the problem of viscous flow past a cone) does of course not invalidate the use of expansions in $1/\sqrt{\mathcal{R}_e}$ (or $\sqrt{\nu}$). It should only be kept in mind that such expansions cannot be uniformly valid in the entire domain.

Existence of a similarity also indicates that a parameter-type expansion may be replaced by a coordinate-type expansion. For instance, for the flow past a semi-infinite flat plate the expansion in orders of the parameter $\sqrt{\nu}$ may be replaced by an expansion in orders of $(\nu/Ux)^{\frac{1}{2}}$. In this case the boundary layer solution may be viewed either as a solution valid for small ν (fixed x) or for large x (fixed ν).

6.2. High Reynolds Numbers. The relationship of the high Reynolds number expansions studied here for simple flow patterns to the corresponding expansion

for more general problems will now be discussed. The question of flow with separation is one of the main problems of high Reynolds number flow. The present paper has no bearing on this problem. The reason is that we have always implicitly assumed the limiting non-viscous flow to be known while the determination of this flow is a basic problem of separated flows. If we, however, restrict ourselves to flow without separation the present case differs from the general case only in two rather inessential ways.

Consider as an example two-dimensional incompressible steady flow without separation. Various considerations then indicate that the stream function ψ has an asymptotic expansion similar to Eq. (5-61):

$$\psi = \psi_L^{(m)} + \psi_T^{(m)} + o(\epsilon^m) \quad (6-5a)$$

where the longitudinal component is

$$\psi_L^{(m)} = \sum_{j=0}^m \nu^{j/2} B_j(z_1, z_2) \quad (6-5b)$$

and the transversal component is

$$\psi_T^{(m)} = \sum_{j=1}^m \nu^{j/2} C_j\left(z_1, \frac{z_2}{\sqrt{\nu}}\right) \quad (6-5c)$$

The dependence on the constants U and L is obvious from dimensional considerations and is not exhibited in the above equations. The coordinate system (z_1, z_2) should be chosen in such a way that z_2 vanishes on the boundary of the solid. Among the possible systems with this property certain preferred coordinates will extend the uniform validity of Eq. (6-5a) into as large a domain as possible. Complete uniformity may be impossible to achieve as discussed in Section 6-1. More generally, the expansion (6-5) may not be uniformly valid near sharp corners, even in the cases when no similarity exists. The transversal component is transcendentally small when $z_2 \neq 0$.

The first difference between the expansion (6-5) and the expansion (5-61) is that in the former case the longitudinal component is more complicated. However, for incompressible flow all the B_j represent potential flow. This may be seen by induction using an argument that generalizes the discussion in connection with Eq. (5-65). The equations for the longitudinal components do not involve the transversal components. This follows by the argument given in Section 5. Furthermore the first longitudinal component, the external flow, must obey the Euler equations. Since no separation occurs, the flow is potential. By insertion of the complete series representing the longitudinal part into the Navier-Stokes equations one then sees that the first component, being potential, contributes nothing to the term expressing the viscous volume force. By combining this argument with mathematical induction one sees that all components of the longitudinal field are potential. This reasoning fails for compressible fluids. For

compressible flow an irrotational velocity field may still contribute to a viscous volume force.

The second difference between the expansions (6-5) and (5-61) is that in the latter case the Prandtl limit may be formed using the "natural" coordinates t and $r - R$. In the former case it may be necessary to use a more complicated system (z_1, z_2) in order to extend the uniform validity to as large a domain as possible (some aspects of the role of the coordinate system are discussed in [2]).

6.3. Low Reynolds Numbers. The classical methods for finding low Reynolds number solutions will be reviewed below. Then the problem will be reexamined from the point of view of finding uniformly valid asymptotic expansions. The special problems studied in Sections 4 and 5 will be compared with more general problems. Analogies with the high Reynolds number case will also be discussed.

