*On the Perturbation of a Periodic Solution when the
Variational System has Non-trivial
Periodic Solutions**

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§1. Introduction. In a previous paper [5 of the Bibliography] a method was given for appraising the range of the parameter in the theory of perturbation of periodic solutions, originated by POINCARÉ. Only the simplest case was considered, in which the variational equations are assumed to have no periodic solutions not identically zero. This condition is rarely satisfied in connection with interesting and important applications. In this paper we present a theory designed to take care of what may be considered the general case in which the variational equations have one or more periodic solutions not identically zero. We again use the method of integral equations and successive approximations of LICHTENSTEIN and E. HÖLDER [4] but in a rather different setting. Namely, we use a Green's matrix (in the ordinary sense) dependent on the parameter μ and having a pole at $\mu = 0$, whereas HÖLDER uses a *generalized* Green's matrix independent of μ and then discusses the "bifurcation equations." In a third paper it is planned to treat in a general way those problems in which the use of a generalized Green's matrix in the manner of HÖLDER is either desirable or necessary. These, roughly speaking, are problems connected with systems having first integrals (valid for all μ) *which are not to be used to lower the order of the system*. In the problems susceptible to present methods preliminary use of all such first integrals to lower the order of the system should be carried out. Our methods are also available for problems having first integrals valid only for $\mu = 0$ or for problems having no first integrals at all.

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The main purpose of this paper is to estimate as explicitly as possible the size of the interval for the parameter μ in which existence of the periodic solution, given for $\mu = 0$, is known. The estimate embodied in Theorem 1 of §4 involves a knowledge of the Green's matrix for a certain linear system with periodic boundary conditions. This linear system depends analytically on μ and reduces to the variational system when $\mu = 0$. The Green's matrix thus has a pole at $\mu = 0$, which, because of a certain assumption, is of the first order. Our estimate is based primarily on the following items:

I. A bound for the norm of the above mentioned Green's matrix after multiplication by μ .

II. A bound on the periodic solution of a certain non-homogeneous linear system which for $\mu = 0$ reduces to a compatible non-homogeneous linear system in which the homogeneous terms are those of the variational system.

III. Bounds and Lipschitz constants for the non-linear terms, although in the light of the remarks of §5, the Lipschitz constants are not involved if the original system is analytic.

Our results clearly imply that the paper contains *a fortiori* a set of sufficient conditions for the existence of periodic solutions when $|\mu| > 0$ is merely assumed to be sufficiently small, but without explicitly estimating *how* small. This "local" problem has been solved by CODDINGTON & LEVINSON [2] in a different form but to approximately the same degree of generality as the present solution. The Coddington-Levinson solution, unlike the present involves the preliminary use of certain transformations so that, among other things, the matrix of coefficients of the variational equations is not only a constant matrix but is displayed in at least partially canonical form. The treatment is then along the lines of the implicit function theorem, in that certain equations are to be solved for certain of the initial values in terms of μ in such wise as to obtain periodicity. The set of sufficient conditions thus involves two essential elements:

(i) Certain equations are satisfied when $\mu = 0$.

(ii) The jacobian of the left members of these equations does not vanish at the appropriate point for which $\mu = 0$.

Conditions of the first type correspond, in the present treatment, to the compatibility of the non-homogeneous system mentioned above and also to the bound placed upon the quantity σ (*cf.* §§2, 4). Conditions of this type are seen to be necessary for the existence of periodic solutions for sufficiently small $|\mu| \neq 0$, tending to the given solution as $\mu \rightarrow 0$. It should be mentioned, however, that we do not guarantee that the bound for σ can not be liberalized.

Conditions of the second type in the Coddington-Levinson treatment correspond to our present assumptions leading to a first order pole for the Green's matrix at $\mu = 0$. The bound on σ is also involved here.

It is not known to what extent the sufficient conditions of the present paper are actually equivalent to those of CODDINGTON & LEVINSON, although in some of the simple cases the non-vanishing of the above mentioned jacobian seems to

be essentially equivalent to the appearance of a first order pole for the above mentioned Green's matrix.

The uniqueness questions discussed in §6 are still in a partially unsatisfactory state because of failure to treat of families of periodic solutions depending on positive fractional powers of μ . We hereby speak somewhat loosely by confining attention to the analytic case. These questions are bypassed in the CODDINGTON-LEVINSON paper, because of an assumed preliminary transformation of the problem in such manner that the non-linear terms are assumed to contain the factor μ .

A simple example to illustrate this situation is the following:

$$\begin{aligned}\frac{dx_1}{dt} &= -\mu x_1 + x_1^3 \\ \frac{dx_2}{dt} &= -\mu x_2 + \sigma x_2^2 + x_1^4 - \mu^2,\end{aligned}$$

in which the right hand sides are to be regarded as periodic in t with period 2π , say. This system possesses the following four families of periodic (constant) solutions, all reducing to the trivial null solution when $\mu = 0$:

- (a) $x_1 = 0, \quad x_2 = \frac{1}{2}\mu\sigma^{-1}(1 - (1 + 4\sigma)^{\frac{1}{2}}).$
- (b) $x_1 = 0, \quad x_2 = \frac{1}{2}\mu\sigma^{-1}(1 + (1 + 4\sigma)^{\frac{1}{2}}).$
- (c) $x_1 = \pm\mu^{\frac{1}{2}}, \quad x_2 = 0, \quad \mu \geq 0.$
- (d) $x_1 = \pm\mu^{\frac{1}{2}}, \quad x_2 = \mu\sigma^{-1}, \quad \mu \geq 0.$

Whereas solutions (a) and (b) are adequately covered by Theorems 1 and 2 respectively, at least for $|\sigma|$ sufficiently small, the solutions (c) and (d), involving fractional powers of μ , are not covered by our theory.

The work of FRIEDRICHS [3] takes a much more general point of view than either the present paper or that of CODDINGTON & LEVINSON. But, since he merely reduces the problem to the solution of the so-called "bifurcation equations," which pose extremely difficult questions in the general case, there is little basis for further comparison.

In §7, we indicate a general transformation which reduces the autonomous case to the non-autonomous case by the introduction of an angular variable. Although such an introduction of an angular variable is a familiar device and although the present transformation is very trivial, I have seen in the literature no mention of such a transformation to the same degree of generality and explicitness.

In our first paper [5], treating the simplest case, we proved some theorems indicating that in certain senses our estimate for the size of the μ -interval could not be improved. Since the same methods of appraisal are used in the present paper with only comparatively slight modifications, it may be suspected

that similar results could be established here. A number of complications, however, have prompted a postponement of any investigation of this interesting question.

§2. Motivation for assuming certain properties of the systems to be considered. The common starting point of this and the previous paper is a system of the following form,

$$(1) \quad \frac{dx_i}{dt} = \sum_{j=1}^n a_{ij}(t)x_j + F_i(x, \mu, t) \quad i = 1, \dots, n,$$

in which the $a_{ij}(t)$ are continuous periodic functions of t with period 2π and in which the functions $F_i(x, \mu, t)$ are also periodic in t with period 2π , are defined for $|\mu| \leq \lambda$ and for $\|x\| = \left[\sum_{i=1}^n x_i^2 \right]^{\frac{1}{2}} \leq r$. Moreover the F 's were assumed to be of class C' in this region and both the F 's and their first partial derivatives with respect to the x 's were assumed to vanish for $x_1 = \dots = x_n = \mu = 0$.

