Uniqueness Theorems for the Tricomi Problem¹. II

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1. Introduction. In a previous paper [7] certain uniqueness theorems were proven for the Tricomi problem for equations of the form

$$K(y)u_{xx} + u_{yy} = 0.$$

The theorems were of two types: in one a certain condition on the behavior of $K(y)$ was imposed while the domain under consideration was left unrestricted, while in the other a restriction is placed on the size of the domain and the behavior of $K(y)$ (except for smoothness and monotonicity) is left free.

In this paper a stronger uniqueness theorem of the first type is established in that the condition imposed on $K(y)$ is less severe. Such a theorem in itself is of limited interest as it seems likely that uniqueness holds for arbitrary monotone $K(y)$ in a general domain. However, FRANKL [2] raised the question of the existence of transonic potential flows out of a jet. He showed that such problems, when considered in the hodograph plane, led to the Tricomi problem with $K(y)$ a specific monotone function. In all previous uniqueness theorems the conditions necessary to establish a general uniqueness theorem were not present. In this paper a general uniqueness theorem is established for this problem. An associated existence theorem under somewhat stronger assumptions is also established.

In Section 2 the gas dynamical problems are stated. Section 3 establishes the necessary uniqueness theorem and Section 4 applies the results of the previous section to the gas dynamical case.

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2. Some gas dynamical problems. We consider a two-dimensional adiabatic potential flow of a perfect gas. As is well known the stream function $\psi = \psi(x, y)$ satisfies the equation

$$(\rho^2 a^2 - \psi_y^2) \psi_{xx} + 2\psi_x \psi_y \psi_{xy} + (\rho^2 a^2 - \psi_x^2) \psi_{yy} = 0 \quad (1)$$

where a is the local velocity of sound and ρ the density of the gas. Equation (1) can be transformed to a linear equation by the introduction of the velocity components as new independent variables; this is the hodograph transformation. Let $u = \rho^{-1} \psi_x$ and $v = -\rho^{-1} \psi_y$ be the rectangular velocity components and $r = \sqrt{u^2 + v^2}$, $\theta = \tan^{-1}(v/u)$, the corresponding components in polar coordinates. It is convenient to normalize r by introducing the dimensionless quantity

$$t = \frac{r^2}{r_0^2}$$

as new independent variable; r_0 is the speed corresponding to zero density. It is given by $k\gamma\rho_0^{\gamma-1}/(\gamma-1)$ where ρ_0 is the density of the gas at zero speed and k and γ are the constants appearing in the adiabatic pressure density relation: $p = k\rho^\gamma$. See [1]. With t and θ as independent variables equation (1) becomes

$$\frac{\partial}{\partial t} \left\{ \frac{2t}{(1-t)^\beta} \frac{\partial \psi}{\partial t} \right\} + \frac{1 - (2\beta + 1)t}{2(1-t)^{\beta+1}} \frac{\partial^2 \psi}{\partial \theta^2} = 0, \quad (2)$$

where

$$\beta = \frac{c_p}{c_p - c_v}, \quad \gamma = \frac{c_p}{c_v}$$

and c_p, c_v are the specific heats at constant pressure and volume respectively.

Equation (2) has a first derivative term which can be eliminated by a further change in independent variables. We set

$$\xi = \theta, \quad \eta = - \int_{1/(2\beta+1)}^t \frac{(1-u)^\beta}{2u} du$$

and in (ξ, η) coordinates equation (2) takes the form

$$\frac{1 - (2\beta + 1)t}{(1-t)^{2\beta+1}} \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0. \quad (3)$$

This equation is a special case of the equation

$$K(\eta) \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0 \quad (4)$$

in which $K(\eta)$ is defined by

$$K(\eta) = \frac{1 - (2\beta + 1)t}{(1-t)^{2\beta+1}}, \quad \eta = - \int_{1/(2\beta+1)}^t \frac{(1-u)^\beta}{2u} du. \quad (5)$$

We note that β is a positive constant (approximately 2.5 for air) and that t is a single-valued function of η defined implicitly by the second relation in (5). The value $\eta = 0$ corresponds to $t = 1/(2\beta + 1)$ and we see that $K(0) = 0$. For positive values of η we have $K'(\eta) = 4\beta(2\beta + 1)t^2/(1 - t)^{3\beta+2} > 0$ with $0 < t < 1/(2\beta + 1)$. Since $\eta \rightarrow \infty$ as $t \rightarrow 0$ it is clear that $K(\eta) \rightarrow 1$ as $\eta \rightarrow \infty$. Negative values of η correspond to values of t larger than $1/(2\beta + 1)$ with the minimum value of η permissible corresponding to $t = 1$ (since $K(\eta) \rightarrow -\infty$ as $t \rightarrow 1$). We designate by η the value $-\int_{1/(2\beta+1)}^1 (1 - u)^\beta / 2u \, du$.