6.3.1. *Method of Stokes and Oseen.* The classical method of STOKES for finding solutions approximately valid for stationary flow at low Reynolds numbers can be described as follows: The Navier-Stokes equations are written in a non-dimensional form using Stokes variables which in this case should be defined by

$$\vec{q}^* = \frac{\vec{q}}{U}, \quad x_i^* = \frac{x_i}{L} \quad \text{and} \quad \mathcal{R}_e = \frac{UL}{\nu} \quad (6-6)$$

where

$\vec{q}$ = velocity vector

L = characteristic length of body

U = free stream velocity

A suitable non-dimensional pressure coefficient must also be defined. Then the flow problem depends on single parameter $\mathcal{R}_e$ which enters only in the momentum equation. In this equation the viscous and pressure forces are $O(1/\mathcal{R}_e)$ while the momentum transport $D\vec{q}/Dt$ is $O(1)$. For low $\mathcal{R}_e$ the momentum transport terms are then neglected and the Stokes equations result. For three-dimensional problems these equations can be solved (in principle) and all the boundary conditions satisfied. However attempts to extend the solution to higher $\mathcal{R}_e$ by iteration lead to difficulties. Furthermore in two-dimensional flows solutions to Stokes equations which satisfy both the boundary condition at the body surface and at infinity $\vec{q}(\infty) = U\vec{i}$ do not exist.

The basic reason for these difficulties was pointed out by OSEEN. He showed that at large distances the neglected transport terms are not uniformly small compared to the viscous or pressure forces. As a remedy OSEEN proposed adding a linearized momentum transport $U(\partial\vec{q}/\partial x_1)$ to the momentum equation which should improve the approximation near infinity without affecting it near the body. The resulting Oseen equations can be solved for two dimensional flows including the boundary condition at infinity. For three-dimensional problems the

Oseen solution has an expansion for low Re the first term of which is the Stokes solution. It is plausible that, in two or three dimensions, the leading term of an expansion of the Oseen solution is also the leading term of a uniformly valid expansion of the exact solution of the Navier-Stokes equations. This leads to the question of what the form of the latter expansion is, how many terms are correctly given by the Oseen (or Stokes) solution and how higher order terms are to be obtained.

6.3.2. *Method of Kaplun.* The problem just posed has been greatly clarified by the work of KAPLUN on expansion for low Reynolds numbers. The discussion below will make extensive use of some ideas due to KAPLUN. However, only some aspects of the problem will be touched upon and full explanations cannot be given here. For a more complete treatment the reader is referred to [3].

From the point of view of expansions, the difficulty inherent in the Stokes equation is that the exact solution does not have a uniformly valid Stokes expansion for low Re . The Stokes limit is defined as the limit where the Stokes variables (Eq. (6-6)) are kept fixed while Re tends to zero. A repeated application of the Stokes limit yields an expansion

$$\vec{q}^* \cong \sum \epsilon_j(Re) \vec{q}_j(\vec{x}^*) \quad (6-7)$$

While such an expansion may be uniformly valid in a finite region enclosing the body, it will fail in the neighborhood of infinity. One may then ask whether it is possible to treat this non-uniformity by methods similar to those used in boundary layer theory. However, as pointed out by KAPLUN, it is advantageous first to transfer the non-uniformity to the boundary by considering the Oseen limit and the corresponding Oseen expansion.

We define the Oseen limit (cf. Eq. (3-11)) as the limit of $\vec{q}^*$ as Re tends to zero while the Oseen coordinates $\tilde{x}_i$ are fixed. Here

$$\tilde{x}_i = \frac{Ux_i}{\nu} = Re x_i^* \quad (6-8)$$

To fix the ideas this limit process may be regarded as the limit where U , ν and x_i are fixed while the characteristic length L tends to zero. The finite body then shrinks to a point which we take to be the origin $x_i = 0$. Since a point cannot create a finite disturbance one expects the Oseen limit of $\vec{q}^*$ to be $U\vec{i}$ at any point except the origin. The limit is, however, non-uniform at the origin since $\vec{q} = 0$ on the body surface. Thus the Oseen limit is analogous to external (Euler) limit in the high Reynolds number case.

