The Differential Equations of the Stream Lines for Compressible Gas Flow

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1. Introduction. In the following paper¹ we have derived the differential equations giving the direct determination of the stream line pattern for the case of the steady flow of a compressible gas in a plane under the assumption that the viscosity and thermal conductivity are zero. This is accomplished by the successive elimination of the pressure and the density by which we obtain a system of two equations in the velocity alone. Following ERICKSEN² the velocity components are now replaced by the inclination ω of the stream lines and the quantity $G = m^2 - 1$ where m is the local mach number. At each stage of this process it is observed that the equations obtained are fully equivalent to the original system governing the flow of the gas.

The condition of integrability of the above system is a cubic equation in the quantity G with coefficients which are scalar differential functions of the inclination ω . Hence G can be expressed as an algebraic function of ω and its derivatives. By elimination of G we arrive at a system of two non-linear differential equations in the single dependent variable ω . However, these equations, as shown in §10, are not fully independent. Either equation can be taken as the equation to determine the stream line pattern after which the other equation will effectively impose only an additional boundary condition.

The solution of the Cauchy problem for the stream line pattern is given in §12. Characteristic curves are discussed and determined in §13 and these turn out, as might be expected, to be the stream lines and those curves making an angle μ , where $\sin \mu = 1/m$, with the stream lines. Finally we have given the formal solution of the problem of determining the stream lines when the data

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² J. L. ERICKSEN, *On the uniqueness of ideal gas flows with given stream line patterns*, Bulletin of the Technical University of Istanbul, vol. 6, 1953, pp. 1-5.

is given along the contour of an obstacle or airfoil. The solution obtained, which is discussed in §14, is quite distinct from the solution of the general Cauchy problem since the contour is a stream line characteristic and hence is not a curve on which the data can be assigned from the standpoint of the Cauchy theory.

2. The basic differential equations. The differential system which is basic in the discussion of the type of flow considered in §1 is the following

$$(2.1) \quad p_{,\alpha} + \rho u_{\sigma} u_{\alpha,\sigma} = 0,$$

$$(2.2) \quad \rho_{,\alpha} u_{\alpha} + \rho u_{\alpha,\alpha} = 0,$$

$$(2.3) \quad \rho u_{\alpha,\beta} u_{\alpha} u_{\beta} - \gamma p u_{\sigma,\sigma} = 0,$$

where p denotes the pressure, ρ the density and the u_{α} are the components of the flow velocity. Also the quantity γ occurring in (2.3) is the usual gas constant defined as the ratio of the two specific heats. Equations (2.1) are the equations of motion, (2.2) the equation of continuity and (2.3) is a form of the condition expressing the fact that entropy is constant along stream lines. Finally we mention to avoid any possible confusion that the above system is referred to rectangular coordinates x^{α} so that the "comma" appearing in these equations denotes partial differentiation with respect to these coordinates.

From the above equations one can deduce that

$$(2.4) \quad \frac{\gamma p}{(\gamma - 1)\rho} + \frac{1}{2}v^2 = H, \quad v^2 = u_{\sigma} u_{\sigma},$$

in which the quantity H is constant along stream lines but will, in general, vary from stream line to stream line. It is assumed in the following discussion, however, that H is an *absolute constant*, i.e., that H has the same constant value over the entire field of flow. This situation exists, for example, when an airfoil is placed in a uniform supersonic flow since the value of H is not affected in fact by the occurrence of shocks. The assumption that H has the same constant value everywhere is valid in many of the important flows that are treated in gas dynamics.

3. Elimination of the pressure. Solving (2.4) for p we have

$$(3.1) \quad p = \frac{\gamma - 1}{2\gamma} (2H - v^2)\rho.$$

By means of (3.1) we can now eliminate p from (2.3). Also by differentiation of (3.1) we can eliminate the derivatives $p_{,\alpha}$ from (2.1). We thus obtain the following equations

$$(3.2) \quad (\gamma - 1)(2H - v^2)\rho_{,\alpha} = 2(\gamma - 1)\rho u_{\sigma} u_{\sigma,\alpha} - 2\gamma\rho u_{\sigma} u_{\alpha,\sigma},$$

$$(3.3) \quad u_{\alpha,\beta} u_\alpha u_\beta - \frac{\gamma - 1}{2} (2H - v^2) u_{\sigma,\sigma} = 0.$$

Equations (3.2) and (3.3) form a system of three equations in three dependent variables ρ and u_α . A solution $\rho = \rho(x)$ and $u_\alpha = u_\alpha(x)$ of this system can be used to determine the quantity p from (3.1) which thus becomes a valid relation. Now from (3.1) and (3.3) we can infer (2.3). Likewise differentiation of (3.1) and elimination of the derivatives $\rho_{,\alpha}$ from the resulting equations and (3.2) leads to the equations (2.1). Next multiply (3.2) by u_α and sum on the repeated index α to obtain

$$(3.4) \quad (\gamma - 1)(2H - v^2) \rho_{,\alpha} u_\alpha = -2\rho u_{\alpha,\beta} u_\alpha u_\beta.$$

By combining (3.4) with (3.3) we are led to the continuity equation (2.2). *Hence the system composed of (3.2) and (3.3) is equivalent to the original system of equations (2.1), (2.2), and (2.3) under the assumption that H is an absolute constant.*

4. Elimination of the density. To eliminate the density ρ from the equations (3.2) it will be helpful to think of these equations as written in the form

$$(4.1) \quad (\log \rho)_{,\alpha} = \frac{2(\gamma - 1)u_\sigma u_{\sigma,\alpha} - 2\gamma u_\sigma u_{\alpha,\sigma}}{(\gamma - 1)(2H - v^2)}.$$