In the previous paper it was then assumed that *the homogeneous linear system* $dx_i/dt = \sum_{j=1}^n a_{ij}(t)x_j$ *had no solution, not identically zero, periodic with period* 2π . As a consequence of this hypothesis, we were enabled to set up the Green's matrix for periodic boundary conditions upon which the rest of the paper was based. In the present paper we discard this last assumption and replace it by some others, the introduction of which we now proceed to motivate.

In (1) the right hand members have already been analyzed into a portion linear in the $x_1, \dots, x_n$ and the residue, including what is linear in μ and non-linear in $x_1, \dots, x_n, \mu$. Evidently a further analysis is called for in the present instance. Actually it turns out to be desirable to carry this analysis up to terms of the second order. We accordingly assume that the F 's are of class C'' , and then set

$$\begin{aligned} \left(\frac{\partial F_i}{\partial \mu} \right)_{x_1=\dots=x_n=\mu=0} &= h_i(t) \\ \left(\frac{\partial^2 F_i}{\partial \mu^2} \right)_{x_1=\dots=x_n=\mu=0} &= S_i(t) \\ \left(\frac{\partial^2 F_i}{\partial x_j \partial \mu} \right)_{x_1=\dots=x_n=\mu=0} &= b_{ij}(t) \\ \left(\frac{\partial^2 F_i}{\partial x_j \partial x_k} \right)_{x_1=\dots=x_n=\mu=0} &= c_{ijk}(t) \end{aligned}$$

$$p_i(x, \mu, t) = F_i(x, \mu, t) - \mu \left[h_i(t) + \sum_{j=1}^n b_{ij}(t)x_j \right] - \frac{1}{2} \sum_{j,k=1}^n c_{ijk}(t)x_j x_k - \frac{1}{2}\mu^2 S_i(t).$$

From these definitions it is seen that $p_i(x, \mu, t)$ together with all of its partial derivatives of the first two orders must vanish when $x_1 = \dots = x_n = \mu = 0$. Moreover (1) can now be rewritten as follows:

$$(2) \quad \frac{dx_i}{dt} = \sum_{j=1}^n [a_{ij}(t) + \mu b_{ij}(t)]x_j + \mu h_i(t) + \frac{1}{2}\mu^2 S_i(t) \\ + \frac{1}{2} \sum_{j,k=1}^n c_{ijk}(t)x_j x_k + p_i(x, \mu, t)$$

where the a 's, b 's, h 's, S 's, c 's and p 's are periodic with period 2π in t .

Now it is clear from examples that a system of this form need not always possess periodic solutions for small values of μ different from 0, which approach the given null solution as μ tends to zero. One can for instance take $n = 1$ and consider the example,

$$\frac{dx}{dt} = -\mu x + \mu h(t) \text{ with the } a\text{'s, } S\text{'s, } c\text{'s and } P\text{'s all } \equiv 0.$$

The general solution of this is

$$x(t) = \mu \int_0^t h(s) e^{\mu(s-t)} ds + x(0) e^{-\mu t},$$

which is periodic with period 2π , if and only if

$$\mu \int_0^{2\pi} h(s) e^{\mu(s-2\pi)} ds + x(0) e^{-2\pi\mu} = x(0);$$

so that

$$x(0) = \frac{\mu}{1 - e^{-2\pi\mu}} \int_0^{2\pi} h(s) e^{\mu(s-2\pi)} ds$$

which approaches 0 as μ tends to 0, if and only if

$$(3) \quad \int_0^{2\pi} h(s) ds = 0.$$

To demand (3) is equivalent to demanding that the system $dx/dt = h(t)$ should have a periodic solution. It is therefore natural to demand, in the more general case (2), that the non-homogeneous linear system

$$\frac{dx_i}{dt} = \sum a_{ij}(t)x_j + h_i(t)$$

should have a periodic solution. The proper generalization of (3) turns out to be

$$(4) \quad \int_0^{2\pi} \sum_{i=1}^n \xi_i(s) h_i(s) ds = 0$$

where $\xi_1(t), \dots, \xi_n(t)$ constitutes the most general periodic solution of the adjoint system $d\xi_i/dt = -\sum_{j=1}^n \xi_j(t) a_{ji}(t)$.

Another enlightening example, in the case $n = 1$, is the following: $dx/dt = \mu x + \mu^2 + \sigma x^2$. This is an integrable Riccati equation, whose only periodic solutions of period 2π are the constant solutions obtained by solving the equation $\sigma x^2 + \mu x + \mu^2 = 0$, namely

$$x = \mu \frac{-1 \pm (1 - 4\sigma)^{\frac{1}{2}}}{2\sigma}.$$

Evidently, then, if $\sigma > \frac{1}{4}$, there are no real periodic solutions at all; and, if $\sigma < \frac{1}{4}$, there are two solutions. There is, however, for $\sigma < \frac{1}{4}$, only one periodic solution, which is continuous simultaneously in μ and σ in the neighborhood of $\mu = 0$, $\sigma = 0$. In order to retain a uniqueness theorem as well as an existence theorem, we therefore introduce a factor σ in connection with the $c_{ijk}(t)$ of (2) and consider in the first instance the existence and uniqueness of periodic solutions of (2) which are continuous in both μ and σ . Solutions not of this type are considered in §7. In other respects, we shall somewhat generalize the system (2) and write it in matrix notation as follows:

$$(5) \quad \frac{dx}{dt} = A(t, \mu)x + \mu h(t, \mu) + \sigma q(x, \mu, t) + p(x, \mu, t).$$

Here $A(t, \mu)$ is a continuous real $n \times n$ matrix function periodic in t with period 2π and analytic in μ for $|\mu| \leq \lambda$. In (2) the elements of $A(t, \mu)$ take, of course, the special form $a_{ij}(t) + \mu b_{ij}(t)$, being linear in μ .

$x = x(t)$ is the unknown vector function.

h , q , and p are also vectors depending periodically on t and of class C' in all their arguments for $|\mu| \leq \lambda$, $\|x\| \leq r$, and for all t .

h is to satisfy the condition,

$$(7) \quad \int_0^{2\pi} \Xi^*(s)h(s, 0) ds = 0,$$

where $\Xi^*(s)$ is a $k \times n$ matrix whose rows constitute a complete set of independent *periodic* solutions of the adjoint system

$$(8) \quad d\xi/ds = -\xi A(s, 0). \quad (k \leq n)$$

Concerning the matrix, $A(t, \mu)$, we shall furthermore assume that an $n \times n$ matrix solution $X(t, \mu)$, depending analytically on μ , of the homogeneous system,

$$(9) \quad \frac{dx}{dt} = A(t, \mu)x,$$

such that $\det X(t, \mu) \neq 0$ (even for $\mu = 0$), also has the property that $\det [X(0, \mu) - X(2\pi, \mu)]$ does not vanish, except at $\mu = 0$, for $|\mu| \leq \lambda$. Moreover, we shall assume that

$$(10) \quad [X(0, \mu) - X(2\pi, \mu)]^{-1} = \mu^{-1}M + N(\mu) = \Lambda(\mu) \quad (\mu \neq 0),$$

where M is a constant $n \times n$ matrix, while $N(\mu)$ is an $n \times n$ matrix whose elements are analytic for $|\mu| \leq \lambda$. We remark parenthetically that the case $M = 0$ need not be excluded. If, however, $M = 0$, $\det [X(0, \mu) - X(2\pi, \mu)] \neq 0$, even at $\mu = 0$; the number of non-trivial periodic solutions of the variational equations, and hence of the adjoint system, is zero; condition (7) becomes vacuous, since $k = 0$; and we are thrown back to the simpler case treated by a slightly different and more suitable method in the previous paper [5].