Equation (3) is elliptic for $\eta > 0$, hyperbolic for $\eta < 0$ with $\eta = 0$ appearing as a parabolic line. The value $\eta = 0$ corresponds to $t = 1/(2\beta + 1)$ and it is easily verified [1] that this occurs at points where the velocity is equal to the local velocity of sound. Thus for $\eta > 0$ or $t < 1/(2\beta + 1)$ we have subsonic velocities while for $\eta < 0$ or $t > 1/(2\beta + 1)$ we have supersonic velocities.

We shall be concerned with the two-dimensional flow between two infinite plane converging walls. Let l_1 and l_2 represent two walls, symmetric with respect to the x -axis and extending to infinity in the negative x -direction. The flow may be thought of as that out of a two-dimensional nozzle. Let the angle between the walls l_1 and l_2 be $2\theta_0 \leq \pi$. A solution ψ of equation (1) is sought subject to the conditions that ψ is prescribed on l_1 and l_2 . It will be supposed that the flow is symmetric and hence we may restrict our considerations to the upper half of the nozzle. Then the boundary conditions become

$$\psi = \text{const. on } l_1$$

$$\psi = 0 \text{ for } y = 0, x \leq C.$$

The value $x = C$ corresponds to the cross-sectional opening of the nozzle. This problem, with the flow partly subsonic and partly supersonic was first considered by FRANKL [2]. In this case the sonic speed will occur at the points A and B , and in the (ξ, η) -plane the above boundary value problem becomes the problem of TRICOMI. The domain D_0 in the (ξ, η) -plane that is to be considered is bounded by the following curves. (1) the half-infinite line $\xi = 0, \eta \geq 0$ (this corresponds to the value $\psi = 0$ for $y = 0, x \leq C$ in the physical or flow plane); (2) the half-infinite line $\xi = \xi_0, \eta \geq 0$ (this corresponds to the condition $\psi = \text{const.}$ on l_1 ; we note that the line $\xi = \xi_0$ represents a velocity making a constant angle ($= \xi_0$) with the x -axis); (3) the characteristic γ_2 of equation (3) which emanates from the point $(\xi_0, 0)$ and extends into the lower half-plane. This entire characteristic corresponds to the point A (see [2] for a discussion of this point). (4) the characteristic γ_1 of (3) emanating from the origin and which intersects γ_2 at the point T . We note that the distance from the origin to the point $(\xi_0, 0)$ is less than $\pi/2$. This fact is used in Section 4 in the proof of uniqueness. The domain D_0 is thus one which extends to infinity in the η direction and is bounded from below by the point T . The value $\eta = \infty$ corresponds to zero velocity ($\eta = \infty$ is equivalent to $t = 0$) which occurs at $x = -\infty$ in the physical plane.

Another problem which may be considered is one in which the walls l_1 and l_2

are finite in length but otherwise satisfy the same conditions of symmetry, *etc.*, as before. In this case the cross-sectional velocity at the entrance of the nozzle would have to be prescribed (*e.g.* uniform flow along the line $x = x_1$ between the walls l_1 and l_2). The domain in the (ξ, η) -plane for the finite nozzle would consist first of the characteristics γ_1 and γ_2 as before; in the subsonic portion of the plane ($\eta < 0$) the domain would be finite in extent consisting of finite segments of the lines $\xi = 0$ and $\xi = \xi_0$ and a curve C connecting these segments. The curve C "corresponds" to the prescribed velocity at the entrance of the nozzle. To obtain the domain just described it is assumed that the velocity prescribed at the entrance of the nozzle is symmetric with respect to the x -axis.

Other problems of this type which lead to the Tricomi problem have been considered. For example the flow in a two-dimensional channel past a wedge has been discussed by GUDERLEY [4].