To study the non-uniformity at the origin one may then look for an internal limit which is the analogue of the Prandtl limit. We introduce variables

$$\bar{x}_i = \tilde{x}_i Re^{-a} = x_i^* Re^{1-a}, \quad a > 0 \quad (6-9)$$

For each value of a we then define the corresponding limit obtained as Re tends to zero with $\bar{x}_i$ kept fixed. The case $a = 0$ corresponds to the Oseen limit. The

inner limit is thus obtained by magnifying the $\tilde{x}_i$ -coordinate by the factor $\mathcal{R}_e^{-a}$. (Note that since a point has no preferred normal direction all coordinates are magnified in the same ratio, whereas in the Prandtl limit only the coordinate in the normal direction is magnified.) To fix the ideas we let $\mathcal{R}_e$ tend to zero by keeping L and U fixed while ν tends to infinity. Thus if $\tilde{x}_i$ is fixed, x_i tends to infinity as ν^{1-a} . In the case $a = 0$ it was seen above that $\vec{q}^*$ tends to $\vec{i}$. For $a = 1$, which is the Stokes limit as defined above, the x_i are fixed and since the influence of the wall increases indefinitely with ν the Stokes limit of $\vec{q}^*$ must be zero as defined above, the x_i are fixed. The influence of the wall at a fixed point in the fluid increases indefinitely with ν . In the two-dimensional case the analogy with steady heat flow shows that $\vec{q}^*$ then tends to zero at every point. As pointed out above this limit is, however, non-uniform at infinity.

We shall now show that the Stokes limit ($a = 1$) is actually the proper inner limit. An example of an inner solution will also be found. The methods will be illustrated with the aid of a special example, namely flow past a circular cylinder defined by $x_1^2 + x_2^2 = L^2$. The free stream velocity is assumed to be $U\vec{i}$. In looking for an inner solution we first observe that for $a > 0$ the limits of the Navier-Stokes equations, keeping $\tilde{x}_i$ fixed, are the Stokes equations. A solution of these equations should then be found. An inner solution should also satisfy the correct boundary conditions at the cylinder. It is then not possible to satisfy the condition that the velocity at infinity be $U\vec{i}$ (note this is not only the correct boundary condition at infinity but also the value of the outer limit arbitrarily near the cylinder). Since a straightforward matching of outer and inner solutions is not possible, we require instead that the velocity given by the Stokes solution should become large at as slow a rate as possible as infinity is approached. It can be shown that this determines the solution uniquely within an arbitrary factor $\epsilon(\mathcal{R}_e)$. For future reference we write the solution using the various coordinates introduced. It should be remembered that the three equations below describe identically the same solution:

$$\tilde{\psi}_S = \mathcal{R}_e^{1-a} \psi^* = \epsilon(\mathcal{R}_e) \left[\tilde{x}_2 \log (\tilde{r} \mathcal{R}_e^{a-1}) + \frac{\tilde{x}_2}{2} (\mathcal{R}_e^{2(1-a)} \tilde{r}^{-2} - 1) \right] \quad (6-10a)$$

$$\psi_S^* = \frac{\psi_S}{UL} = \epsilon(\mathcal{R}_e) \left\{ x_2^* \log r^* + \frac{x_2^*}{2} [(r^*)^{-2} - 1] \right\} \quad (6-10b)$$

$$\tilde{\psi}_S = \frac{\psi_S}{\nu} = \mathcal{R}_e \psi^* = \epsilon(\mathcal{R}_e) \left[\tilde{x}_2 \log (\tilde{r} \mathcal{R}_e^{-1}) + \frac{\tilde{x}_2}{2} (\mathcal{R}_e^2 \tilde{r}^{-2} - 1) \right] \quad (6-10c)$$