After some reduction we find that the condition of integrability of (4.1) can be written

$$(4.2) \quad \phi_{,\sigma} u_\sigma + \gamma \phi u_{\sigma,\sigma} = 0,$$

where we have put $\phi = u_{1,2} - u_{2,1}$. Now suppose that $u_\alpha = u_\alpha(x)$ is a solution of the system (3.3) and (4.2). Inserting these values of u_α into (4.1) these equations, considered as equations in the single dependent variable ρ , will be *completely integrable* since the integrability condition (4.2) will be satisfied identically in the independent variable x^α . Hence (4.1) admits a solution $\rho = \rho(x)$ in accordance with the usual existence theorem for completely integrable systems. When this result is combined with the result at the end of §3 we now see that *the system of two equations (3.3) and (4.2) in the two dependent variables u_α is equivalent to the basic system of equations given in §2 for the flow of the gas.*

5. A condition of integrability involving directional derivatives. Denote by σ the arc length along any stream line and let n denote arc length along the normal trajectories to the stream lines. Also let λ be a unit vector tangent to the stream lines and having the direction of the flow. We suppose that the arc length σ increases algebraically along the stream lines in the direction given by λ . Correspondingly let ν be a unit vector tangent to the normal trajectories and such that the direction of ν coincides with the direction in which the arc

length n increases algebraically. It is assumed that the direction of ν is so chosen that the vector pair λ, ν has the same orientation as the x^1, x^2 coordinate axes.

Now if $f(x)$ is any twice differentiable function it can be shown that³

$$(5.1) \quad \frac{d^2 f}{dn d\sigma} - \frac{d^2 f}{d\sigma dn} = \frac{df}{dn} \frac{d\omega}{dn} + \frac{df}{d\sigma} \frac{d\omega}{d\sigma}.$$

If we denote by K the curvature of the stream lines and by $\bar{K}$ the curvature of the normal trajectories we have by definition

$$(5.2) \quad K = \frac{d\omega}{d\sigma}; \quad \bar{K} = \frac{d(\omega + \pi/2)}{dn} = \frac{d\omega}{dn}.$$

Hence the condition of integrability (5.1) can also be written

$$(5.3) \quad \frac{d^2 f}{dn d\sigma} - \frac{d^2 f}{d\sigma dn} = \frac{df}{dn} \bar{K} + \frac{df}{d\sigma} K.$$

For purpose of reference in the following discussion let us here observe certain simple relations between the components of the above vectors λ and ν and the function $f(x)$. Thus, for example, we have

$$(5.4) \quad \frac{df}{d\sigma} = f_{,\alpha} \lambda_\alpha; \quad \frac{df}{dn} = f_{,\alpha} \nu_\alpha.$$

There are also the relations

$$(5.5) \quad \lambda_1 = \cos \omega = \nu_2; \quad \lambda_2 = \sin \omega = -\nu_1,$$

$$(5.6) \quad u_1 = v \cos \omega = v \lambda_1; \quad u_2 = v \sin \omega = v \lambda_2,$$

where the quantity v , which has appeared in previous formulas, is the magnitude of the velocity. By coordinate differentiation of (5.5) we obtain a set of relations which can readily be written in the compact form

$$(5.7) \quad \lambda_{\alpha,\beta} = \omega_{,\beta} \nu_\alpha; \quad \nu_{\alpha,\beta} = -\omega_{,\beta} \lambda_\alpha.$$

In writing the above and following relations no distinction is made between covariant and contravariant indices, which is permissible since we have based our discussion on the use of rectangular coordinate systems (see §1).

6. Some auxiliary formulae. For the ideal gas under consideration the quantity $\gamma p/\rho = c^2$ where c is the velocity of sound. Hence (2.4) can be written

$$(6.1) \quad c^2 = \frac{\gamma - 1}{2} (2H - v^2).$$

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³ See, T. Y. THOMAS, *The determination of pressure on curved bodies behind shocks*, Communications on Pure and Applied Mathematics, vol. **3**, 1950, p. 122.

Also, introducing the local mach number m defined by v/c , we have from (6.1) that

$$(6.2) \quad v^2 = \frac{2H(\gamma - 1)m^2}{2 + (\gamma - 1)m^2}.$$

Putting $G = m^2 - 1$ we obtain a useful modification of the relation (6.2), namely

$$(6.3) \quad v^2 = \frac{2H(\gamma - 1)(G + 1)}{(\gamma - 1)G + \gamma + 1}.$$

We shall also have occasion to use the two relations which can be derived from (6.3) by directional differentiation along the stream lines and their normal trajectories. These relations are

$$(6.4) \quad \frac{dv}{d\sigma} = \frac{2H(\gamma - 1)}{v[(\gamma - 1)G + \gamma + 1]^2} \frac{dG}{d\sigma},$$

$$(6.5) \quad \frac{dv}{dn} = \frac{2H(\gamma - 1)}{v[(\gamma - 1)G + \gamma + 1]^2} \frac{dG}{dn},$$

where, as in §5, we denote by σ and n the arc lengths along the stream lines and the normal trajectories respectively.

Now consider the two relations (5.6) which can be written in the condensed form $u_\alpha = v\lambda_\alpha$. By coordinate differentiation of these relations and use of (5.7) we obtain

$$(6.6) \quad u_{\alpha,\beta} = v_{,\beta} \lambda_\alpha + v\lambda_{\alpha,\beta} = v_{,\beta} \lambda_\alpha + v\omega_{,\beta} \nu_\alpha.$$

From (6.6) we further deduce, in an obvious manner, that

$$(6.7) \quad u_{\sigma,\sigma} = v_{,\sigma} \lambda_\sigma + v\omega_{,\sigma} \nu_\sigma = \frac{dv}{d\sigma} + v\bar{K},$$

where $\bar{K}$ is the curvature of the normal trajectories, and

$$\begin{aligned} u_{\alpha,\beta} u_\alpha u_\beta &= v_{,\beta} \lambda_\alpha u_\alpha u_\beta + v\omega_{,\beta} \nu_\alpha u_\alpha u_\beta \\ &= v^2 v_{,\beta} \lambda_\alpha \lambda_\alpha \lambda_\beta + v^3 \omega_{,\beta} \nu_\alpha \lambda_\alpha \lambda_\beta \\ &= v^2 v_{,\beta} \lambda_\beta = v^2 \frac{dv}{d\sigma}. \end{aligned} \quad (6.8)$$

From the definition of the quantity ϕ given in §4 we have also that

$$\phi = (v_{,2} \lambda_1 - v_{,1} \lambda_2) + v(\omega_{,2} \nu_1 - \omega_{,1} \nu_2),$$

using (6.6), and hence, using (5.5), we obtain

$$\begin{aligned}
 \phi &= v_{,2} v_2 + v_{,1} v_1 - v(\omega_{,2} \lambda_2 + \omega_{,1} \lambda_1) \\
 (6.9) \quad &= \frac{dv}{dn} - v \frac{d\omega}{d\sigma} = \frac{dv}{dn} - vK,
 \end{aligned}$$

where K is the curvature of the stream lines.