Regarding q and p we shall assume that

$$(11) \quad \begin{aligned} q(y\mu, \mu, t) &= \mu^2 Q(y, \mu, t) \\ p(y\mu, \mu, t) &= \mu^3 P(y, \mu, t) \end{aligned}$$

where Q and P are both vector functions of class C' in the indicated arguments, at least, for $\|y\| \leq r\lambda^{-1}$ and $|\mu| \leq \lambda$. They are, of course, periodic with period 2π in t , because q and p are.

§3. Lemmas on linear non-homogeneous systems. Before proceeding with the precise statement and proof of the main theorem it is convenient to establish a couple of lemmas.

Lemma 1. *Let the homogeneous system (9) have an $n \times n$ matrix solution $X(t, \mu)$ with the properties specified above including the validity of (10). Then, for $0 < |\mu| \leq \lambda$, the non-homogeneous system, $dx/dt = A(t, \mu)x + f(t, \mu)$, where $f(t, \mu)$ is a continuous vector function periodic in t with period 2π , admits a unique periodic solution, which, for $0 \leq t \leq 2\pi$, can be written in the form $\int_0^{2\pi} G(t, s; \mu) f(s, \mu) ds$, where the "Green's matrix," $G(t, s; \mu)$ is independent of f , is continuous in all its arguments except at $t = s$, and is such that*

$$(12) \quad y(t, \mu) = \int_0^{2\pi} G(t, s; \mu) h(s, \mu) ds$$

remains finite as $\mu \rightarrow 0$, provided that (7) is satisfied by the vector function $h(t, \mu)$ of class C' . Moreover the matrix function

$$(13) \quad R(t, s; \mu) = \mu G(t, s; \mu)$$

is bounded, the singularity at $\mu = 0$ being actually removable.

Proof: Using the notation already introduced, we know that the non-homogeneous system $dx/dt = A(t, \mu)x + f(t, \mu)$ has the general solution

$$x(t, \mu) = X(t, \mu)c + \int_0^t X(t, \mu)X(s, \mu)^{-1}f(s, \mu) ds,$$

which comes from the Lagrange method of "variation of parameters." For periodicity we must have $x(0, \mu) = x(2\pi, \mu)$ and hence $c = c(\mu)$ must satisfy

$$[X(0, \mu) - X(2\pi, \mu)]c(\mu) = \int_0^{2\pi} X(2\pi, \mu)X(s, \mu)^{-1}f(s, \mu) ds.$$

We find, therefore, from (10) that

$$c = \int_0^{2\pi} \Lambda(\mu) X(2\pi, \mu) X(s, \mu)^{-1} f(s, \mu) \, ds.$$

Inserting this into the above expression for $x(t, \mu)$, we obtain

$$x(t, \mu) = \int_0^{2\pi} G(t, s; \mu) f(s, \mu) \, ds$$

where

$$\begin{aligned} G(t, s; \mu) &= X(t, \mu) \Lambda(\mu) X(2\pi, \mu) X(s, \mu)^{-1} + X(t, \mu) X(s, \mu)^{-1} && \text{for } s < t. \\ &= X(t, \mu) \Lambda(\mu) X(2\pi, \mu) X(s, \mu)^{-1} && \text{for } t \leq s. \end{aligned}$$

Remembering that $\Lambda(\mu) = \mu^{-1} M + N(\mu)$, we find that

$$x(t, \mu) = \mu^{-1} \int_0^{2\pi} X(t, \mu) M X(2\pi, \mu) X(s, \mu)^{-1} f(s, \mu) \, ds + \dots$$

where the omitted terms are finite as $\mu \rightarrow 0$. Thus $x(t, \mu)$ remains finite if and only if

$$(14) \quad \int_0^{2\pi} M X(2\pi, 0) X(s, 0)^{-1} f(s, 0) \, ds = 0,$$

at least, if $f(s, \mu)$ is of class C' . From (10) we have

$$M\mu^{-1}[X(0, \mu) - X(2\pi, \mu)] + N(\mu)[X(0, \mu) - X(2\pi, \mu)] = I,$$

where I is the unit $n \times n$ matrix. Multiplying by μ and then letting $\mu \rightarrow 0$, we find that $MX(0, 0) = MX(2\pi, 0)$. Hence $MX(2\pi, 0)X(0, 0)^{-1} = M = MX(2\pi, 0)X(2\pi, 0)^{-1}$. Hence $MX(2\pi, 0)X(s, 0)^{-1}$ is periodic in s with period 2π . Moreover the rows of $X(s, 0)^{-1}$, and hence the rows of $MX(2\pi, 0)X(s, 0)^{-1}$, satisfy the adjoint system (8). For putting $\Xi(s) = X(s)^{-1}$, so that $\Xi X = I$ and $\dot{\Xi}X + \Xi\dot{X} = 0$, we recall that $\dot{X}(s) = A(s)X(s)$, where the dot denotes differentiation with respect to s . Hence $\dot{\Xi}X + \Xi AX = 0$. Multiplication on the right by X^{-1} shows at once that the adjoint system is satisfied as stated. Thus, the condition (14) on $f(s, \mu)$ will actually be satisfied if $f(s, \mu) = h(s, \mu)$ in virtue of (7). It is therefore clear that the function $y(t, \mu)$ defined by (12) remains finite as $\mu \rightarrow 0$, as we wished to prove.

It is also clear from the above explicit expression for $G(t, s; \mu)$ that the function $R(t, s; \mu) = \mu G(t, s; \mu)$ is bounded.

Lemma 2. *There exists an integrable bounded function $\Gamma(t, \mu)$ dependent on R but independent of f , such that*

$$\left\| \int_0^{2\pi} R(t, s; \mu) f(s) \, ds \right\|^2 \leq \Gamma(t, \mu) \int_0^{2\pi} \|f(s)\|^2 \, ds.$$

In fact, it is easily seen from Schwarz's inequality that we can always take $\Gamma(t, \mu) = \int_0^{2\pi} \|R(t, s; \mu)\|^2 ds$, although there are cases when $\Gamma(t, \mu)$ can be taken still smaller (cf. Lemmas 3 and 4 of the previous paper). Here it should be noted that $\|a\|^2$ stands for the sum of the squares of the components or elements of a , according as to whether a is a vector or a matrix.

We find it convenient at this point to introduce positive numbers β and M as bounds for $\|y(t, \mu)\|$ and $\Gamma(t, \mu)$ respectively and to record formally that

$$(15) \quad \|y(t, \mu)\| \leq \beta \quad \text{for } |\mu| \leq \lambda$$

$$(16) \quad \Gamma(t, \mu) \leq M \quad \text{for } 0 \leq t \leq 2\pi \text{ and } |\mu| \leq \lambda.$$

§4. The main theorem. Q and P , being of class C' in a closed region, are both bounded and satisfy Lipschitz conditions. We therefore write

$$(17) \quad \begin{aligned} \|Q(y, \mu, t)\| &\leq Q^* & \|P(y, t, \mu)\| &\leq P^* \\ \|Q(y', \mu, t) - Q(y, \mu, t)\| &\leq Q^{**} \|y' - y\| \\ \|P(y', \mu, t) - P(y, \mu, t)\| &\leq P^{**} \|y' - y\| \end{aligned}$$

as long as $\|y\|, \|y'\| \leq r\lambda^{-1}$ and $|\mu| \leq \lambda$. Here P^*, Q^*, P^{**}, Q^{**} are suitably chosen constants. We are now ready to state our main result as follows:

Theorem 1. Assume that $\beta < r\lambda^{-1}$, so that the values of the vector $y(t, \mu)$ (cf. §3, (15)) are within the region of definition of P and Q . Let θ be any positive number less than 1. Next choose positive numbers α, γ , and L satisfying the following inequalities

$$(18) \quad \begin{aligned} \alpha + \beta &\leq r\lambda^{-1} \\ L &\leq \lambda \\ (2\pi M)^{\frac{1}{2}}(\gamma Q^* + LP^*) &\leq \alpha, \\ (19) \quad (2\pi M)^{\frac{1}{2}}(\gamma Q^{**} + LP^{**}) &\leq \theta. \end{aligned}$$

Then there exists a unique continuous vector function $x(t, \mu, \sigma)$ defined for all t , for $|\mu| \leq L$, and for $|\sigma| \leq \gamma$, which is periodic with period 2π in t , which satisfies (5), is such that $\|x(t, \mu, \sigma)\| \leq |\mu| r\lambda^{-1} \leq r$, and reduces to the trivial null solution when $\mu = 0$.