The above problems are particular examples of the following more general situation. Let $K(\eta)$ be a continuous, monotone increasing function of η with $K(0) = 0$. Then the equation

$$K(\eta)\psi_{\xi\xi} + \psi_{\eta\eta} = 0 \quad (6)$$

has real characteristics for negative values of η given by the solutions of the equations

$$\frac{d\eta}{d\xi} = \pm \frac{1}{\sqrt{-K(\eta)}}. \quad (7)$$

Let Γ be a simple rectifiable arc lying in the upper half-plane with end-points on the x -axis. Without loss of generality we place one end-point at the origin. The other end-point will be at some point $P(\xi_0, 0)$. We denote by γ_1 the characteristic of (7) passing through the origin and extending into the portion of the plane $\xi > 0, \eta < 0$; let γ_2 be the characteristic of (7) passing through $P(\xi_0, 0)$ which intersects γ_1 . Then Γ, γ_1 , and γ_2 enclose a domain which we denote by D .

The *Tricomi problem* consists of the determination of a solution of (6) in D assuming prescribed values on Γ and γ_2 . In order to include the gas dynamical cases discussed previously we modify the definition of D as follows: The elliptic (upper) portion of D may extend to infinity in such a way that Γ consists of two infinite arcs Γ_1 and Γ_2 emanating from the origin and $P(\xi_0, 0)$ respectively. It will further be required that the arc passing through the origin have a minimum abscissa, and that certain integrals involving ψ, ψ_x and ψ_y taken over the infinite domain are convergent.

The problem of the transonic flow past a wedge has been considered by FRANKL [2]. This also leads to a problem quite similar to the Tricomi problem, the difference being that over a certain portion of Γ (or Γ_1) a linear combination of ψ and $\partial\psi/\partial\eta$ are prescribed instead of ψ . The results of the following sections apply equally well to this case.

3. A uniqueness theorem. In this section we establish a uniqueness theorem for the Tricomi problem relative to (6) which makes no restriction on the location of Γ or on the size of ξ_0 . Such uniqueness theorems have been obtained previously [2, 3, 5, 7, 8] but in every case some requirement has been made concerning the behavior of $K(\eta)$ or of the domain. FRANKL obtained a uniqueness theorem for the Tricomi problem under the hypothesis that the function

$$F(\eta) \equiv \frac{3K'^2(\eta) - 2K(\eta)K''(\eta)}{K'^2(\eta)} \quad (8)$$

is positive for negative values of η . The results of TRICOMI and GELLERSTEDT for $K(\eta) = \eta$ and $K(\eta) = \eta^\alpha$ respectively satisfy the condition $F(\eta) \geq 0$, $\eta \leq 0$ for any domain. The result given in [7] allows $F(\eta)$ to take on small negative values for arbitrary values of Γ and ξ_0 so long as Γ encloses a *finite* region. The uniqueness theorem given in this section reduces the restriction both on the finiteness of the domain and on the behavior of $F(\eta)$. The simplest statement of the uniqueness theorem requires that $K'''(\eta)$ be negative whenever $F(\eta)$ is negative with no additional hypothesis on Γ . However a closer examination of the proof shows that the result holds under somewhat less restrictive conditions on $K(\eta)$. These less restrictive conditions are needed for the gas dynamical case given in Section 4.

The functions $\psi(x, y)$ that we shall be concerned with are *quasi-regular* solutions of (6). A *quasi-regular* solution of (6) is a function which: (i) satisfies (6) in D ,

$$(ii) \quad \int_0^{\xi_0} \psi(\xi, 0) \psi_\eta(\xi, 0) d\xi \quad \text{exists;}$$

(iii) all applications of Green's theorem to the first and second derivatives of ψ are valid in the sense that the integrals exist over a sequence of domains interior to D and that the limits of these integrals exist as the sequence of domains approaches D .