Here ψ_S is the stream function of the Stokes solution. Note that as $\mathcal{R}_e$ tends to zero, x_i fixed, one expects $\vec{q}^*$ but not necessarily ψ^* to tend to a finite limit. On applying the limit ψ/U should be magnified by the same factor as x_i . We shall now show that the inner solution, represented by Eq. (6-10), may be matched with the outer solution, if $\epsilon(\mathcal{R}_e)$ is chosen properly. The first term of the outer solution is the Oseen limit of the exact solution. Its corresponding velocity field

was shown above to be constant and equal to $U\tilde{z}$. In terms of the stream function ψ this result may be written

$$\lim_{\mathcal{O}} \tilde{\psi} = \tilde{x}_2 \quad (6-11)$$

where $\tilde{\psi}$ is the Oseen form of the ψ , namely, ψ/ν . We now require that the inner solution contain sufficient information about the outer solution in the sense that the Oseen limit of the two solutions agree, *i.e.*

$$\lim_{\mathcal{O}} \tilde{\psi}_s = \lim_{\mathcal{O}} \tilde{\psi} = \tilde{x}_2 \quad (6-12)$$

where $\tilde{\psi}_s$ is the inner solution written in Oseen coordinates (Eq. (6-10c)). It is easily seen that Eq. (6-12) holds if and only if

$$\epsilon(\mathcal{R}_e) = \frac{-1}{\log \mathcal{R}_e} + o\left(\frac{1}{\log \mathcal{R}_e}\right) \quad (6-13)$$

This completes the determination of the inner solution.

To find the limit process which gives the correct inner limit we observe that since the correct outer limit (Oseen limit) of ψ can be obtained from the inner solution ψ_s it should *a fortiori* be possible to find the correct form of the other limits with $a > 0$ from ψ_s . Eq. (6-10a) then shows that the limit of $\bar{q}^*$ keeping $\tilde{x}_i$ fixed is $(1 - a)\tilde{z}$, as long as $(\tilde{x}_i)$ is not a point on the surface of the cylinder. This limit is then non-uniform at the cylinder except when $a = 1$. However, we require an inner limit to be uniform near the body. Hence the correct choice is $a = 1$, *i.e.* the Stokes limit is the correct inner limit. Eqs. (6-10b) and (6-13) then give the first non-vanishing term of the Stokes expansion.

Thus far we have only obtained an inner solution which is valid near the solid and which in particular may be used to find the leading term of an expansion of the exact skin friction formula. In this sense we have found the low $\mathcal{R}_e$ analogue of the boundary layer solution. Pursuing the analogy with the high $\mathcal{R}_e$ case we expect that a uniformly valid asymptotic expansion of the exact solution may be found as a combination of an Oseen series and a Stokes series (*cf.* Eq. (6-5)):

$$\tilde{\psi} \cong \sum \epsilon_j(\mathcal{R}_e) A_j(\tilde{x}_1, \tilde{x}_2) + \mathcal{R}_e \sum \epsilon_j(\mathcal{R}_e) B_j(x_1^*, x_2^*) \quad (6-14)$$

In the examples studied in Section 5 it was shown that the Stokes part of the expansion corresponding to Eq. (6-14) could be omitted. The Oseen expansion was by itself uniformly valid in the sense described in Section 5.1.1. This is, however, not always the case. In the present case the reason for the necessity of a Stokes part may be indicated as follows. Eq. (6-10c) will agree with the beginning of the Stokes expansion for small values of $\tilde{r}$. However, the term

$$\frac{1}{2} \epsilon(\mathcal{R}_e) \tilde{x}_2^2 \mathcal{R}_e^2 / \tilde{r}^2$$

will be transcendentally small and hence not appear in the Oseen expansion. It is negligible everywhere except at the boundary ($\tilde{r} = \mathcal{R}_e$) where it is necessary in

order for the boundary conditions to be satisfied. This term will then instead be supplied by the Stokes part of the expansion where it appears as $\frac{1}{2}\epsilon(R)x_2^*/r^{*2}$ and hence is of a non-transcendental order.