Proof: Restricting ourselves temporarily by requiring that $\mu \neq 0$, we introduce the new unknown vector $z = \mu^{-1}x$. The system (5), with the help of (11) then appears in the form,

$$(20) \quad \frac{dz}{dt} = A(t, \mu)z + h(t, \mu) + \sigma\mu Q(z, \mu, t) + \mu^2 P(z, \mu, t).$$

By Lemma 1, any periodic solution of this for $\mu \neq 0$ must satisfy the system of integral equations

$$z(t, \mu, \sigma) = \int_0^{2\pi} G(t, s; \mu)[h(s, \mu) + \sigma \mu Q(z(s, \mu, \sigma), \mu, s) + \mu^2 P(z(s, \mu, \sigma), \mu, s)] ds,$$

which by (12) and (13) can also be written in the form

$$z(t, \mu, \sigma) = y(t, \mu) + \int_0^{2\pi} R(t, s; \mu)[\sigma Q(z(s, \mu, \sigma), \mu, s) + \mu P(z(s, \mu, \sigma), \mu, s)] ds.$$

We set up formally the system of successive approximations

$$\begin{aligned} z^0(t, \mu, \sigma) &= y(t, \mu) \\ &\vdots \\ z^k(t, \mu, \sigma) &= y(t, \mu) \\ &\quad + \int_0^{2\pi} R(t, s; \mu)[\sigma Q(z^{k-1}(s, \mu, \sigma), \mu, s) + \mu P(z^{k-1}(s, \mu, \sigma), \mu, s)] ds \\ &\hspace{15em} k = 1, 2, \dots \end{aligned}$$

Evidently $z^0(t, \mu, \sigma)$ is effectively defined by these formulas and satisfies the inequality, $\|z^0\| = \|y(t, \mu)\| \leq \beta < r\lambda^{-1}$ by (15). We now assume inductively that $z^{k-1}(t, \mu, \sigma)$ is effectively defined and that

$$\|z^{k-1}(t, \mu, \sigma)\| < r\lambda^{-1} \quad \text{for } 0 \leq t \leq 2\pi, \quad |\mu| \leq L, \quad |\sigma| \leq \gamma.$$

Then by Lemma 2, (16), (17), and (18),

$$\begin{aligned} &\|z^k(t, \mu, \sigma) - y(t, \mu)\|^2 \\ &\leq \Gamma(t) \int_0^{2\pi} \|\sigma Q(z^{k-1}(s, \mu, \sigma), \mu, s) + \mu P(z^{k-1}(s, \mu, \sigma), \mu, s)\|^2 ds \\ &\leq 2\pi M(|\sigma|Q^* + |\mu|P^*)^2 \leq \alpha^2 \quad \text{if } |\sigma| \leq \gamma \quad \text{and } |\mu| \leq L. \end{aligned}$$

Hence $\|z^k(t, \mu, \sigma)\| \leq \|z^k(t, \mu, \sigma) - y(t, \mu)\| + \|y(t, \mu)\| \leq \alpha + \beta \leq r\lambda^{-1}$, since $\alpha \leq r\lambda^{-1} - \beta$. Thus all the successive approximations are effectively defined with values within the region of definition of P and Q .

To prove the uniform convergence as $k \rightarrow \infty$ of $z^k(t, \mu, \sigma)$, we first write

$$\begin{aligned} &z^{k+1}(t, \mu, \sigma) - z^k(t, \mu, \sigma) \\ &= \int_0^{2\pi} R(t, s; \mu)[S(z^k(s, \mu, \sigma), \mu, \sigma, s) - S(z^{k-1}(s, \mu, \sigma), \mu, \sigma, s)] ds \end{aligned}$$

where

$$S(z, \mu, \sigma, s) = \sigma Q(z, \mu, s) + \mu P(z, \mu, s).$$

From the triangle inequality and (17), we know that

$$\|S(z', \mu, \sigma, s) - S(z, \mu, \sigma, s)\| \leq (|\sigma| Q^{**} + |\mu| P^{**}) \|z' - z\|.$$

Hence with the help of Lemma 2, we obtain

$$\begin{aligned} & \|z^{k+1}(t, \mu, \sigma) - z^k(t, \mu, \sigma)\|^2 \\ & \leq \Gamma(t, \mu) (|\sigma| Q^{**} + |\mu| P^{**})^2 \int_0^{2\pi} \|z^k(s, \mu, \sigma) - z^{k-1}(s, \mu, \sigma)\|^2 ds \\ & \leq 2\pi M (|\sigma| Q^{**} + |\mu| P^{**})^2 \max \|z^k(s, \mu, \sigma) - z^{k-1}(s, \mu, \sigma)\|^2. \end{aligned}$$

Thus (19) yields

$$\max \|z^{k+1}(t, \mu, \sigma) - z^k(t, \mu, \sigma)\| \leq \theta \max \|z^k(t, \mu, \sigma) - z^{k-1}(t, \mu, \sigma)\|,$$

from which we establish by induction that

$$\|z^{k+1}(t, \mu, \sigma) - z^k(t, \mu, \sigma)\| \leq \theta^k \max \|z^1(t, \mu, \sigma) - z^0(t, \mu, \sigma)\|.$$

Hence

$$\begin{aligned} & \|z^k(t, \mu, \sigma) - z^h(t, \mu, \sigma)\| \\ & \leq \|z^k(t, \mu, \sigma) - z^{k-1}(t, \mu, \sigma)\| + \|z^{k-1}(t, \mu, \sigma) - z^{k-2}(t, \mu, \sigma)\| \\ & \quad + \dots + \|z^{h+1}(t, \mu, \sigma) - z^h(t, \mu, \sigma)\| \\ & \leq \text{const.} (\theta^h + \theta^{h+1} + \dots + \theta^{k-1}) = \text{const.} \frac{\theta^h - \theta^k}{1 - \theta}, \end{aligned}$$

where h and k are any two positive integers with $k > h$. Since $0 < \theta < 1$, we see at once that we do have convergence of our successive approximations, which is uniform with respect to t, μ , and σ , for $0 \leq t \leq 2\pi$, $|\mu| \leq L$ and for $|\sigma| \leq \gamma$.