Theorem 1. *Let D be the domain described above and suppose $K(\eta)$ has a continuous third derivative satisfying the condition $K'''(\eta) \leq 0$ whenever $F(\eta) < 0$, $\eta < 0$. If ψ is a solution of (6) in D which vanishes on Γ and γ_2 then ψ vanishes identically.*

Proof: We investigate the expression

$$\iint_D (a\psi + b\psi_\xi + c\psi_\eta)(K(\eta)\psi_{\xi\xi} + \psi_{\eta\eta}) d\xi d\eta,$$

where $a = a(\xi, \eta)$, $b = b(\xi, \eta)$, $c = c(\xi, \eta)$ are once continuously differentiable functions which shall be specified below. If ψ is a solution of (6) in D then

obviously the above expression will vanish. An application of Green's theorem to the above expression yields (assuming ψ satisfies (6))

$$\begin{aligned}
 0 = & \frac{1}{2} \iint_D [-2aK\psi_\xi^2 + a_{\xi\xi} K\psi^2 - 2a\psi_\eta^2 + a_{\eta\eta} \psi^2 - Kb_\xi \psi_\xi^2 - 2b_\eta \psi_\xi \psi_\eta + b_\xi \psi_\eta^2 \\
 & + (cK)_\eta \psi_\xi^2 - 2Kc_\xi \psi_\xi \psi_\eta - 2c_\eta \psi_\eta^2] d\xi d\eta + \oint_{\Gamma+\gamma_1+\gamma_2} [aK\psi\psi_\xi d\eta \\
 & - \frac{1}{2}a_\xi K\psi^2 d\eta - a\psi\psi_\eta d\xi + \frac{1}{2}a_\eta \psi^2 d\xi + \frac{1}{2}bK\psi_\xi^2 d\eta - b\psi_\xi \psi_\eta d\xi \\
 & - \frac{1}{2}b\psi_\eta^2 d\eta + cK\psi_\xi \psi_\eta d\eta + \frac{1}{2}cK\psi_\xi^2 d\xi - \frac{1}{2}c\psi_\eta^2 d\xi] \\
 = & I_1 + I_2.
 \end{aligned}$$

By hypothesis ψ vanishes on Γ and γ_2 . Hence I_2 becomes

$$\begin{aligned}
 I_2 = & \int_{\gamma_1} [aK\psi\psi_\xi d\eta - a\psi\psi_\eta d\xi - \frac{1}{2}a_\xi K\psi^2 d\eta + \frac{1}{2}a_\eta \psi^2 d\xi + \frac{1}{2}bK\psi_\xi^2 d\eta \\
 & - b\psi_\xi \psi_\eta d\xi - \frac{1}{2}b\psi_\eta^2 d\eta + cK\psi_\xi \psi_\eta d\eta + \frac{1}{2}cK\psi_\xi^2 d\xi - \frac{1}{2}c\psi_\eta^2 d\xi] \\
 & + \int_{\Gamma+\gamma_2} [\frac{1}{2}bK\psi_\xi^2 d\eta - b\psi_\xi \psi_\eta d\xi - \frac{1}{2}b\psi_\eta^2 d\eta + cK\psi_\xi \psi_\eta d\eta + \frac{1}{2}cK\psi_\xi^2 d\xi \\
 & - \frac{1}{2}c\psi_\eta^2 d\xi].
 \end{aligned}$$

Taking into account the fact that γ_1 and γ_2 are characteristics, *i.e.* that

$$d\eta = -\frac{1}{\sqrt{-K}} d\xi \text{ along } \gamma_1 \text{ and } d\eta = \frac{1}{\sqrt{-K}} d\xi \text{ along } \gamma_2$$

we find

$$\begin{aligned}
 I_2 = & -\frac{1}{2} \int_{\gamma_1} \left(c\sqrt{-K} - b \right) (\sqrt{-K}\psi_\xi^2 - 2\psi_\xi \psi_\eta + \frac{1}{\sqrt{-K}}\psi_\eta^2) d\xi \\
 & + \int_{\gamma_1} [a\sqrt{-K}\psi d\psi - \frac{1}{2}\sqrt{-K}\psi^2(a_\xi d\xi + a_\eta d\eta)] \\
 & - \frac{1}{2} \int_{\gamma_2} (c\sqrt{-K} + b) (\sqrt{-K}\psi_\xi + \psi_\eta)(\psi_\xi d\xi + \psi_\eta d\eta) \\
 & + \int_{\Gamma} [\frac{1}{2}bK\psi_\xi^2 d\eta - b\psi_\xi \psi_\eta d\xi - \frac{1}{2}b\psi_\eta^2 d\eta + cK\psi_\xi \psi_\eta d\eta \\
 & + \frac{1}{2}cK\psi_\xi^2 d\xi - \frac{1}{2}c\psi_\eta^2 d\xi] \\
 = & J_1 + J_2 + J_3 + J_4.
 \end{aligned}$$