Finally we shall derive an equation for the quantity $\phi_{,\alpha} u_\alpha$. We have

$$\begin{aligned}
 \phi_{,\alpha} u_\alpha &= v \phi_{,\alpha} \lambda_\alpha = v \frac{d\phi}{d\sigma} \\
 (6.10) \quad &= v \left(\frac{d^2 v}{d\sigma dn} - K \frac{dv}{d\sigma} - v \frac{dK}{d\sigma} \right),
 \end{aligned}$$

on account of (6.9). But from (5.3) we have

$$\frac{d^2 v}{d\sigma dn} = \frac{d^2 v}{dn d\sigma} - \bar{K} \frac{dv}{dn} - K \frac{dv}{d\sigma}.$$

Making this substitution in the right member of (6.10) we now obtain

$$(6.11) \quad \phi_{,\alpha} u_\alpha = v \left(\frac{d^2 v}{dn d\sigma} - 2K \frac{dv}{d\sigma} - \bar{K} \frac{dv}{dn} - v \frac{dK}{d\sigma} \right).$$

7. Introduction of the variables ω and G . We have shown that the differential equations determining the flow can be reduced to a set of two equations in the two dependent variables u_α , namely

$$(7.1) \quad u_{\alpha,\beta} u_\alpha u_\beta - \frac{\gamma - 1}{2} (2H - v^2) u_{\sigma,\sigma} = 0,$$

$$(7.2) \quad \phi_{,\alpha} u_\alpha + \gamma \phi u_{\sigma,\sigma} = 0.$$

Now from (6.1), (6.7) and (6.8) the equation (7.1) becomes

$$(7.3) \quad v^2 \frac{dv}{d\sigma} - c^2 \left(\frac{dv}{d\sigma} + v \bar{K} \right) = 0,$$

or

$$(7.4) \quad \frac{dv}{d\sigma} = \frac{v \bar{K}}{m^2 - 1} = \frac{v \bar{K}}{G},$$

when we divide (7.3) by c^2 and introduce the local mach number m . Correspondingly we make the substitutions (6.7), (6.9) and (6.11) in (7.2). This gives

$$v \left(\frac{d^2 v}{dn d\sigma} - 2K \frac{dv}{d\sigma} - \bar{K} \frac{dv}{dn} - v \frac{dK}{d\sigma} \right) + \gamma \left(\frac{dv}{dn} - vK \right) \left(\frac{dv}{d\sigma} + v \bar{K} \right) = 0.$$

In this equation we now substitute for the derivative $dv/d\sigma$ the quantity in the right member of (7.4) and for the derivative $d^2 v/dn d\sigma$ the quantity obtained by differentiation of the right member of (7.4). After these substitutions we find

$$(7.5) \quad \frac{d}{dn} \left(\frac{v\bar{K}}{G} \right) - (\gamma + 2) \frac{vK\bar{K}}{G} + (\gamma - 1) \bar{K} \frac{dv}{dn} + \frac{\gamma\bar{K}}{G} \frac{dv}{dn} - v \frac{dK}{d\sigma} - \gamma v K \bar{K} = 0.$$

Let us now replace $dv/d\sigma$ in (7.4) by its value as given by (6.4). Also, after carrying out the differentiation indicated by the first term in (7.5), we substitute for dv/dn in (7.5) the value given by (6.5). After these operations we eliminate v^2 from the resulting equations by the substitution (6.3). The equations so obtained can be written in the form

$$(7.6) \quad \frac{dG}{d\sigma} = \frac{G + 1}{G} [(\gamma - 1)G + \gamma + 1]\bar{K},$$

$$(7.7) \quad \frac{dG}{dn} = \frac{-G(G + 1)}{\bar{K}} \left[G \left(\frac{dK}{d\sigma} + \gamma K \bar{K} \right) - \frac{d\bar{K}}{dn} + (\gamma + 2)K\bar{K} \right].$$

Equations (7.6) and (7.7) can be regarded as a system of two equations in the two dependent variables ω and G . Let $\omega(x)$ and $G(x)$, where $G(x) \geq -1$, be a solution of these equations. The function $\omega(x)$ in this solution then provides us with a determination of the stream line pattern. Now it is obvious that the steps by which we obtained the equations (7.6) and (7.7) can be retraced to yield the equations (7.1) and (7.2). It follows from the result at the end of §4 that the system (7.6) and (7.7) is fully equivalent to the basic system of §2. *In other words there will always exist a flow given by a solution of (2.1), (2.2), and (2.3) for which the inclination ω of the stream lines and the quantity G will be the functions $\omega(x)$ and $G(x)$ determined as a solution of (7.6) and (7.7).*