Let us set $z(t, \mu, \sigma) = \lim_{k \rightarrow \infty} z^k(t, \mu, \sigma)$. From the uniform continuity of P and Q and from the just established uniform convergence of the $z^k(t, \mu, \sigma)$, we have

$$\lim_{k \rightarrow \infty} P(z^k(t, \mu, \sigma), \mu, t) = P(z(t, \mu, \sigma), \mu, t) \text{ uniformly,}$$

with a similar statement about Q . Thus in the formula defining $z^k(t, \mu, \sigma)$ we can pass to the limit under the integral sign, thus showing that

$$\begin{aligned} (21) \quad & z(t, \mu, \sigma) = y(t, \mu) \\ & + \int_0^{2\pi} R(t, s; \mu) [\sigma Q(z(s, \mu, \sigma), \mu, s) + \mu P(z(s, \mu, \sigma), \mu, s)] ds. \end{aligned}$$

Hence from Lemma 1, $z(t, \mu, \sigma)$ is a periodic solution of (20) for $0 < |\mu| \leq L$ and for $|\sigma| \leq \gamma$ such that $\|z(t, \mu, \sigma)\| \leq r\lambda^{-1}$. Hence $x(t, \mu, \sigma) = \mu z(t, \mu, \sigma)$ is a periodic solution of (5) such that $\|x(t, \mu, \sigma)\| \leq |\mu| r\lambda^{-1} \leq r$ and such that $\lim_{\mu \rightarrow 0} x(t, \mu, \sigma) = 0$, as desired.

The uniqueness of our solution $z(t, \mu, \sigma)$, and hence of $x(t, \mu, \sigma)$, may be established in the following way:

Suppose $\bar{z}(t, \mu, \sigma)$ were a second solution. We write (21) with z replaced by $\bar{z}$, and then by subtraction we obtain

$$\begin{aligned} z(t, \mu, \sigma) - \bar{z}(t, \mu, \sigma) &= \int_0^{2\pi} R(t, s; \mu) \sigma [Q(z(s, \mu, \sigma), \mu, s) - Q(\bar{z}(s, \mu, \sigma), \mu, s)] ds \\ &\quad + \int_0^{2\pi} R(t, s; \mu) \mu [P(z(s, \mu, \sigma), \mu, s) - P(\bar{z}(s, \mu, \sigma), \mu, s)] ds. \end{aligned}$$

Handling this as we did the similar expression for $z^{k+1}(t, \mu, \sigma) - z^k(t, \mu, \sigma)$, we find that

$$\max \|z(t, \mu, \sigma) - \bar{z}(t, \mu, \sigma)\| \leq \theta \max \|z(t, \mu, \sigma) - \bar{z}(t, \mu, \sigma)\|.$$

But the only non-negative number not greater than a proper fraction of itself is zero. Thus $\|z(t, \mu, \sigma) - \bar{z}(t, \mu, \sigma)\| \leq 0$ and hence $\bar{z}(t, \mu, \sigma) \equiv z(t, \mu, \sigma)$, showing that $z(t, \mu, \sigma)$ is unique.

§5. Comments on possible refinements of the main theorem. The first thing to notice is that (19) is only used to prove the convergence of the successive approximations and the uniqueness of the solution. In the analytic case, where the functions A , h , g , and p are *analytic* vector functions of the *complex* variables $x_1, \dots, x_n$ and μ , defined for $\|x\|^2 = \sum |x_i|^2 \leq r^2$ and $|\mu| \leq \lambda$, it is only necessary to establish convergence (and uniqueness) in a subdomain of $|\mu| \leq L$ and $|\sigma| \leq \gamma$ in order to conclude that there is also uniform convergence (and uniqueness) in *every* closed subdomain of $|\mu| \leq L$ and $|\sigma| \leq \gamma$. This is because the successive approximations are uniformly bounded in the whole domain and theorems of the Porter-Vitali type on the analytic continuation of uniform convergence are available to prove the convergence of the successive approximations for $|\mu| < L$ and $|\sigma| < \gamma$. The principle of analytic continuation of uniqueness is also applicable, and hence it appears that the theorem of §4 can be modified in the analytic case by omitting (19) from the hypothesis at the small expense of replacing $|\mu| \leq L$ and $|\sigma| \leq \gamma$ by $|\mu| < L$ and $|\sigma| < \gamma$ respectively. Details are left to the reader.

Even in the non analytic case it is probably possible to dispense with (19) provided that we no longer assert uniqueness for the whole domain. This is because our inductive proof of the existence of our successive approximations really involved a proof of the fact that the formula

$$\bar{z}(t, \mu, s) = y(t, \mu) + \int_0^{2\pi} R(t, s; \mu) [\sigma Q(z(s, \mu, \sigma), \mu, s) + \mu P(z(s, \mu, \sigma), \mu, s)] ds$$

may be regarded as a transformation of the function space of vector functions $z(t, \mu, \sigma)$, periodic in t , for which $\|z(t, \mu, \sigma)\| \leq r\lambda^{-1}$, into a part of itself. Theorems of the Birkhoff-Kellogg or Schauder type on fixed points of such transformations [cf. 1 of the Bibliography] are then available to prove the existence (but not the uniqueness) of our periodic solutions. Because there seems to be no hope of obtaining a uniqueness theorem by this method, I have not examined this line of thought in complete detail.

A final observation may be made to the effect that our main theorem may be generalized by allowing the parameter σ to enter in more ways than merely as a coefficient of the term $q(x, \mu, t)$ in (5). Thus we could replace $\sigma q(x, \mu, t) + p(x, \mu, t)$ by $\sigma q(x, \mu, \sigma, t) + p(x, \mu, \sigma, t)$, where the q and p are defined and continuous for $|\sigma| \leq \gamma^*$. The statement and proof of the main theorem need only be modified by adding $\gamma \leq \gamma^*$ to the set (18) of inequalities.

§6. On the uniqueness question. It has already been observed in §2 by means of an example that uniqueness is not to be expected. On the other hand our main theorem asserts uniqueness when attention is confined to the region $\|x\| \leq |\mu| r\lambda^{-1}$ for $|\mu| \leq L \leq \lambda$, and the proof of this uniqueness is seen to be valid for any fixed value of σ such that $|\sigma| \leq \gamma$. These considerations show that there must sometimes be periodic solutions for certain fixed values of σ which reduce to the trivial null solution when $\mu = 0$ and for which $\|x\| > |\mu| r\lambda^{-1}$, at least for $|\mu|$ sufficiently small. The purpose of the present section is to show how our methods can be adapted to provide for these other periodic solutions $x(t, \mu)$, at least in certain cases, where $|\mu|^{-1} \|x(t, \mu)\|$ is bounded, say, $\leq r\lambda_1^{-1}$ ($0 < \lambda_1 < \lambda$) for sufficiently small values of $|\mu|$.

We refer back to equations (5) as our starting point and use also the subsequent developments of §§2 and 3, except that in connection with (11), we assume that Q and P are both vector functions of class C' in $y_1, \dots, y_n$ not only for $\|y\| \leq r\lambda^{-1}$ and for $0 \leq \mu \leq \lambda$ but also for $\|y\| \leq r\lambda_1^{-1}$ and for $0 \leq \mu \leq \lambda_1$. With $\lambda_1 < \lambda$, this gives us more room in the y -space at the expense of less room on the μ -interval, and this is entirely what one would have if the $q(x, \mu, t)$ and $p(x, \mu, t)$ were given *a priori* on the product of the n -sphere $\|x\| \leq r$ and the interval $|\mu| \leq \lambda$. The inequalities (17) are also assumed to be valid for $\|y\| \leq r\lambda_1^{-1}$ and $|\mu| \leq \lambda_1$.