We denote by D' the portion of D above the ξ -axis and by D'' the portion below the ξ -axis. Then I_1 can be broken up as follows:

$$\begin{aligned} I_1 = & - \iint_{D'} a(K\psi_\xi^2 + \psi_\eta^2) d\xi d\eta - \frac{1}{2} \iint_{D''} \{[2aK + Kb_\xi - (cK)_\eta]\psi_\xi^2 \\ & + 2(Kc_\xi + b_\eta)\psi_\xi\psi_\eta + (c_\eta + 2a - b_\xi)\psi_\eta^2\} d\xi d\eta \\ & + \frac{1}{2} \iint_D (Ka_{\xi\xi} + a_{\eta\eta})\psi^2 d\xi d\eta \\ & - \frac{1}{2} \iint_{D'} \{[Kb_\xi - (cK)_\eta]\psi_\xi^2 + 2(Kc_\xi + b_\eta)\psi_\xi\psi_\eta + (c_\eta - b_\xi)\psi_\eta^2\} d\xi d\eta \\ = & J_5 + J_6 + J_7 + J_8. \end{aligned}$$

We shall now select a, b, c in such a manner that each of the integrals $J_1, \dots, J_8$ will be non-positive. Hence each individual integral will vanish. Since the integrands will, in some cases, be negative definite expressions in the first derivatives of ψ it will imply that ψ vanishes throughout D .

We choose

$$b \equiv 0 \quad \text{for } \eta \geq 0, \quad b = c\sqrt{-K} \quad \text{for } \eta \leq 0.$$

This determines b completely in terms of c and we see that since $K(0) = 0$, b will be continuous throughout D . We note that it is sufficient that $a(\xi, \eta)$ possess continuous first derivatives with piecewise continuous second derivatives in D and b and c be merely continuous with piecewise continuous first derivatives in D in order that the application of Green's theorem be valid. This definition of b makes J_1 vanish, while J_3 vanishes from the fact that $d\psi = \psi_\xi d\xi + \psi_\eta d\eta = 0$ along γ_2 .

We choose

$$c \equiv 0, \quad \text{for } \eta \geq 0, \quad c = \frac{4aK}{K'} \quad \text{for } \eta \leq 0.$$

This determines c (and therefore b) in terms of $a(\xi, \eta)$ and we see that c is continuous with piecewise continuous first derivatives in D . The choices for b and c make J_4 and J_8 vanish.

There remain the integrals J_2, J_5, J_6 and J_7 . It is in the determination of these integrals that the sign of $F(\eta)$ plays a role. For simplicity we suppose that $F(\eta)$ is non-negative in the range $\eta_0 \leq \eta \leq 0$ while $F(\eta)$ is negative for $\eta < \eta_0$.

From the definitions of b and c the integral J_6 becomes

$$J_6 = - \iint_{D''} [aF(\eta) + \frac{K}{K'} (a_\eta - \sqrt{-K}a_\xi)(K\psi_\xi + \psi_\eta)^2] d\xi d\eta.$$

To make J_6 non-positive it is necessary to choose $a(\xi, \eta)$ so that the quantity

$$aF(\eta) + \frac{K}{K'}(a_\eta - \sqrt{-K}a_\xi)$$

is non-negative for negative values of η . Let ξ_1 be the smallest abscissa of all points on Γ . We define $a(\xi, \eta)$ as follows:

$$a(\xi, \eta) = \begin{cases} [-K(\eta)]^{-\frac{1}{2}} K'(\eta) [(\xi - \xi_1) + \int_0^\eta \sqrt{-K} d\eta], & \eta \leq \eta_0, \\ [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) [(\xi - \xi_1) + \int_0^\eta \sqrt{-K} d\eta], & \eta_0 \leq \eta \leq 0, \\ [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) (\xi - \xi_1), & \eta \geq 0. \end{cases}$$