We shall not discuss further the determination of the terms in the expansion (6-14). It will only be remarked that since the leading term of the Oseen expansion is $\tilde{x}_2$, (corresponding to free-stream flow) some of the subsequent terms will satisfy the Oseen equation. The Navier-Stokes equations are invariant under the Oseen limit. The Oseen equations make their appearance only when the expansion (6-14) is substituted into the Navier-Stokes equation (*cf.* the discussion of Eq. (5-4)). It is believed that this explanation (differing from OSEEN's own explanation) constitutes a correct method of justifying the Oseen equations.

6.3.3. Sliding Infinite Bodies. The ideas discussed have their obvious analogue for sliding infinite bodies and have in fact been used and illustrated previously in Sections 4 and 5. Under the Oseen limit the cylinder shrinks to a line which has no "pulling power". The Oseen limit is therefore non-uniform at the cylinder. Uniformity may, however, be restored by taking sufficiently many terms of the Oseen expansion without adding a Stokes series (*cf.* Eqs. (5-1), (5-2), and (6-4)). The Stokes limit may, however, be used for finding the skin friction (*cf.* Sections 4.1.3, and 4.2.2 and Eqs. (5-23) *ff.*). Note the special definition of the Stokes limit used for the expanding infinite body (Eq. (3-6)).

A further discussion of sliding infinite cylinders will be given in Section 6.4.

6.3.4. Stationary Flow Past Semi-infinite Bodies. Some of the ideas discussed for finite bodies in Section 6.3.2 will now be generalized to semi-infinite bodies. Let such a body have a characteristic length L . If the Oseen limit is carried out by letting L tend to zero while keeping U and ν fixed the body will shrink to a body without a characteristic length, that is to a conical object. For a finite body this limiting cone is a point whereas a semi-infinite body may shrink to a semi-infinite needle, a semi-infinite plane, a circular cone or some other conical object. The question of the Oseen limit for flow past the body, then, depends on the "arresting power" of the limiting cone. The Oseen limit of the velocity is different from the free stream velocity, if the limiting cone exerts a viscous force strong enough to cause a disturbance in the fluid. This requires that the skin friction per unit length must be greater than zero. This requirement is fulfilled except for the point and the needle. Consider, for example, the case when the limiting cone is a semi-infinite needle parallel to the free stream. For L very small the maximal radius r at a given cross sectional station will be of order L . The area per unit length is $\sim r$ and to obtain a finite skin friction per unit length one must have $\partial u/\partial r \sim 1/r$. This means that $u \sim \log r$ for r small. This however, is inconsistent with the requirement for the limiting needle u will vanish and be differentiable at $r = 0$. When the Oseen limit differs from the free stream velocity the leading term in Eq. (6-14) is not $U\tilde{i}$ and hence the Oseen equation is not valid for any of the subsequent terms.

As an example consider a flow past a paraboloid of revolution where L = radius of curvature at the nose. In the Oseen limit the paraboloid shrinks to a semi-

infinite needle with no arresting power. The Oseen limit of the velocity is then the free stream velocity and hence the solution of the Oseen equation for flow past the paraboloid (which is easily obtained, *cf.* [3]) is a valid approximation for small values of L . Consider on the other hand a flow past a parabola (or rather a parabolic cylinder) and define L as before. The Oseen limit yields a semi-infinite plane which has enough arresting power to cause a disturbance in the flow. The Oseen limit, and hence the first term in the "Oseen expansion", is thus the (as yet unknown) flow past a semi-infinite plane. While the solution of the Oseen equation for flow past a parabola is easily obtained it has no obvious significance as an approximation to the solution of the Navier-Stokes equations.

The generalization of the above reasoning to other objects is simple. We note only that if the original body is say a circular cone or a wedge, one might consider the limit as the opening angle tends to zero. (*Cf.* the analogy between $\mathcal{R}_e$ and β of Eq. (2-6) used in preceding sections.) One then sees that Oseen flow past the circular cone has physical significance as an approximation whereas Oseen flow past a wedge has not.

On the other hand the case of flow past a semi-infinite body has some obvious similarities with the case of sliding infinite bodies. A Stokes limit may be formed by magnifying only coordinates normal to free-stream direction. Furthermore, the uniform validity of the low Reynolds number approximation may be extended in the free-stream direction by the use of coordinates analogous to ξ and η .