8. The cubic equation in G . We now consider the *condition of integrability* of the system (7.6) and (7.7). This condition is obtained precisely in the following manner. First we differentiate the equations (7.6) and (7.7) with respect to n and σ respectively and then subtract corresponding members of the equations obtained by this differentiation. Application of the general relation (5.3) enables us to eliminate the two second directional derivatives and this is followed by the elimination of the derivatives $dG/d\sigma$ and dG/dn , which will occur in certain terms, by the substitution (7.6) and (7.7). The final result, after cancellation of a factor $G + 1$, which we assume to be different from zero, is a cubic relation in G of the form

$$(8.1) \quad C_0 + C_1 G + C_2 G^2 + C_3 G^3 = 0,$$

where the coefficients C are given by the equations

$$(8.2) \quad C_0 = (\gamma + 1) \left[\frac{d\bar{K}}{dn} - (2\gamma + 3)K\bar{K} \right],$$

$$(8.3) \quad C_1 = (\gamma + 1) \left[2 \frac{d\bar{K}}{dn} - 3 \frac{dK}{d\sigma} - 3(2\gamma + 1)K\bar{K} \right],$$

$$(8.4) \quad C_2 = \frac{d}{d\sigma} \left(\frac{1}{\bar{K}} \frac{d\bar{K}}{dn} \right) + \gamma \frac{d\bar{K}}{dn} - 3(2\gamma + 1) \left(\frac{dK}{d\sigma} + \gamma K\bar{K} \right),$$

$$(8.5) \quad C_3 = -\frac{d}{d\sigma} \left(\frac{1}{\bar{K}} \frac{dK}{d\sigma} \right) - (3\gamma - 1) \frac{dK}{d\sigma} - \gamma(2\gamma - 1)K\bar{K}.$$

9. The completely integrable case. When the stream line pattern, which is given by the function $\omega(x)$, is known the system (7.6) and (7.7) becomes a system for the determination of the single quantity G since the curvatures K and $\bar{K}$ and their directional derivatives appearing in the equations will now be definite functions of the coordinates (see §11). Such a system is said to be *completely integrable* if the condition (8.1) is satisfied identically in the coordinates x^a and the quantity G considered as independent variables. For complete integrability we must therefore have

$$(9.1) \quad C_0 = C_1 = C_2 = C_3 = 0.$$

The completely integrable case has been treated by ERICKSEN² who proved the following result: *A flow for which the condition (9.1) is satisfied is of the source-sink type.* This was proved subject to the assumption $\bar{K} \neq 0$. The converse of this result is immediately evident, that is, if the flow is of source-sink type the condition of complete integrability (9.1) is satisfied. For the assertion that the flow is of the source-sink type implies that the stream lines are straight lines and the normal trajectories are circles. Hence $K = 0$ and $d\bar{K}/dn = 0$. Hence we see immediately from the equations (8.2), $\dots$, (8.5) that all quantities C must vanish.

We shall now prove, subject to the above assumption $\bar{K} \neq 0$, that if the stream lines are straight lines the flow will be of the source-sink type. By our hypothesis $K = 0$. Hence the integrability condition (8.1) becomes

$$(9.2) \quad (\gamma + 1) \frac{d\bar{K}}{dn} + 2(\gamma + 1) \frac{d\bar{K}}{dn} G + \left[\frac{d}{d\sigma} \left(\frac{1}{\bar{K}} \frac{d\bar{K}}{dn} \right) + \gamma \frac{d\bar{K}}{dn} \right] G^2 = 0.$$

But from (5.3), we have

$$(9.3) \quad \frac{d\bar{K}}{d\sigma} = \frac{dK}{dn} - K^2 - \bar{K}^2 = -\bar{K}^2, \quad \text{and}$$

$$(9.4) \quad \frac{d}{d\sigma} \frac{d\bar{K}}{dn} = \frac{d}{dn} \frac{d\bar{K}}{d\sigma} - K \frac{d\bar{K}}{d\sigma} - \bar{K} \frac{d\bar{K}}{dn} = -3\bar{K} \frac{d\bar{K}}{dn},$$

when use is made of the first of these relations. Hence using (9.3) and (9.4) we find

$$(9.5) \quad \frac{d}{d\sigma} \left(\frac{1}{\bar{K}} \frac{d\bar{K}}{dn} \right) = \frac{1}{\bar{K}} \frac{d}{d\sigma} \frac{d\bar{K}}{dn} - \frac{1}{\bar{K}^2} \frac{d\bar{K}}{d\sigma} \frac{d\bar{K}}{dn} = -2 \frac{d\bar{K}}{dn}.$$

Making the substitution (9.5) in (9.2) it now follows that

$$(9.6) \quad \gamma + 1 + 2(\gamma + 1)G + (\gamma - 2)G^2 = 0,$$

provided $d\bar{K}/dn \neq 0$. From (9.6) we have $G = \text{const.}$ Hence we see from (7.7) that $G(G + 1) d\bar{K}/dn = 0$. But neither $G = 0$ nor $G = -1$ is a solution of (9.6). Hence we must have $d\bar{K}/dn = 0$ or, in other words, the flow is of the source-sink type as above stated.

10. The differential equations of the stream lines. Since the quantity G is a solution of (8.1) it can be expressed as an algebraic function of the coefficients C and hence as an algebraic function of the curvatures K and $\bar{K}$ and the directional derivatives of these curvatures which appear in the right members of the relations (8.2), $\dots$, (8.5). We denote the function G , when so expressed, by writing $E(\omega)$. The values of the above arguments of $E(\omega)$ must, of course, be subject to the condition that $E(\omega)$ shall be *real*; however, since there is always one real solution of a cubic equation this reality condition can always be satisfied.

Putting $G = E(\omega)$ in the two relations (7.6) and (7.7) we now have

$$(10.1) \quad \frac{dE}{d\sigma} = \frac{E + 1}{E} [(\gamma - 1)E + \gamma + 1]\bar{K},$$

$$(10.2) \quad \frac{dE}{dn} = -\frac{E(E + 1)}{\bar{K}} \left[E \left(\frac{dK}{d\sigma} + \gamma K \bar{K} \right) - \frac{d\bar{K}}{dn} + (\gamma + 2)K\bar{K} \right].$$

These are two differential equations in the single dependent variable ω and they will ostensibly give an overdetermination of ω . We shall now investigate this situation.