We first note that a has continuous first derivatives and piecewise continuous second derivatives in D . The only questionable quantity is a_η at $\eta = 0$ and at $\eta = \eta_0$. At $\eta = 0$ we see that $a_\eta(\xi, 0+) = 0$ while

$$a_\eta(\xi, 0-) = [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) \sqrt{-K(0)}$$

which vanishes since $K(0) = 0$. On the line $\eta = \eta_0$ we have

$$a_\eta(\xi, \eta_0+) = [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) \sqrt{-K(\eta_0)},$$

while

$$\begin{aligned} a_\eta(\xi, \eta_0-) &= [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) \sqrt{-K(\eta_0)} \\ &+ \frac{1}{2} [-K(\eta_0)]^{-\frac{1}{2}} \{3K^2(\eta_0) - 2K(\eta_0)K''(\eta_0)\} (\xi - \xi_1 + \int_0^{\eta_0} \sqrt{-K} d\eta), \end{aligned}$$

and these two values are equal since $F(\eta_0) = 0$. It is clear from the definition that $a \geq 0$ for all ξ, η . For $\eta \geq 0$ this is obvious from the definition of ξ_1 . For negative values of η we note that the quantity $\xi + \int_0^\eta \sqrt{-K} d\eta$ is constant along one of the families of the characteristics (7). Since $\xi - \xi_1 + \int_0^\eta \sqrt{-K} d\eta$ is positive for $\eta = 0$ and all ξ under consideration it remains positive throughout D'' .

With the above definition of $a(\xi, \eta)$ we can evaluate the integrand in J_6 :

$$aF(\eta) + \frac{K}{K'}(a_\eta - \sqrt{-K}a_\xi) = 0 \quad \text{for } \eta \leq \eta_0$$

$$aF(\eta) + \frac{K}{K'}(a_\eta - \sqrt{-K}a_\xi) = aF(\eta) \quad \text{for } \eta_0 \leq \eta \leq 0.$$

Since $F(\eta) \geq 0$ for $\eta_0 \leq \eta \leq 0$ by hypothesis we find that J_6 is non-positive.

The integral J_2 , which is taken along the characteristic γ_1 can be broken up into two parts: γ_1' is that part of γ_1 above the line $\eta = \eta_0$ and γ_1'' the part of γ_1 below $\eta = \eta_0$. Thus we have

$$J_2 = - \int_{\gamma_1'} [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) [\xi - \xi_1 + \int_0^\eta \sqrt{-K} d\eta] \psi^2 d\xi \\ + \int_{\gamma_1''} \left(a_\eta - \sqrt{-K} a_\xi + \frac{aK'}{4K} \right) \psi^2 d\xi.$$

The first of these two integrals is clearly non-positive. In the second integral we have $a_\eta - \sqrt{-K} a_\xi = (K'/(-2K))aF(\eta)$ which is negative as $F(\eta) \leq 0$ for $\eta \leq \eta_0$. Also $aK'/4K$ is negative as K is. Thus J_2 is non-positive.

The integral

$$J_5 = - \iint_{D'} a(K\psi_\xi^2 + \psi_\eta^2) d\xi d\eta$$

is clearly non-positive while the last integral to be considered, J_7 , reduces to

$$J_7 = \frac{1}{2} \iint_{D''} a_{\eta\eta} \psi^2 d\xi d\eta.$$

We break up D'' into D_1'' , the part of D'' above the line $\eta = \eta_0$ and D_2'' , the part of D'' below $\eta = \eta_0$. Then J_7 becomes

$$J_7 = - \frac{1}{4} \iint_{D_1''} [-K(\eta_0)]^{-\frac{1}{2}} K'(\eta_0) \frac{K'}{\sqrt{-K}} \psi^2 d\xi d\eta + \frac{1}{2} \iint_{D_2''} a_{\eta\eta} \psi^2 d\xi d\eta.$$

For $\eta \leq \eta_0$, we have

$$a_{\eta\eta} = \frac{1}{2} (-K)^{-2} (5K'^2 - 4KK'') + \left[(\xi - \xi_1) \right. \\ \left. + \int_0^\eta \sqrt{-K} d\eta \right] (-K)^{-\frac{3}{2}} [K^2 K''' + \frac{3}{4} K' (5K'^2 - 6KK'')]. \quad (10)$$

By hypothesis we know that $3K'^2 \leq 2KK''$ for $\eta \leq \eta_0$. Hence $5K'^2 \leq 4KK''$ and the first term on the right side of (10) is negative. It also follows that $5K'^2 \leq 6KK''$ in this range and the second term on the right will be negative if $K''' \leq 0$ for $\eta \leq \eta_0$. It is clear that the less stringent condition

$$K''' \leq \frac{3K'}{4K^2} (6KK'' - 5K'^2), \quad \eta \leq \eta_0$$

suffices. This condition is weaker since the right side is definitely a positive quantity whenever $K'(\eta) \neq 0$. A still weaker condition is one that merely requires (10) to be negative, and in fact it is this condition that we shall use in the next section.