If now the system (5), for a fixed σ , satisfying

$$(22) \quad |\sigma| (2\pi M)^{\frac{1}{2}} Q^{**} < 1,$$

has a periodic solution $x(t, \mu)$, such that $\|x(t, \mu)\| \leq |\mu| r\lambda_1^{-1}$ for $|\mu| \leq L_1 \leq \lambda_1$, and if we write $x = z\mu$, we find that, for $0 < |\mu| \leq L_1$, $z = z(t, \mu)$ satisfies the system $dz/dt = A(t, \mu)z + h(t, \mu) + \sigma\mu Q(z, \mu, t) + \mu^2 P(z, \mu, t)$. Hence from Lemma 1, we find that

$$(23) \quad z(t, \mu) = y(t, \mu) + \int_0^{2\pi} R(t, s; \mu) [\sigma Q(z(s, \mu), \mu, s) + \mu P(z(s, \mu), \mu, s)] ds.$$

Hence, assuming the existence of $\lim_{\mu \rightarrow 0} z(t, \mu) = z^0(t)$, say, we obtain a periodic vector function $z^0(t)$ which is known to satisfy the integral system,

$$(24) \quad z^0(t) = y(t, 0) + \int_0^{2\pi} R(t, s; 0) \sigma Q(z^0(s), 0, s) ds$$

as well as the non-homogeneous variational equation,

$$(25) \quad \frac{dz^0}{dt} = A(t, 0)z^0 + h(t, 0),$$

and the inequality

$$(26) \quad \|z^0(t)\| \leq r\lambda_1^{-1}.$$

We now state the main result of this section as follows:

Theorem 2. *Conversely, if there is a periodic vector function $z^0(t)$ satisfying (24), (25), and (26), with the equality sign excluded from (26), then a positive number L_1 can be constructed such that for $|\mu| \leq L_1$ there exists a periodic solution of (5) such that $\|x(t, \mu)\| \leq |\mu| r\lambda_1^{-1}$ and such that $\lim_{\mu \rightarrow 0} \mu^{-1}x(t, \mu) = z^0(t)$. The explicit construction of L_1 is as follows: Define the vector function $z^1(t, \mu)$ by means of the equation*

$$(27) \quad z^1(t, \mu) = y(t, \mu) + \int_0^{2\pi} R(t, s; \mu) [\sigma Q(z^0(s), \mu, s) + \mu P(z^0(s), \mu, s)] ds.$$

This is possible because of (26). Moreover, $z^1(t, \mu)$ is continuous in the closed region $0 \leq t \leq 2\pi$, $|\mu| \leq \lambda_1$, so that we also have uniform continuity. Hence, upon setting $\omega(\mu) = \max \|z^1(t, \mu) - z^1(t, 0)\| = \max \|z^1(t, \mu) - z^0(t)\|$, it is clear that $\lim_{\mu \rightarrow 0} \omega(\mu) = 0$. Next choose a number θ and a number $L^{**} > 0$ such that

$$(28) \quad (|\sigma| Q^{**} + L^{**} P^{**})(2\pi M)^{\frac{1}{2}} \leq \theta < 1.$$

This is possible because of (22). Finally choose a positive number L^ so small that*

$$(29) \quad \frac{\omega(\mu)}{1 - \theta} \leq r\lambda_1^{-1} - \max \|z^0(t)\| \text{ for } |\mu| \leq L^*.$$

This is possible because we are assuming (26) with the equality sign excluded. For L_1 we take the smaller of the three numbers L^ , L^{**} , and λ_1 .*

Before proceeding with the proof of this theorem, it is interesting to remark that Theorem 1 implies the existence of at least one such $z^0(t)$, at least for $|\sigma| \leq \gamma$ and always with the understanding that (7) is satisfied. This is because of the fact that $r\lambda^{-1} < r\lambda_1^{-1}$. Theorem 2 then gives an alternative estimate of the μ -interval for the same family of periodic solutions as the family treated in Theorem 1. It is much harder to apply, however, because $z^0(t)$ must not only be produced first, but also the related $z^1(t)$ and $\omega(\mu)$. Theorem 2 is thus useful only because of the possibility of having a multiplicity of $z^0(t)$'s.

All possible $z^0(t)$'s may be found by finding all possible solutions of a certain system of equations in k unknowns. In fact, the most general periodic solution of (25) involves k arbitrary constants, and when this general solution is inserted in (24), we obtain the system referred to, which is, of course, to be solved for the k constants.

To prove Theorem 2, we try to set up the successive approximations

$$\begin{aligned} z^0(t, \mu) &\equiv z^0(t) \\ &\dots \\ (30) \quad z^k(t, \mu) &= y(t, \mu) \\ &\quad + \int_0^{2\pi} R(t, s; \mu) [\sigma Q(z^{k-1}(s, \mu), \mu, s) + \mu P(z^{k-1}(s, \mu), \mu, s)] ds. \end{aligned}$$

First, of all, this definition is effective as far as $z^1(t, \mu)$ is concerned; in fact, with $k = 1$, (30) becomes (27); so that $z^1(t, \mu)$ has already been introduced previously. We now wish to prove by induction that the definition is effective also for $k = 2, 3, \dots$ for $|\mu| \leq L_1$ in such manner that

$$(31) \quad \|z^k(t, \mu) - z^{k-1}(t, \mu)\| \leq \theta^{k-1} \omega(\mu)$$

and

$$(32) \quad \|z^k(t, \mu) - z^0(t)\| \leq \omega(\mu) \frac{1 - \theta^k}{1 - \theta}.$$

These facts are obvious from the definition of $\omega(\mu)$ when $k = 1$. If now we assume an inductive hypothesis to the effect that $z^k(t, \mu)$ and $z^{k-1}(t, \mu)$ are defined and satisfy (31) and (33) for $|\mu| \leq L_1$, we merely need to show $z^{k+1}(t, \mu)$ is effectively defined for $0 \leq t \leq 2\pi$ and $|\mu| \leq L_1$ in such a manner that (31) and (32) remain valid when k is replaced by $k + 1$. Since

$$\begin{aligned} \|z^k(t, \mu)\| &\leq \|z^k(t, \mu) - z^0(t)\| + \|z^0(t)\| \leq \omega(\mu) \frac{1}{1 - \theta} \\ &\quad + \max \|z^0(t)\| \leq r\lambda_1^{-1} \end{aligned}$$

by (29), $z^k(t, \mu)$ is in the domain of definition of Q and P . Hence in (30) we may replace k by $k + 1$. By subtraction we therefore obtain

$$z^{k+1}(t, \mu) - z^k(t, \mu) = \int_0^{2\pi} R(t, s; \mu) [S(z^k(s, \mu), \mu, s) - S(z^{k-1}(s, \mu), \mu, s)] ds$$

where $S(z, \mu, s) = \sigma Q(z, \mu, s) + \mu P(z, \mu, s)$. From (17), we know that

$$\begin{aligned} \|S(z', \mu, s) - S(z, \mu, s)\| &\leq |\sigma| Q^{**} \|z' - z\| + |\mu| P^{**} \|z' - z\|, \\ &= (|\sigma| Q^{**} + |\mu| P^{**}) \|z' - z\|. \end{aligned}$$

Hence we find from Lemma 2 that

$$\|z^{k+1}(t, \mu) - z^k(t, \mu)\|^2 \leq (|\sigma| Q^{**} + |\mu| P^{**})^2 \Gamma(t, \mu) \cdot \int_0^{2\pi} \|z^k(s, \mu) - z^{k-1}(s, \mu)\|^2 ds$$

and hence from (16), (28), and (31) we have

$$\begin{aligned} \max \|z^{k+1}(t, \mu) - z^k(t, \mu)\| \\ \leq (|\sigma| Q^{**} + |\mu| P^{**})(2\pi M)^{\frac{1}{2}} \max \|z^k(s, \mu) - z^{k-1}(s, \mu)\| \\ \leq \theta \max \|z^k(t, \mu) - z^{k-1}(t, \mu)\| \leq \theta^k \omega(\mu) \end{aligned}$$

while from (32) we have

$$\begin{aligned} \|z^{k+1}(t, \mu) - z^0(t)\| &\leq \|z^{k+1}(t, \mu) - z^k(t, \mu)\| + \|z^k(t, \mu) - z^0(t)\| \\ &\leq \omega(\mu) \left[\frac{1 - \theta^k}{1 - \theta} + \theta^k \right] = \omega(\mu) \left[\frac{1 - \theta^{k+1}}{1 - \theta} \right]. \end{aligned}$$

Thus we have obtained (31) and (32) with k replaced by $k + 1$, so that the proof of these formulas by induction is complete.