Let us define the differential expressions A and B by writing

$$(10.3) \quad A = \frac{dE}{d\sigma} - \frac{E + 1}{E} [(\gamma - 1)E + \gamma + 1]\bar{K},$$

$$(10.4) \quad B = \frac{dE}{dn} + \frac{E(E + 1)}{\bar{K}} \left[E \left(\frac{dK}{d\sigma} + \gamma K \bar{K} \right) - \frac{d\bar{K}}{dn} + (\gamma + 2)K\bar{K} \right].$$

By differentiation of (10.3) and (10.4), followed by subtraction, we now have

$$(10.5) \quad \begin{aligned} \frac{dA}{dn} - \frac{dB}{d\sigma} &= \frac{d}{dn} \frac{dE}{d\sigma} - \frac{d}{d\sigma} \frac{dE}{dn} + \dots \\ &= K \frac{dE}{d\sigma} + \bar{K} \frac{dE}{dn} + \dots, \end{aligned}$$

when use is made of the relation (5.3). When we carry out the operation indicated by the dots in (10.5), eliminate the derivatives $dE/d\sigma$ and dE/dn by

the substitutions (10.3) and (10.4) and finally make use of the fact that $G = E(\omega)$ is a solution of (8.1) we arrive at an equation having the form

$$(10.6) \quad \frac{dA}{dn} - \frac{dB}{d\sigma} = PA + QB,$$

where P and Q depend on the curvatures K and $\bar{K}$ and their directional derivatives.

The differential relations (10.1) and (10.2) can be expressed by writing $A = 0$ and $B = 0$ respectively. Now suppose we have a solution $\omega(x)$ of the differential equation $A = 0$ in a region R such that $B = 0$ on a line L which cuts all the stream lines in R , these stream lines being determined by the solution function $\omega(x)$. Since $A = 0$ in R the relation (10.6) becomes

$$(10.7) \quad \frac{dB}{d\sigma} = -QB$$

in R . Now on each stream line (10.7) is an ordinary linear differential equation for B since the quantity Q is determined in R by the solution $\omega(x)$. Hence B is determined uniquely on any stream line when its value is known at a point of this line.⁴ But $B = 0$ by hypothesis at the point where the stream line cuts L . Hence $B = 0$ along each stream line and hence $B = 0$ in R .

We can now state the following result. *If $\omega(x)$ is a solution of the differential equation (10.1) in a region R such that $\omega(x)$ satisfies (10.2) on a line L in R which cuts all stream lines, as determined by the function $\omega(x)$, then the relation (10.2) is satisfied over R . Similarly we have that a solution $\omega(x)$ of (10.2) in a region R which satisfies (10.1) on a line L cutting all normal trajectories of the stream lines in R will satisfy the differential equation (10.1) over R .* Thus we see that either of the differential equations (10.1) or (10.2) essentially determines the stream line pattern. After the equation (10.1) or (10.2) has been chosen the other equation supplies only a supplementary boundary condition.

11. Coordinate form of the differential equations. The quantities K , $\bar{K}$ and their directional derivatives can be expressed in terms of the function ω and its coordinate derivatives. Substitution of these expressions into (10.1) and (10.2) will lead to the coordinate form of these differential equations.

From the definition of the curvatures K and $\bar{K}$ we have

$$(11.1) \quad K = \omega_{,\alpha} \lambda_{\alpha}; \quad \bar{K} = \omega_{,\alpha} \nu_{\alpha},$$

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⁴ It is assumed implicitly that the conditions are such that Q is continuous in R so that the usual existence theorem for ordinary linear differential equations is applicable. An examination of the explicit form of the expressions for P and Q shows that it is sufficient for the continuity of these quantities that the curvatures K and $\bar{K}$ be continuous, non-vanishing, and have continuous first directional derivatives in the region R .

where the unit vectors λ and ν are defined in §5. Differentiating (11.1) and using (5.7) we find

$$(11.2) \quad \frac{dK}{d\sigma} = \omega_{,\alpha\beta} \lambda_\alpha \lambda_\beta + \omega_{,\alpha} \omega_{,\beta} \nu_\alpha \lambda_\beta,$$

$$(11.3) \quad \frac{dK}{dn} = \omega_{,\alpha\beta} \lambda_\alpha \nu_\beta + \omega_{,\alpha} \omega_{,\beta} \nu_\alpha \nu_\beta,$$

$$(11.4) \quad \frac{d\bar{K}}{d\sigma} = \omega_{,\alpha\beta} \nu_\alpha \lambda_\beta - \omega_{,\alpha} \omega_{,\beta} \lambda_\alpha \lambda_\beta,$$

$$(11.5) \quad \frac{d\bar{K}}{dn} = \omega_{,\alpha\beta} \nu_\alpha \nu_\beta - \omega_{,\alpha} \omega_{,\beta} \lambda_\alpha \nu_\beta.$$

Now the equations (11.1) with the λ_α and the ν_α defined as functions of ω by (5.5) give the expressions for K and $\bar{K}$ in terms of ω and its coordinate derivatives. Such an expression will be referred to as a *coordinate expression* for brevity. Correspondingly the coordinate expressions for the directional derivatives of K and $\bar{K}$ are given by (11.2), $\dots$, (11.5). This process can be continued to yield the coordinate expressions for the directional derivatives of K and $\bar{K}$ of any order.

To determine the explicit form of the relations (10.1) and (10.2) in the curvatures K and $\bar{K}$ and their directional derivatives let us consider the identity,

$$(11.6) \quad C_0 + C_1 E + C_2 E^2 + C_3 E^3 = 0.$$

By differentiation of (11.6) with respect to the arc lengths σ and n we have

$$(11.7) \quad J \frac{dE}{d\sigma} + S = 0; \quad J \frac{dE}{dn} + T = 0,$$

where

$$(11.8) \quad J = C_1 + 2C_2 E + 3C_3 E^2,$$

$$(11.9) \quad S = \frac{dC_0}{d\sigma} + E \frac{dC_1}{d\sigma} + E^2 \frac{dC_2}{d\sigma} + E^3 \frac{dC_3}{d\sigma},$$

$$(11.10) \quad T = \frac{dC_0}{dn} + E \frac{dC_1}{dn} + E^2 \frac{dC_2}{dn} + E^3 \frac{dC_3}{dn}.$$