6.4. Sliding Circular Cylinders with General Expansion Rate. We shall now indicate how the concepts and the analytical methods used in previous sections may be applied to the case when the cylinder radius is a general function of time. Thus we consider the problem formulated at the beginning of Section 2 without the specializing assumption $R = At^n$ (Eq. (2-4)).

If a length L is defined from the function $R(t)$, one may introduce r^* , $\tilde{r}$, ξ , η *etc.* as in Section 3. The following notation will also be used:

$$R^* = f(t^*) \quad (6-15)$$

and

$$h = \frac{t^* f'(t^*)}{f(t^*)} \quad (6-16)$$

Note that if $R^* = (t^*)^n$ then $h = n$. In the general case h is not a constant but a function of t^* , or, equivalently, a function of $\eta^* = \eta/\sqrt{\mathcal{R}_e}$.

As an example of an equation for u^* in the general case we give the generalization of Eq. (3-32):

$$u_{\xi\xi}^* + \left[2\xi + \frac{1}{\xi} (1 - 4h(t^*)\eta^2) \right] u_{\xi}^* + [2 - 4h(t^*)]\eta u_{\eta}^* = 0 \quad (6-17)$$

Here ξ and η are defined by Eq. (3-30) (except that the versions of Eq. (3-30b) which depend on the assumption $R^* = (t^*)^n$ are of course no longer valid).

The following special case will occasionally be used as an illustrative example below:

$$R^* = f(t^*) = \frac{\sqrt{t^*}}{1 + \sqrt{t^*}} \quad (6-18)$$

In this case

$$\eta = \frac{\sqrt{\mathcal{R}_e}}{2(1 + \sqrt{t^*})} \quad (6-19a)$$

and

$$0 \leq h = \frac{1}{2(1 + \sqrt{t^*})} = \eta^* = \frac{\eta}{\sqrt{\mathcal{R}_e}} \leq \frac{1}{2} \quad (6-19b)$$

Initially the cylinder then expands at a parabolic rate ($h = \frac{1}{2}$). For large values of time R^* approaches the constant value unity and h tends to zero. For $\mathcal{R}_e$ small, η is uniformly small. For $\mathcal{R}_e$ large η varies between $\frac{1}{2}\sqrt{\mathcal{R}_e}$ and zero.

6.4.1. *Small Reynolds Numbers.* We shall first consider the case when the cross sectional Reynolds number η is very small whereas $\mathcal{R}_e$ has an arbitrary value. A comparison with Section 5.2 leads us to expect that u^* has an asymptotic expansion of the form similar to Eq. (5-45),

$$u^* \cong \sum_{j=0}^{\infty} \epsilon_j(\eta) u_j(\xi; \mathcal{R}_e) \quad (6-20a)$$

where

$$\epsilon_j(\eta) = \left(\frac{-1}{\log \sqrt{\gamma_0 \eta}} \right)^j \quad (6-20b)$$

Because of the absence of similarity it has not been assumed *a priori* that u_j is independent of $\mathcal{R}_e$.

Now if $\mathcal{R}_e$ is fixed η can be very small only if t^* is very small or very large. (We disregard the uninteresting case when R^* vanishes for some finite non-zero values of t^*). Consider the second case. Again avoiding needless generality we then assume that for t^* large $f(t^*)$ has an expansion

$$f(t^*) = (t^*)^n (1 + A(t^*)^{-a} + \dots) \quad (6-21)$$

where

$$0 \leq n < \frac{1}{2}, \quad a > 0$$

(Cf. the concrete example given by Eq. (6-18)). Then

$$\eta = \frac{1}{2} \sqrt{\mathcal{R}_e} (t^*)^{n-\frac{1}{2}} (1 + \dots) \quad (6-22a)$$

and

$$h = n - nA(t^*)^{-a} + \cdots = n - nA \left(\frac{2\eta}{\sqrt{\mathcal{R}_e}} \right)^{a/(1-n)} \quad (6-22b)$$