Because of (31), (32), (29), and the fact that $0 < \theta < 1$, we see that the successive approximations converge uniformly for $0 \leq t \leq 2\pi$ and for $|\mu| \leq L_1$ to a vector function $z(t, \mu)$ such that $\|z(t, \mu)\| \leq r\lambda_1^{-1}$. The integrand in (30) also converges uniformly because of the uniform continuity of P and Q . Hence letting $k \rightarrow \infty$ in (30) we find that $z(t, \mu)$ satisfies (23). It then follows at once from Lemma 1, that $x(t, \mu) = \mu z(t, \mu)$ is a periodic solution of (5), such that $\|x(t, \mu)\| \leq |\mu| r\lambda_1^{-1}$ for $|\mu| \leq L_1$. This completes the proof of Theorem 2.

No claim is made to the effect that our methods give *all* periodic solutions $x(t, \mu)$ of (5) such that $\lim_{\mu \rightarrow 0} x(t, \mu) = 0$, but only those for which $\lim_{\mu \rightarrow 0} \mu^{-1}x(t, \mu)$ exists. A counter example to illustrate this was given in the introduction. A complete discussion of uniqueness in the general case would therefore be quite complicated.

It is worth stating, however, that, *in case the non-linear terms are divisible by μ , the periodic solution $x(t, \mu)$ of (5) given by Theorem 1, is the only periodic solution such that $\lim_{\mu \rightarrow 0} x(t, \mu) = 0$, at least for $|\mu|$ sufficiently small.*

To prove this italicized statement we note that in the present instance we may write (5) in the form

$$(33) \quad \frac{dx}{dt} = A(t, \mu)x + \mu h(t, \mu) + \mu \varphi(x, \mu, t)$$

where the vector φ , non linear in x , must satisfy a Lipschitz condition of the form,

$$(34) \quad \| \varphi(x, \mu, t) - \varphi(x', \mu, t) \| \leq B(\|x\|, \|x'\|) \cdot \|x' - x\|,$$

where $B \rightarrow 0$ as $\|x\|$ and $\|x'\| \rightarrow 0$. Suppose now that there were two solutions of (33), $x(t, \mu)$ and $\bar{x}(t, \mu)$, periodic in t and lying within the domain of definition of φ for $|\mu| \leq L$. We have only to prove that $x(t, \mu) \equiv \bar{x}(t, \mu)$ for $|\mu| \leq L'$, where L' is a sufficiently small positive number $\leq L$.

Lemma 1 shows that $x(t, \mu)$ satisfies the integral system

$$x(t, \mu) = \mu y(t, \mu) + \int_0^{2\pi} R(t, s; \mu) \varphi(x(s, \mu), \mu, s) ds$$

with a similar equation for $\bar{x}(t, \mu)$. By subtraction we have

$$\bar{x}(t, \mu) - x(t, \mu) = \int_0^{2\pi} R(t, s; \mu) (\varphi(\bar{x}(s, \mu), \mu, s) - \varphi(x(s, \mu), \mu, s)) ds.$$

Hence, from Lemma 2, (16), and (34) we have

$$\begin{aligned} \|\bar{x}(t, \mu) - x(t, \mu)\|^2 &\leq \Gamma(t) \int_0^{2\pi} \|\varphi(\bar{x}(s, \mu), \mu, s) - \varphi(x(s, \mu), \mu, s)\|^2 ds \\ &\leq 2\pi MB^2 \max \|\bar{x}(t, \mu) - x(t, \mu)\|^2. \end{aligned}$$

But since $\lim_{\mu \rightarrow 0} \|x(t, \mu)\| = \lim_{\mu \rightarrow 0} \|\bar{x}(t, \mu)\| = 0$ by hypothesis, we see that $B = B(\|x\|, \|\bar{x}\|) = B(\|x(t, \mu)\|, \|\bar{x}(t, \mu)\|) \rightarrow 0$ as $\mu \rightarrow 0$. Hence we may choose L' positive but so small that

$$2\pi MB^2 = \theta^2 < 1 \quad \text{for } |\mu| \leq L'.$$

For such values of μ , we therefore have

$$\max \|\bar{x}(t, \mu) - x(t, \mu)\| \leq \theta \max \|\bar{x}(t, \mu) - x(t, \mu)\|, \quad 0 < \theta < 1.$$

Hence $\max \|\bar{x}(t, \mu) - x(t, \mu)\| = 0$, and the italicized statement is proved.

The uniqueness property just established is of essentially the same type as the uniqueness established by CODDINGTON & LEVINSON [2], and the fact that their solutions do not necessarily reduce to the trivial null solution when $\mu = 0$ is of no particular importance from this point of view, since a simple transformation can produce the situation considered here. On the other hand, from a different point of view, one sees that the vector function $\bar{x}(t, \mu, \sigma) = z(t, \mu, \sigma) - y(t, \mu)$ of (21), or $\bar{x}(t, \mu) = z(t, \mu) - y(t, \mu)$ of (23), satisfy systems whose non-linear terms are divisible by μ . The Coddington-Levinson results on existence of periodic solutions for sufficiently small $|\mu|$ are thus (through appropriate transformations) of the same type as the corresponding results contained in Theorems 1 and 2.

§7. Autonomous systems. If $y_i = \varphi_i(t)$ is a known periodic solution with period 2π , when $\mu = 0$, of the system,

$$(35) \quad \frac{dy_i}{dt} = Y_i(y_1, \dots, y_n, \mu), \quad i = 1, \dots, n,$$

in which the right hand members do not contain t explicitly, we know that the variational equations admit a periodic solution consisting of the derivatives of the φ 's. Since, however, we need not expect the perturbed periodic solutions to have exactly the same period (conventionally chosen as 2π) as the unperturbed solution, it is clear that the methods of this paper are not likely to work merely by writing $x_i = y_i - \varphi_i(t)$,

$$\frac{dx_i}{dt} = Y_i(x_1 + \varphi_1(t), \dots, x_n + \varphi_n(t), \mu) - \varphi_i'(t),$$

so that the given periodic solution appears as the trivial null solution $x_i = 0$.

Nevertheless, if the φ 's are of class C'' and at least two of them have first derivatives which do not vanish simultaneously (as is generally the case) we can always introduce the x 's in another manner, eliminate the t , use an angular variable θ as a new independent variable, and at the same time reduce the order of the system by 1. Periodic solutions of the reduced system with period 2π lead in an obvious way (by the inversion of a quadrature) to periodic solutions of the original system (35) with a period not necessarily exactly 2π .