The equations (11.8), (11.9) and (11.10) enable us to determine the quantities J , S and T in terms of the K , $\bar{K}$ and their directional derivatives when use is made of the relations (8.2), $\dots$, (8.5). Hence we can determine the derivatives $dE/d\sigma$ and dE/dn in terms of the two curvatures and their directional derivatives by (11.7). Making these substitutions the equations (10.1) and (10.2) become

$$(11.11) \quad S = \frac{E+1}{-E} [(\gamma-1)E + \gamma + 2] \bar{K} J,$$

$$(11.12) \quad T = \frac{E(E+1)}{\bar{K}} \left[E \left(\frac{dK}{d\sigma} + \gamma K \bar{K} \right) - \frac{d\bar{K}}{dn} + (\gamma+2) K \bar{K} \right] J.$$

Now it is immediately seen that the directional derivatives of the curvatures K and $\bar{K}$ of highest order in (11.11) and (11.12) result from the derivatives of the quantities C_2 and C_3 in the right members of the expressions (11.9) and (11.10) for S and T . But from (8.4) and (8.5) we have

$$(11.13) \quad \frac{dC_2}{d\sigma} = \frac{1}{\bar{K}} \frac{d^3 \bar{K}}{d\sigma^2 dn} = \frac{1}{\bar{K}} \omega_{,\alpha\beta\gamma\delta} \nu_\alpha \nu_\beta \lambda_\gamma \lambda_\delta,$$

$$(11.14) \quad \frac{dC_2}{dn} = \frac{1}{\bar{K}} \frac{d^3 \bar{K}}{d\sigma dn^2} = \frac{1}{\bar{K}} \omega_{,\alpha\beta\gamma\delta} \nu_\alpha \nu_\beta \nu_\gamma \lambda_\delta,$$

$$(11.15) \quad \frac{dC_3}{d\sigma} = -\frac{1}{\bar{K}} \frac{d^3 K}{d\sigma^3} = -\frac{1}{\bar{K}} \omega_{,\alpha\beta\gamma\delta} \lambda_\alpha \lambda_\beta \lambda_\gamma \lambda_\delta,$$

$$(11.16) \quad \frac{dC_3}{dn} = -\frac{1}{\bar{K}} \frac{d^3 K}{dn d\sigma^2} = -\frac{1}{\bar{K}} \omega_{,\alpha\beta\gamma\delta} \lambda_\alpha \lambda_\beta \lambda_\gamma \nu_\delta,$$

where we have omitted terms involving derivatives of order lower than those which have been written explicitly. In deriving the above relations (11.13), $\dots$, (11.16) it is helpful to observe from (5.3) that the order of the directional differentiation is immaterial as far as these highest derivatives are concerned.

Using the above relations (11.13), $\dots$, (11.16) the equations (11.11) and (11.12) now become

$$(11.17) \quad (E\lambda_\alpha \lambda_\beta - \nu_\alpha \nu_\beta) \lambda_\gamma \lambda_\delta \omega_{,\alpha\beta\gamma\delta} = \dots,$$

$$(11.18) \quad (E\lambda_\alpha \lambda_\beta - \nu_\alpha \nu_\beta) \lambda_\gamma \nu_\delta \omega_{,\alpha\beta\gamma\delta} = \dots,$$

where the dots represent terms involving derivatives of order lower than those which appear in the left members of the equations.

The above procedure enables us to obtain the complete explicit form of the equations (10.1) and (10.2) of the stream lines in terms of the inclination ω and its coordinate derivatives. However, the form (11.17) and (11.18) of these equations, in which the terms containing the coordinate derivatives of ω of highest order only are exhibited, will suffice for the purpose of the following discussion.

12. The Cauchy problem. The problem of finding a unique analytic solution of a differential equation, *e.g.* the solution $\omega(x)$ of (11.17) or (11.18), such that the solution function and certain of its derivatives are assigned analytic functions of the parameter of a regular analytic curve L is commonly referred to as the *Cauchy problem*. To solve this problem for the equations (11.17) and (11.18),

here under consideration, it is convenient to suppose that the curve L is defined by an equation $\phi(x^1, x^2) = 0$ where $\phi(x)$ is an analytic function and then to effect a coordinate transformation

$$(12.1) \quad y^1 = \phi(x^1, x^2); \quad y^2 = \psi(x^1, x^2),$$

where $\psi(x)$ is also an analytic function of the coordinates x^α . The statement that (12.1) is a coordinate transformation implies that $\phi(x)$ and $\psi(x)$ are functionally independent. This condition can be satisfied, in particular, by choosing the two congruences $\phi(x) = \text{const.}$ and $\psi(x) = \text{const.}$ to be orthogonal.

In the y coordinate system defined by (12.1) the curve L has the equation $y^1 = 0$ and the coordinate y^2 can be taken as the parameter for the specification of the data on L . We now have relations of the type

$$(12.2) \quad \left\{ \begin{array}{l} \omega_{,\alpha} = \frac{\partial \omega}{\partial y^i} y_{,\alpha}^i, \\ \omega_{,\alpha\beta} = \frac{\partial^2 \omega}{\partial y^i \partial y^j} y_{,\alpha}^i y_{,\beta}^j + \frac{\partial \omega}{\partial y^i} y_{,\alpha\beta}^i, \\ \dots \\ \dots \\ \dots \end{array} \right.$$

Making the substitutions (12.2) the equations (11.17) and (11.18) become

$$(12.3) \quad [y_{,\alpha}^i y_{,\beta}^j y_{,\gamma}^k y_{,\delta}^m (E \lambda_\alpha \lambda_\beta - \nu_\alpha \nu_\beta) \lambda_\gamma \lambda_\delta] \frac{\partial^4 \omega}{\partial y^i \partial y^j \partial y^k \partial y^m} = \dots,$$

$$(12.4) \quad [y_{,\alpha}^i y_{,\beta}^j y_{,\gamma}^k y_{,\delta}^m (E \lambda_\alpha \lambda_\beta - \nu_\alpha \nu_\beta) \lambda_\gamma \nu_\delta] \frac{\partial^4 \omega}{\partial y^i \partial y^j \partial y^k \partial y^m} = \dots,$$

where the dots denote expressions which involve derivatives of ω with respect to the coordinates y^α of order lower than those appearing in the left members of the equations.