Consider now Eq. (6-17). As in Section 5.1.2 we see that the transport term is transcendentally small because of the factor η^2 . It may hence be neglected in determining the expansion (6-20). Furthermore, if $n < \frac{1}{2}$ and $\mathcal{R}_e$ is fixed the second term (as well as subsequent terms) in the expansion for h (Eq. (6-22b)) will make a transcendentally small contribution to the term $-4h\eta u_\eta^*$. Hence one may replace h by its limiting value n . Then $\mathcal{R}_e$ disappears entirely from the equations for the u_j . Since the boundary conditions do not involve $\mathcal{R}_e$ either, the u_j are independent of $\mathcal{R}_e$ and identical with the functions found in Section 5.1.2. (Cf. Eqs. (5-47), (5-54), etc.)

The argument just presented means that the expansion (6-20) is valid only for values of t^* large enough so that the deviation of R^* from its limiting form $(t^*)^n$ is negligible. This statement is of course no longer valid if one proceeds with correction terms beyond the infinite series (6-20).

Similarly one sees that if the dominant term of $f(t^*)$ near $t^* = 0$ is $(t^*)^n$, $n > \frac{1}{2}$, then h may be replaced by n . For small values of t^* the η -expansion derived in Section 5.1.2 is then valid. Thus there may be cases when this expansion is valid both for very small and for very large values of time. The two expansions differ only by the value of n which must be $> \frac{1}{2}$ for small t^* and $< \frac{1}{2}$ for large t^* .

Consider now the special example given by Eq. (6-18). An η -expansion with $h = n = 0$ is always valid for sufficiently large times. However, if $\mathcal{R}_e$ is very small η is uniformly small ($\eta \leq \frac{1}{2}\sqrt{\mathcal{R}_e}$). In this case there should exist an expansion for small η which is uniformly valid for all t^* . Then the variation of h with t^* must necessarily be taken into account. Formally this is seen by the fact that if η is not small compared to $\sqrt{\mathcal{R}_e}$ the second term in Eq. (6-22b), as well as subsequent terms, are of order unity and no longer transcendentally small with regard to the ϵ_j defined by Eq. (6-20b).

When the variation of h is taken into account the calculations become considerably more complicated. One may assume an expansion of the form

$$u^* \cong \sum_{j=1}^{\infty} \epsilon_j(\eta) u_j(\xi, \eta^*) \quad (6-23)$$

The first term can be seen to be independent of h and hence of η^* , and hence agrees with the function obtained in Section 5.1.2 (Eqs. (5-47) and (5-49)). The result is thus

$$\epsilon_1(\eta) u_1(\xi, \eta^*) = \frac{-1}{\log \sqrt{\gamma_0} \eta} \frac{E_1(\xi^2)}{2} \quad (6-24)$$

The shape of the function R^* thus enters only indirectly by way of the definition of η .

The subsequent terms of Eq. (6-23) will, however, depend on η^* and satisfy partial rather than ordinary equations.

6.4.2. *Large Reynolds Number.* We shall now consider the case of large values of $\mathcal{R}_e$. As in Section 5.2 it is seen that u^* has only a transversal component, which may be expected to have a Prandtl expansion of the form

$$u^*(\bar{r}, t^*; \mathcal{R}_e) \cong \sum_{j=0}^{\infty} \left(\frac{1}{\sqrt{\mathcal{R}_e}} \right)^j u_j(\bar{r}, t^*) \quad (6-25)$$