Assuming that $\varphi_{n-1}'(t)$ and $\varphi_n'(t)$ do not vanish simultaneously, a perfectly explicit way of carrying out this program is to write the following transformation from variables $x_1, \dots, x_{n-1}, \theta$ to the variables $y_1, \dots, y_n$:

$$(36) \quad \begin{aligned} y_i &= x_i + \varphi_i(\theta), & i &= 1, \dots, n-2 \\ y_{n-1} &= \varphi_{n-1}(\theta) + x_{n-1} \varphi_n'(\theta) \\ y_n &= -x_{n-1} \varphi_{n-1}'(\theta) + \varphi_n(\theta). \end{aligned}$$

The jacobian of $y_1, \dots, y_n$, with respect to $x_1, \dots, x_{n-1}, \theta$ is seen to be

$$[\varphi_{n-1}'(\theta)]^2 + [\varphi_n'(\theta)]^2 - x_{n-1}[\varphi_n'(\theta)\varphi_{n-1}''(\theta) - \varphi_n''(\theta)\varphi_{n-1}'(\theta)]$$

which can not vanish if

$$|x_{n-1}| < \frac{\min(\varphi_{n-1}'^2 + \varphi_n'^2)}{\max|\varphi_n'\varphi_{n-1}'' - \varphi_n''\varphi_{n-1}'|}.$$

Equations (35) appear in the new variables $x_1, \dots, x_{n-1}, \theta$ as follows:

$$\begin{aligned} dx_i/dt &= X_i^*(x_1, \dots, x_{n-1}, \theta, \mu) & i &= 1, \dots, n-1 \\ d\theta/dt &= \Theta(x_1, \dots, x_{n-1}, \theta, \mu). \end{aligned}$$

Here the right hand members are continuous and periodic with period 2π in θ , and the system admits the solution $x_1 = x_2 = \dots = x_{n-1} = 0, \theta = t$ when $\mu = 0$, corresponding to the original given periodic solution of (35). Thus, when

$x_1, \dots, x_{n-1}$ and μ are small, Θ must be close to unity. Hence there is a definite region enclosing the given periodic solution in which $\Theta \neq 0$. We can thus effect the elimination of t by writing $X_i = X_i^*/\Theta$, so that

$$(37) \quad \frac{dx_i}{d\theta} = X_i(x_1, \dots, x_{n-1}, \theta, \mu), \quad i = 1, \dots, n-1.$$

We may now generally appraise the μ interval for the analytic continuation of the trivial null solution of (37) by the methods of the previous paper [5] or by those of the present paper (replacing t by θ and n by $n-1$) according as to whether the variational equations of the trivial null solution of (37) do not have or do have a non-trivial periodic solution.

Many interesting problems lead in this way to the simple case treated in the previous paper. Such is the analytic continuation of the circular orbits in the two body problem to obtain periodic solutions of the restricted problem of three bodies (at least in general) provided that the integral of JACOBI is used to eliminate one of the dependent variables. Other problems lead to the more difficult situation which can be handled by the theorem of §4 of the present paper. An example of this latter type is discussed in the next section.

§8. Applications to systems of the Van der Pol type. By a system of the Van der Pol type we mean a second order system of the form,

$$(38) \quad \frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = -y_1 + \mu f(y_1, y_2).$$

The Van der Pol equation itself is obtained by eliminating y_2 in the special case that $f(y_1, y_2) = (1 - y_1^2)y_2$. In any case, when $\mu = 0$, we have the periodic solution $y_1 = a \sin t$, $y_2 = a \cos t$. Upon introducing the transformation,

$$y_1 = a(1-x) \sin \theta, \quad y_2 = a(1-x) \cos \theta, \quad a \neq 0,$$

which is non singular for $x \neq 1$, the equations appear in the form,

$$(39) \quad \frac{dx}{dt} = X^*(x, \theta, \mu), \quad \frac{d\theta}{dt} = \Theta(x, \theta, \mu);$$

and, upon eliminating t , we obtain

$$(40) \quad \frac{dx}{d\theta} = \mu F(x, \theta, \mu), \quad \text{with} \quad \mu F = \frac{X^*}{\Theta}.$$

The reader can readily work out the explicit expressions for X^* , Θ , and F . The latter turns out to be

$$(41) \quad F(x, \theta, \mu) = - \frac{(1-x)f[a(1-x) \sin \theta, a(1-x) \cos \theta] \cos \theta}{a(1-x) - \mu f[a(1-x) \sin \theta, a(1-x) \cos \theta] \sin \theta},$$

which is periodic with period 2π in θ . A periodic solution $x(\theta, \mu)$ of (40), with period 2π , gives rise to a periodic solution of (39) or (38), with period

$$\int_0^{2\pi} (\Theta[x(\theta, \mu), \theta, \mu])^{-1} d\theta,$$

and, for $\mu = 0$, the solution, $x(\theta) \equiv 0$, corresponds to $y_1 = a \sin t$, $y_2 = a \cos t$. Thus we are interested in investigating the possibility of analytic continuation of this trivial null solution; and, if analytic continuation is possible at all, we want to find explicitly a positive number L such that we can be sure that there is a periodic solution for every μ whose absolute value does not exceed L .

To show that these questions can be answered by applying the results of §§2-4, we begin by developing F in powers of x . We will also change the notation slightly, using t instead of θ in order to conform with the notation of §§2-4.

$$F(x, t, \mu) = h(t, \mu) + g(t, \mu)x + m(x, \mu, t)x^2.$$

And we further develop g in powers of μ :

$$g(t, \mu) = b(t) + c(t, \mu)\mu.$$

Thus the equation (40) appears in the form

$$(42) \quad \frac{dx}{dt} = b(t)\mu x + \mu h(t, \mu) + p(x, \mu, t),$$

where $p(x, \mu, t) = c(t, \mu)\mu^2 x + m(x, \mu, t)\mu x^2$ can evidently be regarded as a vector in one dimensional space satisfying the property indicated by (11). Thus (42) is a special case of (5), with $n = 1$, $A(t, \mu) = b(t)\mu$, and $q \equiv 0$.

The adjoint equation (8) reduces to $d\xi/ds = 0$; and hence the condition (7) reduces to

$$(43) \quad \int_0^{2\pi} h(s, 0) ds = 0 \quad \text{or} \quad \int_0^{2\pi} F(0, t, 0) dt = 0,$$

which need not be satisfied except, possibly, for particular values of a , that is, for a particular choice of the given periodic solution $y_1 = a \sin t$, $y_2 = a \cos t$. We can regard (43) as an equation for the determination of a . As is well known, it is satisfied in the Van der Pol case if and only if $a = \pm 2$.

The only remaining thing to be examined is the behaviour of a solution $X(t, \mu)$ of the homogeneous equation (9), whereby we are interested in the condition $\det [X(0, \mu) - X(2\pi, \mu)] \neq 0$ for $\mu \neq 0$. Clearly we may take $X(t, \mu) = \exp \int_0^t \mu b(s) ds$, and our condition becomes $\exp. \int_0^{2\pi} \mu b(s) ds \neq 1$, which is satisfied if and only if

$$(44) \quad \int_0^{2\pi} b(s) ds \neq 0 \quad \text{or} \quad \int_0^{2\pi} F_x(0, t, 0) dt \neq 0.$$

The reader may also verify that, if (44) is satisfied, we find that (10) is satisfied. In the Van der Pol case, with $a = 2$, a simple calculation shows that

$$\int_0^{2\pi} F_x(0, t, 0) dt = -2\pi.$$

The hypothesis of our main theorem that $\|y(t, \mu)\| \leq \beta < \lambda^{-1}r$ can make essential trouble neither in this application nor any other in which the P and Q are defined by (11). For in cutting down on the size of λ we are not forced to increase β , whereas we can make $\lambda^{-1}r$ as large as we wish.

Thus, systems of the Van der Pol type are in general comprised in our general theory, and that too in a very simple way, since not only is $n = 1$ but the q of (5) is identically zero.

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