Let us now define the following quantities:

$$(12.5) \quad \omega; \quad \frac{\partial \omega}{\partial y^1}; \quad \frac{\partial^2 \omega}{\partial y^1 \partial y^1}; \quad \frac{\partial^3 \omega}{\partial y^1 \partial y^1 \partial y^1}$$

on the curve L as analytic functions of the variable y^2 . By differentiation of these quantities with respect to y^2 it is seen that all derivatives of ω of orders one to four will be known with the single exception of the derivative

$$(12.6) \quad \frac{\partial^4 \omega}{\partial y^1 \partial y^1 \partial y^1 \partial y^1} = \frac{\partial^4 \omega}{\partial \phi^4}.$$

This derivative must now be determined by the equation (12.3) or (12.4). For this purpose we write (12.3) and (12.4) in the form

$$(12.7) \quad (\phi, \alpha \lambda_\alpha)^2 [E(\phi, \beta \lambda_\beta)^2 - (\phi, \beta \nu_\beta)^2] \frac{\partial^4 \omega}{\partial \phi^4} = R,$$

$$(12.8) \quad (\phi, \alpha \lambda_\alpha)(\phi, \sigma \nu_\sigma) [E(\phi, \beta \lambda_\beta)^2 - (\phi, \beta \nu_\beta)^2] \frac{\partial^4 \omega}{\partial \phi^4} = S,$$

in which (12.6) is the only derivative appearing in the left members of the equations.

Now suppose that the coefficient

$$(12.9) \quad (\phi, \alpha \lambda_\alpha)^2 [E(\phi, \beta \lambda_\beta)^2 - (\phi, \beta \nu_\beta)^2]$$

of the derivative $\partial^4 \omega / \partial \phi^4$ in the left member of (12.7) does not vanish at any point of L for the assigned data (12.5). Suppose further that the quantity

$$(12.10) \quad \frac{R}{(\phi, \alpha \lambda_\alpha)^2 [E(\phi, \beta \lambda_\beta)^2 - (\phi, \beta \nu_\beta)^2]},$$

which is equal to the derivative $\partial^4 \omega / \partial \phi^4$ from (12.7) is an analytic function in the neighborhood of the values of its arguments at each point of L when these values are determined by the data (12.5). Then (12.7) will determine the derivative $\partial^4 \omega / \partial \phi^4$ on L and it is easily seen moreover that the derivatives of ω of higher order can be determined successively by differentiation of (12.7). The power series for the function $\omega(\phi, \psi)$ will therefore have a unique formal determination at each point of L and by the Cauchy-Kowalewski result these series will converge and will thus yield an analytic solution of the differential equation (12.3) in the neighborhood of the curve L .

As shown in §10 the above solution ω of (12.3) will determine a possible stream line pattern if the relation (12.4) is satisfied along L . To secure this condition we eliminate the derivative $\partial^4 \omega / \partial \phi^4$ from the two equations (12.7) and (12.8) to obtain

$$(12.11) \quad (\phi, \alpha \nu_\alpha) R = (\phi, \alpha \lambda_\alpha) S.$$

This relation imposes a condition on the choice of the data (12.5) on the curve L . If the data is chosen to satisfy the condition (12.11) on L , then (12.8) will be satisfied on L , and it will follow from the result in §10 that the above solution ω of the equation (12.7) will also satisfy (12.8) in the neighborhood of L and hence will give the stream lines of an actual gas flow.

By transforming the data (12.5) subject to the condition (12.11) from the y to the x coordinate system we can obtain the direct conditions on the function ω and its x coordinate derivatives for the Cauchy-Kowalewski solution of the system (11.17) and (11.18). A further transformation by means of equations of the type (11.1), (11.2) and (11.3) will of course enable us to state these conditions in terms of directional derivatives.

13. Characteristics. Let us now suppose that the quantity (12.9) vanishes on the curve L in §12 so that the conditions for the solution of the Cauchy problem do not apply. Such a curve is called a *characteristic* relative to the assigned data.

If the characteristic condition is satisfied by the vanishing of the quantity $\phi_{,\alpha} \lambda_\alpha$ on L it is immediately evident that L is a stream line.

We now examine the case for which L is a characteristic in virtue of the fact that

$$(13.1) \quad E(\phi_{,\alpha} \lambda_\alpha)^2 - (\phi_{,\alpha} \nu_\alpha)^2 = 0.$$

Denote by μ the angle (positive or negative) which L makes with an intersecting stream line. Now the angle determined by the vector with components $\phi_{,\alpha}$, which is normal to L , and the unit vector ν normal to the stream lines, is equal to μ . Hence

$$(13.2) \quad \cos^2 \mu = \frac{(\phi_{,\alpha} \nu_\alpha)^2}{\phi_{,1}^2 + \phi_{,2}^2},$$

from the well known formula for the cosine of the angle determined by two vectors. Similarly

$$(13.3) \quad \sin^2 \mu = \cos^2 \left(\mu + \frac{\pi}{2} \right) = \frac{(\phi_{,\alpha} \lambda_\alpha)^2}{\phi_{,1}^2 + \phi_{,2}^2}.$$

Hence from (13.1), (13.2) and (13.3) we have

$$(13.4) \quad \tan^2 \mu = \frac{(\phi_{,\alpha} \lambda_\alpha)^2}{(\phi_{,\alpha} \nu_\alpha)^2} = \frac{1}{E} = \frac{1}{m^2 - 1},$$

when we introduce the local mach number m . Frequently one writes $\sin \mu = \pm 1/m$ as the equation for the angle μ in place of (13.4). This result shows of course that real characteristics L , other than stream lines, will exist only when the flow is supersonic. *Hence for supersonic flow there exist three congruences of characteristic curves consisting of the stream lines and those curves which make an angle μ , given by $\sin \mu = \pm 1/m$, with the stream lines.*

14. Assignment of data on stream lines. The determination of the stream line pattern when the data is assigned on a stream line L , which is a characteristic by the result of §13, is of especial interest since such a curve may in particular be the contour of a given profile or airfoil in the flow and hence known from the conditions of the problem. In the following discussion we shall assume that L is the contour of a profile so that the curvature K and the successive derivatives of K with respect to arc length will be determined functions on the curve L .