The integrals J_2 , J_5 , J_6 , and J_7 are all non-positive and since their sum is zero each must vanish. An examination of J_5 shows that this can happen only if $\psi_\xi = \psi_\eta = 0$ throughout D' . Since $\psi = 0$ on Γ this implies that ψ vanishes identically in D' and hence on the ξ -axis. From this it follows that $\psi = 0$ in D'' and the proof is complete.

4. Existence and uniqueness for gas dynamical problems. In this section we apply the uniqueness theorem to the problems discussed in Section 2. The function $K(\eta)$ is given by (5):

$$K(\eta) = \frac{1 - (2\beta + 1)t}{(1 - t)^{2\beta+1}}, \quad \eta = - \int_{1/(2\beta+1)}^t \frac{(1 - u)^\beta}{2u} du \quad (5)$$

with the derivatives (with respect to η)

$$K'(\eta) = 4\beta(2\beta + 1) \frac{t^2}{(1 - t)^{3\beta+2}}, \quad K''(\eta) = -8\beta(2\beta + 1)t^2 \frac{3\beta t + 2}{(1 - t)^{4\beta+3}}$$

$$K'''(\eta) = 16\beta(2\beta + 1)t^2 \frac{12\beta^2 t^2 + (17\beta + 2)t + 4}{(1 - t)^{5\beta+4}}.$$

The value $\eta = 0$ corresponds to $t = 1/(2\beta + 1)$ and the range $1/(2\beta + 1) \leq t \leq 1$ corresponds to negative values of η . It is important to note that $K(\eta)$ does not satisfy the condition

$$3K'^2 - 2KK'' \geq 0$$

for all negative values of η . A simple computation shows that

$$3K'^2 - 2KK'' \geq 0 \quad \text{for} \quad \frac{1}{2\beta + 1} \leq t \leq \frac{2}{\beta + 2}$$

$$3K'^2 - 2KK'' \leq 0 \quad \text{for} \quad \frac{2}{\beta + 2} \leq t < 1.$$

Thus FRANKL'S proof of uniqueness [2] is valid in the range $1/(2\beta + 1) \leq t \leq 2/(\beta + 2)$. On the other hand Theorem 1 cannot be applied directly as $K'''(\eta)$ is positive throughout the range $1/(2\beta + 1) \leq t < 1$. It turns out that the stronger form of the theorem, which makes use of inequality (10) is applicable.

To apply Theorem 1 to the case of a flow out of a jet we must verify the fact that (10) holds throughout the range under consideration. In this case we have $\xi_1 = 0$ and $0 \leq \xi \leq \pi/2$. For the case of air we have

$$\frac{c_p}{c_v} = 1.4 \quad \text{and} \quad \beta = 2.5.$$

Inserting these values for $K(\eta)$ and its derivatives in (10) and employing the value $\beta = 2.5$, we find

$$(8 - 7t - 90t^2 + 189t^3) \left(\xi + \int_0^\eta \sqrt{-K} d\eta \right) + (4 - 9t - 15t^2)(6t - 1)^{\frac{1}{2}}(1 - t)^{\frac{1}{2}} \leq 0 \quad (11)$$

which must hold in the range $2/(2 + \beta) = \frac{4}{9} \leq t \leq t_0$. Here t_0 is the maximum value of t corresponding to the minimum value of η , say η_0 , which occurs. The quantity η_0 is determined from the maximum value $\xi_0 = \pi/2$ by the fact that we select the characteristics γ_1 and γ_2 through the origin and $(\pi/2, 0)$.