The first term of this expansion then satisfies the boundary layer equation

$$u_{0,t^*} - \frac{\bar{r}f'(t^*)}{f(t^*)} u_{0,\bar{r}} = u_{0,\bar{r}\bar{r}} \quad (6-26)$$

with boundary conditions

$$u_0(0, t^*) = 1 \quad (6-27a)$$

$$u_0(\infty, t^*) = u_0(\bar{r}, 0) = 0 \quad (6-27b)$$

This equation is a generalization of Eq. (5-69a). Even in the general case it is possible to reduce it to an ordinary differential equation with an easy solution. By a change of variables one may reduce it to the heat equation. The similarity property of the problem then becomes obvious and one obtains a simple solution which is a function of one variable only. The following transformation is used:

$$r_1 = \bar{r}f \quad (6-28a)$$

$$t_1 = \int_0^{t^*} f^2(\sigma) d\sigma \quad (6-28b)$$

Eq. (6-26) then reduces to

$$u_{0,t_1} = u_{0,r_1 r_1} \quad (6-29)$$

The boundary conditions then give the similarity solution

$$u_0 = \operatorname{erfc} \frac{r_1}{2\sqrt{t_1}} = \operatorname{erfc} \left[\frac{\bar{r}f(t^*)}{2 \left(\int_0^{t^*} f^2(\sigma) d\sigma \right)^{\frac{1}{2}}} \right] \quad (6-30)$$

The skin friction coefficient is then

$$C_f = \frac{1}{\sqrt{\pi}} \left[\frac{f(t^*)}{\left(\int_0^{t^*} f^2(\sigma) d\sigma \right)^{\frac{1}{2}}} \right] + O \left(\frac{1}{\sqrt{\mathcal{R}_e}} \right) \quad (6-31)$$

The last two formulas include Eqs. (5-74) and (5-85) (first term only!) as special cases. Unfortunately, the corrections to the boundary layer solution cannot be worked out as simply as in Section 5; they will not be discussed here.

It was seen in Section 6.4.1 that ξ and η are useful parameters in the general low Reynolds case. Eq. (6-30) shows that this is no longer the case for high Reynolds numbers. The situation in this case may be analyzed in terms of the boundary layer thickness δ defined by

$$\frac{\delta}{L} = \frac{1}{L} \int_R^\infty u_0 dr = \text{constant} \times \frac{l(t^*)}{\sqrt{\mathcal{R}_e}} \quad (6-32a)$$

where

$$l(t^*) = \frac{2}{f(t^*)} \left(\int_0^{t^*} f^2(\sigma) d\sigma \right)^{\frac{1}{2}} \quad (6-32b)$$

Boundary layer theory should apply when the boundary layer thickness δ is small compared to R . It is then natural to consider the parameters

$$\xi_1 = \frac{r}{\delta}, \quad \eta_1 = \frac{R}{\delta} \quad (6-33)$$

When $f(t^*) = (t^*)^n$ then $l(t^*) = 2\sqrt{t^*}/\sqrt{2n+1}$ and ξ_1 and η_1 are within constants identical with ξ and η as defined by Eq. (3-30). In the general case u_0 is a function of $\xi_1 - \eta_1$ only as shown by Eq. (6-30). It is to be expected that u_0 is a good approximation to u^* only when η_1 is large. The approximation may be uniformly valid in time if f is such that $\mathcal{R}_e$ large makes η_1 uniformly large. For an arbitrary $\mathcal{R}_e$ u_0 may still be a good approximation for large or small t^* if η_1 becomes large.

Note that for $\mathcal{R}_e$ very small or $\mathcal{R}_e$ very large one may find a first approximation which is valid for all functions $f(t^*)$ (Eqs. (6-24) and (6-30)). Computational difficulties arise only when the correction terms are considered.

6.5. Concluding remarks. The analytical methods used in this paper can also be extended to problems other than those studied in Section 6.4. Some obvious generalizations are the problems of sliding cylinders of non-circular cross-section, and sliding rotating cylinders. A more complicated example is a sliding cylinder which also has a component of motion normal to its axis. Another related class of problems concerns steady flow past semi-infinite bodies. Although this last class of problems is still more complex, the analytical methods which should apply are in principle the same as used on the simple examples studied here. In particular if one studies conditions far downstream on a body of revolution some of the results in the present paper may be carried over directly (*cf.* [7]).

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* Part of the material in [1] is contained in two papers presented by G. E. LATTI at the VIII International Congress for Theoretical and Applied Mechanics (Istanbul, 1952).