We define the curvature $\bar{K}$ to be an analytic function, not identically zero, of the arc length along L . Let P_0 be a point on L at which $\bar{K} \neq 0$ and assign the value $G_0 \neq 0$ to the quantity G at P_0 ; we suppose $G_0 > -1$. Then G will

have a unique determination as a solution of (7.6) on a segment L^* containing the point P_0 and such that G and $\bar{K}$ do not vanish on L^* . We see that G cannot have the value -1 at any point of L^* since otherwise it would follow from (7.6) that $G = -1$ everywhere on L^* . The special cases for which $\bar{K} = 0$, $G = 0$ or $G = -1$ at P_0 will not be considered in this article.

Now it suffices to solve the equations (7.7) and (8.1); for the discussion in §10 shows that any solution $\omega(x)$ and $G(x)$ of (7.7) and (8.1) in the neighborhood of L^* will automatically satisfy (7.6) if this equation is satisfied on L^* . But this latter condition is met by the above determination on L^* of the quantity G and hence we can confine our attention to the two equations (7.7) and (8.1) in the following discussion.

With $\bar{K}$ and G now known along L^* let us consider the equation (8.1) which can be written in the form

$$(14.1) \quad \frac{d}{d\sigma} \left(\frac{d\bar{K}}{dn} \right) + J \frac{d\bar{K}}{dn} + W = 0,$$

where J and W depend on G , the curvatures K and $\bar{K}$ and the derivative of K with respect to σ . The functions J and W are therefore known analytic functions of σ on L^* . Hence (14.1) determines the quantity $d\bar{K}/dn$ on L^* as an analytic function of the arc length σ of L^* such that $d\bar{K}/dn$ has an assigned value $(d\bar{K}/dn)_0$ at the above point P_0 . Also from (7.7) the derivative dG/dn is now directly determined, without integration, as an analytic function of the arc length σ on L^* .

By differentiation of (8.1) with respect to n and application of the integrability condition (5.3) we obtain an equation which can be written

$$(14.2) \quad \frac{d}{d\sigma} \left(\frac{d^2 \bar{K}}{dn^2} \right) + U \frac{d^2 \bar{K}}{dn^2} + V = 0,$$

in which U and V are known analytic functions of the arc length on L^* in view of the above determinations. From (14.2) we can now find, by integration, the second derivative $d^2 \bar{K}/dn^2$ along L^* when we specify its value $(d^2 \bar{K}/dn^2)_0$ at the point P_0 . Again by differentiation of (7.7) with respect to n the second derivative $d^2 G/dn^2$ is determined along L^* .

It is evident that this process can be continued to yield the derivatives of $\bar{K}$ and G of any order with respect to n . Hence the quantities appearing in the following three sequences

$$(14.3) \quad \begin{cases} K; & \frac{dK}{d\sigma}; & \frac{d^2 K}{d\sigma^2}; & \dots \\ \bar{K}; & \frac{d\bar{K}}{dn}; & \frac{d^2 \bar{K}}{dn^2}; & \dots \\ G; & \frac{dG}{dn}; & \frac{d^2 G}{dn^2}; & \dots \end{cases}$$

may be assumed to be known to us on L^* . Now the determination of the second and third sequences depends on the assigned values

$$(14.4) \quad \left(\frac{d\bar{K}}{dn} \right)_0; \quad \left(\frac{d^2 \bar{K}}{dn^2} \right)_0; \quad \dots$$

at the point P_0 . These quantities (14.4) will now be chosen as follows. Let C be a regular analytic curve normal to L at P_0 and let n denote arc length along C . We suppose that the direction along C in which n increases algebraically is chosen as described in §5. If $\bar{\omega}$ denotes the inclination of C its curvature $\bar{K} = d\bar{\omega}/dn$ at P_0 is assumed to be equal to the value of $\bar{K}$ at P_0 as previously given by the assignment of $\bar{K}$ along L . With this understanding we now choose the quantities in the above sequence (14.3) to be the successive derivatives of the curvature of C at the point P_0 .

Now it is clear that the quantities in the above three sequences (14.3) yield all directional derivatives of ω and G along L^* . By transformations inverse to those of the type (11.1), (11.2) and (11.3) it follows that all coordinate derivatives of ω and G are determined along L^* . We thus obtain the formal power series expansions of ω and G about any point of L^* and from their derivation these series are seen to constitute formal solutions of the equations (7.7) and (8.1) in the neighborhood of L^* and to satisfy the condition (7.6) along L^* . Hence the series likewise satisfy (7.6) in the neighborhood of L^* .

We do not here wish to concern ourselves with the question of the convergence of the above series. However, anticipating the possibility of a proof of convergence, we state the following result. *Let the contour of a profile be a regular analytic curve L and let the quantity $\bar{K}$ be defined along L as an analytic function, not identically zero, of its arc length. Let P_0 be a point of L at which $\bar{K} \neq 0$ and C a regular analytic curve which is normal to L at P_0 , and such that its curvature at P_0 is equal to the assigned value of $\bar{K}$ at this point. Then there will exist stream lines in a neighborhood N of P_0 , corresponding to a compressible gas flow, for which the quantity G will assume a given value G_0 at P_0 such that $-1 < G_0 \neq 0$. Moreover these stream lines will be defined by a uniquely determined analytic function $\omega(x)$ and will be such that (1) the segment of L lying in the neighborhood N will be a stream line, (2) the curvatures of the normal trajectories will have the above given values $\bar{K}$ on the segment of L in N , and (3) the segment of the curve C in N will be normal to the stream lines.*

We have omitted from the above italicized statement the fact that the function $G(x)$ is likewise determined by our consideration because we are primarily concerned in this paper with results of a geometrical nature. However the above result is not entirely of this character due to the occurrence of the constant G_0 which appears as a dynamical parameter in the determination of the stream line pattern.