To evaluate the expression $\xi + \int_0^\eta \sqrt{-K} d\eta$ we first note that this quantity is constant along each characteristic parallel to γ_1 . Hence along each segment $\eta = \eta_1$ in the domain in the quantity $\xi + \int_0^\eta \sqrt{-K} d\eta$ takes on its maximum value at the point of intersection of the line $\eta = \eta_1$ with the characteristic γ_2 . Inserting the expression for $K(\eta)$ into the above quantity and changing variables from η to t , we find

$$\xi + \int_0^\eta \sqrt{-K} d\eta = \xi - \frac{1}{2} \int_{\frac{1}{3}}^t \frac{1}{t} \sqrt{\frac{6t-1}{1-t}} dt.$$

It is easy to evaluate the above integral. We find

$$R \equiv -\frac{1}{2} \int_{\frac{1}{3}}^t \frac{1}{t} \sqrt{\frac{6t-1}{1-t}} dt = \tan^{-1} \sqrt{\frac{6t-1}{1-t}} - \sqrt{6} \tan^{-1} \sqrt{\frac{6t-1}{6-6t}}.$$

We see that the maximum value of t , t_0 , is that value of t which makes $|R|$ equal to $\pi/4$ (since the maximum value of $|R|$ is half the maximum length of ξ). A computation shows that $t_0 < 0.65$. We therefore have to establish (11) in the range $\frac{4}{9} \leq t \leq 0.65$. Inequality (11) can be written

$$(8 - 7t - 90t^2 + 189t^3)(\xi + R) \leq (15t^2 + 9t - 4)(6t - 1)^{\frac{1}{2}}(1 - t)^{\frac{1}{2}}.$$

It is easy to verify that the strict inequality holds at $t = \frac{4}{9}$ and hence in a neighborhood of $\frac{4}{9}$. To prove the above inequality we first make certain simplifications. We replace $(1 - t)^{\frac{1}{2}}$ on the right by its minimum value $\sqrt{0.35}$ and replace $\xi + R$ on the left by its maximum value, obtaining

$$\begin{aligned} \frac{1}{2}(8 - 7t - 90t^2 + 189t^3) \left(\pi + 2 \tan^{-1} \sqrt{\frac{6t-1}{1-t}} - 2\sqrt{6} \tan^{-1} \sqrt{\frac{6t-1}{6-6t}} \right) \\ \leq (15t^2 + 9t - 4)(6t - 1)^{\frac{1}{2}} \sqrt{0.35}. \end{aligned}$$

A computation shows that the strict inequality holds at $t = \frac{4}{9}$. The right side of this inequality increases with t . On the left side the first quantity increases with t while the second decreases with increasing t . Thus the second quantity

on the left has its maximum value for $t = \frac{4}{9}$ and this value is less than 0.63. Hence it remains to show that

$$0.63(8 - 7t - 90t^2 + 189t^3) \leq \sqrt{0.35} (6t - 1)^{\frac{1}{2}}(15t^2 + 9t - 4).$$

This inequality holds at $t = \frac{4}{9}$. The slope of the left hand member is smaller than the slope of the right side in the range $\frac{4}{9} \leq t \leq 0.65$. Hence the inequality holds throughout this range. Proceeding in the reverse direction from this last inequality we strengthen the inequality at each step and hence (11) is proved.

Since the inequality is strict throughout the range in question it is clear that the result holds in an interval for β about the value 2.5. This means that the ratio of specific heats $\gamma = c_p/c_v$ need not be exactly 1.4 but may lie in a range about 1.4. It would be interesting to investigate how large this range could be and still have the preceding results hold. This would involve determining the exact range in which inequality (10) holds with $K(\eta)$ given by (5).

The existence of a solution of (6) subject to the boundary conditions described in Section 2 can be assured under somewhat more restrictive conditions. It has been shown in [6] that if the curve Γ in the upper half-plane coincides with a *normal curve* (see [6] for the definition of normal curve) of equation (6) in some (arbitrarily small) neighborhood of its end-points then the Fredholm alternative holds. That is, we have

Theorem 2. *Let D be the domain described above and suppose Γ coincides with the normal curve of (6) in some neighborhood of the origin and $P(\xi_0, 0)$. Then either there exists a unique quasi-regular solution of (6) with prescribed values on Γ and γ_2 or there exist (non-identically vanishing) solutions of (6) which vanish on Γ and γ_2 .*

This theorem is proved in [6]. If it is combined with the uniqueness theorem established for equation (6) it yields an existence theorem (in the hodograph plane) for the transonic flow out of a nozzle.

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